



Existence of Extremals for a Fourier Restriction Inequality on the One-Sheeted Hyperboloid

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Abstract

We prove the existence of functions that extremize the endpoint L^2 to L^4 adjoint Fourier restriction inequality on the one-sheeted hyperboloid in Euclidean space \mathbb{R}^4 and that, taking symmetries into consideration, any extremizing sequence has a subsequence that converges to an extremizer.

Keywords Sharp Fourier restriction theory · Sharp Strichartz estimates · Maximizers · Convolution of singular measures · Concentration-compactness

Mathematics Subject Classification 42B10 · 42B37 · 51M16

1 Introduction

In seminal paper [43] R. Strichartz addressed the adjoint restriction problem of the Fourier transform to $d - 1$ dimensional quadric submanifolds of Euclidean space \mathbb{R}^d , establishing the necessary and sufficient conditions on p such that an $L^2 \rightarrow L^p$ estimate holds. Recently, there has been interest in studying the existence of extremizers and the sharp $L^2 \rightarrow L^p$ estimates for adjoint Fourier restriction operators and progress has been made in the case of quadric curves and surfaces: the paraboloid and parabola [22, 28], the cone [5, 22, 40], the sphere and circle [6, 8, 13, 23, 26, 36, 42], the two-sheeted hyperboloid and hyperbola [9, 10, 39] and the saddle [7, 18, 19] (see also [1, 15, 16, 25] for the case of power curves and surfaces). The study of such sharp L^2 to L^p estimates is intimately related to the study of extremizers and sharp constants for Strichartz estimates for classical partial differential equations, such as the Schrödinger, hyperbolic Schrödinger, wave and Klein–Gordon equations. In this

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note we address the case of the one-sheeted (or hyperbolic) hyperboloid in \mathbb{R}^4 , which is related to the so called Klein–Gordon equation with imaginary mass.

1.1 Setting

Let \mathcal{H}^3 denote the upper half of the three dimensional one-sheeted hyperboloid in \mathbb{R}^4 ,

$$\mathcal{H}^3 = \left\{ (x, \sqrt{|x|^2 - 1}) : x \in \mathbb{R}^3, |x| \geq 1 \right\},$$

equipped with the measure μ with density

$$d\mu(x, t) = \mathbb{1}_{\{|x|>1\}}(x) \delta\left(t - \sqrt{|x|^2 - 1}\right) \frac{dt dx}{\sqrt{|x|^2 - 1}}, \tag{1.1}$$

so that for all $g \in \mathcal{S}(\mathbb{R}^4)$ it holds that

$$\int_{\mathcal{H}^3} g(x, t) d\mu(x, t) = \int_{\{y \in \mathbb{R}^3 : |y|>1\}} g(y, \sqrt{|y|^2 - 1}) \frac{dy}{\sqrt{|y|^2 - 1}}. \tag{1.2}$$

A function $f : \mathcal{H}^3 \rightarrow \mathbb{R}$ can be identified with a function from \mathbb{R}^3 to \mathbb{R} , using the orthogonal projection from \mathcal{H}^3 to $\mathbb{R}^3 \times \{0\}$, and in what follows we do so. We denote the $L^p(\mathcal{H}^3, \mu)$ norm of a function f on \mathcal{H}^3 by $\|f\|_{L^p(\mathcal{H}^3)}$, $\|f\|_{L^p(\mu)}$ or simply $\|f\|_{L^p}$, $\|f\|_p$ if it is clear from context.

The Fourier extension operator on the hyperboloid \mathcal{H}^3 , also known as the adjoint Fourier restriction operator, is given by

$$Tf(x, t) = \int_{\{y \in \mathbb{R}^3 : |y|>1\}} e^{ix \cdot y} e^{it\sqrt{|y|^2 - 1}} f(y, \sqrt{|y|^2 - 1}) \frac{dy}{\sqrt{|y|^2 - 1}}, \tag{1.3}$$

where $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^4)$. Note that $Tf(x, t) = \widehat{f\mu}(-x, -t)$, with the Fourier transform in \mathbb{R}^4 defined by $\hat{g}(x, t) = \int_{\mathbb{R}^3 \times \mathbb{R}} e^{-i(x \cdot y + ts)} g(y, s) dy ds$.

Strichartz proved in [43] that for all $\frac{10}{3} \leq p \leq 4$ there exists $C_p < \infty$ such that for all $f \in L^2(\mathcal{H}^3)$ the following estimate for Tf holds

$$\|Tf\|_{L^p(\mathbb{R}^4)} \leq \mathbf{H}_p \|f\|_{L^2(\mathcal{H}^3)}, \tag{1.4}$$

¹ Strictly speaking, it is identified with a function with domain $\{x \in \mathbb{R}^4 : |x| \geq 1\}$ but we will ignore this minor point and, whenever necessary, it will be understood that f is extended to be equal to zero inside the unit ball. We could have chosen to write our operator as acting on a weighted L^2 space of Euclidean space, but we will take this geometric point of view instead.

where $\mathbf{H}_p < \infty$ denotes the best constant in (1.4),

$$\mathbf{H}_p = \sup_{0 \neq f \in L^2(\mathcal{H}^3)} \frac{\|Tf\|_{L^p(\mathbb{R}^4)}}{\|f\|_{L^2(\mathcal{H}^3)}}. \tag{1.5}$$

The (full) one-sheeted hyperboloid is defined by

$$\overline{\mathcal{H}}^3 := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : t^2 = |x|^2 - 1, |x| \geq 1\},$$

and we endow it with the Lorentz invariant measure $\bar{\mu} = \mu_+ + \mu_-$ where $\mu_+ = \mu$ as in (1.1)–(1.2) is supported on \mathcal{H}^3 , and μ_- is given by

$$d\mu_-(x, t) = \mathbb{1}_{\{|x|>1\}} \delta\left(t + \sqrt{|x|^2 - 1}\right) \frac{dt \, dx}{\sqrt{|x|^2 - 1}},$$

so that μ_- equals the reflection of μ via the reflection map $(x, t) \mapsto (-x, -t)$ and is supported on $-\mathcal{H}^3$. The adjoint Fourier restriction operator on $\overline{\mathcal{H}}^3$ is

$$\begin{aligned} \overline{T}f(x, t) &= \widehat{f\bar{\mu}}(-x, -t) = \int_{\overline{\mathcal{H}}^3} e^{i(x \cdot y + ts)} f(y, s) \, d\bar{\mu}(y, s) \\ &= \int_{\{y \in \mathbb{R}^3 : |y| > 1\}} e^{ix \cdot y} e^{it\sqrt{|y|^2 - 1}} f_+(y) \frac{dy}{\sqrt{|y|^2 - 1}} \\ &\quad + \int_{\{y \in \mathbb{R}^3 : |y| > 1\}} e^{ix \cdot y} e^{-it\sqrt{|y|^2 - 1}} f_-(y) \frac{dy}{\sqrt{|y|^2 - 1}}, \end{aligned} \tag{1.6}$$

where $f = f_+ + f_-$, the function f_+ is supported on the upper half of the one-sheeted hyperboloid, \mathcal{H}^3 , and the function f_- , on the lower half, $-\mathcal{H}^3$, and we have identified their domains with \mathbb{R}^3 via the orthogonal projection as stated before. We see that $\overline{T}f(x, t) = Tf_+(x, t) + Tf_-(x, -t)$.

The triangle inequality and (1.4) imply that for $\frac{10}{3} \leq p \leq 4$ the following estimate holds

$$\|\overline{T}f\|_{L^p(\mathbb{R}^4)} \leq \overline{\mathbf{H}}_p \|f\|_{L^2(\overline{\mathcal{H}}^3)}, \tag{1.7}$$

where $\overline{\mathbf{H}}_p < \infty$ is the sharp constant

$$\overline{\mathbf{H}}_p = \sup_{0 \neq f \in L^2(\overline{\mathcal{H}}^3)} \frac{\|\overline{T}f\|_{L^p(\mathbb{R}^4)}}{\|f\|_{L^2(\overline{\mathcal{H}}^3)}}. \tag{1.8}$$

The Lorentz group on \mathbb{R}^4 , denoted \mathcal{L} , preserves $\overline{\mathcal{H}}^3$, $\bar{\mu}$, and acts on functions on $\overline{\mathcal{H}}^3$ by composition: $L^*f(x, t) := f(L(x, t))$, $L \in \mathcal{L}$ (see Sect. 2 for more details). In

particular, we have $\|f\|_{L^q(\overline{\mathcal{H}}^3)} = \|L^* f\|_{L^q(\overline{\mathcal{H}}^3)}$ and $\|Tf\|_{L^p(\mathbb{R}^4)} = \|T(L^* f)\|_{L^p(\mathbb{R}^4)}$, for all $p, q \in [1, \infty]$.

Definition 1.1 An extremizer (or maximizer) for (1.4) is a function $0 \neq f \in L^2(\mathcal{H}^3)$ that satisfies $\|Tf\|_{L^p(\mathbb{R}^4)} = \mathbf{H}_p \|f\|_{L^2(\mathcal{H}^3)}$. An L^2 -normalized extremizing sequence for (1.4) $\{f_n\}_n \subset L^2(\mathcal{H}^3)$ is such that $\|f_n\|_{L^2(\mathcal{H}^3)} = 1$ and $\|Tf_n\|_{L^p(\mathbb{R}^4)} \rightarrow \mathbf{H}_p$, as $n \rightarrow \infty$. A corresponding definition holds for extremizers and extremizing sequences for (1.7).

1.2 Main Results

This paper is devoted to the study of the sharp instances of (1.4) and (1.7) in the endpoint case $p = 4$, that is, the inequalities

$$\|Tf\|_{L^4(\mathbb{R}^4)} \leq \mathbf{H}_4 \|f\|_{L^2(\mathcal{H}^3)}, \tag{1.9}$$

$$\|\overline{T}g\|_{L^4(\mathbb{R}^4)} \leq \overline{\mathbf{H}}_4 \|g\|_{L^2(\overline{\mathcal{H}}^3)}, \tag{1.10}$$

and our main results concern the existence of extremizers as well as the precompactness of extremizing sequences. The statements of the main results of this paper are as follows.

Theorem 1.2 *There exists an extremizer in $L^2(\mathcal{H}^3)$ for inequality (1.9). Moreover, for every L^2 -normalized complex valued extremizing sequence $\{f_n\}_n$ for (1.9), there exist a subsequence $\{f_{n_k}\}_k$ and a sequence $\{(x_k, t_k)\}_k \subset \mathbb{R}^3 \times \mathbb{R}$ such that $\{e^{ix_k \cdot y} e^{it_k \sqrt{|y|^2 - 1}} f_{n_k}\}_k$ is convergent in $L^2(\mathcal{H}^3)$.*

Theorem 1.3 *There exists an extremizer in $L^2(\overline{\mathcal{H}}^3)$ for inequality (1.10). Moreover, for every L^2 -normalized complex valued extremizing sequence $\{f_n\}_n$ for (1.10), there exist a subsequence $\{f_{n_k}\}_k$ and sequences $\{\xi_k\}_k \subset \mathbb{R}^4$ and $\{L_k\}_k \subset \mathcal{L}$ such that $\{e^{i\xi_k \cdot \xi} L_k^* f_{n_k}\}_k$ is convergent in $L^2(\overline{\mathcal{H}}^3)$.*

In the statement of the theorems we are writing $e^{ix_k \cdot y} e^{it_k \sqrt{|y|^2 - 1}} f_{n_k}$ for the function $y \mapsto e^{ix_k \cdot y} e^{it_k \sqrt{|y|^2 - 1}} f_{n_k}(y)$ and $e^{i\xi_k \cdot \xi} L_k^* f_{n_k}$ for the function $\xi \mapsto e^{i\xi_k \cdot \xi} f_{n_k}(L_k \xi)$.

Remark 1.4 Note the qualitative difference regarding existence of extremizers between the one-sheeted hyperboloid and the two-sheeted hyperboloid (or their upper sheets) equipped with its Lorentz invariant measure, which are defined respectively by

$$\{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : t^2 = |x|^2 + 1\}, \quad (\delta(t - \sqrt{|x|^2 + 1}) + \delta(t + \sqrt{|x|^2 + 1})) \frac{dt \, dx}{\sqrt{|x|^2 + 1}},$$

both of which can be considered as ‘‘perturbations’’ of the cone. It was shown in [39] that for the L^2 to $L^4(\mathbb{R}^4)$ adjoint Fourier restriction inequality on the two-sheeted hyperboloid and on its upper sheet, extremizers do not exist and the best constant was calculated explicitly. On the other hand, for the L^2 to $L^4(\mathbb{R}^4)$ adjoint Fourier

restriction inequality on the cone, extremizers exist, their exact form was obtained and the best constant was calculated (see [5]).

We note that the results in [21] do not apply to the case of the hyperboloid due to the lack of scale invariance, but information can be obtained from the arguments therein, although we will not go that route. See the discussion in [39, Sect. 2] for some details in the related context of the two-sheeted hyperboloid.

We take this opportunity to indicate a *correction* to [39, Thm. 1.2, Prop. 7.5], where the value of the best constant for the $L^2 \rightarrow L^6$ adjoint Fourier restriction inequality on the two-sheeted hyperboloid in \mathbb{R}^2 , there denoted $\bar{\mathbf{H}}_{2,6}$, is incorrect. Details can be found in version 3 of [39] available at www.arxiv.org.

The convolution form of inequalities (1.9) and (1.10), obtained via Plancherel's theorem, tells us that in both cases, \mathcal{H}^3 and $\overline{\mathcal{H}^3}$, there exist nonnegative real valued extremizers, and the symmetrization method used in [23], or the one in [35], can be adapted to show that if a function f is a nonnegative real valued extremizer for \bar{T} on $\overline{\mathcal{H}^3}$ then f is necessarily an even function: $f(x, t) = f(-x, -t)$, for $\bar{\mu}$ -a.e. $(x, t) \in \overline{\mathcal{H}^3}$. We discuss the details in Sect. 2.

It would be of interest to treat the endpoint $p = \frac{10}{3}$ as well, and more generally to study the existence of extremizers at the endpoint and non-endpoint cases for all² $d \geq 2$, as was recently done for non-endpoint cases of the two-sheeted hyperboloid in [9, 10]. Our analysis here extends the known results on sharp Fourier extension inequalities for quadric manifolds as studied in Strichartz paper [43].

1.3 Organization of the Paper and Outline of the Proofs of the Main Theorems

From now on, references to the sharp inequalities (1.4) and (1.7) are understood with $p = 4$, unless it is explicitly said otherwise.

An important tool in our proofs is [20, Prop. 1.1] which we include next for the convenience of the reader.

Proposition 1.5 *Let \mathbb{H} be a Hilbert space and $S : \mathbb{H} \rightarrow L^p(\mathbb{R}^d)$ be a continuous linear operator, for some $p \in (2, \infty)$. Let $\{f_n\}_n \subset \mathbb{H}$ be such that:*

- (i) $\|f_n\|_{\mathbb{H}} = 1$,
- (ii) $\lim_{n \rightarrow \infty} \|Sf_n\|_{L^p(\mathbb{R}^d)} = \|S\|_{\mathbb{H} \rightarrow L^p(\mathbb{R}^d)}$,
- (iii) $f_n \rightharpoonup f$ and $f \neq 0$,
- (iv) $Sf_n \rightarrow Sf$ a.e. in \mathbb{R}^d .

Then $f_n \rightarrow f$ in \mathbb{H} . In particular, $\|f\|_{\mathbb{H}} = 1$ and $\|Sf\|_{L^p(\mathbb{R}^d)} = \|S\|_{\mathbb{H} \rightarrow L^p(\mathbb{R}^d)}$.

To prove Theorem 1.2 we apply Proposition 1.5 with $p = d = 4$, \mathbb{H} equals to $L^2(\mathcal{H}^3)$ and S equals T . We need to verify (iii) and (iv), as (i) and (ii) are immediate

² When $d = 1$ the one-sheeted hyperboloid coincides with the two-sheeted hyperboloid after a 90° rotation, and the later has been studied in [9]. They consider only one of the two branches but it is not difficult to see that the existence argument in the non-endpoint cases carries through to the two branches. On the other hand, an argument is needed to settle the endpoint $p = 6$ for two branches (this is also the case when $d = 2$ and $p = 6$ as clarified in the correction to [39] alluded to before).

for a normalized extremizing sequence. We handle (iv) as in [38, Prop. 8.3] and [21]. To prove (iii) we will see that the only way it can fail, the failure being that a weak limiting function equals zero, is that the extremizing sequence concentrates at infinity, which is defined as follows for \mathcal{H}^3 , with an analogous definition for $\overline{\mathcal{H}}^3$.

Definition 1.6 We say that the sequence $\{f_n\}_n \subset L^2(\mathcal{H}^3)$ concentrates at infinity if $\inf_n \|f_n\|_{L^2(\mathcal{H}^3)} > 0$ and for every $\varepsilon, R > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\|f_n \mathbb{1}_{\{|y| \leq R\}}\|_{L^2(\mathcal{H}^3)} < \varepsilon,$$

where, as mentioned before, we are identifying a function on \mathcal{H}^3 with a function on $\{y \in \mathbb{R}^3 : |y| \geq 1\}$.

Finally, to discard the possibility of concentration at infinity we will use a comparison argument with the cone.

In the case of the full one-sheeted hyperboloid $\overline{\mathcal{H}}^3$ there is the addition of Lorentz invariance, and our proof of Theorem 1.3 will require additional steps when compared to the case of the upper half, \mathcal{H}^3 . Because of this, in addition to the use of Proposition 1.5 and a comparison to the double cone, we will use a concentration-compactness argument to be able to discard concentration at infinity.

More in detail, the organization of the paper is as follows. In Sect. 3 we explicitly calculate the double convolution $\mu * \mu$ which we use in Sect. 4 to prove the *strict* lower bounds

$$\mathbf{H}_4 > (2\pi)^{5/4}, \quad \overline{\mathbf{H}}_4 > \left(\frac{3}{2}\right)^{1/4} (2\pi)^{5/4}, \tag{1.11}$$

which tell us that the best constant for the adjoint Fourier restriction operator on the (resp. full) one-sheeted hyperboloid is strictly greater than that for the (resp. double) cone.

In Sect. 5 we prove Theorem 1.2 by collecting the necessary ingredients to use Proposition 1.5. Here the first inequality in (1.11) is used to show that the L^2 mass of an extremizing sequence can not tend to infinity (i.e. there is no concentration at infinity).

From Sect. 6 onward we focus on the full one-sheeted hyperboloid $\overline{\mathcal{H}}^3$. As mentioned before, the existence of Lorentz invariance adds complexity to the proof of Theorem 1.3, compared to the much simpler proof of Theorem 1.2. We will use a concentration-compactness type argument that we discuss in Sect. 9. In short, given an L^2 normalized extremizing sequence $\{f_n\}_n$ for $\overline{\mathcal{H}}^3$, three possibilities hold (possibly after passing to a further subsequence): *compactness*, *vanishing* or *dichotomy*. In Sect. 10 we prove bilinear estimates at (radial) dyadic scales and show that they imply that *dichotomy* can not occur. In Sect. 11 we obtain a (radial) dyadic refinement of (1.7) and use it to show that *vanishing* can not occur.

To treat the *compactness* case, it will be necessary to study so called ‘‘cap bounds’’ or refinements of the $L^2 \rightarrow L^4$ estimate for the adjoint Fourier restriction operators T and \overline{T} and this we achieve in Sect. 8 by ‘‘lifting’’ to the hyperboloid the results for the

sphere in \mathbb{R}^3 , as proved in [13], and recalled in Sect. 7 (more precisely we study so called δ -quasi-extremals and their relationship with caps). By doing this lifting of the cap refinements available for the sphere, we do not have to develop bilinear estimates in the angular variable, but only in the radial variable.

In Sect. 12 we study some limiting relationships between the hyperboloid and the cone. The results of this section together with the second strict inequality in (1.11) are used to study the *compactness* alternative in the case of an extremizing sequence concentrating at infinity, discarding some possible behaviors.

Finally, in Sect. 13 we put together all the preliminary results of previous sections to show that if an extremizing sequence satisfies *compactness* then it is precompact in $L^2(\overline{\mathcal{H}}^3)$, modulo multiplication by characters and composition with Lorentz transformations, completing the proof of Theorem 1.3.

Although our approach to the proof of Theorem 1.3 depends on the Lebesgue exponent “4” being an even integer, which for other works in this field has meant to restrict to nonnegative (and possibly symmetric) extremizing sequences, we point out that we are able to handle the case of general complex valued extremizing sequences. Besides the fact that some arguments are simpler if one works with an even integer as we can multiply out some expressions, they could (in principle) be reworked for general real Lebesgue exponents. In the view of the author, the crucial step where evenness is used is in the inequality $\|T(f)\|_{L^4(\mathbb{R}^4)} \leq \|T(|f|)\|_{L^4(\mathbb{R}^4)}$, which may not hold for non even exponents. This is used in the proof of Theorem 1.3, Case 1.

Having explained our methods, we now mention a different possible path to two aspects of our proof. As stated earlier, in this work we obtain a relationship between quasi-extremals and caps by lifting the known results for the sphere but we mention that there is the alternative route through bilinear estimates to obtain cap refinements of inequalities (1.9) and (1.10). The works [9, 10] treat the related two-sheeted hyperboloid in the non-endpoint cases and of particular interest is the development of bilinear estimates in the angular and radial variables which offer a template to obtain similar results for the one-sheeted hyperboloid (see also [2–4]).

A second aspect of our proof is the use of a concentration-compactness type argument. There is a different possible approach, the missing mass method³ (MMM). This is a general framework to address the problem of existence in optimization problems; in this particular setting of maximizers for adjoint Fourier restriction inequalities it was first introduced by Frank et al. [26] for the case of the sphere, and later also successfully applied to power curves and (hyper-)surfaces [15, 25]. It has the advantage of allowing complex valued functions in the setting of general Lebesgue exponents, which could be useful when addressing the remaining cases (specially the endpoint cases) of (1.4) and (1.7), that is, when $10/3 \leq p < 4$ and the ambient space is \mathbb{R}^4 , as well as the remaining Strichartz estimates for the one-sheeted hyperboloid in \mathbb{R}^{d+1} , where $2(d+2)/d \leq p \leq 2(d+1)/(d-1)$, $d \geq 2$, and $6 \leq p < \infty$ if $d = 1$.

³ It is also possible to use profile decompositions but we will not discuss that alternative here. For the MMM, see the introduction in [26] for some historical references and the main idea of the method.

1.4 Notation and Some Definitions

The set of natural numbers is $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, 3, \dots\}$.

For $s > 0$, we let $\mathcal{H}_s^3 := \{(x, t) : x \in \mathbb{R}^3, t = \sqrt{|x|^2 - s^2}\}$, equipped with the measure

$$d\mu_s(x, t) = \mathbb{1}_{\{|x|>s\}} \delta\left(t - \sqrt{|x|^2 - s^2}\right) \frac{dx dt}{\sqrt{|x|^2 - s^2}}, \tag{1.12}$$

and adjoint Fourier restriction operator T_s ,

$$T_s f(x, t) = \widehat{f\mu_s}(-x, -t) = \int_{\{y \in \mathbb{R}^3 : |y|>s\}} e^{ix \cdot y} e^{it\sqrt{|y|^2 - s^2}} f(y) \frac{dy}{\sqrt{|y|^2 - s^2}}. \tag{1.13}$$

There are corresponding definitions of $\overline{\mathcal{H}}_s^3$, $\overline{\mu}_s$ and \overline{T}_s in analogy with the case $s = 1$.

The cone in \mathbb{R}^4 is denoted $\Gamma^3 := \{(y, |y|) : y \in \mathbb{R}^3\}$ which comes with its Lorentz and scale invariant measure σ_c ,

$$\int_{\Gamma^3} f d\sigma_c = \int_{\mathbb{R}^3} f(y, |y|) \frac{dy}{|y|}.$$

The adjoint Fourier restriction operator on the cone, T_c , is given by the expression

$$T_c f(x, t) = \int_{\mathbb{R}^3} e^{ix \cdot y} e^{it|y|} f(y) \frac{dy}{|y|}, \tag{1.14}$$

which acts, a priori, on functions $f \in \mathcal{S}(\mathbb{R}^3)$. The adjoint Fourier restriction operator on the double cone, $\overline{\Gamma}^3 := \Gamma^3 \cup -\Gamma^3$, denoted by \overline{T}_c , is given by the expression

$$\overline{T}_c f(x, t) = \int_{\mathbb{R}^3} e^{ix \cdot y} e^{it|y|} f(y, |y|) \frac{dy}{|y|} + \int_{\mathbb{R}^3} e^{ix \cdot y} e^{-it|y|} f(y, -|y|) \frac{dy}{|y|}, \tag{1.15}$$

$f \in \mathcal{S}(\mathbb{R}^4)$. We let $\mathbf{C}_4, \overline{\mathbf{C}}_4 < \infty$ denote the best constants in the inequalities

$$\|T_c f\|_{L^4(\mathbb{R}^4)} \leq \mathbf{C}_4 \|f\|_{L^2(\Gamma^3)}, \quad \|\overline{T}_c f\|_{L^4(\mathbb{R}^4)} \leq \overline{\mathbf{C}}_4 \|f\|_{L^2(\overline{\Gamma}^3)},$$

respectively. We sometimes use the alternative notation $\|T\| = \mathbf{H}_4$, $\|\overline{T}\| = \overline{\mathbf{H}}_4$, $\|T_c\| = \mathbf{C}_4$ and $\|\overline{T}_c\| = \overline{\mathbf{C}}_4$.

The sphere of radius $r > 0$ in \mathbb{R}^3 is $\mathbb{S}_r^2 := \{y \in \mathbb{R}^3 : |y| = r\}$. The sphere of radius 1 is denoted simply \mathbb{S}^2 . On \mathbb{S}_r^2 we consider the measure σ_r ,

$$\int_{\mathbb{S}_r^2} f \, d\sigma_r = \int_{\mathbb{S}^2} f(r\omega)r \, d\sigma(\omega), \tag{1.16}$$

where σ is the surface measure on \mathbb{S}^2 . With this choice, $\sigma_r(\mathbb{S}_r^2) = r\sigma(\mathbb{S}^2)$, for all $r > 0$. For $r > 0$ and a function $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ we set $f_r : \mathbb{S}^2 \rightarrow \mathbb{R}$ by $f_r(\cdot) = f(r \cdot)$.

We let \mathbf{S} denote the best constant in the convolution form of the Tomas–Stein inequality for the sphere \mathbb{S}^2 ,

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leq \mathbf{S}^2 \|f\|_{L^2(\mathbb{S}^2)}^2.$$

We also use the following convention. For $f : \overline{\mathcal{H}^3} \rightarrow \mathbb{R}$ we write $f = f_+ + f_-$, where f_+ is supported on \mathcal{H}^3 and f_- on the reflection of \mathcal{H}^3 with respect to the origin, $-\mathcal{H}^3$, and we further identify their domains with \mathbb{R}^3 via the orthogonal projection. For $A \subseteq \mathbb{R}^3$ we define

$$\int_A f \, d\mu := \int_{\{(x,t) \in \mathcal{H}^3 : x \in A\}} f \, d\mu.$$

$f \in L^1(\mathcal{H}^3)$, while for $\overline{\mathcal{H}^3}$,

$$\int_A f \, d\bar{\mu} := \int_{\{(x,t) \in \overline{\mathcal{H}^3} : x \in A\}} f \, d\bar{\mu} :$$

$f \in L^1(\overline{\mathcal{H}^3})$, so that in both cases the integral over $A \subset \mathbb{R}^3$ equals to the integral over the “lift” of A to \mathcal{H}^3 or $\overline{\mathcal{H}^3}$, as it corresponds.

An element $R \in SO(4)$ that preserves the t -axis, $R(0, 0, 0, 1) = (0, 0, 0, 1)$, is canonically identified with an element of $SO(3)$, and as such we will just write $R \in SO(3)$.

We let $\psi_s(r) = \sqrt{r^2 - s^2} \mathbb{1}_{\{r \geq s\}}$, $\phi_s(t) = \psi_s^{-1}(t) = \sqrt{t^2 + s^2} \mathbb{1}_{\{t \geq 0\}}$. The (closed) ball of radius $r > 0$ centered at $y \in \mathbb{R}^3$ is $B(y, r)$. For a set A , $\mathbb{1}_A$ denotes the characteristic function of A and A^c , the complement of A with respect to a set containing A that will be understood from context, usually \mathbb{R}^3 , \mathcal{H}^3 or $\overline{\mathcal{H}^3}$. We sometimes slightly abuse notation and use $|A|$ to denote the measure of a set A , where the measure used must be understood from context, for instance, if A is an interval it refers to the Lebesgue measure, if $A \subseteq \mathbb{S}^2$, it refers to the surface measure, etc. The support of a function f is denoted $\text{supp}(f)$.

We will use the usual asymptotic notation $X \lesssim Y$, $Y \gtrsim X$ if there exists a constant C (independent of X, Y) such that $|X| \leq CY$; we use $X \asymp Y$ if $X \lesssim Y$ and $Y \lesssim X$; when such constants depend on parameters of the problem that we want to make

explicit, such as α, \dots etc., we write $\lesssim_{\alpha, \dots}, \gtrsim_{\alpha, \dots}$ and $\asymp_{\alpha, \dots}$. At times we will use the common asymptotic notation $o(\cdot)$ and $O(\cdot)$. Thus, $g_n = o(f_n)$ if $g_n/f_n \rightarrow 0$ as $n \rightarrow \infty$, while $g_n = O(f_n)$ if $|g_n| \leq C|f_n|$ for all n . If there is more than one parameter, say $n \in \mathbb{N}$ and $s > 0$, then $g_n(s) = o_n(f_n(s))$ means the limit of $g_n/f_n \rightarrow 0$ is taken with respect to n and is uniform in s , that is $\sup_s |g_n(s)/f_n(s)| \rightarrow 0$ as $n \rightarrow \infty$.

2 Lorentz Invariance, Symmetrization and Caps

2.1 Lorentz Invariance

Recall that the Lorentz group on \mathbb{R}^4 , denoted \mathcal{L} , is defined as the group of invertible linear transformations in \mathbb{R}^4 that preserve the bilinear form

$$B(x, y) = x_4y_4 - x_3y_3 - x_2y_2 - x_1y_1,$$

for $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. If $L \in \mathcal{L}$ then $|\det L| = 1$. Given that we can write $\overline{\mathcal{H}}^3 = \{(x, t) \in \mathbb{R}^{3+1} : B((x, t), (x, t)) = -1\}$ it is direct that \mathcal{L} preserves the hyperboloid: $L(\overline{\mathcal{H}}^3) = \overline{\mathcal{H}}^3$, for every $L \in \mathcal{L}$. Moreover, \mathcal{L} preserves the measure $\bar{\mu}$, in the sense that for every $f \in L^1(\overline{\mathcal{H}}^3)$ and $L \in \mathcal{L}$

$$\int_{\overline{\mathcal{H}}^3} f(x, t) d\bar{\mu}(x, t) = \int_{\overline{\mathcal{H}}^3} f(L(x, t)) d\bar{\mu}(x, t). \tag{2.1}$$

To see this, note that a simple calculation shows that we can write

$$d\bar{\mu}(x, t) = \delta(t^2 - |x|^2 + 1) dx dt$$

so that

$$\int_{\mathbb{R}^4} f(x, t) d\bar{\mu}(x, t) = \int_{\mathbb{R}^4} f(x, t) \delta(t^2 - |x|^2 + 1) dt dx.$$

Then, if L is a Lorentz transformation and $f \in L^1(\overline{\mathcal{H}}^3)$, (2.1) can be seen to hold by the change of variable formula.

For $t \in (-1, 1)$ the Lorentz boost $L^t \in \mathcal{L}$ is the linear map

$$L^t(\xi_1, \xi_2, \xi_3, \tau) = \left(\frac{\xi_1 + t\tau}{\sqrt{1-t^2}}, \xi_2, \xi_3, \frac{t\xi_1 + \tau}{\sqrt{1-t^2}} \right), \tag{2.2}$$

while L_t denotes the rescaling $L_t := (1-t^2)^{1/2}L^t$, so that $(L_t)^{-1} = (1-t^2)^{-1/2}L^{-t}$.

2.2 Convolution Form

With the Fourier transform in \mathbb{R}^d normalized as $\widehat{F}(x) = \int_{\mathbb{R}^d} e^{-ix \cdot y} F(y) dy$ we have the identities

$$\widehat{F * G} = \widehat{F} \widehat{G}, \quad \|\widehat{F}\|_{L^2(\mathbb{R}^d)} = (2\pi)^{d/2} \|F\|_{L^2(\mathbb{R}^d)},$$

so that using $Tf(x, t) = \widehat{f\mu}(-x, -t)$ and $\overline{T}g(x, t) = \widehat{g\bar{\mu}}(-x, -t)$ we find the equalities

$$\|Tf\|_{L^4(\mathbb{R}^4)} = 2\pi \|f\mu * f\mu\|_{L^2(\mathbb{R}^4)}^{1/2}, \quad \|\overline{T}g\|_{L^4(\mathbb{R}^4)} = 2\pi \|g\bar{\mu} * g\bar{\mu}\|_{L^2(\mathbb{R}^4)}^{1/2}. \tag{2.3}$$

Using this *convolution form* of the L^4 norm and the triangle inequality we see that $\|Tf\|_{L^4(\mathbb{R}^4)} \leq \|T|f|\|_{L^4(\mathbb{R}^4)}$ and $\|\overline{T}g\|_{L^4(\mathbb{R}^4)} \leq \|\overline{T}|g|\|_{L^4(\mathbb{R}^4)}$, so that if f is an extremizer for (1.4) (resp. g for (1.7)), then so is $|f|$ (resp. $|g|$), showing that if extremizers exist then there are nonnegative real valued extremizers.

2.3 Symmetrization

Let $f \in L^2(\overline{\mathcal{H}}^3)$ be a complex valued function. Denote the reflection of f by $\widetilde{f}(x, t) = f(-x, -t)$ and the nonnegative L^2 -symmetrization of f by

$$f_{\sharp}(x, t) = \left(\frac{|f(x, t)|^2 + |f(-x, -t)|^2}{2} \right)^{1/2}.$$

Regarding the relationship between f and f_{\sharp} we have the following lemma.

Lemma 2.1 *Let $f \in L^2(\overline{\mathcal{H}}^3)$ be a complex valued function. Then*

$$\|f\bar{\mu} * f\bar{\mu}\|_{L^2(\mathbb{R}^4)} \leq \|f_{\sharp}\bar{\mu} * f_{\sharp}\bar{\mu}\|_{L^2(\mathbb{R}^4)}. \tag{2.4}$$

Proof As in [23, Proof of Prop. 3.2] we write

$$\begin{aligned} f\bar{\mu} * \widetilde{f\bar{\mu}}(\xi, \tau) &= \int f(y, s)f(-x, -t) \delta((\xi, \tau) - (y, s) - (x, t)) d\bar{\mu}(y, s) d\bar{\mu}(x, t) \\ &= \frac{1}{2} \int (f(y, s)f(-x, -t) + f(-y, -s)f(x, t)) \\ &\quad \times \delta((\xi, \tau) - (y, s) - (x, t)) d\bar{\mu}(y, s) d\bar{\mu}(x, t), \end{aligned}$$

and apply the Cauchy–Schwarz inequality

$$|f(y, s)f(-x, -t) + f(-y, -s)f(x, t)| \leq 2f_{\sharp}(y, s)f_{\sharp}(x, t),$$

to obtain that for all $(\xi, \tau) \in \mathbb{R}^4$

$$|f\bar{\mu} * \tilde{f}\bar{\mu}(\xi, \tau)| \leq f_{\sharp}\bar{\mu} * f_{\sharp}\bar{\mu}(\xi, \tau).$$

Then

$$\|f\bar{\mu} * f\bar{\mu}\|_{L^2(\mathbb{R}^4)} = \|f\bar{\mu} * \tilde{f}\bar{\mu}\|_{L^2(\mathbb{R}^4)} \leq \|f_{\sharp}\bar{\mu} * f_{\sharp}\bar{\mu}\|_{L^2(\mathbb{R}^4)}.$$

□

Since we also have

$$\|f\|_{L^2(\bar{\mu})} = \|f_{\sharp}\|_{L^2(\bar{\mu})},$$

it follows that if extremizers exist for \bar{T} , then there exist real valued extremizers for \bar{T} which are nonnegative even functions on $\bar{\mathcal{H}}^3$. Moreover, any nonnegative real valued extremizer is *necessarily* even. This can be explained by studying the cases of equality in (2.4) by following the proof of the inequality (see [8] for a detailed discussion in the case of the sphere) or, alternatively, by using the same method as in the proof of [35, Lemma 6.1] where a different approach to symmetrization is used and the cases of equality were studied. Therefore, we have the following result.

Proposition 2.2 *If $f \in L^2(\bar{\mathcal{H}}^3)$ is a nonnegative real valued extremizer for (1.7), then $f(x, t) = f(-x, -t)$ for $\bar{\mu}$ -a.e. $(x, t) \in \bar{\mathcal{H}}^3$.*

There are some interesting problems that we do not address in this article:

- (i) the nonnegativity of *all* real valued extremizers,
- (ii) the relationship between complex and real valued extremizers,
- (iii) the smoothness of extremizers.

We provide the following comments in the context of the $L^2(\mathbb{S}^{d-1}) \rightarrow L^p(\mathbb{R}^d)$ adjoint Fourier restriction inequality on the sphere. Christ and Shao [14] showed that for the case of the sphere \mathbb{S}^2 in \mathbb{R}^3 and $p = 4$ each complex valued extremizer is of the form $x \mapsto ce^{ix \cdot \xi} F(x)$, for some $\xi \in \mathbb{R}^3$, some $c \in \mathbb{C}$ and some nonnegative extremizer F , and that extremizers are of class C^∞ ; these results were later expanded to all dimensions $d \geq 2$ and even integers p in [36, Lemma 2.2, Thm. 1.2] and [37]. Note that the answer obtained for (ii) resolves (i). By using the outline in [14, 36, 37], the Euler–Lagrange equation, which can be obtained as in [12], and the results in [11] we expect similar relationships for the case of \mathcal{H}^3 and $\bar{\mathcal{H}}^3$, but have not investigated the extent to which the arguments would need to be changed.

A related question is that of the rate of decay at infinity of an extremizer for which the argument in [27] gives a possible route; see also [35].

We remark that Theorems 1.2 and 1.3 are stated for general (possibly complex valued) extremizing sequences, that is, we do not assume nonnegativity and/or symmetry.

2.4 Caps

A (closed) spherical cap $\mathcal{C} \subseteq \mathbb{S}^2$ is a set of the form $\mathcal{C} = \{x \in \mathbb{S}^2 : |x - x_0| \leq t\}$ for some $x_0 \in \mathbb{S}^2$ and $t > 0$. If we want to be explicit about the dependence on x_0 and t we write $\mathcal{C}(x_0, t)$.

A cap \mathcal{C} of \mathcal{H}_s^3 is a set of the form

$$\mathcal{C} = \left\{ (r\omega, \sqrt{r^2 - s^2}) : r \in [a, b], \omega \in \mathcal{C} \right\}, \tag{2.5}$$

where $s \leq a < b \leq \infty$ and $\mathcal{C} \subseteq \mathbb{S}^2$ is a spherical cap. When $a = s 2^k$ and $b = s 2^{k+1}$ for some $k \in \mathbb{Z}$ we say that \mathcal{C} is a dyadic cap. We identify a cap \mathcal{C} as before with its orthogonal projection to $\mathbb{R}^3 \times \{0\}$, and moreover we use spherical coordinates and write the cap in (2.5) as $\mathcal{C} = [a, b] \times \mathcal{C}$, where the hyperboloid it belongs to will be understood from context. A cap \mathcal{C} of $\overline{\mathcal{H}}_s^3$ is such that either $\mathcal{C} \subseteq \mathcal{H}_s^3$ or its reflection with respect to the origin $(-\mathcal{C}) \subseteq \mathcal{H}_s^3$ is a cap on \mathcal{H}_s^3 .

The μ_s -measure of a cap is easily calculated

$$\mu_s(\mathcal{C}) = \sigma(\mathcal{C}) \int_a^b \frac{r^2}{\sqrt{r^2 - s^2}} dr = \frac{\sigma(\mathcal{C})}{2} \left(s^2 \ln(r + \sqrt{r^2 - s^2}) + r\sqrt{r^2 - s^2} \right) \Big|_a^b. \tag{2.6}$$

For a cap $\mathcal{C} = [a, b] \times \mathcal{C}$ in \mathcal{H}_s^3 and $t > 0$ we define the rescaled cap $t\mathcal{C} = [ta, tb] \times \mathcal{C}$ as the cap in \mathcal{H}_{ts}^3 given by

$$t\mathcal{C} = \left\{ (r\omega, \sqrt{r^2 - (ts)^2}) : r \in [ta, tb], \omega \in \mathcal{C} \right\},$$

and note that

$$\mu_{ts}(t\mathcal{C}) = t^2 \mu_s(\mathcal{C}). \tag{2.7}$$

We also note that for such a cap $\mathcal{C} \subset \mathcal{H}_s^3$ there exist $R \in SO(3)$ and $\varepsilon \in [0, \pi]$ such that

$$\begin{aligned} R^{-1}(\mathcal{C}) &= \{(r\omega, \sqrt{r^2 - s^2}) : a \leq r \leq b, \\ &\omega = (\cos \varphi, \cos \theta \sin \varphi, \sin \theta \sin \varphi), \theta \in [0, 2\pi], \varphi \in [0, \varepsilon]\}. \end{aligned} \tag{2.8}$$

Keeping this notation in mind for the rest of the section we study the use of Lorentz transformations and scaling in the regimes where $\bar{\mu}(\mathcal{C})$ is large and small. The following two lemmas will be useful in Sect. 13 when dealing with the full one-sheated hyperboloid $\overline{\mathcal{H}}^3$. To motivate them, let us see how their need arises as we try to prove

the precompactness of an extremizing sequence. Let $\{f_n\}_n \subset L^2(\overline{\mathcal{H}^3})$ be an extremizing sequence for \overline{T} . Because of “cap refinements” of (1.10) (Lemma 8.1), for each f_n we can find a dyadic cap $C_n = [2^{N_n}, 2^{N_n+1}] \times \mathcal{C}_n \subset \overline{\mathcal{H}^3}$, $\mathcal{C}_n \subseteq \mathbb{S}^2$, such that

$$\int_{C_n} |f_n| \, d\bar{\mu} \gtrsim \bar{\mu}(C_n)^{1/2}.$$

If we could find Lorentz transformations $L_n \in \mathcal{L}$ such that for each $n \in \mathbb{N}$, $L_n^{-1}(C_n)$ is contained in a fixed ball of \mathbb{R}^4 , independent of n , then $\{L_n^* f_n\}_n$ does not concentrate at infinity and then its precompactness modulo multiplication by characters $\xi \mapsto e^{i\xi_n \cdot \xi}$ would easily follow (this is the content of Proposition 5.2 below). For this reason, it is useful to study when such Lorentz transformations can be found. As noted in [9, Lemma 4] for the two-sheeted hyperboloid, there are Lorentz transformations that can map certain caps of bounded measure into a ball whose radius depends only on the value of the measure of the cap. We record this property for the one-sheeted hyperboloid in the next lemma.

Lemma 2.3 *Let $s > 0$, $k \in \mathbb{N}$ and $C_k \subset \overline{\mathcal{H}_s^3}$ be a dyadic cap of the form $C_k = [s2^k, s2^{k+1}] \times \mathcal{C}_k$, for some spherical cap $\mathcal{C}_k \subseteq \mathbb{S}^2$. Let R and ε be associated to C_k as in (2.8), then:*

(i) *The $\bar{\mu}_s$ -measure of C_k satisfies*

$$\begin{aligned} \bar{\mu}_s(C_k) &= 3\pi s^2(1 + o_k(1))2^{2k}(1 - \cos \varepsilon) \\ &= \frac{3\pi s^2}{1 + \cos \varepsilon}(1 + o_k(1))2^{2k} \sin^2 \varepsilon. \end{aligned} \tag{2.9}$$

(ii) *Suppose $\varepsilon \in [0, \frac{\pi}{2}]$. Then, there exists $t \in [0, 1)$ such that the orthogonal projection of $L^{-t}R^{-1}(C_k) \subset \overline{\mathcal{H}_s^3}$ to \mathbb{R}^3 is contained in a ball of \mathbb{R}^3 of radius comparable to $s + s^{-1}\bar{\mu}_s(C_k) + \bar{\mu}_s(C_k)^{1/2}$.*

Proof Without loss of generality, we may assume that C_k is contained in the upper half \mathcal{H}_s^3 . For part (i), (2.6) implies that the $\bar{\mu}_s$ -measure of C_k is given by the expression

$$\bar{\mu}_s(C_k) = \pi s^2(1 - \cos \varepsilon) \left(\ln \left(\frac{2^{k+1} + \sqrt{2^{2(k+1)} - 1}}{2^k + \sqrt{2^{2k} - 1}} \right) + 2^{k+1}\sqrt{2^{2(k+1)} - 1} - 2^k\sqrt{2^{2k} - 1} \right).$$

The expression involving the logarithm converges to $\ln(2)$ as $k \rightarrow \infty$, while

$$2^{k+1}\sqrt{2^{2(k+1)} - 1} - 2^k\sqrt{2^{2k} - 1} = 3 \cdot 2^{2k}(1 + o_k(1)).$$

For part (ii), let $R \in SO(3)$ and $\varepsilon \in [0, \frac{\pi}{2}]$ be such that (2.8) holds. The image of $R^{-1}(C_k)$ under the Lorentz boost L^{-t} is

$$L^{-t}R^{-1}(C_k) = \left\{ \left(\frac{r \cos \varphi - t\sqrt{r^2 - s^2}}{\sqrt{1 - t^2}}, r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, \frac{\sqrt{r^2 - s^2} - tr \cos \varphi}{\sqrt{1 - t^2}} \right) \right\}.$$

$$r \in [s2^k, s2^{k+1}], \theta \in [0, 2\pi], \varphi \in [0, \varepsilon] \}. \tag{2.10}$$

Let $t = \sqrt{1 - 2^{-2(k+1)}}$, so that the first coordinate of a point in the set on the right hand side of (2.10) is bounded as follows

$$\begin{aligned} \left| \frac{r \cos \varphi - t\sqrt{r^2 - s^2}}{\sqrt{1 - t^2}} \right| &= 2^{k+1}r | \cos \varphi - \sqrt{1 - 2^{-2(k+1)}}\sqrt{1 - (s/r)^2} | \\ &\leq 2^{2(k+1)}s(1 - \cos \varphi) + 2^{2(k+1)}s(1 - (1 - 2^{-2(k+1)})) \\ &= 2^{2(k+1)}s(1 - \cos \varepsilon) + s \\ &\lesssim \frac{\bar{\mu}_s(C_k)}{s} + s, \end{aligned}$$

where in the last line we used (2.9). The second and third coordinates are bounded as follows

$$|r \cos \theta \sin \varphi|, |r \sin \theta \sin \varphi| \leq 2^{k+1}s \sin \varepsilon \lesssim \sqrt{\bar{\mu}(C_k)}.$$

Then $L^{-t}R^{-1}(C_k)$ is contained in the set

$$\left\{ (x, t) \in \bar{\mathcal{H}}_s^3 : |x| \leq C \left(\sqrt{\bar{\mu}_s(C_k)} + \frac{\bar{\mu}_s(C_k)}{s} + s \right) \right\},$$

for some constant C independent of k and s . □

Continuing with the comment before Lemma 2.3, suppose now that the measure of the caps C_n is such that $\lim_{n \rightarrow \infty} \bar{\mu}(C_n) = \infty$, and set $s_n = 2^{-N_n} \rightarrow 0$ as $n \rightarrow \infty$, so that if R_n, ε_n are related to C_n as in (2.8), then, (2.9) implies $\bar{\mu}(C_n) \asymp s_n^{-2} \sin^2(\varepsilon_n) \rightarrow \infty$ as $n \rightarrow \infty$. We rescale and define $\tilde{C}_n := s_n C_n = [1, 2] \times C_n \subset \bar{\mathcal{H}}_{s_n}^3$ so that $\bar{\mu}_{s_n}(\tilde{C}_n) = s_n^2 \bar{\mu}(C_n)$. We may also rescale the sequence $\{f_n\}_n$ by setting $g_n := s_n^{-1} f_n(s_n^{-1} \cdot) \in L^2(\bar{\mathcal{H}}_{s_n}^3)$, which then satisfies

$$\int_{\tilde{C}_n} |g_n| d\bar{\mu}_{s_n} \gtrsim \bar{\mu}_{s_n}(\tilde{C}_n)^{1/2}. \tag{2.11}$$

If the sequence $\{\bar{\mu}_{s_n}(\tilde{C}_n)\}_n$ (possesses a subsequence that) is bounded below away from zero, then we will be able to use a comparison argument with the cone, as in a sense the $\bar{\mathcal{H}}_{s_n}^3$'s are approaching the cone $\bar{\Gamma}^3$, as $n \rightarrow \infty$. In this way, it will be established that this possibility does not arise and here the strict inequality between the best constants of this two manifolds comes into play. We are then lead to consider the complementary case, that is, when $\{\bar{\mu}_{s_n}(\tilde{C}_n)\}_n$ converges to zero. In this scenario we would like to use Lorentz transformations together with dilations in the following way. We want to find a sequence $\{L_n\}_n \subset \mathcal{L}$ such that $\tilde{f}_n := L_n^* f_n$ can be appropriately rescaled so that $\tilde{g}_n := a_n^{-1} \tilde{f}_n(a_n^{-1} \cdot) \in L^2(\bar{\mathcal{H}}_{a_n}^3)$, for some sequence $a_n \rightarrow 0$ as

$n \rightarrow \infty$, satisfies (2.11) with the corresponding sequence $\{\bar{\mu}_{a_n}(\tilde{C}_n)\}_n$ bounded below away from zero. In this way we will also be able to rule out this scenario. The following lemma will tell us how to find the L_n 's and the a_n 's.

Lemma 2.4 *Let $s \leq \frac{1}{2}$, $\mathcal{C} \subseteq \mathbb{S}^2$ be a spherical cap and $\mathcal{C} = [1, 2] \times \mathcal{C}$ be a cap in the hyperboloid \mathcal{H}_s^3 . Let R and ε be as in (2.8) and suppose that $\varepsilon \in [0, \frac{\pi}{2}]$ and $s^{-2} \sin^2 \varepsilon \geq 8$. Then there exist $0 \leq t < 1$ such that $L_t^{-1}R^{-1}(\mathcal{C}) \subset \mathcal{H}_{\frac{s}{\sqrt{1-t^2}}}^3$ satisfies*

$$\mu_{\frac{s}{\sqrt{1-t^2}}}(L_t^{-1}R^{-1}(\mathcal{C})) \geq \frac{\pi}{2} \text{ and } L_t^{-1}R^{-1}(\mathcal{C}) \subseteq [\frac{7}{16}, \frac{33}{16}] \times \mathbb{S}^2. \tag{2.12}$$

Moreover, if $\varepsilon \in [0, \frac{\pi}{3}]$, we can take $t = \cos \varepsilon$, while if $\varepsilon \in (\frac{\pi}{3}, \frac{\pi}{2}]$ we can take $t = 0$.

We point out that the value “8” in the inequality $s^{-2} \sin^2 \varepsilon \geq 8$ is meant to mean “large” and can be change to any other positive constant with the understanding that the values in (2.12) will change accordingly. Note that in the comment before the statement of the previous lemma we had $s_n^{-2} \sin^2(\varepsilon_n) \rightarrow \infty$ as $n \rightarrow \infty$ so that in the application that condition will surely be fulfilled. We will then take $t_n = \cos \varepsilon_n$ and $a_n = s_n/\sqrt{1-t_n^2} = (s_n^{-1} \sin \varepsilon_n)^{-1} \rightarrow 0$, as $n \rightarrow \infty$.

Proof of Lemma 2.4 With $R \in SO(3)$ and $\varepsilon \in [0, \frac{\pi}{2}]$ satisfying (2.8), note that $L_t^{-1}R^{-1}(\mathcal{C}) = (1-t^2)^{-1/2}L^{-t}R^{-1}(\mathcal{C}) \subseteq \mathcal{H}_{s(1-t^2)^{-1/2}}^3$, for every $t \in (-1, 1)$. According to (2.6), the μ_s -measure of \mathcal{C} satisfies

$$\begin{aligned} \mu_s(\mathcal{C}) &= 2\pi(1 - \cos \varepsilon) \left(\frac{s^2}{2} \ln \left(\sqrt{r^2 - s^2} + r \right) + \frac{r}{2} \sqrt{r^2 - s^2} \right) \Big|_1 \\ &\geq \pi(1 - \cos \varepsilon) (\sqrt{4 - s^2} - \sqrt{1 - s^2}) \geq \pi(1 - \cos \varepsilon), \end{aligned}$$

so that in what follows we can assume $\cos \varepsilon \geq 1/2$, otherwise we are done by taking $t = 0$. From (2.7), for $t \in (0, 1)$,

$$\mu_{\frac{s}{\sqrt{1-t^2}}}(L_t^{-1}R^{-1}(\mathcal{C})) = (1 - t^2)^{-1} \mu_s(\mathcal{C}),$$

so that choosing $t = \cos \varepsilon$ gives $\mu_{s(1-t^2)^{-1/2}}(L_t^{-1}R^{-1}(\mathcal{C})) \geq \frac{\pi}{1+\cos \varepsilon} \geq \frac{\pi}{2}$. On the other hand, we have

$$\begin{aligned} L_t^{-1}R^{-1}(\mathcal{C}) &= \left\{ (1-t^2)^{-1/2} \left(\frac{r \cos \varphi - t\sqrt{r^2 - s^2}}{\sqrt{1-t^2}}, \right. \right. \\ &\quad \left. \left. r \cos \theta \sin \varphi, r \sin \theta \sin \varphi, \frac{\sqrt{r^2 - s^2} - tr \cos \varphi}{\sqrt{1-t^2}} \right) : r \in [1, 2], \right. \\ &\quad \left. \theta \in [0, 2\pi], \varphi \in [0, \varepsilon] \right\}, \end{aligned}$$

and since $\cos \varphi \geq \cos \varepsilon$ and $1 \leq r \leq 2$ we obtain that the fourth coordinate of any point in $L_t^{-1}R^{-1}(C)$ is bounded as follows

$$\frac{\sqrt{r^2 - s^2} - tr \cos \varphi}{1 - t^2} = \frac{r(\sqrt{1 - (s/r)^2} - t \cos \varphi)}{1 - t^2} \leq 2 \frac{1 - \cos^2 \varepsilon}{1 - \cos^2 \varepsilon} = 2,$$

and

$$\begin{aligned} \frac{r(\sqrt{1 - (s/r)^2} - t \cos \varphi)}{1 - t^2} &= \frac{r}{\sin^2 \varepsilon} \left(\sqrt{1 - (s/r)^2} - \cos \varepsilon \cos \varphi \right) \\ &\geq \frac{r}{\sin^2 \varepsilon} \left(\sqrt{1 - (s/r)^2} - \cos \varepsilon \right) \\ &= \frac{r}{\sqrt{1 - (s/r)^2} + \cos \varepsilon} \left(1 - \frac{1}{r^2 s^{-2} \sin^2 \varepsilon} \right) \\ &\geq \frac{r}{2} \left(1 - \frac{1}{8r^2} \right) \geq \frac{7}{16}. \end{aligned}$$

Therefore

$$L_t^{-1}R^{-1}(C) \subseteq [\phi_{\frac{s}{\sqrt{1-t^2}}}(\frac{7}{16}), \phi_{\frac{s}{\sqrt{1-t^2}}}(2)] \times \mathbb{S}^2.$$

Now, from the definition of t and the assumption that $s^{-2} \sin^2 \varepsilon \geq 8$ we obtain

$$\frac{s}{\sqrt{1-t^2}} = \frac{s}{\sin \varepsilon} \leq \frac{\sqrt{2}}{4},$$

so that the following inequalities hold

$$r \leq \phi_{\frac{s}{\sqrt{1-t^2}}}(r) = \sqrt{r^2 + s^2(1-t^2)^{-1}} \leq \sqrt{r^2 + 1/8},$$

from where $\phi_{\frac{s}{\sqrt{1-t^2}}}(\frac{7}{16}) \geq \frac{7}{16}$ and $\phi_{\frac{s}{\sqrt{1-t^2}}}(2) \leq \frac{33}{16}$ and then we find $L_t^{-1}R^{-1}(C) \subseteq [\frac{7}{16}, \frac{33}{16}] \times \mathbb{S}^2$. □

3 Calculation of a Double Convolution

In previous studies of quadric surfaces and curves and their perturbations it has become clear the importance of the double or triple, and more generally the n -th fold, convolution of the underlying measure. Its properties may determine existence or nonexistence of extremizers and in some cases it can be used to find their explicit form and/or the value of the best constant in the corresponding adjoint Fourier restriction inequality. In the case of the one-sheeted hyperboloid and its upper half, the double convolution will be used to prove that extremizing sequences do not concentrate at infinity.

Let $\mu_s * \mu_s$ denote the double convolution of μ_s with itself, defined by duality

$$\langle \mu_s * \mu_s, f \rangle = \int_{(\mathbb{R}^4)^2} f(x + x', t + t') \, d\mu_s(x, t) \, d\mu_s(x', t'),$$

for all $f \in \mathcal{S}(\mathbb{R}^4)$. It is not difficult to see that $\mu_s * \mu_s$ is absolutely continuous with respect to the Lebesgue measure in \mathbb{R}^4 , indeed this follows from (1.4) since $e^{-\tau}(\mu_s * \mu_s) \in L^2(\mathbb{R}^4)$, it being the (inverse) Fourier transform of the $L^2(\mathbb{R}^4)$ function $(e^{-\tau} \widehat{\mu_s})^2$ (see also [34, Prop. 2.1]). In what follows we identify $\mu_s * \mu_s$ with its Radon–Nicolodym derivative with respect to the Lebesgue measure in \mathbb{R}^4 .

Proposition 3.1 *Let μ_s be the measure on \mathcal{H}_s^3 defined in (1.12). Then*

(i) *The support of the convolution measure $\mu_s * \mu_s$ is*

$$\text{supp}(\mu_s * \mu_s) = \{(\xi, \tau) \in \mathbb{R}^4 : \tau \geq 0, |\xi| \leq \sqrt{\tau^2 + s^2} + s\}.$$

(ii) *For every $(\xi, \tau) \in \mathbb{R}^4$ with $\tau \geq 0$ we have the formula*

$$\begin{aligned} \mu_s * \mu_s(\xi, \tau) = \frac{2\pi}{|\xi|} & \left(|\xi| \left(1 + \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{\frac{1}{2}} \mathbb{1}_{\{|\xi| < \sqrt{\tau^2 + s^2} - s\}} + \tau \mathbb{1}_{\left\{ \sqrt{\tau^2 + s^2} - s \leq |\xi| \leq \sqrt{\tau^2 + (2s)^2} \right\}} \right. \\ & \left. + (\tau - |\xi|) \left(1 + \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{\frac{1}{2}} \mathbb{1}_{\left\{ \sqrt{\tau^2 + (2s)^2} < |\xi| \leq \sqrt{\tau^2 + s^2} + s \right\}} \right). \end{aligned} \tag{3.1}$$

When $\xi = 0$ and $\tau > 0$ we understand that in (3.1) $\mu_s * \mu_s(0, \tau) = 2\pi \left(1 + \frac{4s^2}{\tau^2} \right)^{1/2}$.

We postpone the proof of Proposition 3.1 and study the behavior of $\mu_s * \mu_s(\xi, \tau)$ for large τ .

Lemma 3.2 *For all $\tau > 0$,*

$$2\pi \left(1 + \frac{4s^2}{\tau^2} \right)^{1/2} \leq \sup_{\xi \in \mathbb{R}^3} \mu_s * \mu_s(\xi, \tau) \leq 2\pi \left(1 + \frac{2s}{\tau} \right).$$

In particular

$$\lim_{\tau \rightarrow \infty} \sup_{\xi \in \mathbb{R}^3} \mu_s * \mu_s(\xi, \tau) = 2\pi.$$

Proof We start by noting that

$$\mu_s * \mu_s(s\xi, s\tau) = \mu * \mu(\xi, \tau),$$

hence it is enough to consider the case $s = 1$. We analyze the different cases in formula (3.1).

Case 1: $|\xi| < \sqrt{\tau^2 + 1} - 1$. Then

$$\left(1 + \frac{4}{\tau^2}\right)^{1/2} \leq \left(1 + \frac{4}{\tau^2 - |\xi|^2}\right)^{1/2} \leq \left(\frac{\sqrt{\tau^2 + 1} + 1}{\sqrt{\tau^2 + 1} - 1}\right)^{1/2} = \frac{\tau}{\sqrt{\tau^2 + 1} - 1}.$$

Case 2: $\sqrt{\tau^2 + 1} - 1 \leq |\xi| \leq \sqrt{\tau^2 + 4}$. Then

$$\frac{\tau}{\sqrt{\tau^2 + 4}} \leq \frac{\tau}{|\xi|} \leq \frac{\tau}{\sqrt{\tau^2 + 1} - 1}.$$

Case 3: $\sqrt{\tau^2 + 4} < |\xi| \leq \sqrt{\tau^2 + 1} + 1$. Then $|\xi|^2 - \tau^2 > 4$ and

$$|\xi| \mapsto \frac{\tau}{|\xi|} - \left(1 + \frac{4}{\tau^2 - |\xi|^2}\right)^{1/2},$$

is a decreasing function of $|\xi|$. Then

$$\frac{\tau}{|\xi|} - \left(1 + \frac{4}{\tau^2 - |\xi|^2}\right)^{1/2} \leq \frac{\tau}{\sqrt{\tau^2 + 4}},$$

and

$$\frac{\tau}{|\xi|} - \left(1 + \frac{4}{\tau^2 - |\xi|^2}\right)^{1/2} \geq \frac{\tau}{\sqrt{\tau^2 + 1} + 1} - \left(1 - \frac{2}{\sqrt{\tau^2 + 1} + 1}\right)^{1/2} = 0.$$

As a conclusion, for all $\tau > 0$ and $\xi \in \mathbb{R}^3$

$$\mu * \mu(\xi, \tau) \leq \frac{2\pi\tau}{\sqrt{\tau^2 + 1} - 1} = 2\pi \left(\left(1 + \frac{1}{\tau^2}\right)^{1/2} + \frac{1}{\tau} \right) \leq 2\pi \left(1 + \frac{2}{\tau}\right),$$

and for $\tau > 0$

$$\sup_{\xi \in \mathbb{R}^3} \mu * \mu(\xi, \tau) \geq 2\pi \left(1 + \frac{4}{\tau^2}\right)^{1/2}.$$

□

We now turn to the proof of Proposition 3.1.

Proof of Proposition 3.1 Part (i) is a simple calculation and is left to the reader. For part (ii) we start by discussing a change of coordinates that was used in the proof of [22, Lemma 5.1] in the arxiv’s second version of [22]; see also Appendix 3 on the arxiv’s version of [39] where an outline of the computation of the double convolution of the Lorentz invariant measure on the two-sheeted hyperboloid was given using the same technique.

For each fixed $\xi \neq 0$ we consider a spherical coordinate system with axis ξ , that is, each $\eta \in \mathbb{R}^3$ is described as $\eta = (\rho \cos \theta \sin \varphi, \rho \sin \theta \sin \varphi, \rho \cos \varphi)$, where

$\rho = |\eta| \geq 0$, $\varphi \in [0, \pi]$ is the angle between ξ and η and $\theta \in [0, 2\pi]$ is a polar coordinate angle on the plane orthogonal to ξ . Then $d\eta = \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi$.

Define the new variable $\zeta = |\xi - \eta|$, which corresponds to the size of the side opposite to the origin, 0, in the triangle whose vertices are located at 0, ξ and η . Then

$$\zeta^2 = |\xi|^2 + \rho^2 - 2|\xi|\rho \cos \varphi.$$

Changing variables from φ to ζ , gives $\zeta \, d\zeta = |\xi| \rho \sin \varphi \, d\varphi$, so that in the variables (ρ, ζ, θ) we have $d\eta = \frac{\rho \zeta}{|\xi|} \, d\rho \, d\zeta \, d\theta$. The range of ζ can be seen by using that ζ , $|\xi|$ and ρ are the sizes of the sides of a triangle, so $|\rho - \zeta| \leq |\xi| \leq \rho + \zeta$, which translates into $||\xi| - \rho| \leq \zeta \leq |\xi| + \rho$.

Using delta calculus (see for instance the survey article [24]) and the previous change of variables we have

$$\begin{aligned} \mu_s * \mu_s(\xi, \tau) &= \int_{\substack{\eta \in \mathbb{R}^3 \\ |\eta| \geq s \\ |\xi - \eta| \geq s}} \frac{\delta\left(\tau - \sqrt{|\xi - \eta|^2 - s^2} - \sqrt{|\eta|^2 - s^2}\right)}{\sqrt{|\xi - \eta|^2 - s^2} \sqrt{|\eta|^2 - s^2}} \, d\eta \\ &= \frac{2\pi}{|\xi|} \int_{\substack{|\rho - \zeta| \leq |\xi| \\ \rho + \zeta \geq |\xi| \\ \rho \geq s, \zeta \geq s}} \frac{\delta\left(\tau - \sqrt{\zeta^2 - s^2} - \sqrt{\rho^2 - s^2}\right)}{\sqrt{\zeta^2 - s^2} \sqrt{\rho^2 - s^2}} \rho \zeta \, d\rho \, d\zeta \\ &= \frac{2\pi}{|\xi|} \int_{R_s} \delta(\tau - u - v) \, du \, dv, \end{aligned}$$

where we changed variables $u = \sqrt{\rho^2 - s^2}$, $v = \sqrt{\zeta^2 - s^2}$ and $R_s = R_s(\xi)$ is the image of the region $\{(\rho, \zeta) : |\rho - \zeta| \leq |\xi|, \rho + \zeta \geq |\xi|, \rho \geq s, \zeta \geq s\}$ under the transformation $(\rho, \zeta) \mapsto (u, v)$. Using the change of variables $a = u - v$, $b = u + v$, so that $2 \, du \, dv = da \, db$, we obtain

$$\mu_s * \mu_s(\xi, \tau) = \frac{\pi}{|\xi|} \int_{\tilde{R}_s} \delta(\tau - b) \, da \, db = \frac{\pi}{|\xi|} |\tilde{R}_s \cap \tilde{\ell}_\tau| = \frac{\pi}{|\xi|} \sqrt{2} |R_s \cap \ell_\tau|, \tag{3.2}$$

where $\tilde{R}_s = \tilde{R}_s(\xi)$ is the image of $R_s(\xi)$ under the map $(u, v) \mapsto (a, b)$, $\tilde{\ell}_\tau$ is the horizontal line $\{(a, b) \in \mathbb{R}^2 : b = \tau\}$, ℓ_τ is the line $\{(u, v) \in \mathbb{R}^2 : u + v = \tau\}$ and $|R_s \cap \ell_\tau|$ denotes the measure of $R_s \cap \ell_\tau$ as a subset of ℓ_τ with the induced Lebesgue measure. In order to calculate $|R_s \cap \ell_\tau|$ we divide the analysis into two cases.

Case 1: $|\xi| \leq 2s$. The boundary of the region

$$\left\{ (\rho, \zeta) : |\rho - \zeta| \leq |\xi|, \rho + \zeta \geq |\xi|, \rho \geq s, \zeta \geq s \right\},$$

consists of two (bounded) line segments and two half lines. Its image in the (u, v) -plane, R_s , is bounded by two line segments and two curves and is symmetric with respect to the diagonal $u = v$. The line segments have equations

$$\left\{ (u, v): u = 0, 0 \leq v \leq \sqrt{(|\xi| + s)^2 - s^2} \right\}, \left\{ (u, v): 0 \leq u \leq \sqrt{(|\xi| + s)^2 - s^2}, v = 0 \right\},$$

and the curves have equations

$$\left\{ (u, v): u \geq 0, v = \left((\sqrt{u^2 + s^2} + |\xi|)^2 - s^2 \right)^{1/2} \right\},$$

$$\left\{ (u, v): u \geq \left((|\xi| + s)^2 - s^2 \right)^{1/2}, v = \left((\sqrt{u^2 + s^2} - |\xi|)^2 - s^2 \right)^{1/2} \right\}. \tag{3.3}$$

Then $|R_s \cap \ell_\tau|$ is given by

$$|R_s \cap \ell_\tau| = \begin{cases} \sqrt{2}\tau & , \text{ if } 0 \leq \tau \leq \sqrt{(|\xi| + s)^2 - s^2} \\ \sqrt{2}|u - v| & , \text{ if } \tau > \sqrt{(|\xi| + s)^2 - s^2}, \end{cases}$$

where in the last expression u and v are related to (ξ, τ) by the equations $u + v = \tau$ and $v = \left((\sqrt{u^2 + s^2} + |\xi|)^2 - s^2 \right)^{1/2}$. Therefore

$$\begin{aligned} \sqrt{2}|R_s \cap \ell_\tau| &= 2\tau \mathbb{1}_{\{\tau \leq \sqrt{(|\xi| + s)^2 - s^2}\}} \\ &+ 2\left((\sqrt{u_1(\xi, \tau)^2 + s^2} + |\xi|)^2 - s^2 \right)^{1/2} - u_1(\xi, \tau) \mathbb{1}_{\{\tau > \sqrt{(|\xi| + s)^2 - s^2}\}}, \end{aligned}$$

where $u_1(\xi, \tau)$ and (ξ, τ) are related by the expression

$$\tau = u_1(\xi, \tau) + \left((\sqrt{u_1(\xi, \tau)^2 + s^2} + |\xi|)^2 - s^2 \right)^{1/2}, \tag{3.4}$$

and $0 \leq u_1(\xi, \tau) \leq \frac{\tau}{2}$.

Case 2: $|\xi| > 2s$. Now the boundary of the region $\{(\rho, \varsigma): |\rho - \varsigma| \leq |\xi|, \rho + \varsigma \geq |\xi|, \rho \geq s, \varsigma \geq s\}$ consists of three (bounded) line segments and two half lines and the region R_s is now bounded by two line segments and three curves. The line segments have equations

$$\left\{ (u, v): u = 0, \sqrt{(|\xi| - s)^2 - s^2} \leq v \leq \sqrt{(|\xi| + s)^2 - s^2} \right\},$$

$$\left\{ (u, v): \sqrt{(|\xi| - s)^2 - s^2} \leq u \leq \sqrt{(|\xi| + s)^2 - s^2}, v = 0 \right\}.$$

The next two curves have equations as in (3.3). The last boundary curve is the image of the segment $\{(\rho, \varsigma) : \rho + \varsigma = |\xi|, s \leq \rho \leq |\xi| - s\}$. Its equation is

$$\left\{ (u, v) : 0 \leq u \leq ((|\xi| - s)^2 - s^2)^{1/2}, v = ((|\xi| - \sqrt{u^2 + s^2})^2 - s^2)^{1/2} \right\},$$

and note that it is the graph of a strictly decreasing and concave function of u . It follows that

$$|R_s \cap \ell_\tau| = \begin{cases} \sqrt{2}(\tau - |u_2 - v_2|) & , \text{ if } \sqrt{(|\xi| - s)^2 - s^2} \leq \tau \leq \sqrt{|\xi|^2 - (2s)^2}, \\ \sqrt{2}\tau & , \text{ if } \sqrt{|\xi|^2 - (2s)^2} \leq \tau \leq \sqrt{(|\xi| + s)^2 - s^2}, \\ \sqrt{2}|u_1 - v_1| & , \text{ if } \tau \geq \sqrt{(|\xi| + s)^2 - s^2}, \end{cases}$$

where $(u_1, v_1), (u_2, v_2)$ are the solutions to the equations $u_1 + v_1 = \tau, u_2 + v_2 = \tau, v_1 = \left(\left(\sqrt{u_1^2 + s^2} + |\xi| \right)^2 - s^2 \right)^{1/2}$ and $v_2 = \left(\left(|\xi| - \sqrt{u_2^2 + s^2} \right)^2 - s^2 \right)^{1/2}$.

Then

$$\begin{aligned} \sqrt{2}|R_s \cap \ell_\tau| &= 2 \left(\tau - \left(\left(\left(|\xi| - \sqrt{u_2(\xi, \tau)^2 + s^2} \right)^2 - s^2 \right)^{1/2} \right. \right. \\ &\quad \left. \left. - u_2(\xi, \tau) \right) \right) \mathbb{1}_{\{\sqrt{(|\xi|-s)^2-s^2} \leq \tau < \sqrt{|\xi|^2-(2s)^2}\}} \\ &\quad + 2\tau \mathbb{1}_{\{\sqrt{|\xi|^2-(2s)^2} \leq \tau \leq \sqrt{(|\xi|+s)^2-s^2}\}} \\ &\quad + 2 \left(\left(\left(\sqrt{u_1(\xi, \tau)^2 + s^2} + |\xi| \right)^2 - s^2 \right)^{1/2} - u_1(\xi, \tau) \right) \mathbb{1}_{\{\tau > \sqrt{(|\xi|+s)^2-s^2}\}}, \end{aligned}$$

where $u_1(\xi, \tau)$ is as in (3.4) and $u_2(\xi, \tau)$ and (ξ, τ) are related by the expression

$$\tau = u_2(\xi, \tau) + \left(\left(\sqrt{u_2(\xi, \tau)^2 + s^2} - |\xi| \right)^2 - s^2 \right)^{1/2},$$

and $0 \leq u_2(\xi, \tau) \leq \frac{\tau}{2}$. Algebraic manipulation shows that for (ξ, τ) in their respective domains of definition

$$\tau - 2u_i(\xi, \tau) = |\xi| \left(1 + \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{1/2}, \quad i = 1, 2. \tag{3.5}$$

Collecting all in one expression we have

$$\begin{aligned} \sqrt{2}|R_s \cap \ell_\tau| &= 2\tau \mathbb{1}_{\{\tau \leq \sqrt{(|\xi|+s)^2-s^2}\}} \mathbb{1}_{\{|\xi| \leq 2s\}} \\ &\quad + 4u_2(\xi, \tau) \mathbb{1}_{\{\sqrt{(|\xi|-s)^2-s^2} \leq \tau < \sqrt{|\xi|^2-(2s)^2}\}} \mathbb{1}_{\{|\xi| > 2s\}} \\ &\quad + 2\tau \mathbb{1}_{\{\sqrt{|\xi|^2-(2s)^2} \leq \tau \leq \sqrt{(|\xi|+s)^2-s^2}\}} \mathbb{1}_{\{|\xi| > 2s\}} \end{aligned}$$

$$+ 2(\tau - 2u_1(\xi, \tau)) \mathbb{1}_{\{\tau > \sqrt{(|\xi|+s)^2 - s^2}\}}.$$

Replacing $u_1(\xi, \tau)$ and $u_2(\xi, \tau)$ using (3.5) we obtain using (3.2)

$$\begin{aligned} \mu_s * \mu_s(\tau, \xi) &= \frac{2\pi}{|\xi|} \left(\tau \mathbb{1}_{\{\tau \leq \sqrt{(|\xi|+s)^2 - s^2}\}} \mathbb{1}_{\{|\xi| \leq 2s\}} \right. \\ &\quad + \left(\tau - |\xi| \left(1 + \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{1/2} \right) \mathbb{1}_{\{\sqrt{(|\xi|-s)^2 - s^2} \leq \tau < \sqrt{|\xi|^2 - (2s)^2}\}} \mathbb{1}_{\{|\xi| > 2s\}} \\ &\quad + \tau \mathbb{1}_{\{\sqrt{|\xi|^2 - (2s)^2} \leq \tau \leq \sqrt{(|\xi|+s)^2 - s^2}\}} \mathbb{1}_{\{|\xi| > 2s\}} \\ &\quad \left. + |\xi| \left(1 + \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{1/2} \mathbb{1}_{\{\tau > \sqrt{(|\xi|+s)^2 - s^2}\}} \right). \end{aligned} \tag{3.6}$$

Rearranging (3.6) we find that $\mu_s * \mu_s$ can be written in the equivalent form (3.1). □

More generally, the same method used in the proof of Proposition 3.1 allows us to write an explicit formula for $\mu_s * \mu_t$, for any $s, t \geq 0$. For instance, as it will be useful in Sect. 12, we have

$$\mu_s * \mu_0(\xi, \tau) = \frac{\pi}{|\xi|} \int_{\tilde{Q}_s(\xi)} \delta(\tau - b) da db, \tag{3.7}$$

where $\tilde{Q}_s(\xi)$ is the image of the set $\{(\rho, \varsigma) : |\rho - \varsigma| \leq |\xi|, \rho + \varsigma \geq |\xi|, \rho \geq 0, \varsigma \geq s\}$ under the transformations $(\rho, \varsigma) \mapsto (u, v) = (\rho, \sqrt{\varsigma^2 - s^2}) \mapsto (a, b) = (u - v, u + v)$. Here μ_0 equals σ_c , the Lorentz invariant measure on the cone. A calculation similar to the one for $\mu_s * \mu_s$ gives the following explicit formula

$$\begin{aligned} \mu_s * \sigma_c(\xi, \tau) &= \frac{2\pi}{|\xi|} \left(\frac{|\xi|(\tau^2 - |\xi|^2 + s^2)}{\tau^2 - |\xi|^2} \mathbb{1}_{\{\tau \geq s\}} \mathbb{1}_{\{|\xi| < \tau - s\}} \right. \\ &\quad + \frac{(\tau + |\xi|)^2 - s^2}{2(\tau + |\xi|)} \mathbb{1}_{\{\tau \geq 0\}} \mathbb{1}_{\{|\tau - s| \leq |\xi| < \sqrt{\tau^2 + s^2}\}} \\ &\quad \left. + \frac{s^2 - (|\xi| - \tau)^2}{2(|\xi| - \tau)} \mathbb{1}_{\{\tau \geq 0\}} \mathbb{1}_{\{\sqrt{\tau^2 + s^2} \leq |\xi| \leq \tau + s\}} \right). \end{aligned} \tag{3.8}$$

Using (3.8) we see that for each $\tau \geq 0$

$$\sup_{\xi \in \mathbb{R}^3} \mu_s * \sigma_c(\xi, \tau) = 2\pi \cdot \begin{cases} \frac{\tau}{\sqrt{\tau^2 + s^2}}, & 0 \leq \tau \leq \frac{s}{2}(2(\sqrt{5} - 1))^{1/2}, \\ 1 + \frac{(s - \sqrt{s^2 - \tau^2})^2}{\tau^2}, & \frac{s}{2}(2(\sqrt{5} - 1))^{1/2} \leq \tau \leq s, \\ 1 + \frac{s}{2\tau - s}, & \tau \geq s, \end{cases} \tag{3.9}$$

and $\|\mu_s * \sigma_c\|_{L^\infty(\mathbb{R}^4)} = 4\pi$.

The methods introduced in this section allow us to write explicit formulas for double convolutions of the form $f\mu_s * f\mu_s$, for f regular enough (continuous, for instance) similar to those for the sphere [13, pp. 282]. Indeed, unwinding the changes of variables leading to (3.2) in the proof of Proposition 3.1 (as well as the ones leading to (3.7)), for $\xi \in \mathbb{R}^3 \setminus \{0\}$ we let

$$\begin{aligned} \alpha_s(a, b, \theta) &= \frac{|\xi|^2 + ab}{|\xi|\sqrt{(a+b)^2 + 4s^2}}, & \beta_s(a, b, \theta) &= \frac{|\xi|^2 + ab - s^2}{|\xi|(a+b)}, \\ \omega_s(a, b, \theta) &= (\sqrt{1 - \alpha_s(a, b, \theta)^2} \cos \theta, \sqrt{1 - \alpha_s(a, b, \theta)^2} \sin \theta, \alpha_s(a, b, \theta)), \\ \vartheta_s(a, b, \theta) &= (\sqrt{1 - \beta_s(a, b, \theta)^2} \cos \theta, \sqrt{1 - \beta_s(a, b, \theta)^2} \sin \theta, \beta_s(a, b, \theta)), \end{aligned}$$

and

$$\begin{aligned} F_s(a, b) &= \int_0^{2\pi} f\left(\xi - \sqrt{\frac{(a+b)^2}{2} + s^2} \omega_s(a, b, \theta)\right) f\left(\sqrt{\frac{(a+b)^2}{2} + s^2} \omega_s(a, b, \theta)\right) d\theta, \\ G_s(a, b) &= \int_0^{2\pi} f\left(\xi - \frac{a+b}{2} \vartheta_s(a, b, \theta)\right) f\left(\frac{a+b}{2} \vartheta_s(a, b, \theta)\right) d\theta, \\ H_0(a, b) &= \int_0^{2\pi} f\left(\xi - \frac{a+b}{2} \omega_0(a, b, \theta)\right) f\left(\frac{a+b}{2} \omega_0(a, b, \theta)\right) d\theta. \end{aligned}$$

Recalling the sets $\tilde{R}_s(\xi)$ and $\tilde{Q}_s(\xi)$ from (3.2) and (3.7) we have

$$f\mu_s * f\mu_s(\xi, \tau) = \frac{1}{2|\xi|} \int_{\tilde{R}_s(\xi)} F_s(a, b) \delta(\tau - b) da db = \frac{1}{2|\xi|} \int_{\mathbb{R}} F_s(a, \tau) \mathbb{1}_{\tilde{R}_s(\xi)}(a, \tau) da, \tag{3.10}$$

$$f\mu_s * f\sigma_c(\xi, \tau) = \frac{1}{2|\xi|} \int_{\tilde{Q}_s(\xi)} G_s(a, b) \delta(\tau - b) da db = \frac{1}{2|\xi|} \int_{\mathbb{R}} G_s(a, \tau) \mathbb{1}_{\tilde{Q}_s(\xi)}(a, \tau) da, \tag{3.11}$$

and

$$\begin{aligned} f\sigma_c * f\sigma_c(\xi, \tau) &= \frac{1}{2|\xi|} \int_{\{(a,b): |a| \leq |\xi| \leq b\}} H_0(a, b) \delta(\tau - b) da db \\ &= \frac{1}{2|\xi|} \int_{\mathbb{R}} H_0(a, \tau) \mathbb{1}_{\{(a,b): |a| \leq |\xi| \leq b\}}(a, \tau) da. \end{aligned} \tag{3.12}$$

It is worth noting that $\mathbb{1}_{\tilde{Q}_s(\xi)} \rightarrow \mathbb{1}_{\{(a,b): |a| \leq |\xi| \leq b\}}$ and $\mathbb{1}_{\tilde{R}_s(\xi)} \rightarrow \mathbb{1}_{\{(a,b): |a| \leq |\xi| \leq b\}}$ pointwise in \mathbb{R}^2 as $s \rightarrow 0^+$. Moreover, when f is continuous, $F_s \rightarrow H_0$ and $G_s \rightarrow H_0$ pointwise in the region $\{(a, b) : a + b \geq 0\} \subset \mathbb{R}^2$, as $s \rightarrow 0^+$.

4 Comparison with the Cone

Recall that σ_c denotes the scale and Lorentz invariant measure on the cone Γ^3 and T_c denotes its associated adjoint Fourier restriction operator. From [5] we know the value of the sharp constant

$$\sup_{0 \neq f \in L^2(\sigma_c)} \frac{\|f\sigma_c * f\sigma_c\|_{L^4(\mathbb{R}^4)}^2}{\|f\|_{L^2(\sigma_c)}^4} = 2\pi. \tag{4.1}$$

We had defined the numerical constants

$$\begin{aligned} \mathbf{C}_4 &= \sup_{0 \neq f \in L^2(\sigma_c)} \frac{\|T_c f\|_{L^4(\mathbb{R}^4)}}{\|f\|_{L^2(\sigma_c)}} = 2\pi \sup_{0 \neq f \in L^2(\sigma)} \frac{\|f\sigma_c * f\sigma_c\|_{L^4(\mathbb{R}^4)}^{1/2}}{\|f\|_{L^2(\sigma_c)}}, \\ \mathbf{H}_4 &= \sup_{0 \neq f \in L^2(\mu)} \frac{\|T f\|_{L^4(\mathbb{R}^4)}}{\|f\|_{L^2(\mu)}} = 2\pi \sup_{0 \neq f \in L^2(\mu)} \frac{\|f\mu * f\mu\|_{L^4(\mathbb{R}^4)}^{1/2}}{\|f\|_{L^2(\mu)}}. \end{aligned}$$

The next proposition gives a comparison between \mathbf{C}_4 and \mathbf{H}_4 and its role is the analog of the comparison of the best constant for the sphere and the paraboloid in \mathbb{R}^3 as used in [13] where a strict inequality was needed to rule out concentration at a pair of antipodal points. In our present case, a strict inequality will rule out concentration at infinity.

Proposition 4.1 $\mathbf{H}_4 > \mathbf{C}_4$.

Proof For $s > 0$ we consider the family of trial functions $f_a(y) = e^{-\frac{a}{2}\sqrt{|y|^2 - s^2}}$, $a > 0$, and claim that

$$\sup_{a > 0} \frac{\|T_s f_a\|_{L^4(\mathbb{R}^4)}}{\|f_a\|_{L^2(\mathcal{H}_s^3)}} > \sup_{0 \neq f \in L^2(\sigma_c)} \frac{\|T_c f\|_{L^4(\mathbb{R}^4)}}{\|f\|_{L^2(\sigma_c)}}.$$

Using spherical coordinates, the $L^2(\mathcal{H}_s^3)$ -norm of f_a is given by the expression

$$\begin{aligned} \|f_a\|_{L^2(\mathcal{H}_s^3)}^2 &= \int_{\mathbb{R}^3} e^{-a\sqrt{|x|^2 - s^2}} \frac{dx}{\sqrt{|x|^2 - s^2}} = 4\pi \int_s^\infty e^{-a\sqrt{r^2 - s^2}} \frac{r^2}{\sqrt{r^2 - s^2}} dr \\ &= 4\pi \int_0^\infty e^{-a\tau} \sqrt{\tau^2 + s^2} d\tau. \end{aligned}$$

It is easier to estimate $\|T_s f_a\|_{L^4(\mathbb{R}^4)}$ if we use the convolution form (2.3),

$$\|T_s f_a\|_{L^4(\mathbb{R}^4)} = 2\pi \|f_a \mu_s * f_a \mu_s\|_{L^2(\mathbb{R}^4)}^{1/2}.$$

As in [39, Appendix 2], using that f_a is the restriction to \mathcal{H}_s^3 of the exponential of the linear function in \mathbb{R}^4 , $(\xi, \tau) \mapsto e^{-\frac{a}{2}\tau}$, we obtain

$$f_a \mu_s * f_a \mu_s(\xi, \tau) = e^{-\frac{a}{2}\tau} (\mu_s * \mu_s(\xi, \tau)).$$

It will be enough for our purpose to use

$$\mu_s * \mu_s(\xi, \tau) \geq \frac{2\pi}{|\xi|} \left(|\xi| \left(1 + \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{\frac{1}{2}} \mathbb{1}_{\{|\xi| < \sqrt{\tau^2 + s^2} - s\}} + \tau \mathbb{1}_{\{\sqrt{\tau^2 + s^2} - s \leq |\xi| \leq \sqrt{\tau^2 + (2s)^2}\}} \right),$$

as obtained from (3.1). In this way

$$\begin{aligned} & f_a \mu_s * f_a \mu_s(\xi, \tau) \\ & \geq \frac{2\pi}{|\xi|} e^{-\frac{a}{2}\tau} \left(|\xi| \left(1 + \frac{4s^2}{\tau^2 - |\xi|^2} \right)^{\frac{1}{2}} \mathbb{1}_{\{|\xi| < \sqrt{\tau^2 + s^2} - s\}} + \tau \mathbb{1}_{\{\sqrt{\tau^2 + s^2} - s \leq |\xi| \leq \sqrt{\tau^2 + (2s)^2}\}} \right), \end{aligned}$$

so that using spherical coordinates we obtain

$$\begin{aligned} \|f_a \mu_s * f_a \mu_s\|_{L^2(\mathbb{R}^4)}^2 & \geq (2\pi)^2 \int_{\mathbb{R}^3 \times \mathbb{R}} e^{-a\tau} \left(|\xi|^2 \left(1 + \frac{4s^2}{\tau^2 - |\xi|^2} \right) \mathbb{1}_{\{|\xi| < \sqrt{\tau^2 + s^2} - s\}} \right. \\ & \quad \left. + \tau^2 \mathbb{1}_{\{\sqrt{\tau^2 + s^2} - s \leq |\xi| \leq \sqrt{\tau^2 + (2s)^2}\}} \right) d\tau \frac{d\xi}{|\xi|^2} \\ & = 16\pi^3 \int_0^\infty e^{-a\tau} \left(\tau^2 \sqrt{\tau^2 + 4s^2} - \frac{2}{3}(\tau^2 + 4s^2)\sqrt{\tau^2 + s^2} + \frac{8s^3}{3} \right. \\ & \quad \left. + 2s^2\tau \log\left(\frac{\tau + \sqrt{\tau^2 + s^2}}{s}\right) \right) d\tau. \end{aligned}$$

Since by scaling it is enough to consider $s = 1$ (see Sect. 14) we let

$$\begin{aligned} I(a) & = 16\pi^3 \int_0^\infty e^{-a\tau} \left(\tau^2 \sqrt{\tau^2 + 4} - \frac{2}{3}(\tau^2 + 4)\sqrt{\tau^2 + 1} + \frac{8}{3} \right. \\ & \quad \left. + 2\tau \log(\tau + \sqrt{\tau^2 + 1}) \right) d\tau, \\ II(a) & = 16\pi^2 \left(\int_0^\infty e^{-a\tau} \sqrt{\tau^2 + s^2} d\tau \right)^2, \end{aligned}$$

then

$$\frac{\|f_a \mu * f_a \mu\|_{L^2(\mathbb{R}^4)}^2}{\|f_a\|_{L^2(\mu)}^4} \geq \frac{I(a)}{II(a)}.$$

From Lemma A.1 in the Appendix, we conclude that for all $a > 0$ small enough

$$\frac{\|f_a \mu * f_a \mu\|_{L^2(\mathbb{R}^4)}^2}{\|f_a\|_{L^2(\mu)}^4} > 2\pi. \tag{4.2}$$

This finishes the proof in view of (4.1). □

Remark 4.2 The easy lower bound we can obtain for $\|f_a \mu * f_a \mu\|_{L^2(\mathbb{R}^4)}^2 \|f_a\|_{L^2(\mu)}^{-4}$ using the analog of [34, Lemma 6.1] is not good enough in this case to obtain (4.2).

Let us now move to the full one-sheeted hyperboloid $\overline{\mathcal{H}}^3$. Recall that \overline{T}_c denotes the adjoint Fourier restriction operator on the double cone $\overline{\Gamma}^3$. An argument in [22] can be used to show that

$$\overline{C}_4 = \left(\frac{3}{2}\right)^{\frac{1}{4}} C_4, \tag{4.3}$$

see for instance [39, Prop. 7.3]. We now compare the best constants for \overline{T} and \overline{T}_c .

Proposition 4.3 $\overline{H}_4 > \overline{C}_4$.

Proof Let $f_a(y) = e^{-\frac{a}{2}\sqrt{|y|^2-1}}$ be as in the proof of Proposition 4.1 and set $g_a = f_{a,+} + f_{a,-}$, where $f_{a,+} = cf_a$ and $f_{a,-} = cf_a$ (here there are domain identifications through projections to \mathbb{R}^3), in other words, $g_a(\xi, \tau) = ce^{-\frac{a}{2}|\tau|} \mathbb{1}_{\overline{\mathcal{H}}^3}(\xi, \tau)$, where c is such that g_a is L^2 normalized. Expanding and using the positivity of $f_{a,+}$ and $f_{a,-}$ (which for brevity we simply call f_+ and f_- , respectively) we see that

$$\begin{aligned} \|\overline{T}g_a\|_{L^4}^4 &= \|Tf_+\|_{L^4(\mathbb{R}^3)}^4 + \|Tf_-\|_{L^4(\mathbb{R}^3)}^4 + 4\|(Tf_+)(Tf_-(\cdot, -\cdot))\|_{L^2}^2 \\ &\quad + 4(2\pi)^4 \langle f_+\mu * f_+\mu, f_+\mu * f_-\mu \rangle \\ &\quad + 4(2\pi)^4 \langle f_+\mu * f_-\mu, f_-\mu * f_-\mu \rangle \\ &\geq \|Tf_+\|_{L^4(\mathbb{R}^3)}^4 + \|Tf_-\|_{L^4(\mathbb{R}^3)}^4 + 4\|(Tf_+)(Tf_-(\cdot, -\cdot))\|_{L^2}^2. \end{aligned}$$

On the other hand $Tf_-(\cdot, -\cdot) = \overline{Tf_+}$, the complex conjugate, since $f_-(y) = f_+(-y)$. Then $\|(Tf_+)(Tf_-(\cdot, -\cdot))\|_{L^2}^2 = \|Tf_+\|_{L^4(\mathbb{R}^3)}^4 = \|Tf_-\|_{L^4(\mathbb{R}^3)}^4$ and we obtain

$$\|\overline{T}g_a\|_{L^4}^4 \geq 6\|Tf_{a,+}\|_{L^4(\mathbb{R}^3)}^4.$$

If $a > 0$ is small enough, then from (4.2) in the proof of Proposition 4.1 and using $\|f_{a,+}\|_{L^2(\mu)} = \sqrt{2}/2$, we obtain

$$\overline{H}_4^4 \geq \|\overline{T}g_a\|_{L^4}^4 \geq 6\|Tf_{a,+}\|_{L^4(\mathbb{R}^3)}^4 > 6C_4^4 \|f_{a,+}\|_{L^2(\mu)}^4 = \frac{3}{2}C_4^4.$$

The conclusion follows using (4.3). □

5 The Upper Half of the One-Sheeted Hyperboloid

In this section we present the proof of Theorem 1.2. The proof of precompactness of extremizing sequences, modulo multiplication by characters, is much simpler for the upper half of the one-sheeted hyperboloid as the full Lorentz invariance of $\overline{\mathcal{H}^3}$ is absent for \mathcal{H}^3 .

In what follows we collect the necessary results to invoke Proposition 1.5 and the first such step is to show that, with enumeration as in Proposition 1.5, (i) and (iii) imply (iv), possibly after passing to a subsequence.

Proposition 5.1 *Let $\{f_n\}_n$ be a sequence in $L^2(\mathcal{H}^3)$ satisfying $\sup_n \|f_n\|_{L^2(\mathcal{H}^3)} < \infty$. Suppose that there exists $f \in L^2(\mathcal{H}^3)$ such that $f_n \rightharpoonup f$ as $n \rightarrow \infty$. Then, there exists a subsequence $\{f_{n_k}\}_k$ such that $Tf_{n_k} \rightarrow Tf$ a.e. in \mathbb{R}^4 .*

The previous result implies an analogous one for the full two-sheeted hyperboloid $\overline{\mathcal{H}^3}$. Recall the Fourier multiplier notation

$$e^{it\sqrt{-\Delta-s^2}}u(x) = \frac{1}{(2\pi)^3} \int_{\{y \in \mathbb{R}^3 : |y| \geq s\}} e^{ix \cdot y} e^{it\sqrt{|y|^2-s^2}} \hat{u}(y) dy, \tag{5.1}$$

and the homogeneous $\dot{H}^{1/2}(\mathbb{R}^3)$ Sobolev norm and inner product

$$\|u\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^2 := \int_{\mathbb{R}^3} |\hat{u}(y)|^2 |y| dy, \quad \langle u, v \rangle_{\dot{H}^{1/2}(\mathbb{R}^3)} := \int_{\mathbb{R}^3} \hat{u}(y) \overline{\hat{v}(y)} |y| dy. \tag{5.2}$$

Proof of Proposition 5.1 The proof follows similar lines to the proofs of [21, Thm. 1.1] and [38, Prop. 8.3]. We start by splitting $f_n = f_n \mathbb{1}_{B(0,2)} + f_n \mathbb{1}_{B(0,2)^c} =: f_{n,1} + f_{n,2}$, respectively, and $f = f \mathbb{1}_{B(0,2)} + f \mathbb{1}_{B(0,2)^c} =: f_1 + f_2$. The conclusion of the proposition will follow if we show that there exists a subsequence $\{f_{n_k}\}_k$ such that $Tf_{n_k,1} \rightarrow Tf_1$ and $Tf_{n_k,2} \rightarrow Tf_2$ a.e. in \mathbb{R}^4 .

Since $f_{n,1} \rightharpoonup f_1$ in $L^2(\mathcal{H}^3)$ and the support of $f_{n,1}$ is contained on the compact set $B(0, 2)$, it follows that $Tf_{n,1}(x, t) \rightarrow Tf_1(x, t)$ for all $(x, t) \in \mathbb{R}^4$ provided that the function $(y, s) \mapsto e^{ix \cdot y} e^{its} \mathbb{1}_{B(0,2)}(y)$ belongs to $L^2(\mathcal{H}^3)$, which is the case.

To study the pointwise convergence of $Tf_{n,2}$ define g_n and g by their Fourier transforms as follows

$$\hat{g}_n(y) = \frac{f_{n,2}(y)}{\sqrt{|y|^2 - 1}}, \quad \hat{g}(y) = \frac{f_2(y)}{\sqrt{|y|^2 - 1}}.$$

Because

$$\|f_{n,2}\|_{L^2(\mathcal{H}^3)}^2 = \int_{\{y \in \mathbb{R}^3 : |y| \geq 2\}} |f_n(y)|^2 \frac{dy}{\sqrt{|y|^2 - 1}} \leq \sup_k \|f_k\|_{L^2(\mathcal{H}^3)}^2 \lesssim 1,$$

we see that the norms of the g_n 's in the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^3)$ are uniformly bounded

$$\|g_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |\hat{g}_n(y)|^2 |y| \, dy \leq \frac{2}{\sqrt{3}} \int_{\{y \in \mathbb{R}^3: |y| \geq 2\}} |f_n(y)|^2 \frac{dy}{\sqrt{|y|^2 - 1}} \lesssim 1.$$

The weak convergence of $\{f_{n,2}\}_n$ to f_2 in $L^2(\mathcal{H}^3)$ easily implies $g_n \rightharpoonup g$ as $n \rightarrow \infty$ in $\dot{H}^{1/2}(\mathbb{R}^3)$. On the other hand

$$(2\pi)^3 \|g_n\|_{L^2(\mathbb{R}^3)}^2 = \|\hat{g}_n\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{1}{\sqrt{3}} \int_{\{y \in \mathbb{R}^3: |y| \geq 2\}} |f_{n,2}(y)|^2 \frac{dy}{\sqrt{|y|^2 - 1}} \lesssim 1,$$

so $\{g_n\}_n$ is uniformly bounded in $L^2(\mathbb{R}^3)$.

The operator T applied to $f_{n,2}$ equals $(2\pi)^3 e^{it\sqrt{-\Delta-1}} g_n$, where the operator $e^{it\sqrt{-\Delta-1}}$ is understood in the Fourier multiplier sense as in (5.1). Let $t \in \mathbb{R}$ be fixed. By the continuity of $e^{it\sqrt{-\Delta-1}}$ in $\dot{H}^{1/2}(\mathbb{R}^3)$ we obtain

$$e^{it\sqrt{-\Delta-1}} g_n \rightharpoonup e^{it\sqrt{-\Delta-1}} g,$$

weakly in $\dot{H}^{1/2}(\mathbb{R}^3)$, as $n \rightarrow \infty$. Then, by the Rellich–Kondrashov Theorem ([17, Thm. 7.1]), for any $R > 0$

$$e^{it\sqrt{-\Delta-1}} g_n \rightarrow e^{it\sqrt{-\Delta-1}} g,$$

strongly in $L^2(B(0, R))$, as $n \rightarrow \infty$. Denote by

$$F_n(t) := \int_{|x| < R} \left| e^{it\sqrt{-\Delta-1}} (g_n - g) \right|^2 dx = \|e^{it\sqrt{-\Delta-1}} (g_n - g)\|_{L^2(B(0,R))}^2.$$

By Hölder’s inequality and Sobolev embedding, [17, Thm. 6.5], we obtain

$$\begin{aligned} F_n(t) &= \|e^{it\sqrt{-\Delta-1}} (g_n - g)\|_{L^2(B(0,R))}^2 \leq CR \|e^{it\sqrt{-\Delta-1}} (g_n - g)\|_{L^3(B(0,R))}^2 \\ &\leq CR \|e^{it\sqrt{-\Delta-1}} (g_n - g)\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^2 \lesssim R, \end{aligned}$$

then, by the Fubini and Dominated Convergence Theorems we have that

$$\int_{-R}^R F_n(t) dt = \int_{-R}^R \int_{|x| < R} \left| e^{it\sqrt{-\Delta-1}} (g_n - g) \right|^2 dx dt \rightarrow 0,$$

as $n \rightarrow \infty$. This implies that, up to a subsequence,

$$e^{it\sqrt{-\Delta-1}}g_n(x) - e^{it\sqrt{-\Delta-1}}g(x) \rightarrow 0 \text{ a.e. } (x, t) \in B(0, R) \times (-R, R).$$

Repeating the argument on a discrete sequence of radii R_n such that $R_n \rightarrow \infty$, as $n \rightarrow \infty$ we conclude, by a diagonal argument, that there exists a subsequence $\{g_{n_k}\}_k$ of $\{g_n\}_n$ such that

$$e^{it\sqrt{-\Delta-1}}g_{n_k}(x) - e^{it\sqrt{-\Delta-1}}g(x) \rightarrow 0 \text{ a.e. for } (x, t) \in \mathbb{R}^4,$$

or equivalently, in terms of the sequence $\{f_{n,2}\}_n$ and the operator T ,

$$Tf_{n_k,2}(x, t) \rightarrow Tf_2(x, t) \text{ a.e. } (x, t) \in \mathbb{R}^4.$$

□

We now show that the only obstruction to precompactness of extremizing sequences is the possibility of concentration at infinity, as in Definition 1.6.

Proposition 5.2 *Let $\{f_n\}_n \subset L^2(\mathcal{H}^3)$ be an L^2 normalized extremizing sequence for T . Suppose that $\{f_n\}_n$ does not concentrate at infinity. Then there exist a subsequence $\{f_{n_k}\}_k$ and a sequence $\{(x_k, t_k)\}_k \subset \mathbb{R}^4$ such that $\{e^{ix_k \cdot y} e^{it_k \sqrt{|y|^2-1}} f_{n_k}\}_k$ is convergent in $L^2(\mathcal{H}^3)$.*

Proof If $\{f_n\}_n$ does not concentrate at infinity, then there exist $\varepsilon, R > 0$ with the property that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that

$$\|f_n \mathbb{1}_{B(0,R)}\|_{L^2(\mathcal{H}^3)} \geq \varepsilon.$$

We can generate a subsequence, $\{f_{n_k}\}_k$, such that $\|f_{n_k} \mathbb{1}_{B(0,R)}\|_{L^2(\mathcal{H}^3)} \geq \varepsilon$ for all $k \in \mathbb{N}$. Rename the subsequence as $\{f_n\}_n$, if necessary. Writing $f_n = f_n \mathbb{1}_{B(0,R)} + f_n \mathbb{1}_{B(0,R)^c} =: f_{n,1} + f_{n,2}$, respectively, we have

$$\begin{aligned} \|Tf_{n,1}\|_{L^4(\mathbb{R}^4)} &= \|T(f_n - f_{n,2})\|_{L^4(\mathbb{R}^4)} \geq \|Tf_n\|_{L^4(\mathbb{R}^4)} - \|Tf_{n,2}\|_{L^4(\mathbb{R}^4)} \\ &\geq \|Tf_n\|_{L^4(\mathbb{R}^3)} - \mathbf{H}_4 \|f_{n,2}\|_{L^2(\mathcal{H}^3)} \\ &= \|Tf_n\|_{L^4(\mathbb{R}^3)} - \mathbf{H}_4 \left(1 - \|f_{n,1}\|_{L^2(\mathcal{H}^3)}^2\right)^{1/2} \\ &\geq \|Tf_n\|_{L^4(\mathbb{R}^3)} - \mathbf{H}_4 \sqrt{1 - \varepsilon^2}. \end{aligned} \tag{5.3}$$

As the right hand side in (5.3) converges to $c := \mathbf{H}_4 - \mathbf{H}_4 \sqrt{1 - \varepsilon^2} > 0$ as $n \rightarrow \infty$ we see that

$$\|Tf_{n,1}\|_{L^4(\mathbb{R}^4)} \geq \frac{c}{2} > 0, \tag{5.4}$$

for all large n .

We may use the argument in the proof of [20, Thm. 1.1] to construct the sequence $\{(x_n, t_n)\}_n$. In brief, the argument goes as follows. Taking any $\bar{p} \in [\frac{10}{3}, 4)$, interpolating the L^4 norm of $Tf_{n,1}$ between $L^{\bar{p}}$ and L^∞ and using (5.4) together with the boundedness of T in $L^{\bar{p}}$ imply that $\|Tf_{n,1}\|_{L^\infty(\mathbb{R}^4)} \gtrsim 1$, so that there exists a sequence $\{(x_n, t_n)\}_n \subset \mathbb{R}^4$ such that $|Tf_{n,1}(x_n, t_n)| \geq C > 0$, that is, $|(T(e^{ix_n \cdot y} e^{it_n \sqrt{|y|^2-1}} f_{n,1}))(0, 0)| \geq C > 0$. The compact support of $f_{n,1}$ implies that $Tf_{n,1}$ belongs to $C^\infty(\mathbb{R}^4)$ and $\|Tf_{n,1}\|_{L^\infty(\mathbb{R}^4)} \lesssim \|f_{n,1}\|_{L^1} \lesssim 1$, $\|\nabla_{x,t} Tf_{n,1}\|_{L^\infty(\mathbb{R}^4)} \lesssim \|f_{n,1}\|_{L^1} \lesssim 1$. By the Arzelá-Ascoli Theorem, it follows that $\{T(e^{ix_n \cdot y} e^{it_n \sqrt{|y|^2-1}} f_{n,1})\}_n$ is precompact in the space of continuous functions on the unit ball of \mathbb{R}^4 . On the other hand, passing to a subsequence, we may assume $F_n := e^{ix_n \cdot y} e^{it_n \sqrt{|y|^2-1}} f_{n,1} \rightharpoonup f_1$ weakly in $L^2(\mathcal{H}^3)$, for some $f_1 \in L^2(\mathcal{H}^3)$, and then $T(F_n)(x, t) \rightarrow Tf_1(x, t)$ for all $(x, t) \in \mathbb{R}^4$. Moreover, $T(F_n) \rightarrow Tf_1$ uniformly in the unit ball, so that $|Tf_1(0, 0)| \geq C > 0$, which implies that $f_1 \neq 0$.

Compactness of the unit ball in $L^2(\mathcal{H}^3)$ in the weak topology implies that, after passing to a further subsequence, $e^{ix_n \cdot y} e^{it_n \sqrt{|y|^2-1}} f_n \rightharpoonup f$, for some $f \in L^2(\mathcal{H}^3)$. Since $f_1 = f \mathbb{1}_{B(0,R)}$ a.e. in \mathbb{R}^3 we conclude that $f \neq 0$. Therefore condition (iii) of Proposition 1.5 is satisfied. Proposition 5.1 implies that condition (iv) is also satisfied. As (i) and (ii) are immediate, we conclude that $e^{ix_n \cdot y} e^{it_n \sqrt{|y|^2-1}} f_n \rightarrow f$ in $L^2(\mathcal{H}^3)$, and we are done. \square

To conclude the precompactness of extremizing sequences we need to discard the possibility of concentration at infinity. For this we use a comparison argument with the cone where the upper bound for $\mu_s * \mu_s$ as found in Lemma 3.2 will be useful.

Lemma 5.3 *Let $a > 1$ and $f \in L^2(\mathcal{H}^3)$ be supported in the region where $|y| \geq a$. Then*

$$\|f\mu * f\mu\|_{L^2(\mathbb{R}^4)}^2 \leq 2\pi \left(1 + \frac{1}{\sqrt{a^2-1}}\right) \|f\|_{L^2(\mu)}^4.$$

Proof If f is supported where $|y| \geq a$, then the support of $f\mu * f\mu$ is contained in the region $\{(\xi, \tau) \in \mathbb{R}^4 : \tau \geq 2\sqrt{a^2-1}\}$. The Cauchy-Schwarz inequality provides the a.e. pointwise bound

$$|f\mu * f\mu|^2(\xi, \tau) \leq (|f|^2\mu * |f|^2\mu)(\xi, \tau)(\mu * \mu)(\xi, \tau),$$

which together with the upper bound in Lemma 3.2 imply

$$|f\mu * f\mu|^2(\xi, \tau) \leq 2\pi \left(1 + \frac{1}{\sqrt{a^2-1}}\right) (|f|^2\mu * |f|^2\mu)(\xi, \tau),$$

for a.e. $(\xi, \tau) \in \mathbb{R}^4$. Integrating over $(\xi, \tau) \in \mathbb{R}^4$ yields the result. \square

It is now direct to prove our first main theorem.

Proof of Theorem 1.2 We start by noting that if an L^2 -normalized sequence $\{f_n\}_n$ concentrates at infinity, then for any $a > 1$, $\|f_n \mathbb{1}_{B(0,a)}\|_{L^2(\mu)} \rightarrow 0$ as $n \rightarrow \infty$, therefore, for such a sequence we obtain, using Lemma 5.3, that

$$\limsup_{n \rightarrow \infty} \frac{\|f_n \mu * f_n \mu\|_{L^2(\mathbb{R}^4)}^2}{\|f_n\|_{L^2(\mu)}^4} \leq 2\pi.$$

Using Proposition 4.1 we conclude that an extremizing sequence for T does not concentrate at infinity. We apply Proposition 5.2 to conclude. \square

6 The Full One-Sheeted Hyperboloid

Our task in the sections to come is to prove Theorem 1.3, the existence of extremals for the adjoint Fourier restriction inequality on the one-sheeted hyperboloid \mathcal{H}^3 . In the L^4 case, there is an argument available for the cone Γ^3 that allows to relate the best constant and extremizers for Γ^3 with that for the double cone $\bar{\Gamma}^3$. It relies on the observation that the algebraic sums $\Gamma^3 + \Gamma^3$ and $\Gamma^3 + (-\Gamma^3)$ intersect on a null set of \mathbb{R}^3 , namely, $(\Gamma^3 + \Gamma^3) \cap (\Gamma^3 + (-\Gamma^3)) = \Gamma^3$, so that for any $f_+, g_+, h_+ \in L^2(\Gamma^3)$ and $f_- \in L^2(-\Gamma^3)$ one has

$$\langle f_+ \sigma_c * g_+ \sigma_c, h_+ \sigma_c * f_- \tilde{\sigma}_c \rangle_{L^2(\mathbb{R}^4)} = 0,$$

where $\tilde{\sigma}_c$ denotes the reflection of σ_c , supported on $-\Gamma^3$. An analogous property in the L^4 case applies to the two-sheeted hyperboloid in \mathbb{R}^4 and allows one to obtain its best constant from that of the upper sheet only (see [39, Prop. 7.3, Cor. 7.4]). This approach is not applicable to $\bar{\mathcal{H}}^3$ because here $\mathcal{H}^3 + \mathcal{H}^3$ and $\mathcal{H}^3 + (-\mathcal{H}^3)$ intersect on a set of infinite Lebesgue measure.

The argument we use to prove precompactness of extremizing sequences (modulo multiplication by characters and Lorentz transformations) is close to that of Brocchi, Oliveira e Silva and the author [1] and of [38] by the author using a concentration-compactness argument, a refined cap estimate, comparison to the cone and the use of Lorentz invariance. It borrows from the Christ–Shao argument [13] the cap refinement of the Tomas–Stein inequality for \mathbb{S}^2 to obtain a similar refinement for the hyperboloid, as well as the understanding that it will be necessary to compare to a “limiting” manifold, in our case, the cone.

In the next section we review the aforementioned cap refinement for the Tomas–Stein inequality for \mathbb{S}^2 that will be used in the subsequent section to obtain a corresponding cap refinement for the adjoint Fourier restriction inequality on the hyperboloid via a lifting method. In later sections we consider the concentration-compactness argument.

7 The Tomas–Stein Inequality for \mathbb{S}^2 and Refinements

The sharp convolution form of the Tomas–Stein inequality for \mathbb{S}^2 states that for all $f \in L^2(\mathbb{S}^2)$ we have

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leq \mathbf{S}^2 \|f\|_{L^2(\mathbb{S}^2)}^2, \tag{7.1}$$

where $\mathbf{S} = (2\pi)^{1/4}$ is the sharp constant, as obtained in [22].

In this section we review some refinements of (7.1) that will be useful in the next section. The exposition here follows that of [13, Sect. 6]. We start by setting things up to define the X_p spaces, $p \in [1, \infty)$, and the first step is to generate increasingly finer “grids” for the sphere \mathbb{S}^2 . With this in mind, for each integer $k \geq 0$ choose a maximal subset $\{y_k^j\}_j \subset \mathbb{S}^2$ satisfying $|y_k^j - y_k^l| \geq 2^{-k}$, for all $j \neq l$. Then, for each $k \geq 0$, the spherical caps $\mathcal{C}_k^j := \mathcal{C}(y_k^j, 2^{-k+1})$ have finite overlap and cover \mathbb{S}^2 , that is, $\cup_j \mathcal{C}_k^j = \mathbb{S}^2$, and there exists a constant C , independent of k , such that any point in \mathbb{S}^2 belongs to no more than C caps in $\{\mathcal{C}_k^j\}_j$, for every $k \geq 0$. For $p \in [1, \infty)$, the X_p norm of f is defined by the expression

$$\|f\|_{X_p}^4 = \sum_{k \geq 0} \sum_j 2^{-4k} \left(\frac{1}{|\mathcal{C}_k^j|} \int_{\mathcal{C}_k^j} |f|^p d\sigma \right)^{4/p}. \tag{7.2}$$

Moyua et al. showed in [31] that there is a continuous inclusion $L^2(\mathbb{S}^2) \subset X_p$ for all $p \in (1, 2)$ and that for any $p \geq \frac{12}{7}$ there exists $C < \infty$ such that for all $f \in L^2(\mathbb{S}^2)$

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leq C \|f\|_{X_p}. \tag{7.3}$$

Let us define

$$\Lambda_{k,j}(f) = \left(|\mathcal{C}_k^j|^{-1} \int_{\mathcal{C}_k^j} |f| d\sigma \right) \left(|\mathcal{C}_k^j|^{-1} \int_{\mathbb{S}^2} |f|^2 d\sigma \right)^{-1/2},$$

which by Hölder’s inequality satisfies $\Lambda_{k,j}(f) \leq 1$. It was shown in [13, Lemma 6.1] that for any $p \in [1, 2)$, there exists $C < \infty$ and $\gamma > 0$ such that for any $f \in L^2(\mathbb{S}^2)$,

$$\|f\|_{X_p} \leq C \|f\|_{L^2(\mathbb{S}^2)} \sup_{k,j} (\Lambda_{k,j}(f))^\gamma. \tag{7.4}$$

Combining the two results, (7.3) and (7.4), by choosing any $p \in [\frac{12}{7}, 2)$, we obtain the following “cap refinement” of (7.1): there exists $C < \infty$ and $\gamma > 0$ such that for all $f \in L^2(\mathbb{S}^2)$

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{S}^2)} \sup_{k,j} (\Lambda_{k,j}(f))^\gamma. \tag{7.5}$$

A δ -quasi-extremal for the sphere is a function $f \neq 0$ that satisfies $\|f\sigma * f\sigma\|_{L^4(\mathbb{R}^3)} \geq \delta^2 \mathbf{S}^2 \|f\|_{L^2(\mathbb{S}^2)}^2$. With the aid of the previous inequality, Christ and Shao proved the following result regarding δ -quasi-extremals.

Lemma 7.1 ([13, Lemma 2.9]) *For any $\delta > 0$ there exists $C_\delta > 0$ and $\eta_\delta > 0$ with the following property. If $f \in L^2(\mathbb{S}^2)$ satisfies $\|f\sigma * f\sigma\|_2 \geq \delta^2 \mathbf{S}^2 \|f\|_2^2$ then there exist a decomposition $f = g + h$ and a spherical cap $\mathcal{C} \subseteq \mathbb{S}^2$ satisfying*

$$0 \leq |g|, |h| \leq |f|, \tag{7.6}$$

$$g, h \text{ have disjoint supports,} \tag{7.7}$$

$$|g(x)| \leq C_\delta \|f\|_2 |\mathcal{C}|^{-1/2} \mathbb{1}_{\mathcal{C}}(x) \text{ for all } x, \tag{7.8}$$

$$\|g\|_2 \geq \eta_\delta \|f\|_2. \tag{7.9}$$

Moreover (7.8) and (7.9) hold with constants that satisfy $C_\delta \asymp \delta^{-1/\gamma}$ and $\eta_\delta \asymp \delta^{1/\gamma}$, where $\gamma > 0$ is a universal constant⁴.

It will be our task in the next section to obtain an analogous result for the hyperboloid and for this it will be convenient to briefly discuss the construction of the function g and the cap \mathcal{C} in the conclusion of the previous lemma. Letting $f \in L^2(\mathbb{S}^2)$ be a δ -quasi-extremal, inequality (7.5) implies that there is a constant $c_0 \in (0, \infty)$, independent of f , such that

$$\sup_{k,j} \Lambda_{k,j}(f) \geq 2c_0 \delta^{1/\gamma}.$$

It follows that there exist k and j such that $\Lambda_{k,j}(f) \geq c_0 \delta^{1/\lambda}$. Let $\mathcal{C} := \mathcal{C}_k^j$. Then,

$$\int_{\mathcal{C}} |f| \, d\sigma \geq c_0 \delta^{1/\gamma} |\mathcal{C}|^{1/2} \|f\|_{L^2(\mathbb{S}^2)}. \tag{7.10}$$

Let $R = (\frac{1}{2}c_0 \delta^{1/\gamma} |\mathcal{C}|^{1/2})^{-1} \|f\|_{L^2(\mathbb{S}^2)}$, $A = \{x \in \mathcal{C} : |f| \leq R\}$, $g = f \mathbb{1}_A$ and $h = f - f \mathbb{1}_A$. It is now a simple exercise to prove that g, h and \mathcal{C} satisfy the conditions stated in the lemma.

Remark 7.2 Let us consider the following scenario: a measurable set $E \subseteq \mathbb{R}$ and a measurable function $F : E \times \mathbb{S}^2 \rightarrow \mathbb{C}$ that satisfies $F \in L^2(E \times \mathbb{S}^2)$, $\|F_r \sigma * F_r \sigma\|_{L^2(\mathbb{R}^3)} \geq \delta^2 \mathbf{S}^2 \|F_r\|_{L^2(\mathbb{S}^2)}^2 > 0$ for all $r \in E$, where $F_r(x) = F(r, x)$, $(r, x) \in E \times \mathbb{S}^2$. Applying Lemma 7.1 to F_r for each $r \in E$ generates caps $\mathcal{C}_r \subseteq \mathbb{S}^2$ and functions G_r and H_r , and in this way functions $G, H : E \times \mathbb{S}^2 \rightarrow \mathbb{C}$, which a priori may not be measurable in the product space $E \times \mathbb{S}^2$. This can be overcome if we are careful with the choice of the caps as we now proceed to explain. For a collection of spherical caps $\{\mathcal{C}_r\}_{r \in E}$ satisfying (7.10) with $\mathcal{C} = \mathcal{C}_r$ and $f = F_r$, for all $r \in E$,

⁴ The power dependence of C_δ and η_δ on δ can be found in the proof of the lemma in [13, pp. 277–278]

denote

$$\mathcal{G}_0 = \{(r, x) : r \in E, x \in \mathcal{C}_r\},$$

$$\mathcal{G}_1 = \left\{ (r, x) \in \mathcal{G}_0 : |F_r(x)| \leq \left(\frac{1}{2} c_0 \delta^{1/\gamma} |\mathcal{C}_r|^{1/2} \right)^{-1} \|F_r\|_{L^2(\mathbb{S}^2)} \right\}.$$

Then, as explained following (7.10), we can take $G = F \mathbb{1}_{\mathcal{G}_1}$ and $H = F - F \mathbb{1}_{\mathcal{G}_1}$. We need to argue that we can have \mathcal{G}_0 and \mathcal{G}_1 measurable, by choosing the collection $\{\mathcal{C}_r\}_{r \in E}$ appropriately. When $r \in E$, then $\sup_{k,j} \Lambda_{k,j}(F_r) \geq 2c(\delta)$, for some universal constant $c(\delta)$. The cap $\mathcal{C}_r = \mathcal{C}_k^j$ is to be chosen so that $\Lambda_{k,j}(F_r) \geq c(\delta)$, that is,

$$|\mathcal{C}_r|^{-1/2} \int_{\mathcal{C}_r} |F_r| \, d\sigma \geq c(\delta) \|F_r\|_{L^2(\mathbb{S}^2)}.$$

The set of caps $\{\mathcal{C}_k^j : k, j\}$ in \mathbb{S}^2 is parametrized by indices k and j where $k \in \mathbb{N}$ and $j \in \{1, 2, \dots, J_k\}$, for some $J_k < \infty$. Let $\mathcal{Z} = \{(k, j) : k \in \mathbb{N}, j \in \{1, \dots, J_k\}\}$ and define the function $\Theta : E \times \mathcal{Z} \rightarrow \mathbb{R}$ by

$$\Theta(r, k, j) = |\mathcal{C}_k^j|^{-1/2} \|F_r\|_{L^2(\mathbb{S}^2)}^{-1} \int_{\mathcal{C}_k^j} |F_r| \, d\sigma.$$

By Fubini’s theorem, for each fixed $(k, j) \in E \times \mathcal{Z}$, $\Theta(\cdot, k, j)$ is a measurable function. By assumption, for each $r \in E$, $\sup_{k,j} \Theta(r, k, j) \geq 2c(\delta)$. We want to find a measurable function $\tau : E \rightarrow \mathcal{Z}$ such that

$$\Theta(r, \tau(r)) \geq \sup_{k,j} \Theta(r, k, j) - c(\delta) \geq c(\delta),$$

a so called $c(\delta)$ -maximizer. That this is possible is a consequence of measurable selection theorems, see for instance [41, Thm. 4.1].

For such a measurable selection function τ write $\tau(r) = (k(r), j(r)) \in \mathcal{Z}$, then the function $E \rightarrow \mathbb{S}^2, r \mapsto y_{k(r)}^{j(r)}$, is measurable and we can write $\mathcal{G}_0 = \{(x, r) : r \in E, |x - y_{k(r)}^{j(r)}| \leq 2^{-k(r)+1}\}$. We can therefore assume that the sets \mathcal{G}_0 and \mathcal{G}_1 are measurable sets in $E \times \mathbb{S}^2$, so that G and H are measurable functions. In this way, we have the following lemma.

Lemma 7.3 *Let $E \subseteq \mathbb{R}$ be a measurable set and $F : E \times \mathbb{S}^2 \rightarrow \mathbb{C}$ be a measurable function satisfying $F \in L^2(E \times \mathbb{S}^2), \|F_r \sigma * F_r \sigma\|_{L^2(\mathbb{R}^3)} \geq \delta^2 \mathbf{S}^2 \|F_r\|_{L^2(\mathbb{S}^2)}^2 > 0$ for all $r \in E$, where $F_r(x) = F(r, x), (r, x) \in E \times \mathbb{S}^2$. Then, there are spherical caps $\{\mathcal{C}_r\}_{r \in E}$ and measurable functions G, H satisfying: $F = G + H, G$ and H have disjoint supports, $0 \leq |G|, |H| \leq |F|$, and for all $r \in E$:*

$$|G_r(x)| \leq C_\delta \|F_r\|_2 |\mathcal{C}_r|^{-1/2} \mathbb{1}_{\mathcal{C}_r}(x), \quad x \in \mathbb{S}^2 \text{ and } \|G_r\|_2 \geq \eta_\delta \|F_r\|_2.$$

We now prove a slight improvement of Lemma 7.1 that adds one more restriction to the function g . It tells us that we can replace a δ -quasi-extremal for the sphere for a better controlled one at the expense of powers of δ .

Lemma 7.4 *For any $\delta > 0$ there exists $C_\delta > 0$, $\eta_\delta > 0$ and $\lambda_\delta > 0$ with the following property. If $f \in L^2(\mathbb{S}^2)$ satisfies $\|f\sigma * f\sigma\|_2 \geq \delta^2 \mathbf{S}^2 \|f\|_2^2$ then there exist a decomposition $f = g + h$ and a spherical cap \mathcal{C} satisfying (7.6), (7.7), (7.8), 7.9 and*

$$\|g\sigma * g\sigma\|_2 \geq \lambda_\delta \mathbf{S}^2 \|f\|_2^2. \tag{7.11}$$

Moreover (7.8), (7.9) and (7.11) hold with constants that satisfy $C_\delta \asymp \delta^{-1/\gamma}$, $\eta_\delta \asymp \delta^{1+1/\gamma}$ and $\lambda_\delta \asymp \delta^{6+4/\gamma}$, where $\gamma > 0$ is a universal constant.

Remark 7.5 It is not difficult to see (e.g. [38, Lemma 6.2]) that for a function g satisfying (7.8) and (7.9) there is a lower bound for the L^1 norm of the form

$$\int_{\mathcal{C}} |g| \, d\sigma \geq \frac{\eta_\delta^2}{C_\delta} \|f\|_2 |\mathcal{C}|^{1/2}. \tag{7.12}$$

Note that the sharp estimate (7.1) for \mathbb{S}^2 implies that for g satisfying (7.11) we have

$$\mathbf{S} \|g\|_2 \geq \|g\sigma * g\sigma\|_2^{1/2} \geq \lambda_\delta^{1/2} \mathbf{S} \|f\|_2,$$

so that

$$\|g\|_{L^2(\mathbb{S}^2)} \geq \lambda_\delta^{1/2} \|f\|_{L^2(\mathbb{S}^2)}. \tag{7.13}$$

Proof of Lemma 7.4 Take C_δ and η_δ as in the conclusion of Lemma 7.1. We claim that the lemma at hand holds with respective constants C_δ , $\delta\eta_\delta/\sqrt{2}$ and $\lambda_\delta = (\delta^3\eta_\delta^2/8)^2$. To see this we employ a decomposition algorithm, reminiscent of that in [13, Sect. 8, step 6A], defined in the following inductive way.

Let $G_0 = f$ and $f_0 = 0$ and suppose that for $N \geq 0$ we have defined G_N and f_k , for $0 \leq k \leq N$, satisfying:

$$f = G_N + f_0 + \dots + f_N, \tag{7.14}$$

$$\text{supp}(G_N), \text{supp}(f_0), \dots, \text{supp}(f_N) \text{ are pairwise disjoint,} \tag{7.15}$$

$$\|G_N\sigma * G_N\sigma\|_2 \geq \frac{1}{2} \delta^2 \mathbf{S}^2 \|f\|_2^2. \tag{7.16}$$

The previous conditions are satisfied if $N = 0$. We now define the inductive step of the algorithm. If (7.14), (7.15) and (7.16) hold for N we define G_{N+1} and f_{N+1} in the following way.

Given that $\|G_N\sigma * G_N\sigma\|_2 \geq \frac{1}{2} \delta^2 \mathbf{S}^2 \|f\|_2^2 \geq \frac{1}{2} \delta^2 \mathbf{S}^2 \|G_N\|_2^2$ we can apply Lemma 7.1 to G_N to obtain a decomposition $G_N = g_N + h_N$ and a cap \mathcal{C}_N . Define

$G_{N+1} = h_N$ and $f_{N+1} = g_N$. The functions G_{N+1} and f_{N+1} therefore have disjoint supports and satisfy

$$|f_{N+1}(x)| \leq C_\delta \|G_N\|_2 |\mathcal{C}_N|^{-1/2} \mathbb{1}_{\mathcal{C}_N}(x) \leq C_\delta \|f\|_2 |\mathcal{C}_N|^{-1/2} \mathbb{1}_{\mathcal{C}_N}(x) \text{ for all } x, \tag{7.17}$$

$$\|f_{N+1}\|_2 \geq \eta_\delta \|G_N\|_2 \geq \frac{1}{\sqrt{2}} \eta_\delta \delta \|f\|_2, \tag{7.18}$$

where the second inequality in (7.18) follows as in (7.13).

The algorithm terminates at $N \geq 1$ when either $\|f_N \sigma * f_N \sigma\|_2 \geq \lambda_\delta \mathbf{S}^2 \|f\|_2^2$ or $\|G_N \sigma * G_N \sigma\|_2 < \frac{1}{2} \delta^2 \mathbf{S}^2 \|f\|_2^2$. In the former case we say the algorithm stops in a win and set $g = f_N, h = G_N + f_0 + \dots + f_{N-1}, \mathcal{C} = \mathcal{C}_N$ and the Lemma is proved.

Let $N_\delta := \lceil 2\eta_\delta^{-2} \delta^{-2} \rceil$. We claim that the algorithm stops in a win for some $N \leq N_\delta$. We first show that the algorithm can not run for more than N_δ steps, otherwise, using (7.18) we have

$$\|f\|_2 \geq \left(\sum_{k=1}^{N_\delta+1} \|f_k\|_2^2 \right)^{1/2} \geq \frac{1}{\sqrt{2}} (N_\delta + 1)^{1/2} \eta_\delta \delta \|f\|_2 > \|f\|_2,$$

which is impossible.

Second, we show that if the algorithm has not stopped in a win during the first N steps for some $N \leq 2N_\delta$, then we can perform the step $N + 1$. More precisely, if $\|f_k \sigma * f_k \sigma\|_2 < \lambda_\delta \mathbf{S}^2 \|f\|_2^2$ for all $1 \leq k \leq N$, for some $N \leq 2N_\delta$, then $\|G_N \sigma * G_N \sigma\|_2 \geq \frac{1}{2} \delta^2 \mathbf{S}^2 \|f\|_2^2$. Indeed, using Plancherel’s theorem and then the triangle inequality we obtain

$$\begin{aligned} \|G_N \sigma * G_N \sigma\|_2^{1/2} &\geq \|f \sigma * f \sigma\|_2^{1/2} - \sum_{k=1}^N \|f_k \sigma * f_k \sigma\|_2^{1/2} \geq \delta \mathbf{S} \|f\|_2 - N \lambda_\delta^{1/2} \mathbf{S} \|f\|_2 \\ &\geq (\delta - 2N \lambda_\delta^{1/2}) \mathbf{S} \|f\|_2 \\ &\geq \frac{1}{2} \delta \mathbf{S} \|f\|_2. \end{aligned}$$

It follows that the algorithm stops in a win for some $N \leq N_\delta$. This finishes the proof. □

The next topic we review is that of “weak interaction between distant caps”. For spherical caps $\mathcal{C}, \mathcal{C}' \subseteq \mathbb{S}^2$ there is a notion of distance. Let $(y, a), (y', a') \in \mathbb{S}^2 \times (0, \infty)$ denote the centers and radii of the spherical caps $\mathcal{C}, \mathcal{C}'$,

$$\mathcal{C} = \{x \in \mathbb{S}^2 : |x - y| \leq a\}, \quad \mathcal{C}' = \{x \in \mathbb{S}^2 : |x - y'| \leq a'\}.$$

The distance between \mathcal{C} and \mathcal{C}' is defined by the expression

$$\varrho(\mathcal{C}, \mathcal{C}') = \min(d(\mathcal{C}, \mathcal{C}'), d(\mathcal{C}, -\mathcal{C}')), \tag{7.19}$$

where, as in [33], we can take d to be the hyperbolic distance between (y, a) and (y', a') in the upper half space model, that is⁵

$$d(\mathcal{C}, \mathcal{C}') = \operatorname{arc\,cosh}\left(1 + \frac{(a - a')^2 + |y - y'|^2}{2aa'}\right).$$

The following lemma quantifies the notion of weak interaction between distant caps.

Lemma 7.6 ([13, Lemma 7.6]) *For any $\varepsilon > 0$ there exists $\rho < \infty$ such that*

$$\|\mathbb{1}_{\mathcal{C}}\sigma * \mathbb{1}_{\mathcal{C}'}\sigma\|_{L^2(\mathbb{R}^3)} < \varepsilon|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}, \quad \text{whenever } \varrho(\mathcal{C}, \mathcal{C}') > \rho.$$

An inspection of the proof of the previous statement in [13, Lemma 7.6] shows that an analog result holds if we have caps $\mathcal{C} \subseteq \mathbb{S}_r^2$ and $\mathcal{C}' \subseteq \mathbb{S}_t^2$, with $r, t \in [1, 2]$, that is, denoting $\frac{1}{r}\mathcal{C}$ the rescale of \mathcal{C} to \mathbb{S}^2 ,

$$\frac{1}{r}\mathcal{C} = \{x \in \mathbb{R}^3 : rx \in \mathcal{C}\},$$

we have the following lemma.

Lemma 7.7 *Let $r, t \in [1, 2]$, $\mathcal{C} \subseteq \mathbb{S}_r^2$ and $\mathcal{C}' \subseteq \mathbb{S}_t^2$. Then for any $\varepsilon > 0$ there exists $\rho < \infty$ such that $\|\mathbb{1}_{\mathcal{C}}\sigma_r * \mathbb{1}_{\mathcal{C}'}\sigma_t\|_{L^2(\mathbb{R}^3)} < \varepsilon|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}$, whenever $\varrho(\frac{1}{r}\mathcal{C}, \frac{1}{t}\mathcal{C}') > \rho$.*

8 Lifting to the Hyperboloid the Inequality for the Sphere

The aim of this section is to use the Tomas–Stein inequality for the sphere \mathbb{S}^2 to obtain qualitative properties of δ -quasi-extremals for the hyperboloid. The connection here between the hyperboloid and the sphere is that the latter corresponds to horizontal traces of the former. This connection between the adjoint Fourier restriction operator on a hypersurface and on its traces appears, for instance, in the work of Nicola [32]. An alternative approach to the methods in this section can be developed using refined bilinear estimates, but we choose to give a different and new argument. The main result of this section is the following lemma.

Lemma 8.1 *Let $0 \leq s \leq \frac{1}{2}$. For any $\delta > 0$ there exists $C_\delta > 0, \eta_\delta > 0$ and $\nu_\delta > 0$ with the following property. If $f(x, t) \in L^2(\mathcal{H}_s^3)$ supported where $1 \leq |x| \leq 2$ satisfies*

⁵ We point out that for the two lemmas that follow we don't need d to be a distance. It would be perfectly fine to consider instead the expression

$$\frac{(a - a')^2}{aa'} + \frac{|y - y'|^2}{a^2} + \frac{|y - y'|^2}{(a')^2},$$

so that caps are far apart if either a/a' or a'/a is large or the distance from y to y' is much larger than either a or a' .

$\|f \mu_s * f \mu_s\|_{L^2(\mathbb{R}^4)} \geq \delta^2 \mathbf{H}_4^2 \|f\|_{L^2}^2$ then there exist a decomposition $f = g + h$, a spherical cap $\mathcal{C} \subseteq \mathbb{S}^2$ and a cap $\mathcal{C} = [1, 2] \times \mathcal{C} \subset \mathcal{H}_s^3$ satisfying

$$0 \leq |g|, |h| \leq |f|, \tag{8.1}$$

$$g, h \text{ have disjoint supports,} \tag{8.2}$$

$$\text{supp}(g) \subseteq \mathcal{C}, \tag{8.3}$$

$$|g(x, t)| \leq C_\delta \|f\|_{L^2} \mu_s(\mathcal{C})^{-1/2} \mathbb{1}_{\mathcal{C}}(x, t) \text{ for all } (x, t), \tag{8.4}$$

$$\|g\|_{L^2} \geq \eta_\delta \|f\|_{L^2}, \tag{8.5}$$

$$\|g\|_{L^1} \geq \nu_\delta \mu_s(\mathcal{C})^{1/2} \|f\|_{L^2}. \tag{8.6}$$

The constants C_δ , η_δ and ν_δ are uniform in $s \leq \frac{1}{2}$.

Remark 8.2 The previous lemma is equivalent to the analog result for $\overline{\mathcal{H}}_s^3$. Indeed, that the result for $\overline{\mathcal{H}}_s^3$ implies a similar one for \mathcal{H}_s^3 is immediate. On the other direction, if $f \in L^2(\overline{\mathcal{H}}_s^3)$ is a δ -quasi-extremal for (1.10), that is

$$\|\overline{T}_s f\|_{L^4(\mathbb{R}^4)}^4 = (2\pi)^4 \|f \bar{\mu}_s * f \bar{\mu}_s\|_{L^2(\mathbb{R}^4)}^2 \geq (2\pi)^4 \delta^4 \overline{\mathbf{H}}_4^4 \|f\|_{L^2(\overline{\mathcal{H}}_s^3)}^4,$$

then, writing $f = f_+ + f_-$ so that $\overline{T}_s f = T_s f_+ + T_s f_-(\cdot, -\cdot)$ and $\|f\|_{L^2(\overline{\mathcal{H}}_s^3)}^2 = \|f_+\|_{L^2(\mathcal{H}_s^3)}^2 + \|f_-\|_{L^2(\mathcal{H}_s^3)}^2$ we obtain that

$$\|f_\epsilon \mu_s * f_\epsilon \mu_s\|_{L^2(\mathbb{R}^4)}^2 \geq 2^{-4} \delta^4 \overline{\mathbf{H}}_4^4 \|f_\epsilon\|_{L^2(\mathcal{H}_s^3)}^4,$$

for $\epsilon = +$ or for $\epsilon = -$, so that if both $\|f_+\|_{L^2(\mathcal{H}_s^3)}^2 \geq \delta^2 \|f\|_{L^2(\overline{\mathcal{H}}_s^3)}^2$ and $\|f_-\|_{L^2(\mathcal{H}_s^3)}^2 \geq \delta^2 \|f\|_{L^2(\overline{\mathcal{H}}_s^3)}^2$, then we obtain the conclusions in Lemma 8.1 for f from the ones for f_+ or f_- , as it corresponds. On the other hand, if say $\|f_-\|_{L^2(\mathcal{H}_s^3)}^2 < \delta^2 \|f\|_{L^2(\overline{\mathcal{H}}_s^3)}^2$, then $\|f_+\|_{L^2(\mathcal{H}_s^3)}^2 \geq (1 - \delta^2) \|f\|_{L^2(\overline{\mathcal{H}}_s^3)}^2$ and

$$\|Tf_+\|_{L^4} \geq \|\overline{T}f\|_{L^4} - \|Tf_-\|_{L^4} \geq 2\pi\delta(\overline{\mathbf{H}}_4 - \mathbf{H}_4) \|f\|_{L^2(\overline{\mathcal{H}}_s^3)} \geq c\delta\mathbf{H}_4 \|f_+\|_{L^2(\mathcal{H}_s^3)},$$

so that Lemma 8.1 applied to f_+ yields the result for f .

The support condition $1 \leq |x| \leq 2$ can be changed to $a \leq |x| \leq b$ for any $a \geq s$ and $b < \infty$, understanding that the implicit constants may depend on a, b . We can alternatively state the previous lemma for $f \in L^2(\mathcal{H}^3)$ supported where $2^N \leq |x| \leq 2^{N+1}$, $N \geq 1$, the implicit constants independent of N , as can be easily checked by the use of scaling.

Recall that we write $\psi_s(r) = \sqrt{r^2 - s^2} \mathbb{1}_{\{r \geq s\}}$ and $\phi_s(t) = \psi_s^{-1}(t) = \sqrt{t^2 + s^2} \mathbb{1}_{\{t \geq 0\}}$ and for $f \in \mathcal{S}(\mathbb{R}^3)$ and $r > 0$ we denote by $f\sigma_r$ the measure

supported on $\mathbb{S}_r^2 := \{y \in \mathbb{R}^3 : |y| = r\}$ given by

$$\langle f\sigma_r, \varphi \rangle = \int_{\mathbb{S}^2} f(ry)\varphi(ry)r \, d\sigma(y).$$

We denote f_r the function $x \mapsto f(rx)$, which we consider as a function from \mathbb{S}^2 to \mathbb{C} .

In the next lemma we show that we can write the double convolution of functions on the hyperboloid \mathcal{H}_s^3 as an integral of convolutions of sliced spheres.

Lemma 8.3 *Let $s \geq 0$. For $f, g \in L^2(\mathcal{H}_s^3)$ we have the representation formula*

$$(f\mu_s * g\mu_s)(x, t) = \int_0^t (f\sigma_{\phi_s(t')} * g\sigma_{\phi_s(t-t')})(x) \, dt', \tag{8.7}$$

for a.e. $(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+$.

Proof Let $\varphi \in C_c^\infty(\mathbb{R}^4)$. Using spherical coordinates we have

$$\begin{aligned} \langle f\mu_s * g\mu_s, \varphi \rangle &= \int_{|x|, |y| \geq s} \varphi(x + y, \psi_s(x) + \psi_s(y)) f(x)g(y) \frac{dx \, dy}{\sqrt{|x|^2 - s^2}\sqrt{|y|^2 - s^2}} \\ &= \int_s^\infty \int_s^\infty \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \varphi(r\omega + r'\omega', \psi_s(r) + \psi_s(r')) f(r\omega)g(r'\omega') \frac{r^2 r'^2 \, d\omega \, d\omega' \, dr \, dr'}{\sqrt{r^2 - s^2}\sqrt{r'^2 - s^2}}. \end{aligned}$$

We change variables $(r, r') \mapsto (u, u') = (\psi_s(r), \psi_s(r')) = (\sqrt{r^2 - s^2}, \sqrt{r'^2 - s^2})$ and obtain

$$\begin{aligned} \langle f\mu_s * g\mu_s, \varphi \rangle &= \int_0^\infty \int_0^\infty \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \varphi(\phi_s(u)\omega + \phi_s(u')\omega', u + u') \\ &\quad \times f(\phi_s(u)\omega)g(\phi_s(u')\omega')\phi_s(u)\phi_s(u') \, d\omega \, d\omega' \, du \, du'. \end{aligned}$$

We change variables $(u, u') \mapsto (t, t') = (u + u', u)$ and obtain

$$\begin{aligned} \langle f\mu_s * g\mu_s, \varphi \rangle &= \int_0^\infty \int_0^t \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \varphi(\phi_s(t')\omega + \phi_s(t-t')\omega', t) \\ &\quad \times f(\phi_s(t')\omega)g(\phi_s(t-t')\omega')\phi_s(t')\phi_s(t-t') \, d\omega \, d\omega' \, dt' \, dt \\ &= \int_0^\infty \int_0^t \left(\int_{\mathbb{R}^3} \varphi(x, t)(f\sigma_{\phi_s(t')} * g\sigma_{\phi_s(t-t')})(x) \, dx \right) dt' \, dt \end{aligned}$$

$$= \left\langle \int_0^t (f \sigma_{\phi_s(t')} * g \sigma_{\phi_s(t-t')})(x) dt', \varphi \right\rangle,$$

where we used Fubini’s Theorem and that for any $r, r' > 0$,

$$\begin{aligned} \langle f \sigma_r * g \sigma_{r'}, \varphi(\cdot, t) \rangle &= \int_{\mathbb{R}^3} \varphi(x, t) (f \sigma_r * g \sigma_{r'})(x) dx \\ &= \int_{\mathbb{S}_r^2 \times \mathbb{S}_{r'}^2} \varphi(x + x', t) f(x) g(x') d\sigma_r(x) d\sigma_{r'}(x') \\ &= \int_{\mathbb{S}^2 \times \mathbb{S}^2} \varphi(r\omega + r'\omega', t) f(r\omega) g(r'\omega') r r' d\sigma(\omega) d\sigma(\omega'). \end{aligned}$$

□

Next, we record a formula for the $L^p(\mathcal{H}_s^3)$ norm in terms of the L^p norm of the slices.

Lemma 8.4 *Let $f \in L^p(\mathcal{H}_s^3)$. Then*

$$\|f\|_{L^p(\mathcal{H}_s^3)}^p = \int_0^\infty \|f\phi_s(t)\|_{L^p(\mathbb{S}^2)}^p \phi_s(t) dt. \tag{8.8}$$

Proof Using spherical coordinates we have

$$\begin{aligned} \|f\|_{L^p(\mathcal{H}_s^3)}^p &= \int_s^\infty \int_{\mathbb{S}^2} |f(r\omega)|^p \frac{r^2}{\sqrt{r^2 - s^2}} d\omega dr = \int_0^\infty \int_{\mathbb{S}^2} |f(\phi_s(t)\omega)|^p \phi_s(t) d\omega dt \\ &= \int_0^\infty \|f\phi_s(t)\|_{L^p(\mathbb{S}^2)}^p \phi_s(t) dt. \end{aligned}$$

□

We now analyze the dependence of $\|f \sigma_r * g \sigma_{r'}\|_{L^2(\mathbb{R}^3)}$ in (r, r') . We start with the scaling property of $\widehat{f \sigma_r}$ as a function of r . We have

$$\widehat{(f \sigma_r)}(x) = \int_{\mathbb{S}_r^2} e^{-ix \cdot y} f(y) d\sigma_r(y) = \int_{\mathbb{S}^2} e^{-irx \cdot y} f(ry) r d\sigma(y) = r \widehat{(f_r \sigma)}(rx).$$

Thus

$$\|\widehat{f \sigma_r}\|_{L^4(\mathbb{R}^3)} = r^{1/4} \|\widehat{f_r \sigma}\|_{L^4(\mathbb{R}^3)} \leq (2\pi)^{3/4} r^{1/4} \mathbf{S} \|f_r\|_{L^2(\mathbb{S}^2)}.$$

Then, the Cauchy–Schwarz inequality implies that for any $r, r' > 0$

$$\|\widehat{f\sigma_r} \widehat{g\sigma_{r'}}\|_{L^2(\mathbb{R}^3)} \leq \|\widehat{f\sigma_r}\|_{L^4} \|\widehat{g\sigma_{r'}}\|_{L^4} \leq (2\pi)^{3/2} \mathbf{S}^2(rr')^{1/4} \|f_r\|_{L^2(\mathbb{S}^2)} \|g_{r'}\|_{L^2(\mathbb{S}^2)},$$

so that

$$\|f\sigma_r * g\sigma_{r'}\|_{L^2(\mathbb{R}^3)} \leq \mathbf{S}^2(rr')^{1/4} \|f_r\|_{L^2(\mathbb{S}^2)} \|g_{r'}\|_{L^2(\mathbb{S}^2)}, \tag{8.9}$$

and in particular, when $r = r'$ we obtain

$$\|f\sigma_r * g\sigma_r\|_{L^2(\mathbb{R}^3)} = r^{1/2} \|f_r\sigma * g_r\sigma\|_{L^2(\mathbb{R}^3)} \leq \mathbf{S}^2 r^{1/2} \|f_r\|_{L^2(\mathbb{S}^2)} \|g_r\|_{L^2(\mathbb{S}^2)}. \tag{8.10}$$

Definition 8.5 A quasi-cap of \mathcal{H}_s^3 is a measurable set $\mathcal{G} \subseteq \mathcal{H}_s^3$ for which there exist $E \subseteq \mathbb{R}$ and spherical caps $\mathcal{C}_t \subseteq \mathbb{S}^2$, for $t \in E$, such that

$$\mathcal{G} = \{(x, t) \in \mathbb{R}^4 : t \in E, x \in \phi_s(t)\mathcal{C}_t\}. \tag{8.11}$$

We note that a cap is also a quasi-cap; the difference in a generic quasi-cap is that the spherical caps may not be the same as in the case of a cap, and the set E may not be an interval.

In our main result of the section, Lemma 8.1, we want to obtain an analog of Lemma 7.1 for a compact subset of the hyperboloid. The idea is to use the cap Lemma 7.1 for the sphere on horizontal slices of the hyperboloid via (8.7) in a measurable way (recall Remark 7.2), and show that there are enough aligned sliced caps of similar size to obtain a cap for the hyperboloid. We do it for the upper sheet as the full one-sheeted hyperboloid follows from this as already noted in Remark 8.2. The proof of Lemma 8.1 is accomplished in the following way. First, we show that on a large subset of t 's in $[\psi_s(1), \psi_s(2)]$ we can apply Lemma 7.4 to the function $x \in \mathbb{S}^2 \mapsto f(\phi_s(t)x)$ in a measurable way. This will allow us to prove a version of Lemma 8.1 where instead of a cap we have a quasi-cap. Next, we show that a subset of the quasi-cap of large relative measure is comparable to a cap and satisfies the requirements of Lemma 8.1, which then are shown to be satisfied by the cap itself. To prove this last point, we will make use of the quantitative version of the statement that “distant spherical caps interact weakly” as stated in Lemmas 7.6 and 7.7.

Proof of Lemma 8.1 In what follows, $c(\delta)$ denotes a constant that depends only on δ and is allowed to change from line to line.⁶ Recall from Remark 7.5 that (8.6) can be obtained from (8.4) and (8.5) with $v_\delta = \eta_\delta^2/C_\delta$.

We first argue that we can assume that the support of $f(\cdot, t)$ does not contain antipodal points for each $t \in [\psi_s(1), \psi_s(2)]$. We can cover \mathbb{S}^2 as the union of finitely many spherical caps $\{\mathcal{C}_k\}_{k=1, \dots, \kappa}$ each of radius $\frac{1}{4}$, whose centers form a maximally $\frac{1}{4}$ -separated set on \mathbb{S}^2 , and induce a decomposition of \mathcal{H}_s^3 as the union of the caps

⁶ Reviewing the argument one can see that such constants can be taken to depend only on powers, positive and negative, of δ .

$\{[s, \infty) \times \mathbb{C}_k\}_{k=1, \dots, \kappa}$. By the triangle inequality we can therefore assume that f is supported on the cap $[s, \infty) \times \mathbb{C}_k$, for some k , at the expense of changing δ by δ/κ . The reason for doing this is to ensure that there are no nearly antipodal spherical caps later on.

Let us start by noting that for (x, t) in the support of f and $s \in [0, \frac{1}{2}]$ we have $|x| \in [1, 2]$ and $t = \psi_s(x) \in [\psi_s(1), \psi_s(2)] = [\sqrt{1-s^2}, \sqrt{4-s^2}] \subseteq [\frac{\sqrt{3}}{2}, 2]$, and that from Lemma 8.4

$$\int_{\psi_s(1)}^{\psi_s(2)} \|f_{\phi_s(t)}\|_{L^2(\mathbb{S}^2)}^2 dt \leq \|f\|_{L^2(\mathcal{H}_s^3)}^2 \leq 2 \int_{\psi_s(1)}^{\psi_s(2)} \|f_{\phi_s(t)}\|_{L^2(\mathbb{S}^2)}^2 dt.$$

On the other hand $(f\mu_s * f\mu_s)(x, t)$ is supported where $2\psi_s(1) \leq t \leq 2\psi_s(2)$. From Lemma 8.3 for a.e. $(x, t) \in \mathbb{R}^4$ we have

$$f\mu_s * f\mu_s(x, t) = \int_{\psi_s(1)}^{\psi_s(2)} (f\sigma_{\phi_s(t')} * f\sigma_{\phi_s(t-t')})(x) dt', \tag{8.12}$$

(recall that $\phi_s(\tau) = 0$ for $\tau < 0$). Let

$$E_\gamma = \left\{ t \in [\psi_s(1), \psi_s(2)] : \begin{array}{l} \|f\sigma_{\phi_s(t)} * f\sigma_{\phi_s(t)}\|_{L^2(\mathbb{R}^3)} \geq \gamma^2 \delta^2 \mathbf{H}_4^2 \mathbf{S}^2 \|f_{\phi_s(t)}\|_2^2, \\ \|f_{\phi_s(t)}\|_2 \geq \gamma \delta \mathbf{H}_4 \|f\|_2 \end{array} \right\},$$

and

$$E_{\gamma, \lambda} = \left\{ t \in [\psi_s(1), \psi_s(2)] : \begin{array}{l} \|f\sigma_{\phi_s(t)} * f\sigma_{\phi_s(t)}\|_{L^2(\mathbb{R}^3)} \geq \gamma^2 \delta^2 \mathbf{H}_4^2 \mathbf{S}^2 \|f_{\phi_s(t)}\|_2^2, \\ \lambda \delta \mathbf{H}_4 \|f\|_2 \geq \|f_{\phi_s(t)}\|_2 \geq \gamma \delta \mathbf{H}_4 \|f\|_2 \end{array} \right\}.$$

Here, $\|f_{\phi_s(t)}\|_2 = \|f(\phi_s(t) \cdot, t)\|_{L^2(\mathbb{S}^2)}$, while $\|f\|_2 = \|f\|_{L^2(\mathcal{H}_s^3)}$. We claim that $|E_\gamma| \geq c(\delta)$ and $|E_{\gamma, \lambda}| \geq c(\delta)$ if γ and λ are chosen small and large enough depending on δ , respectively. Let us first analyze $|E_\gamma|$. From (8.12), using Fubini's theorem and Minkowski's integral inequality we have

$$\begin{aligned} \delta^2 \mathbf{H}_4^2 \|f\|_2^2 &\leq \left\| \int_{\psi_s(1)}^{\psi_s(2)} (f\sigma_{\phi_s(t')} * f\sigma_{\phi_s(t-t')})(x) dt' \right\|_{L^2_{t,x}} \\ &\leq \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|f\sigma_{\phi_s(t')} * f\sigma_{\phi_s(t-t')}\|_{L^2_x} \mathbb{1}_{E_\gamma^c}(t') dt' \right\|_{L^2_t} \\ &\quad + \left\| \int_{\psi_s(1)}^{\psi_s(2)} (f\sigma_{\phi_s(t')} * f\sigma_{\phi_s(t-t')})(x) \mathbb{1}_{E_\gamma}(t') dt' \right\|_{L^2_{x,t}}. \end{aligned}$$

Plancherel’s theorem and the Cauchy–Schwarz inequality give

$$\|f\sigma_{\phi_s(t')} * f\sigma_{\phi_s(t-t')}\|_{L_x^2} \leq \|f\sigma_{\phi_s(t')} * f\sigma_{\phi_s(t')}\|_{L_x^2}^{1/2} \|f\sigma_{\phi_s(t-t')} * f\sigma_{\phi_s(t-t')}\|_{L_x^2}^{1/2},$$

so that using the sharp estimate for $\|f\sigma_{\phi_s(t-t')} * f\sigma_{\phi_s(t-t')}\|_{L_x^2}$ as in (8.10), recalling that $\phi_s(t')$, $\phi_s(t - t') \in [1, 2]$, we obtain

$$\begin{aligned} & \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|f\sigma_{\phi_s(t')} * f\sigma_{\phi_s(t-t')}\|_{L_x^2} \mathbb{1}_{E_\gamma}(t') dt' \right\|_{L_t^2} \\ & \leq 2\gamma \delta \mathbf{H}_4 \mathbf{S}^2 \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|f\phi_s(t')\|_2 \|f\phi_s(t-t')\|_2 dt' \right\|_{L_t^2} \\ & \quad + 2\gamma \delta \mathbf{H}_4 \mathbf{S}^2 \|f\|_2 \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|f\phi_s(t-t')\|_2 dt' \right\|_{L_t^2} \\ & \leq 8\gamma \delta \mathbf{H}_4 \mathbf{S}^2 \|f\|_2^2. \end{aligned}$$

Therefore, choosing $\gamma = \delta \mathbf{H}_4 / (16\mathbf{S}^2)$ we obtain

$$\left\| \int_{\psi_s(1)}^{\psi_s(2)} (f\sigma_{\phi_s(t')} * f\sigma_{\phi_s(t-t')})(x) \mathbb{1}_{E_\gamma}(t') dt' \right\|_{L_{x,t}^2} \geq \frac{1}{2} \delta^2 \mathbf{H}_4^2 \|f\|_2^2.$$

For this choice of γ we then obtain

$$\begin{aligned} \frac{1}{2} \delta^2 \mathbf{H}_4^2 \|f\|_2^2 & \leq \left\| \int_{\psi_s(1)}^{\psi_s(2)} (f\sigma_{\phi_s(t')} * f\sigma_{\phi_s(t-t')})(x) \mathbb{1}_{E_\gamma}(t') dt' \right\|_{L_{x,t}^2} \\ & \leq \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|f\sigma_{\phi_s(t')} * f\sigma_{\phi_s(t-t')}\|_{L_x^2} \mathbb{1}_{E_\gamma}(t') dt' \right\|_{L_t^2} \\ & \leq 2\mathbf{S}^2 \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|f\phi_s(t')\|_{L_x^2} \|f\phi_s(t-t')\|_{L_x^2} \mathbb{1}_{E_\gamma}(t') dt' \right\|_{L_t^2} \\ & \leq 2\mathbf{S}^2 |E_\gamma|^{1/2} \int_{\psi_s(1)}^{\psi_s(2)} \|f\phi_s(t)\|_{L_x^2}^2 dt \leq 2\mathbf{S}^2 \|f\|_2^2 |E_\gamma|^{1/2}, \end{aligned}$$

and therefore $|E_\gamma| \geq \mathbf{H}_4^4 \delta^4 / (16\mathbf{S}^4)$.

To analyze $|E_{\gamma,\lambda}|$ we use

$$E_{\gamma,\lambda} = E_\gamma \cap \{t \in [\psi_s(1), \psi_s(2)] : \|f_{\phi_s(t)}\|_2 \leq \lambda\delta\mathbf{H}_4\|f\|_2\}.$$

Chebyshev’s and Hölder’s inequalities imply

$$\begin{aligned} |\{t \in [\psi_s(1), \psi_s(2)] : \|f_{\phi_s(t)}\|_2 > \lambda\delta\mathbf{H}_4\|f\|_2\}| &\leq \frac{1}{\lambda\delta\mathbf{H}_4\|f\|_2} \int_{\psi_s(1)}^{\psi_s(2)} \|f_{\phi_s(t)}\|_2 dt \\ &\leq \frac{2}{\lambda\delta\mathbf{H}_4}. \end{aligned}$$

Therefore, choosing $\lambda = 64\mathbf{S}^4/(\mathbf{H}_4^5\delta^5)$ we obtain

$$|E_{\gamma,\lambda}| \geq |E_\gamma| - |\{t \in [\psi_s(1), \psi_s(2)] : \|f_{\phi_s(t)}\|_2 > \lambda\delta\mathbf{H}_4\|f\|_2\}| \geq \frac{\mathbf{H}_4^4}{32\mathbf{S}^4}\delta^4.$$

From now on, let us fix such values of γ and λ and let $E := E_{\gamma,\lambda}$. From the definition of E and (8.10), we have that for $t \in E$

$$\|f_{\phi_s(t)}\sigma * f_{\phi_s(t)}\sigma\|_{L^2(\mathbb{R}^3)} \geq c\phi_s(t)^{-1/2}\delta^4\|f_{\phi_s(t)}\|_{L^2(\mathbb{S}^2)}^2,$$

so that Lemma 7.1 imply that for $t \in E$ there are caps $\mathcal{C}_t \subseteq \mathbb{S}^2$ and a decomposition $f_{\phi_s(t)} = G_{\phi_s(t)} + H_{\phi_s(t)}$. In this way we obtain a decomposition $f = g + h$, where $g(\phi_s(t)x, t) = G_{\phi_s(t)}(x)\mathbb{1}_E(t)$, $x \in \mathbb{S}^2$, $t \in [\psi_s(1), \psi_s(2)]$. As argued in Remark 7.2 and recorded in Lemma 7.3, by using a measurable selection theorem we can perform this decomposition in such a way that g and h are measurable functions and $\mathcal{G}_0 := \{(x, t) \in \mathbb{R}^4 : t \in E, x \in \phi_s(t)\mathcal{C}_t\}$ is a measurable subset of \mathcal{H}_s^3 , so that \mathcal{G}_0 is a quasicap. According to Lemma 7.1, g and h satisfy the following conditions: $f = g + h$, $0 \leq |g|, |h| \leq |f|$, g and h have disjoint supports, $g(x, t) = 0$ if $t \notin E$,

$$\begin{aligned} |g(\phi_s(t)x, t)| &\leq C_\delta\|f_{\phi_s(t)}\|_2|\mathcal{C}_t|^{-1/2}\mathbb{1}_{\mathcal{C}_t}(x), \text{ for all } t \in E, x \in \mathbb{S}^2, \\ \|g_{\phi_s(t)}\|_2 &\geq \eta_\delta\|f_{\phi_s(t)}\|_2, \quad \|g_{\phi_s(t)}\|_1 \geq \frac{\eta_\delta^2}{C_\delta}|\mathcal{C}_t|^{1/2}\|f_{\phi_s(t)}\|_2, \text{ for all } t \in E. \end{aligned} \tag{8.13}$$

Note that Lemma 8.4 and (8.13) imply

$$\|g\|_2 \geq \eta_\delta\|f\|_2.$$

Given that for $t \in E$ we have $\delta^2\mathbf{H}_4\|f\|_2 \lesssim \|f_{\phi_s(t)}\|_2 \lesssim \delta^{-4}\mathbf{H}_4\|f\|_2$ we conclude, possibly by changing the constants that depend on δ , that the function g satisfies

$$|g(\phi_s(t)x, t)| \leq C_\delta\|f\|_2|\mathcal{C}_t|^{-1/2}\mathbb{1}_{\mathcal{C}_t}(x)\mathbb{1}_E(t), \text{ for all } t \in [\psi_s(1), \psi_s(2)], x \in \mathbb{S}^2,$$

$$(8.14)$$

and

$$\|g_{\phi_s(t)}\|_{L^2(\mathbb{S}^2)} \geq \eta_\delta \|f\|_2 \text{ and } \|g_{\phi_s(t)}\|_{L^1(\mathbb{S}^2)} \geq \frac{\eta_\delta^2}{C_\delta} |\mathcal{C}_t|^{1/2} \|f\|_2, \text{ for each } t \in E. \tag{8.15}$$

Summing up, we can restate what has been done so far in the following way: If $f \in L^2(\mathcal{H}_s^3)$ satisfies $\|f\mu_s * f\mu_s\|_2 \geq \delta^2 \mathbf{H}_4^2 \|f\|_2^2$ and is supported where $1 \leq |x| \leq 2$ then there exist a decomposition $f = g + h$, a set $E \subseteq [\psi_s(1), \psi_s(2)]$ satisfying $|E| \gtrsim \delta^4$ and a quasi-cap \mathcal{G}_0 (associated to E as in (8.11)) such that g and h have disjoint supports,

$$|g(x, t)| \leq C_\delta \|f\|_2 |\mathcal{C}_t|^{-1/2} \mathbb{1}_{\mathcal{G}_0}(x, t), \text{ for all } (x, t) \in \mathcal{H}_s^3,$$

and (8.15) holds. This is the analog of Lemma 8.1 with a quasi-cap instead of a cap.

Using the quasi-cap analog of Lemma 8.1, as described in the previous paragraph, we can argue exactly as in Lemma 7.4 for the sphere to ensure, possibly after changing the constants that depend on δ , that there exist a quasi-cap, which we continue to denote \mathcal{G}_0 , associated to a set $E \subseteq [\psi_s(1), \psi_s(2)]$ with $|E| \gtrsim \delta^4$, and functions g and h with the properties of the previous paragraph and additionally

$$\|g\mu_s * g\mu_s\|_{L^2(\mathbb{R}^4)} \geq c_\delta \|f\|_2^2. \tag{8.16}$$

The next and final step is to show that the caps $\mathcal{C}_t, t \in E$, which define \mathcal{G}_0 are aligned for a large fraction of the t 's, and by this we mean that they have close radii and centers, up to powers of δ .

Recall that for caps $\mathcal{C}, \mathcal{C}' \subseteq \mathbb{S}^2$ there is a distance function $\varrho(\mathcal{C}, \mathcal{C}')$, defined in (7.19), that is relevant in Lemmas 7.6 and 7.7. For $\rho > 0$ define

$$A_\rho = \{(t, t') \in E \times E : \varrho(\mathcal{C}_t, \mathcal{C}_{t'}) \leq \rho\}.$$

Then, starting from (8.16) we have the estimate

$$\begin{aligned} c_\delta \|f\|_2^2 &\leq \|g\mu_s * g\mu_s\|_2 = \left\| \int_{\psi_s(1)}^{\psi_s(2)} (g\sigma_{\phi_s(t')} * g\sigma_{\phi_s(t-t')})(x) dx dt' \right\|_{L_{x,t}^2} \\ &\leq \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|g\sigma_{\phi_s(t')} * g\sigma_{\phi_s(t-t')}\|_{L_x^2} \mathbb{1}_{A_\rho}(t', t-t') dt' \right\|_{L_t^2} \\ &\quad + \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|g\sigma_{\phi_s(t')} * g\sigma_{\phi_s(t-t')}\|_{L_x^2} \mathbb{1}_{A_\rho^c}(t', t-t') dt' \right\|_{L_t^2} \end{aligned}$$

$$\begin{aligned}
 &\leq \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|g\sigma_{\phi_s(t')} * g\sigma_{\phi_s(t-t')}\|_{L_x^2} \mathbb{1}_{A_\rho}(t', t-t') dt' \right\|_{L_t^2} + C_\delta^2 \|f\|_2^2 \left\| \int_{\psi_s(1)}^{\psi_s(2)} |\mathcal{C}_{t'}|^{-1/2} \right. \\
 &\quad \times |\mathcal{C}_{t-t'}|^{-1/2} \|\mathbb{1}_{\phi_s(t')\mathcal{C}_{t'}} \sigma_{\phi_s(t')} * \mathbb{1}_{\phi_s(t-t')\mathcal{C}_{t-t'}} \sigma_{\phi_s(t-t')}\|_{L_x^2} \mathbb{1}_{(E \times E) \cap A_\rho}(t', t-t') dt' \left. \right\|_{L_t^2} \\
 &\leq \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|g\sigma_{\phi_s(t')} * g\sigma_{\phi_s(t-t')}\|_{L_x^2} \mathbb{1}_{A_\rho}(t', t-t') dt' \right\|_{L_t^2} + \frac{C_\delta}{2} \|f\|_2^2,
 \end{aligned}$$

where in the second to last line we used (8.14) and the last line holds if ρ is large enough as a function of δ , by the use of Lemma 7.7. For such choice of ρ we can therefore ensure that

$$\left\| \int_{\psi_s(1)}^{\psi_s(2)} \|g\sigma_{\phi_s(t')} * g\sigma_{\phi_s(t-t')}\|_{L_x^2} \mathbb{1}_{A_\rho}(t', t-t') dt' \right\|_{L_t^2} \geq \frac{C_\delta}{2} \|f\|_2^2. \tag{8.17}$$

Note that (8.14) implies $\|g\phi_s(t)\|_2 \leq C_\delta \|f\|_2$ for all $t \in E$. This and (8.17) imply that

$$\begin{aligned}
 \frac{C_\delta}{2} \|f\|_2^2 &\leq 2\mathbf{S}^2 \left\| \int_{\psi_s(1)}^{\psi_s(2)} \|g\phi_s(t')\|_2 \|g\phi_s(t-t')\|_2 \mathbb{1}_{A_\rho}(t', t-t') dt' \right\|_{L_t^2} \\
 &\leq 2\mathbf{S}^2 C_\delta^2 \|f\|_2^2 \int_{\psi_s(1)}^{\psi_s(2)} \|\mathbb{1}_{A_\rho}(t', t-t')\|_{L_t^2} dt' \\
 &\leq 4\mathbf{S}^2 C_\delta^2 \|f\|_2^2 |A_\rho|^{1/2},
 \end{aligned}$$

where $\rho = \rho(\delta)$ is the already fixed function of δ and $|A_\rho|$ denotes the Lebesgue measure of $A_\rho \subseteq \mathbb{R}^2$. As $|A_\rho| \leq 2$ we conclude that $|A_\rho| \asymp c(\delta)$. By Fubini’s theorem, the fibers $A_\rho(t) := \{t' \in E : (t, t') \in A_\rho\} = \{t' \in E : \varrho(\mathcal{C}_t, \mathcal{C}_{t'}) \leq \rho\}$ are a.e. measurable, the function $t \in E \mapsto |A_\rho(t)| = |\{t' \in E : \varrho(\mathcal{C}_t, \mathcal{C}_{t'}) \leq \rho\}|$ is measurable and $|A_\rho| \leq 2 \operatorname{ess\,sup}_{t \in E} |A_\rho(t)|$. We then obtain the following estimate

$$c(\delta) \asymp \operatorname{ess\,sup}_{t \in E} |\{t' \in E : \varrho(\mathcal{C}_t, \mathcal{C}_{t'}) \leq \rho\}| \leq \sup_{(y,a) \in \mathbb{S}^2 \times (0,\infty)} |\{t' \in E : \varrho(\mathcal{C}(y, a), \mathcal{C}_{t'}) \leq \rho\}|,$$

from where we conclude the existence of a spherical cap $\mathcal{C}(y_0, a_0)$ such that

$$|\{t \in E : \varrho(\mathcal{C}(y_0, a_0), \mathcal{C}_t) \leq \rho\}| \asymp c(\delta).$$

Denote $\mathcal{C}_0 = \mathcal{C}(y_0, a_0)$ and $B_\rho = \{t \in E : \varrho(\mathcal{C}_0, \mathcal{C}_t) \leq \rho\}$. For $t \in B_\rho$, the radii and the distance between the centers of the caps \mathcal{C}_0 and \mathcal{C}_t are of the same order

⁷ From the proof of Lemma 7.6 in [13] one can see that $\cosh \rho$ can be taken to be a power of δ^{-1} .

modulo powers of δ . More precisely, if we let (y, a) denote the center and radius of a cap $\mathcal{C}_t, t \in B_\rho$, then the definition of the distance function ϱ ensures that

$$c(\delta)a_0 \leq a \leq c'(\delta)a_0, \text{ and } |y_0 - y| \leq c''(\delta)a_0. \tag{8.18}$$

This is the only place where we used the assumption that f is supported on a cap $[1, 2] \times \mathcal{C}$, were the radius of \mathcal{C} is $\frac{1}{4}$, because this implies that the centers of the caps associated to $g_{\phi_s(t)}, t \in E$, can be chosen to be at distance at most $\frac{1}{2}$ from each other and therefore any two caps $\mathcal{C}_t, \mathcal{C}_{t'}$ for $t, t' \in E$ are not nearly antipodal.

From (8.18) we conclude that for $t \in B_\rho$ we have $|\mathcal{C}_t| \asymp_\delta |\mathcal{C}_0|$ and there exists $c(\delta) \geq 1$ such that the $c(\delta)$ -enlargement of \mathcal{C}_0 , denoted \mathcal{C}_0^δ and defined by

$$\mathcal{C}_0^\delta := \{x \in \mathbb{S}^2 : |x - y_0| \leq c(\delta)a_0\},$$

contains \mathcal{C}_t for all $t \in B_\rho$, and hence the cap $\mathcal{C} := [1, 2] \times \mathcal{C}_0^\delta \subseteq \mathcal{H}_s^3$ contains the quasi-cap $\mathcal{G}_1 := \{(x, t) \in \mathcal{G}_0 : t \in B_\rho\}$. Note also that $|\mathcal{C}_t| \asymp_\delta |\mathcal{C}_0^\delta|$, for all $t \in B_\rho$.

Now, for each $t \in E, g_{\phi_s(t)}$ is supported on \mathcal{C}_t and $\int_{\mathcal{C}_t} |g_{\phi_s(t)}| d\sigma \geq c(\delta)|\mathcal{C}_t|^{1/2} \|f\|_2$, as stated in (8.15). If in addition $t \in B_\rho$, then

$$\int_{\mathcal{C}_0^\delta} |g_{\phi_s(t)}| d\sigma = \int_{\mathcal{C}_t} |g_{\phi_s(t)}| d\sigma \geq c(\delta)|\mathcal{C}_t|^{1/2} \|f\|_2 \geq c'(\delta)|\mathcal{C}_0^\delta|^{1/2} \|f\|_2,$$

and so integrating in $t \in B_\rho$ and using that $\phi_s(t) \geq 1$ if $t \geq \psi_s(1)$ gives

$$\int_{B_\rho} \int_{\mathcal{C}_0^\delta} |g_{\phi_s(t)}| \phi_s(t) d\sigma dt \geq c(\delta)|\mathcal{C}_0^\delta|^{1/2} |B_\rho| \|f\|_2 \geq c'(\delta)|\mathcal{C}_0^\delta|^{1/2} \|f\|_2.$$

Given that $\mu_s(\mathcal{C}) = \mu_s([1, 2] \times \mathcal{C}_0^\delta) \asymp |\mathcal{C}_0^\delta|$ we obtain

$$\int_{\mathcal{C}} |g \mathbb{1}_{\mathcal{G}_1}| d\mu_s = \int_{B_\rho} \int_{\mathcal{C}_0^\delta} |g_{\phi_s(t)}| \phi_s(t) d\sigma dt \geq c(\delta)\mu_s(\mathcal{C})^{1/2} \|f\|_2.$$

Then $g \mathbb{1}_{\mathcal{G}_1}, f - g \mathbb{1}_{\mathcal{G}_1}$ and \mathcal{C} satisfy all of our requirements, given that $\text{supp}(g \mathbb{1}_{\mathcal{G}_1}) \subseteq \overline{\mathcal{G}_1} \subseteq \mathcal{C}, \mathcal{G}_1 \subseteq \mathcal{G}_0, |\mathcal{C}_t| \asymp_\delta \mu_s(\mathcal{C})$ for all $t \in B_\rho$, and thus

$$\begin{aligned} |g \mathbb{1}_{\mathcal{G}_1}(x, t)| &\leq c(\delta) \|f\|_{L^2(\mathcal{H}_s^3)} \mu_s(\mathcal{C})^{1/2} \mathbb{1}_{\mathcal{C}}(x, t), \text{ for all } (x, t), \\ \|g \mathbb{1}_{\mathcal{G}_1}\|_{L^2(\mathcal{H}_s^3)} &\geq c(\delta) \|f\|_{L^2(\mathcal{H}_s^3)}, \\ \|g \mathbb{1}_{\mathcal{G}_1}\|_{L^1(\mathcal{H}_s^3)} &\geq c(\delta) \mu_s(\mathcal{C})^{1/2} \|f\|_{L^2(\mathcal{H}_s^3)}. \end{aligned}$$

□

9 A Concentration-Compactness Lemma

The result of this section is stated for $\overline{\mathcal{H}}_s^3$ but a similar statement and proof also hold for \mathcal{H}_s^3 .

Lemma 9.1 *Let $\{\rho_n\}_n$ be a sequence in $L^2(\overline{\mathcal{H}}_s^3)$ satisfying*

$$\int_{\overline{\mathcal{H}}_s^3} |\rho_n|^2 d\bar{\mu}_s = \lambda,$$

where $\lambda > 0$ is fixed. Then there exists a subsequence $\{\rho_{n_k}\}_k$ such that $\{|\rho_{n_k}|^2\}_k$ satisfies one of the following three possibilities:

(i) (compactness) there exists $\ell_k \in \mathbb{N}$ such that

$$\forall \varepsilon > 0, \exists R < \infty, \int_{\{s2^{\ell_k-R} \leq |y| \leq s2^{\ell_k+R}\}} |\rho_{n_k}|^2 d\bar{\mu}_s \geq \lambda - \varepsilon;$$

(ii) (vanishing) $\lim_{k \rightarrow \infty} \sup_{\ell \in \mathbb{N}} \int_{\{s2^{\ell-R} \leq |y| \leq s2^{\ell+R}\}} |\rho_{n_k}|^2 d\bar{\mu}_s = 0$, for all $R < \infty$;

(iii) (dichotomy) There exists $\alpha \in (0, \lambda)$ such that for all $\varepsilon > 0$, there exist $R \in \mathbb{N}$, $k_0 \geq 1$ and nonnegative functions $\rho_{k,1}, \rho_{k,2} \in L^2(\overline{\mathcal{H}}_s^3)$ satisfying for $k \geq k_0$:

$$\|\rho_{n_k} - (\rho_{k,1} + \rho_{k,2})\|_{L^2(\overline{\mathcal{H}}_s^3)} \leq \varepsilon, \tag{9.1}$$

$$\left| \int_{\overline{\mathcal{H}}_s^3} |\rho_{k,1}|^2 d\bar{\mu}_s - \alpha \right| \leq \varepsilon, \quad \left| \int_{\overline{\mathcal{H}}_s^3} |\rho_{k,2}|^2 d\bar{\mu}_s - (\lambda - \alpha) \right| \leq \varepsilon, \tag{9.2}$$

$$\text{supp}(\rho_{k,1}) \subseteq \{y \in \mathbb{R}^3 : s2^{\ell_k-R} \leq |y| \leq s2^{\ell_k+R}\}, \tag{9.3}$$

$$\text{supp}(\rho_{k,2}) \subseteq \{y \in \mathbb{R}^3 : |y| \leq s2^{\ell_k-R_k}\} \cup \{y \in \mathbb{R}^3 : |y| \geq s2^{\ell_k+R_k}\}, \tag{9.4}$$

for certain sequences $\{\ell_k\}_k$ and $\{R_k\}_k$, where $R_k \rightarrow \infty$ as $k \rightarrow \infty$.

Proof The proof is identical to the proof of Lemma I.1 in [30], by defining the sequence of functions

$$Q_n : [0, \infty) \rightarrow \mathbb{R}_+, \quad Q_n(t) = \sup_{\ell \in \mathbb{N}} \int_{\{s2^{\ell-t} \leq |y| \leq s2^{\ell+t}\}} |\rho_n(y)|^2 d\bar{\mu}_s(y).$$

We omit the details. □

In the forthcoming sections, we will be working with an L^2 normalized extremizing sequence $\{f_n\}_n$ and will apply the preceding lemma with $\lambda = 1$. We will slightly abuse notation and say that $\{f_n\}_n$ satisfies either *concentration*, *vanishing* or *dichotomy*, when the sequence $\{|f_n|^2\}_n$ satisfies the respective alternative.

10 Bilinear Estimates and Discarding Dichotomy

In this section we show that an extremizing sequence for \bar{T} can not satisfy the dichotomy condition (iii) of Lemma 9.1. This will be a consequence of bilinear estimates at dyadic scales.

Proposition 10.1 *There exists a constant $C < \infty$ with the following property. Let $s > 0$, $k, k' \in \mathbb{N}$ and $f, g \in L^2(\mathcal{H}_s^3)$ supported where $2^k s \leq |y| \leq 2^{k+1} s$ and $2^{k'} s \leq |y| \leq 2^{k'+1} s$ respectively. Then*

$$\|T_s f \cdot T_s g\|_{L^2(\mathbb{R}^4)} \leq C 2^{-\frac{1}{4}|k-k'|} \|f\|_{L^2(\mathcal{H}_s^3)} \|g\|_{L^2(\mathcal{H}_s^3)}.$$

Proof Without loss of generality we can assume $k' \geq k$. Using Lemma 8.3 we write

$$f \mu_s * g \mu_s(x, t) = \int_0^t (f \sigma_{\phi_s(t')} * g \sigma_{\phi_s(t-t')})(x) dt',$$

so that by Minkowski’s integral inequality

$$\|f \mu_s * g \mu_s\|_{L^2_{x,t}} \leq \left\| \int_0^t \|f \sigma_{\phi_s(t')} * g \sigma_{\phi_s(t-t')}\|_{L^2_x} dt' \right\|_{L^2_t}. \tag{10.1}$$

Recalling (8.9), the right hand side of (10.1) satisfies

$$\begin{aligned} & \left\| \int_0^t \|f \sigma_{\phi_s(t')} * g \sigma_{\phi_s(t-t')}\|_{L^2_x} dt' \right\|_{L^2_t} \\ & \leq C \left\| \int_0^t \phi_s(t')^{1/4} \|f_{\phi_s(t')}\|_2 \phi_s(t-t')^{1/4} \|g_{\phi_s(t-t')}\|_2 dt' \right\|_{L^2_t} \\ & \leq C \int_0^\infty \phi_s(t')^{1/4} \|f_{\phi_s(t')}\|_2 \left\| \mathbb{1}_{\{t \geq t'\}}(t') \phi_s(t-t')^{1/4} \|g_{\phi_s(t-t')}\|_2 \right\|_{L^2_t} dt' \\ & \leq C \left\| \phi_s(t)^{1/4} \|g_{\phi_s(t)}\|_2 \right\|_{L^2_t} \int_{\psi_s(2^k s)}^{\psi_s(2^{k+1} s)} \phi_s(t')^{1/4} \|f_{\phi_s(t')}\|_2 dt', \end{aligned}$$

where in the last line we used the support condition for f . Recalling the support condition for g

$$\begin{aligned} \left\| \phi_s(t)^{1/4} \|g_{\phi_s(t)}\|_{L^2(\mathbb{S}^2)} \right\|_{L^2_t}^2 &= \int_{\psi_s(2^{k's})}^{\psi_s(2^{k'+1}s)} \phi_s(t)^{1/2} \|g_{\phi_s(t)}\|_{L^2(\mathbb{S}^2)}^2 dt \\ &\leq (\phi_s(\psi_s(2^{k's})))^{-1/2} \int_0^\infty \phi_s(t) \|g_{\phi_s(t)}\|_{L^2(\mathbb{S}^2)}^2 dt \\ &= (2^{k's})^{-1/2} \|g\|_{L^2(\mathcal{H}_s^3)}^2, \end{aligned}$$

where in the last line we used Lemma 8.4. Similarly

$$\begin{aligned} \int_{\psi_s(2^k s)}^{\psi_s(2^{k+1}s)} \phi_s(t')^{1/4} \|f_{\phi_s(t')}\|_2 dt' &\leq \left(\int_{\psi_s(2^k s)}^{\psi_s(2^{k+1}s)} \phi_s(t')^{1/2} \|f_{\phi_s(t')}\|_2^2 dt' \right)^{1/2} \\ &\quad \times \left(\int_{\psi_s(2^k s)}^{\psi_s(2^{k+1}s)} 1 dt' \right)^{1/2} \\ &\leq (2^k s)^{-1/4} (\psi_s(2^{k+1}s) - \psi_s(2^k s))^{1/2} \|f\|_{L^2(\mathcal{H}_s^3)} \\ &\asymp (2^k s)^{-1/4} (2^k s)^{1/2} \|f\|_{L^2(\mathcal{H}_s^3)} \\ &= (2^k s)^{1/4} \|f\|_{L^2(\mathcal{H}_s^3)}. \end{aligned}$$

We conclude that

$$\begin{aligned} \|f \mu_s * g \mu_s\|_{L^2_{x,t}} &\leq \left\| \int_0^t \|f \sigma_{\phi_s(t')} * g \sigma_{\phi_s(t-t')}\|_{L^2_x} dt' \right\|_{L^2_t} \lesssim 2^{k/4} \|f\|_{L^2(\mathcal{H}_s^3)} 2^{-k'/4} \|g\|_{L^2(\mathcal{H}_s^3)} \\ &= 2^{-\frac{1}{4}|k'-k|} \|f\|_{L^2(\mathcal{H}_s^3)} \|g\|_{L^2(\mathcal{H}_s^3)}. \end{aligned}$$

□

Proposition 10.2 *Let $f, g \in L^2(\mathcal{H}^3)$ and suppose that their supports are separated in the sense that there exist $k, k' \in \mathbb{N}$, $k \leq k'$, such that $\text{supp}(f) \subseteq \{|y| \leq 2^k\}$ and $\text{supp}(g) \subseteq \{|y| \geq 2^{k'}\}$. Then*

$$\|Tf \cdot Tg\|_{L^2(\mathbb{R}^4)} \leq C 2^{-\frac{1}{4}|k-k'|} \|f\|_{L^2(\mathcal{H}^3)} \|g\|_{L^2(\mathcal{H}^3)}.$$

Similarly, if there exist $k, R, R' \in \mathbb{N}, R \leq R'$, such that $\text{supp}(f) \subseteq \{2^{k-R} \leq |y| \leq 2^{k+R}\}$ and $\text{supp}(g) \subseteq \{|y| \leq 2^{k-R'}\} \cup \{|y| \geq 2^{k+R'}\}$, then

$$\|Tf \cdot Tg\|_{L^2(\mathbb{R}^4)} \leq C2^{-\frac{1}{4}|R-R'|} \|f\|_{L^2(\mathcal{H}^3)} \|g\|_{L^2(\mathcal{H}^3)}.$$

Proof We decompose $f = \sum_{m \in \mathbb{N}} f_m$ and $g = \sum_{m' \in \mathbb{N}} g_{m'}$ where f_m, g_m are supported where $2^m \leq |y| \leq 2^{m+1}, m \geq 0$. Then

$$\begin{aligned} \|Tf \cdot Tg\|_{L^2(\mathbb{R}^4)} &= \left\| \sum_{m,m'} Tf_m \cdot Tg_{m'} \right\|_{L^2} \leq \sum_{m,m'} \|Tf_m \cdot Tg_{m'}\|_{L^2} \\ &\leq \sum_{m,m'} 2^{-\frac{1}{4}|m-m'|} \|f_m\|_{L^2} \|g_{m'}\|_{L^2} \\ &= 2^{-\frac{1}{4}|k'-k+1|} \sum_{m \leq 0, m' \geq 0} 2^{-\frac{1}{4}|m-m'|} \|f_{m+k-1}\|_{L^2} \|g_{m'+k'}\|_{L^2} \\ &\leq C2^{-\frac{1}{4}|k'-k|} \|f\|_{L^2(\mathcal{H}^3)} \|g\|_{L^2(\mathcal{H}^3)}. \end{aligned}$$

The second part of the proposition follows from the first and the triangle inequality. □

Decomposing a function $f \in L^2(\overline{\mathcal{H}}^3)$ as the sum of a function $f_+ \in L^2(\mathcal{H}^3)$ and $f_- \in L^2(-\mathcal{H}^3)$, $f = f_+ + f_-$, using that $\overline{T}f(\cdot, \cdot) = Tf_+(\cdot, \cdot) + Tf_-(\cdot, -\cdot)$ and the triangle inequality we can obtain a statement analogous to the previous proposition for functions on the full one-sheeted hyperboloid $\overline{\mathcal{H}}^3$: if f, g belong to $L^2(\overline{\mathcal{H}}^3)$ and satisfy for some $k, R, R' \in \mathbb{N}, R \leq R'$:

$$\text{supp}(f) \subseteq \{2^{k-R} \leq |y| \leq 2^{k+R}\}, \text{supp}(g) \subseteq \{|y| \leq 2^{k-R'}\} \cup \{|y| \geq 2^{k+R'}\},$$

then

$$\|\overline{T}f \cdot \overline{T}g\|_{L^2(\mathbb{R}^4)} \leq C2^{-\frac{1}{4}|R-R'|} \|f\|_{L^2(\overline{\mathcal{H}}^3)} \|g\|_{L^2(\overline{\mathcal{H}}^3)}. \tag{10.2}$$

Proposition 10.3 *An extremizing sequence for the adjoint Fourier restriction inequality (1.10) on $\overline{\mathcal{H}}^3$ does not satisfy dichotomy.*

Proof Let us argue by contradiction. Let $\{f_n\}_n$ be an extremizing sequence such that $\{|f_n|^2\}_n$ satisfies condition (iii), *dichotomy*, in Lemma 9.1. Let $\varepsilon > 0$ be given and $f_{n,1}, f_{n,2}, n_0$ be as in the conclusion of the dichotomy condition. Then, for $n \geq n_0$

$$\|\overline{T}f_n - \overline{T}f_{n,1} - \overline{T}f_{n,2}\|_{L^4} \leq \overline{\mathbf{H}}_4 \|f_n - (f_{n,1} + f_{n,2})\|_{L^2} \leq \overline{\mathbf{H}}_4 \varepsilon,$$

therefore

$$\|\overline{T}f_n\|_{L^4} \leq \overline{\mathbf{H}}_4 \varepsilon + \|\overline{T}(f_{n,1} + f_{n,2})\|_{L^4}. \tag{10.3}$$

Expanding, using Proposition 10.2 (or the comment thereafter) and the support condition for $f_{n,1}$ and $f_{n,2}$ as in (9.1)–(9.4), there exists $C < \infty$ independent of ε such that for all n large enough

$$\begin{aligned} \|\overline{T}(f_{n,1} + f_{n,2})\|_{L^4}^4 &= \|(\overline{T}(f_{n,1} + f_{n,2}))^2\|_{L^2}^2 = \|(\overline{T}f_{n,1})^2 + 2\overline{T}f_{n,1} \cdot \overline{T}f_{n,2} + (\overline{T}f_{n,2})^2\|_{L^2}^2 \\ &= \|\overline{T}f_{n,1}\|_{L^4}^4 + \|\overline{T}f_{n,2}\|_{L^4}^4 + 2\langle (\overline{T}f_{n,1})^2, (\overline{T}f_{n,2})^2 \rangle \\ &\quad + 4\langle (\overline{T}f_{n,1})^2, \overline{T}f_{n,1} \cdot \overline{T}f_{n,2} \rangle + 4\langle (\overline{T}f_{n,2})^2, \overline{T}f_{n,1} \cdot \overline{T}f_{n,2} \rangle \\ &\leq \|\overline{T}f_{n,1}\|_{L^4}^4 + \|\overline{T}f_{n,2}\|_{L^4}^4 + \varepsilon \\ &\leq \overline{\mathbf{H}}_4^4 \|f_{n,1}\|_2^4 + \overline{\mathbf{H}}_4^4 \|f_{n,2}\|_2^4 + \varepsilon \\ &\leq \overline{\mathbf{H}}_4^4 (\alpha^2 + (1 - \alpha)^2) + C\varepsilon, \end{aligned}$$

so that using (10.3) and taking $n \rightarrow \infty$ we find that for any $\varepsilon > 0$

$$\overline{\mathbf{H}}_4^4 \leq \overline{\mathbf{H}}_4^4 (\alpha^2 + (1 - \alpha)^2) + C\varepsilon,$$

for some constant $C < \infty$ independent of ε .

We conclude $1 \leq \alpha^2 + (1 - \alpha)^2$. We reach a contradiction since $\alpha \in (0, 1)$ and the numerical inequality $\alpha^2 + (1 - \alpha)^2 < 1$ holds. \square

The proof we just gave to discard *dichotomy* can be seen in the context of the *strict superadditivity condition* as proposed by Lions [30, Sect. I.2]; see for instance the comment at the end of Appendix A in [35].

11 Dyadic Refinements and Discarding Vanishing

In this section we prove a dyadic improvement of the $L^2 \rightarrow L^4$ inequality (1.4) that will imply that extremizing sequences for \overline{T} do not satisfy the *vanishing* condition (ii) of Lemma 9.1. We start with the following proposition.

Proposition 11.1 *There exists a constant $C < \infty$ with the following property. Let $f \in L^2(\mathcal{H}^3)$ and for $k \in \mathbb{N}$ let $f_k(y) = f(y) \mathbb{1}_{\{2^k \leq |y| < 2^{k+1}\}}$. Then*

$$\|Tf\|_{L^4(\mathbb{R}^4)} \leq C \left(\sum_{k \geq 0} \|f_k\|_{L^2(\overline{\mathcal{H}}^3)}^3 \right)^{1/3}. \tag{11.1}$$

Proof We follow [38, Proof of Prop. 3.4]. We have

$$\begin{aligned} \|Tf\|_{L^4(\mathbb{R}^4)}^3 &= \|Tf \cdot Tf \cdot Tf\|_{L^{4/3}} = \left\| \sum_{k,l,m} Tf_k \cdot Tf_l \cdot Tf_m \right\|_{L^{4/3}} \\ &\leq \sum_{k,l,m} \|Tf_k \cdot Tf_l \cdot Tf_m\|_{L^{4/3}}. \end{aligned}$$

Fix a triplet (k, l, m) . We can assume without loss of generality that $|k - l| = \max\{|k - l|, |k - m|, |l - m|\}$ so that the use of Hölder’s inequality and Proposition 10.1 give

$$\begin{aligned} \|Tf_k \cdot Tf_l \cdot Tf_m\|_{L^{4/3}} &\leq \|Tf_k \cdot Tf_l\|_{L^2} \|Tf_m\|_{L^4} \\ &\lesssim 2^{-\frac{1}{4}|k-l|} \|f_k\|_{L^2} \|f_l\|_{L^2} \|f_m\|_{L^2} \\ &\leq 2^{-|k-l|/12} 2^{-|k-m|/12} 2^{-|l-m|/12} \|f_k\|_{L^2} \|f_l\|_{L^2} \|f_m\|_{L^2}. \end{aligned}$$

We conclude that

$$\|Tf\|_{L^4(\mathbb{R}^4)}^3 \lesssim \sum_{k,l,m} 2^{-|k-l|/12} 2^{-|k-m|/12} 2^{-|l-m|/12} \|f_k\|_{L^2} \|f_l\|_{L^2} \|f_m\|_{L^2}.$$

Applying Hölder’s inequality to the last estimate we obtain

$$\|Tf\|_{L^4(\mathbb{R}^4)}^3 \lesssim \sum_{k,l,m} 2^{-|k-l|/12} 2^{-|k-m|/12} 2^{-|l-m|/12} \|f_k\|_{L^2}^3 \lesssim \sum_k \|f_k\|_{L^2}^3.$$

□

As an application we have the following corollary.

Corollary 11.2 *There exists a constant $C < \infty$ with the following property. Let $f \in L^2(\mathcal{H}^3)$ and for $k \in \mathbb{N}$ let $f_k(y) = f(y)\mathbb{1}_{\{2^k \leq |y| < 2^{k+1}\}}$. Then*

$$\|Tf\|_{L^4(\mathbb{R}^4)} \leq C \sup_{k \in \mathbb{N}} \|f_k\|_{L^2(\mathcal{H}^3)}^{1/3} \|f\|_{L^2(\mathcal{H}^3)}^{2/3}. \tag{11.2}$$

Proof From Proposition 11.1 we obtain

$$\begin{aligned} \|Tf\|_{L^4(\mathbb{R}^4)} &\leq C \left(\sum_{k \geq 0} \|f_k\|_{L^2(\mathcal{H}^3)}^3 \right)^{1/3} = C \left(\sum_{k \geq 0} \|f_k\|_{L^2(\mathcal{H}^3)} \cdot \|f_k\|_{L^2(\mathcal{H}^3)}^2 \right)^{1/3} \\ &\leq C \sup_{k \in \mathbb{N}} \|f_k\|_{L^2(\mathcal{H}^3)}^{1/3} \left(\sum_{k \geq 0} \|f_k\|_{L^2(\mathcal{H}^3)}^2 \right)^{1/3} \\ &= C \sup_{k \in \mathbb{N}} \|f_k\|_{L^2(\mathcal{H}^3)}^{1/3} \|f\|_{L^2(\mathcal{H}^3)}^{2/3}. \end{aligned}$$

□

The same previous argument and (10.2) give

$$\|\overline{T}f\|_{L^4(\mathbb{R}^4)} \lesssim \sup_{k \in \mathbb{N}} \|f_k\|_{L^2(\overline{\mathcal{H}}^3)}^{1/3} \|f\|_{L^2(\overline{\mathcal{H}}^3)}^{2/3}, \tag{11.3}$$

and thus it is immediate that for an extremizing sequence for \overline{T} the vanishing alternative does not hold.

Proposition 11.3 *Extremizing sequences for the adjoint Fourier restriction inequality (1.10) on $\overline{\mathcal{H}^3}$ do not satisfy vanishing.*

12 Convergence to the Cone

The content of this section is important in the study of the compactness alternative of Lemma 9.1, in the case in which, in addition, the extremizing sequences concentrate at infinity.

Formally, we can write $\Gamma^3 = \mathcal{H}_0^3$, $\sigma_c = \mu_0$ and $T_c = T_0$. It is natural then to study relationships between the adjoint Fourier restriction operator on the cone (Γ^3, σ_c) and on each member of the family $\{(\mathcal{H}_s^3, \mu_s)\}_{s>0}$, in the limit $s \rightarrow 0^+$, and this is the content of this section (see also [29, Lemma 2.9] for related results for the case of the two-sheeted hyperboloid).

Note that if $0 \leq t \leq s$ and $|y| \geq s$, then the inequality $\sqrt{|y|^2 - s^2} \leq \sqrt{|y|^2 - t^2}$ implies that for $f \in L^2(\mu_s)$

$$\|f \mathbb{1}_{\{|y| \geq s\}}\|_{L^2(\sigma_c)} \leq \|f \mathbb{1}_{\{|y| \geq s\}}\|_{L^2(\mu_t)} \leq \|f\|_{L^2(\mu_s)},$$

and for $f \in L^2(\mu_s)$, extended to be zero in the region where $|y| \leq s$,

$$\lim_{t \rightarrow 0^+} \|f\|_{L^2(\mu_t)} = \|f\|_{L^2(\sigma_c)}.$$

Throughout this section we will commonly encounter the situation of having $f \in L^2(\mathcal{H}_s^3)$ and regard it as a function in $L^2(\mathcal{H}_t^3)$, $0 \leq t \leq s$, via the orthogonal projection to $\mathbb{R}^3 \times \{0\}$. In this case, it will be understood that f is extended by zero in the region where⁸ $|y| \leq s$.

Let us consider the following situation. Let $a > 0$, $\{s_n\}_n \subset \mathbb{R}$ satisfying $s_n \rightarrow 0$ as $n \rightarrow \infty$. Let $\{f_n\}_n$ be a family of functions with $f_n \in L^2(\mathcal{H}_{s_n}^3)$, supported where $|y| \geq a$ and satisfying $\sup_n \|f_n\|_{L^2(\mu_{s_n})} < \infty$. As already noted, $\|f_n\|_{L^2(\mu_{s_n})} \geq \|f_n\|_{L^2(\sigma_c)}$, therefore $\{f_n \mathbb{1}_{\{|y| \geq s_n\}}\}_n$ is a bounded sequence in $L^2(\sigma_c)$. We can assume, possibly after passing to a subsequence, that $f_n \rightharpoonup f$ in $L^2(\sigma_c)$. The aim of this section is to compare $\|f \sigma_c * f \sigma_c\|_2$ and the limiting behavior of $\|f_n \mu_{s_n} * f_n \mu_{s_n}\|_2$, as $n \rightarrow \infty$, in the case when $f \neq 0$. We have some preliminary results.

Lemma 12.1 *Let $a > 0$ and $f \in L^2(\mathcal{H}_s^3)$ for all small $s > 0$ and supported where $|y| \geq a$, then*

$$\|T_s f - T_c f\|_{L^4(\mathbb{R}^4)} \rightarrow 0 \text{ as } s \rightarrow 0^+.$$

Proof From the uniform in s bound $\|T_s\| = \|T\|$ and density arguments, it suffices to consider the case when $f \in C_c^\infty(\mathbb{R}^3)$. Let $b \in (a, \infty)$ be such that the support of f is contained in the region where $a \leq |y| \leq b$.

⁸ Alternatively, we can think of f as a function living in $L^2(\mathbb{R}^3, w \, dx)$, for different weights w .

By Plancherel’s theorem, to show $T_s f \rightarrow Tf$ in $L^4(\mathbb{R}^4)$, as $s \rightarrow 0^+$, it suffices to show that $f\mu_s * f\mu_s \rightarrow f\sigma_c * f\sigma_c$ and $f\mu_s * f\sigma_c \rightarrow f\sigma_c * f\sigma_c$ in $L^2(\mathbb{R}^4)$, as $s \rightarrow 0^+$.

First, we claim that there is pointwise convergence $f\mu_s * f\mu_s(\xi, \tau) \rightarrow f\sigma_c * f\sigma_c(\xi, \tau)$ and $f\mu_s * f\sigma_c(\xi, \tau) \rightarrow f\sigma_c * f\sigma_c(\xi, \tau)$, a.e. $(\xi, \tau) \in \mathbb{R}^4$, as $s \rightarrow 0^+$. Indeed, as in the proof of the explicit formula for $\mu_s * \mu_s$ in Sect. 3, we can write integral formulas for $f\mu_s * f\mu_s, f\mu_s * f\sigma_c$ and $f\sigma_c * f\sigma_c$ for any $s \geq 0$ as in (3.10)–(3.12). Given that $\tilde{R}_s(\xi)$ and $\tilde{Q}_s(\xi)$ are explicit, we can spell out (3.10) and (3.11) from where it becomes clear that there is a.e. pointwise convergence to $f\sigma_c * f\sigma_c$ as $s \rightarrow 0^+$. Note that for each fixed $\xi \neq 0, \mathbb{1}_{\tilde{R}_s(\xi)}(u, v) \rightarrow \mathbb{1}_{\{|u| \leq |\xi| \leq v\}}(u, v)$ and $\mathbb{1}_{\tilde{Q}_s(\xi)}(u, v) \rightarrow \mathbb{1}_{\{|u| \leq |\xi| \leq v\}}(u, v)$ a.e. pointwise as $s \rightarrow 0^+$.

By the Dominated Convergence Theorem, to finish it suffices to show that there exists $F \in L^2(\mathbb{R}^4)$ such that $|f\mu_s * f\mu_s(\xi, \tau)| \leq F(\xi, \tau)$ and $|f\mu_s * f\sigma_c(\xi, \tau)| \leq F(\xi, \tau)$, for a.e. $(\xi, \tau) \in \mathbb{R}^4$. We use the inequalities

$$|f\mu_s * f\mu_s(\xi, \tau)|^2 \leq \|f\|_{L^\infty}^4 (\mu_s * \mu_s)^2(\xi, \tau),$$

$$|f\mu_s * f\sigma_c(\xi, \tau)|^2 \leq \|f\|_{L^\infty}^4 (\mu_s * \sigma_c)^2(\xi, \tau).$$

On the supports of $f\mu_s * f\mu_s$ and $f\mu_s * f\sigma_c$, the functions $\mu_s * \mu_s$ and $\mu_s * \sigma_c$ are uniformly bounded in $s \in (0, 1)$, as can be seen from Lemma 3.2 and formula (3.9). It follows that we can take

$$F(\xi, \tau) = 4\pi \|f\|_{L^\infty}^2 (1 + a^{-1}) \mathbb{1}_{\{a \leq \tau \leq 2b\}} \mathbb{1}_{\{|\xi| \leq 2b\}}(\xi, \tau).$$

□

Remark 12.2 Another possible way to prove Lemma 12.1, which does not rely on the exponent being an even integer, can be to follow the outline in the proof of [29, Lemma 2.9 (d)] which makes use of the analysis of oscillatory integrals through the method of stationary phase. More in detail, we could proceed as follows. As in the proof above, we can restrict attention to the case when $f \in C_c^\infty(\mathbb{R}^3)$, supported in the region where $a \leq |y| \leq b$, for some $b < \infty$.

We first consider the pointwise convergence $T_s f(x, t) \rightarrow T_c f(x, t)$, as $s \rightarrow 0$ for a.e. $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$. Recall the definitions of $T_s f(x, t)$ and $T_c f(x, t)$ in (1.13) and (1.14) and note that there is pointwise convergence of their integrands, that is

$$e^{ix \cdot y} e^{it\sqrt{|y|^2 - s^2}} f(y) \frac{\mathbb{1}_{\{|y| > s\}}}{\sqrt{|y|^2 - s^2}} \rightarrow e^{ix \cdot y} e^{it|y|} f(y) \frac{\mathbb{1}_{\{|y| > 0\}}}{|y|}, \text{ as } s \rightarrow 0,$$

for all $(x, t) \in \mathbb{R}^4, y \in \mathbb{R}^3$. On the other hand, as the support of f is contained in the region where $|y| \geq a$, for all $s \in (0, a/2)$ we have

$$\left| e^{ix \cdot y} e^{it\sqrt{|y|^2 - s^2}} f(y) \frac{\mathbb{1}_{\{|y| > s\}}}{\sqrt{|y|^2 - s^2}} \right| \leq |f(y)| \frac{1}{\sqrt{|a|^2 - s^2}} \leq \frac{2}{\sqrt{3}a} |f(y)|,$$

so that as $|f| \in L^2(\mathbb{R}^3)$, we can use the dominated convergence theorem to conclude that $T_s f \rightarrow T_c f$ pointwise in \mathbb{R}^4 .

Let us take $M \in [1, \infty)$ and $s \leq a/2$. We have the identity

$$|x|^2 T_s f(x, t) = - \int_{\mathbb{R}^3} e^{ix \cdot y} \Delta_y \left(\frac{e^{it\sqrt{|y|^2 - s^2}} f(y)}{\sqrt{|y|^2 - s^2}} \right) dy,$$

as can be seen by integration by parts, so that if $|t| \leq M$, we obtain

$$|T_s f(x, t)| \lesssim_{a,b} \frac{1 + M^2}{1 + |x|^2} \|f\|_{H^2(\mathbb{R}^3)} \in L^4(\mathbb{R}^3 \times [-M, M]),$$

where $H^2(\mathbb{R}^3)$ denotes the inhomogeneous Sobolev space with norm $\|f\|_{H^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |\hat{f}(x)|^2 (1 + |x|^2)^2 dx$. By the dominated convergence theorem we conclude that $T_s f \rightarrow T_c f$ in $L^4(\mathbb{R}^3 \times [-M, M])$, as $s \rightarrow 0^+$, for each $M < \infty$.

To treat the region where $|t| \geq M$, recall the dispersive estimates

$$\|T_s g(\cdot, t)\|_{L_x^\infty(\mathbb{R}^3)} \lesssim_{a,b} \frac{1}{t^{3/2}} \|g\|_{L^1(\mathbb{R}^3)} \quad \text{and} \quad \|T_c g(\cdot, t)\|_{L_x^2(\mathbb{R}^3)} \lesssim_{a,b} \frac{1}{t} \|g\|_{L^1(\mathbb{R}^3)},$$

valid for any $g \in L^1(\mathbb{R}^3)$ supported where $a \leq |y| \leq b$. They can be proved using the method of stationary phase or by studying the fundamental solutions of the respective underlying classical partial differential equation as mentioned in the Introduction. Since we also have the L^2 -norm conservation $\|T_s g\|_{L_x^2(\mathbb{R}^3)} = \|T_c g\|_{L_x^2(\mathbb{R}^3)} = \|g\|_{L^2(\mathbb{R}^3)}$ we obtain the interpolated estimates

$$\|T_s g\|_{L_x^4(\mathbb{R}^3)} \lesssim_{a,b} \frac{1}{t^{3/4}} \|g\|_{L^{4/3}(\mathbb{R}^3)} \quad \text{and} \quad \|T_c g\|_{L_x^4(\mathbb{R}^3)} \lesssim_{a,b} \frac{1}{t^{1/2}} \|g\|_{L^{4/3}(\mathbb{R}^3)}.$$

In this way

$$\|T_s f - T_c f\|_{L_{x,t}^4(|t| \geq M)} \leq \|T_s f\|_{L_{x,t}^4(|t| \geq M)} + \|T_c f\|_{L_{x,t}^4(|t| \geq M)} \lesssim_{a,b} \frac{1}{M^{1/4}} \|f\|_{L^{4/3}(\mathbb{R}^3)}.$$

The previous estimate in the region $\{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : |t| \geq M\}$ and the L^4 convergence in the region $\mathbb{R}^3 \times [-M, M]$, valid for any $M \in [1, \infty)$, imply the desired result.

Recall the Fourier multiplier notation and the $\dot{H}^{1/2}(\mathbb{R}^3)$ homogeneous Sobolev norm and inner product from (5.1) and (5.2). We have the following lemma.

Lemma 12.3 *Let $a > 0$, then for each fixed $t \in \mathbb{R}$*

$$\lim_{s \rightarrow 0} \sup_{\substack{u \in \dot{H}^{1/2}(\mathbb{R}^3) \\ \text{supp}(\hat{u}) \subseteq \{\xi \in \mathbb{R}^3 : |\xi| \geq a\}}} \frac{\|e^{it\sqrt{-\Delta - s^2}} u - e^{it\sqrt{-\Delta}} u\|_{\dot{H}^{1/2}(\mathbb{R}^3)}}{\|u\|_{\dot{H}^{1/2}(\mathbb{R}^3)}} = 0.$$

Proof For any $s \geq 0$ we have $\|e^{it\sqrt{-\Delta-s^2}}u\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = \|u\|_{\dot{H}^{1/2}}$. Now

$$e^{it\sqrt{|y|^2-s^2}} - e^{it|y|} = \int_0^s \frac{d}{dr} e^{it\sqrt{|y|^2-r^2}} dr = -it \int_0^s e^{it\sqrt{|y|^2-r^2}} \frac{r}{\sqrt{|y|^2-r^2}} dr.$$

Then,

$$\begin{aligned} \|(e^{it\sqrt{-\Delta-s^2}} - e^{it\sqrt{-\Delta}})u\|_{\dot{H}^{1/2}(\mathbb{R}^3)} &\leq |t| \int_0^s \left\| e^{it\sqrt{-\Delta-r^2}} \frac{r}{\sqrt{-\Delta-r^2}} u \right\|_{\dot{H}^{1/2}(\mathbb{R}^4)} dr \\ &= |t| \int_0^s \left\| \frac{r}{\sqrt{-\Delta-r^2}} u \right\|_{\dot{H}^{1/2}(\mathbb{R}^3)} dr. \end{aligned}$$

If $0 \leq s < a$ and $\text{supp}(\hat{u}) \subseteq \{|\xi| \geq a\}$, then

$$\left\| \frac{r}{\sqrt{-\Delta-r^2}} u \right\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq \frac{r}{\sqrt{a^2-r^2}} \|u\|_{\dot{H}^{1/2}(\mathbb{R}^3)},$$

so that

$$\|(e^{it\sqrt{-\Delta-s^2}} - e^{-it\sqrt{-\Delta}})u\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq |t|(a - \sqrt{a^2-s^2}) \|u\|_{\dot{H}^{1/2}(\mathbb{R}^3)},$$

and the conclusion follows. □

We now address the pointwise convergence of $T_{s_n} f_n$ to $T_c f$.

Lemma 12.4 *Let $a > 0$ and $\{s_n\}_n$ be a sequence of positive real numbers converging to zero. Let $f \in L^2(\Gamma^3)$ and $\{f_n\}_n$ be a sequence satisfying $f_n \in L^2(\mathcal{H}_{s_n}^3)$, $\sup_n \|f_n\|_{L^2(\mu_{s_n})} < \infty$ and supported where $|y| \geq a$, for all n . Suppose that $f_n \rightarrow f$ in $L^2(\Gamma^3)$, as $n \rightarrow \infty$. Then, as $n \rightarrow \infty$*

$$T_{s_n} f_n(x, t) \rightarrow T_c f(x, t) \text{ for a.e. } (x, t) \in \mathbb{R}^4.$$

Proof Following the argument in the proof of Proposition 5.1, we start by defining v_n and v by their Fourier transforms

$$\hat{v}_n(y) = \frac{f_n(y)}{\sqrt{|y|^2 - s_n^2}}, \quad \hat{v}(y) = \frac{f(y)}{|y|}.$$

Since $\sup_n \|f_n\|_{L^2(\Gamma^3)} \leq \sup_n \|f_n\|_{L^2(\mu_{s_n})} < \infty$ and the functions are supported where $|y| \geq a > 0$ we see that

$$\sup_n \|v_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^2 = \sup_n \int_{\mathbb{R}^3} |\hat{v}_n(y)|^2 |y| dy \leq \sup_n \frac{a}{\sqrt{a^2 - s_n^2}} \|f_n\|_{L^2(\mu_{s_n})}^2 < \infty,$$

and

$$\sup_n \|v_n\|_{L^2(\mathbb{R}^3)}^2 = (2\pi)^{-3} \sup_n \int_{\mathbb{R}^3} |\hat{v}_n(y)|^2 dy \leq (2\pi)^{-3} \sup_n \frac{1}{\sqrt{a^2 - s_n^2}} \|f_n\|_{L^2(\mu_{s_n})}^2 < \infty.$$

If $\varphi \in \dot{H}^{1/2}(\mathbb{R}^3)$, then $\hat{\varphi}(\cdot)|\cdot| \in L^2(\Gamma^3)$, from where we can deduce that $v_n \rightharpoonup v$ in $\dot{H}^{1/2}(\mathbb{R}^3)$, as $n \rightarrow \infty$. The operator T_{s_n} applied to f_n equals $(2\pi)^3 e^{it\sqrt{-\Delta - s_n^2}} v_n$. Fix $t \in \mathbb{R}$. From Lemma 12.3 we know $\|(e^{it\sqrt{-\Delta - s_n^2}} - e^{it\sqrt{-\Delta}})\mathbb{1}_{\{\sqrt{-\Delta} \geq a\}}\| \rightarrow 0$ as $n \rightarrow \infty$, the norm being as operators on $\dot{H}^{1/2}(\mathbb{R}^3)$. This, added to the continuity of $e^{it\sqrt{-\Delta}}$ in $\dot{H}^{1/2}(\mathbb{R}^3)$ implies

$$e^{it\sqrt{-\Delta - s_n^2}} v_n \rightharpoonup e^{it\sqrt{-\Delta}} v,$$

weakly in $\dot{H}^{1/2}(\mathbb{R}^3)$, as $n \rightarrow \infty$. Then, by the Rellich–Kondrashov Theorem, for any $R > 0$

$$e^{it\sqrt{-\Delta - s_n^2}} v_n \rightarrow e^{it\sqrt{-\Delta}} v,$$

strongly in $L^2(B(0, R))$, as $n \rightarrow \infty$. Denote by

$$F_n(t) := \int_{|x| < R} \left| e^{it\sqrt{-\Delta - s_n^2}} v_n - e^{it\sqrt{-\Delta}} v \right|^2 dx = \|e^{it\sqrt{-\Delta - s_n^2}} v_n - e^{it\sqrt{-\Delta}} v\|_{L^2(B(0, R))}^2.$$

Since we have $\|\hat{v}_n\|_{L^2(\mathbb{R}^3)} \lesssim_a \|f_n\|_{L^2(\mu_{s_n})}$ and $\|\hat{v}\|_{L^2(\mathbb{R}^3)} \lesssim_a \|f\|_{L^2(\sigma_c)}$, we obtain

$$\begin{aligned} F_n(t) &\leq \|e^{it\sqrt{-\Delta - s_n^2}} v_n - e^{it\sqrt{-\Delta}} v\|_{L^2(\mathbb{R}^3)}^2 \leq (\|v_n\|_{L^2(\mathbb{R}^3)} + \|v\|_{L^2(\mathbb{R}^3)})^2 \\ &\lesssim \|f_n\|_{L^2(\mu_s)}^2 + \|f\|_{L^2(\sigma_c)}^2. \end{aligned}$$

We can now finish exactly as in the proof of Proposition 5.1 and conclude that there exists a subsequence $\{n_k\}_k$ such that

$$T_{s_{n_k}} f_{n_k} - T_c f \rightarrow 0 \quad \text{a.e. in } \mathbb{R}^4.$$

□

Finally, we prove that the existence of an extremizing sequence that concentrates at infinity with a nonzero weak limit, after appropriate rescaling, implies that the operator norm of T is upper bounded by that of T_c (which in the end we will pair with Proposition 4.1 to rule out this scenario).

Lemma 12.5 *Let $\{s_n\}_n$ be a sequence of positive real numbers converging to zero. Let $f \in L^2(\Gamma^3)$ be a nonzero function and $\{f_n\}_n$ be a sequence satisfying $f_n \in L^2(\mathcal{H}_{s_n}^3)$, for all n . Suppose that:*

- (i) $\|f_n\|_{L^2(\mu_{s_n})} = 1,$
- (ii) $\|T_{s_n} f_n\|_{L^4} \rightarrow \|T\| (= \|T_1\|),$
- (iii) $f_n \rightharpoonup f \neq 0$ in $L^2(\Gamma^3),$

If there exists $a > 0$ such that

- (iv) $\text{supp}(f), \text{supp}(f_n) \subseteq \{y \in \mathbb{R}^3 : |y| \geq a\},$ for all $n,$

then

$$\|T\| \leq \|T_c\|.$$

If condition (iv) is relaxed to

- (v) $\sup_{n \in \mathbb{N}} \|f_n \mathbb{1}_{\{|y| \leq a\}}\|_{L^2(\mu_{s_n})} \leq \varepsilon,$ for some $\varepsilon > 0,$

then

$$\|T\|^2 \|f \mathbb{1}_{\{|y| \geq a\}}\|_{L^2(\sigma_c)}^2 \leq \|T_c\|^2 \|f \mathbb{1}_{\{|y| \geq a\}}\|_{L^2(\sigma_c)}^2 + C\varepsilon,$$

for some universal constant $C.$ In particular, if we have $\sup_{n \in \mathbb{N}} \|f_n \mathbb{1}_{\{|y| \leq a\}}\|_{L^2(\mu_{s_n})} \rightarrow 0$ as $a \rightarrow 0^+,$ then $\|T\| \leq \|T_c\|.$

An analog statement applies if we change T and T_c by \bar{T} and $\bar{T}_c,$ respectively, the proof being identical.

Proof We argue as in [20]. By the weak convergence condition (iii),

$$\|f_n - f\|_{L^2(\sigma_c)}^2 = \|f_n\|_{L^2(\sigma_c)}^2 - \|f\|_{L^2(\sigma_c)}^2 + o(1). \tag{12.1}$$

Now consider that (iv) holds. By (i) and (iv), $\|f_n\|_{L^2(\sigma_c)}^2 - \|f_n\|_{L^2(\mu_{s_n})}^2 \rightarrow 0.$ Indeed,

$$\begin{aligned} 0 &\leq \|f_n\|_{L^2(\mu_{s_n})}^2 - \|f_n\|_{L^2(\sigma_c)}^2 = \int_{|y| \geq a} |f_n(y)|^2 \left| \frac{1}{\sqrt{|y|^2 - s_n^2}} - \frac{1}{|y|} \right| dy \\ &\leq \|f_n\|_{L^2(\mu_{s_n})}^2 \left\| \frac{|y| - \sqrt{|y|^2 - s_n^2}}{|y|} \mathbb{1}_{\{|y| \geq a\}} \right\|_{L^\infty(\mathbb{R}^3)} \\ &= \left(1 - \sqrt{1 - s_n^2 a^{-2}}\right) \rightarrow 0, \end{aligned} \tag{12.2}$$

as $n \rightarrow \infty.$ Then, (12.1) implies

$$\|f_n - f\|_{L^2(\sigma_c)}^2 = \|f_n\|_{L^2(\mu_{s_n})}^2 - \|f\|_{L^2(\sigma_c)}^2 + o(1). \tag{12.3}$$

Because of conditions (iii) and (iv) and Lemma 12.4, $T_{s_n} f_n \rightarrow T_c f$ a.e. pointwise in $\mathbb{R}^4,$ as $n \rightarrow \infty,$ and we can apply the Brézis–Lieb lemma to the sequence $\{T_{s_n} f_n\}_n \subset L^4(\mathbb{R}^4)$ to obtain

$$\|T_{s_n} f_n - T_c f\|_{L^4(\mathbb{R}^4)}^4 = \|T_{s_n} f_n\|_{L^4(\mathbb{R}^4)}^4 - \|T_c f\|_{L^4(\mathbb{R}^4)}^4 + o(1).$$

Because by scaling the norm of the operator T_{s_n} is independent of n (see Sect. 14) and by (ii) $\|T_{s_n} f_n\|_{L^4(\mathbb{R}^4)} \rightarrow \|T\|$ as $n \rightarrow \infty$, we obtain

$$\begin{aligned} \|T_{s_n}\|^2 &= \|T\|^2 = \frac{\|T_{s_n} f_n\|_{L^4(\mathbb{R}^4)}^2}{\|f_n\|_{L^2(\mu_{s_n})}^2} + o(1) \\ &= \frac{(\|T_{s_n} f_n - T_c f\|_{L^4}^4 + \|T_c f\|_{L^4}^4 + o(1))^{1/2}}{\|f_n - f\|_{L^2(\sigma_c)}^2 + \|f\|_{L^2(\sigma_c)}^2 + o(1)} + o(1) \\ &\leq \frac{\|T_{s_n} f_n - T_c f\|_{L^4}^2 + \|T_c f\|_{L^4}^2 + o(1)}{\|f_n - f\|_{L^2(\sigma_c)}^2 + \|f\|_{L^2(\sigma_c)}^2 + o(1)} + o(1) \\ &\leq \frac{\|T_{s_n} f_n - T_{s_n} f\|_{L^4}^2 + \|T_c f\|_{L^4}^2 + o(1)}{\|f_n - f\|_{L^2(\sigma_c)}^2 + \|f\|_{L^2(\sigma_c)}^2 + o(1)} + o(1), \end{aligned} \tag{12.4}$$

where in the last inequality we used the triangle inequality and that $\|T_{s_n} f - T_c f\|_{L^4} \rightarrow 0$ as $n \rightarrow \infty$, from Lemma 12.1. Then

$$\|T_{s_n}\|^2 \leq \frac{\|T_{s_n}\|^2 \|f_n - f\|_{L^2(\mu_{s_n})}^2 + \|T_c f\|_{L^4}^2 + o(1)}{\|f_n - f\|_{L^2(\sigma_c)}^2 + \|f\|_{L^2(\sigma_c)}^2 + o(1)} + o(1),$$

and hence

$$\|T_{s_n}\|^2 (\|f_n - f\|_{L^2(\sigma_c)}^2 + \|f\|_{L^2(\sigma_c)}^2 + o(1)) \leq \|T_{s_n}\|^2 \|f_n - f\|_{L^2(\mu_{s_n})}^2 + \|T_c f\|_{L^4}^2 + o(1),$$

which is equivalent to

$$\|T_{s_n}\|^2 \|f\|_{L^2(\sigma_c)}^2 \leq \|T_c f\|_{L^4}^2 + \|T_{s_n}\|^2 (\|f_n - f\|_{L^2(\mu_{s_n})}^2 - \|f_n - f\|_{L^2(\sigma_c)}^2) + o(1).$$

Arguing as in (12.2) we obtain $\|f_n - f\|_{L^2(\mu_{s_n})}^2 - \|f_n - f\|_{L^2(\sigma_c)}^2 \rightarrow 0$, and therefore,

$$\|T\| = \|T_{s_n}\| \leq \frac{\|T_c f\|_{L^4}}{\|f\|_{L^2(\sigma_c)}} \leq \|T_c\|.$$

Finally, if we relax the support condition (iv) to the ε -small norm condition (v), it will be enough if in (12.4) we use

$$\frac{\|T_{s_n} f_n\|_{L^4(\mathbb{R}^4)}^2}{\|f_n\|_{L^2(\mu_{s_n})}^2} \leq \frac{\|T_{s_n}(f_n \mathbb{1}_{\{|y|\geq a\}})\|_{L^4(\mathbb{R}^4)}^2}{\|f_n \mathbb{1}_{\{|y|\geq a\}}\|_{L^2(\mu_{s_n})}^2} + C\varepsilon,$$

where $C < \infty$ is independent of n and a , together with $f_n \mathbb{1}_{\{|y|\geq a\}} \rightarrow f \mathbb{1}_{\{|y|\geq a\}}$ in $L^2(\Gamma^3)$ and $T_{s_n}(f_n \mathbb{1}_{\{|y|\geq a\}}) \rightarrow T_c(f \mathbb{1}_{\{|y|\geq a\}})$ a.e. in \mathbb{R}^4 , as $n \rightarrow \infty$, the latter property being a consequence of the former and Lemma 12.4. \square

13 Proof of Theorem 1.3

In previous Sects. 10 and 11, we proved that if $\{f_n\}_n$ is an extremizing sequence for \bar{T} , then subsequences of $\{|f_n|^2\}_n$ can not satisfy vanishing nor dichotomy of Lemma 9.1, which as we saw, were a consequence of bilinear estimates for \bar{T} . In this section we prove that, as a consequence of the compactness alternative and Lemma 12.5 of the previous section, extremizing sequences possess convergent subsequences and, as a result, extremizers exist.

Proof of Theorem 1.3 Let $\{f_n\}_n \subset L^2(\bar{\mathcal{H}}^3)$ be an L^2 normalized complex valued extremizing sequence for \bar{T} . After passing to a subsequence if necessary we can assume that alternative (i) in Lemma 9.1 holds for $\{|f_n|^2\}_n$, that is, there exists $\{\ell_n\}_n \subset \mathbb{N}$ with the property that for all $\varepsilon > 0$ there exists $R_\varepsilon < \infty$ such that for all $R \geq R_\varepsilon$ and $n \in \mathbb{N}$

$$\int_{\{2^{\ell_n-R} \leq |y| \leq 2^{\ell_n+R}\}} |f_n(y)|^2 d\bar{\mu}(y) \geq 1 - \varepsilon. \tag{13.1}$$

If there exists a subsequence $\{n_k\}_k \subset \mathbb{N}$ such that $\{\ell_{n_k}\}_k$ is bounded above, then we can apply the same method provided in the proof of Proposition 5.2 for the upper half of the hyperboloid, \mathcal{H}^3 , to conclude that there exists a subsequence $\{f_{n_k}\}_k$ that satisfies the conclusion of the theorem with all L_{n_k} 's equal to the identity matrix. Therefore, in what follows we will assume that $\ell_n \rightarrow \infty$ as $n \rightarrow \infty$.

Passing to a subsequence if necessary we can assume then that $\{f_n\}_n$ satisfies the following: $\|f_n\|_{L^2} = 1$, $\|\bar{T} f_n\|_{L^4} \rightarrow \bar{\mathbf{H}}_4$ and there exists a sequence $\{\ell_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ such that $\ell_n \rightarrow \infty$ as $n \rightarrow \infty$ and for any $\varepsilon > 0$ there exists $R_\varepsilon < \infty$ such that for all $R \geq R_\varepsilon$ and all $n \in \mathbb{N}$ equation (13.1) holds. Therefore, with R_ε as before, we have that for all $R \geq R_\varepsilon$ there exists $N_n \in [\ell_n - R, \ell_n + R] \cap \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$\int_{\{2^{N_n} \leq |y| \leq 2^{N_n+1}\}} |f_n(y)|^2 d\bar{\mu}(y) \geq \frac{1 - \varepsilon}{2R}.$$

Denote P_N the dyadic cut off at scale 2^N , that is, $P_N f(y) := f(y) \mathbb{1}_{\{2^N \leq |y| < 2^{N+1}\}}$. Using the continuity of \bar{T} and the triangle inequality we obtain

$$\begin{aligned} \|\bar{T}(P_{N_n} f_n)\|_{L^4} &\geq \|\bar{T} f_n\|_{L^4} - \bar{\mathbf{H}}_4 \|f_n - P_{N_n} f_n\|_{L^2(\bar{\mu})} \geq \|\bar{T} f_n\|_{L^4} - \bar{\mathbf{H}}_4 \left(1 - \frac{1 - \varepsilon}{2R}\right)^{1/2} \\ &= \bar{\mathbf{H}}_4 - \bar{\mathbf{H}}_4 \left(1 - \frac{1 - \varepsilon}{2R}\right)^{1/2} + o_n(1). \end{aligned}$$

Choosing $\varepsilon = \varepsilon_0$ close to 0 and $R = R_{\varepsilon_0} + 1$, we obtain a sequence $\{N_n\}_n \subset \mathbb{N}$, with $|N_n - \ell_n| \leq R_{\varepsilon_0} + 1$, so that $N_n \rightarrow \infty$ as $n \rightarrow \infty$, and a constant $c > 0$ such

that for all n large enough⁹

$$\|P_{N_n} f_n\|_{L^2(\bar{\mu})} > c, \quad \|\bar{T}(P_{N_n} f_n)\|_{L^4} > c.$$

We rescale f_n defining g_n by $g_n(y) = 2^{N_n} f(2^{N_n} y)$. Letting $s_n = 2^{-N_n}$ we have $s_n \rightarrow 0$ as $n \rightarrow \infty$, $g_n \in L^2(\bar{\mathcal{H}}_{s_n}^3)$,

$$\begin{aligned} \|g_n\|_{L^2(\bar{\mu}_{s_n})} &= \|f_n\|_{L^2(\bar{\mu})} = 1, \\ \|\bar{T}_{s_n} g_n\|_{L^4} &= \|\bar{T} f_n\|_{L^4} \rightarrow \bar{\mathbf{H}}_4 \text{ as } n \rightarrow \infty, \\ \|P_1 g_n\|_{L^2(\bar{\mu}_{s_n})} &= \|P_{N_n} f_n\|_{L^2(\bar{\mu})} > c \text{ and} \\ \|\bar{T}_{s_n}(P_1 g_n)\|_{L^4} &= \|\bar{T}(P_{N_n} f_n)\|_{L^4} > c, \end{aligned} \tag{13.2}$$

$$\tag{13.3}$$

as obtained by simple scaling (see Sect. 14). Moreover, from (13.1) for any small $\varepsilon > 0$, $R > 2R_\varepsilon$ and $n \in \mathbb{N}$

$$\int_{\{2^{-R} \leq |y| \leq 2^R\}} |g_n(y)|^2 d\bar{\mu}_{s_n}(y) \geq 1 - \varepsilon, \tag{13.4}$$

so that the g_n 's are ‘‘localized at scale 1’’. Using Lemma 8.1 applied to \bar{T}_{s_n} and $P_1 g_n$, which is possible given (13.2) and (13.3), we obtain that for all $n \in \mathbb{N}$ there exist caps $\mathcal{C}_n \subset \bar{\mathcal{H}}_{s_n}^3$, which we may consider all to be contained in the upper half, $\mathcal{H}_{s_n}^3$, possibly after passing to a subsequence,¹⁰ $\mathcal{C}_n = [1, 2] \times \mathcal{C}_n \subset \mathcal{H}_{s_n}^3$, for some spherical caps $\mathcal{C}_n \subseteq \mathbb{S}^2$, such that

$$\int_{\mathcal{C}_n} |g_n(y)| d\bar{\mu}_{s_n}(y) = \int_{\mathcal{C}_n} |P_1 g_n(y)| d\bar{\mu}_{s_n}(y) \geq c \bar{\mu}_{s_n}(\mathcal{C}_n)^{1/2} \|P_1 g_n\|_{L^2(\bar{\mu}_{s_n})} \gtrsim \bar{\mu}_{s_n}(\mathcal{C}_n)^{1/2},$$

as a consequence of (8.6). Equivalently

$$\int_{2^{N_n} \mathcal{C}_n} |f_n(y)| d\bar{\mu}(y) \gtrsim \bar{\mu}(2^{N_n} \mathcal{C}_n)^{1/2}. \tag{13.5}$$

Let $\alpha = \limsup_{n \rightarrow \infty} \bar{\mu}_{s_n}(\mathcal{C}_n)$. Two cases arise.

Case 1: $\alpha > 0$. Passing to a subsequence if necessary, we can assume that there exists a constant $c > 0$ such that for all n

$$\int_{\mathcal{C}_n} |g_n(y)| d\bar{\mu}_{s_n}(y) \geq c > 0.$$

⁹ By redefining the sequence $\{f_n\}_n$, if necessary, we will assume that the property holds for all $n \geq 1$.

¹⁰ Otherwise we reflect the f_n 's and g_n 's with respect to the origin, as necessary, by considering the sequences $\{L^* f_n\}_n$ and $\{L^* g_n\}_n$ where $L \in \mathcal{L}$ is the reflection map $L(x, t) = (-x, -t)$

We can view g_n as a function on the double cone via the usual identification using the orthogonal projection onto \mathbb{R}^3 , where we extend it to be zero in the region where $|y| \leq s_n$. Since $\|g_n\|_{L^2(\bar{\sigma}_c)} \leq \|g_n\|_{L^2(\bar{\mu}_{s_n})} = 1$ and

$$0 < c < \int_{C_n} |g_n(y)| \, d\bar{\mu}_{s_n}(y) \lesssim \int_{C_n} |g_n(y)| \, d\bar{\sigma}_c(y), \tag{13.6}$$

for all n large enough (as a consequence of (13.4)), there is weak convergence of $\{|g_n|\}_n$ in $L^2(\bar{\sigma}_c)$ after the possible extraction of a subsequence, $|g_n| \rightharpoonup g$, for some $g \in L^2(\bar{\sigma}_c)$ which satisfies $g \neq 0$ by (13.6). Inequality (13.4) implies that

$$\lim_{a \rightarrow 0^+} \sup_{n \in \mathbb{N}} \|g_n \mathbb{1}_{\{|y| \leq a\}}\|_{L^2(\bar{\mu}_{s_n})} = 0.$$

Because $\|\bar{T}_{s_n}(g_n)\|_{L^4} \leq \|\bar{T}_{s_n}(|g_n|)\|_{L^4}$, it is then also the case that $\|\bar{T}_{s_n}(|g_n|)\|_{L^4} \rightarrow \bar{\mathbf{H}}_4$, so that we can use part (v) of Lemma 12.5 applied to $\{|g_n|\}_n$ to conclude

$$\|\bar{T}\| \leq \|\bar{T}_c\|,$$

which is in contradiction with Proposition 4.3. Therefore, this case does not arise.

Case 2: $\alpha = 0$. Let $\{\gamma_n\}_n \subset [0, \pi]$ and $\{R_n\}_n \subset SO(3)$ be such that

$$R_n^{-1}(C_n) = \{(r\omega, \sqrt{r^2 - s_n^2}) : 1 \leq r \leq 2, \\ \omega = (\cos \varphi, \cos \theta \sin \varphi, \sin \theta \sin \varphi), \theta \in [0, 2\pi], \varphi \in [0, \gamma_n]\}.$$

The condition $\alpha = 0$ implies $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$. Let $\beta = \limsup_{n \rightarrow \infty} \bar{\mu}(2^{N_n} C_n) = \limsup_{n \rightarrow \infty} 2^{2N_n} \bar{\mu}_{s_n}(C_n)$. Two subcases arise.

Subcase 2a: $\beta < \infty$. This implies that the sequence $\{\bar{\mu}(2^{N_n} C_n)\}_n$ is bounded. We may assume that the angles γ_n are less than $\pi/2$ as $\{\gamma_n\}_n$ tends to zero. From Lemma 2.3 with $s = 1$, there exists $\{t_n\}_n \subset [0, 1)$ such that the caps $\{L^{-t_n} R_n^{-1}(2^{N_n} C_n) : n \in \mathbb{N}\}$ are contained in a fixed bounded ball of \mathbb{R}^4 . It therefore follows from (13.5) and the Cauchy–Schwarz inequality that $\{(R_n L^{t_n})^* f_n\}_n \subset L^2(\bar{\mathcal{H}}^3)$ is an extremizing sequence with L^2 norm uniformly bounded below by a constant $c > 0$ in a fixed ball. We can then conclude the precompactness modulo characters of the sequence $\{(R_n L^{t_n})^* f_n\}_n$ using the argument in the proof of Proposition 5.2.

Subcase 2b: $\beta = \infty$. From (2.9) in Lemma 2.3 with $s = 1$, after passing to a subsequence if necessary, $\lim_{n \rightarrow \infty} 2^{2N_n} \sin^2(\gamma_n) = \infty$. Set $t_n = \cos \gamma_n$, so that $t_n \rightarrow 1$ as $n \rightarrow \infty$. From Lemma 2.4 with $s = s_n$, the set $\tilde{C}_n := L_{t_n}^{-1} R_n^{-1}(C_n) \subset \bar{\mathcal{H}}_{s_n(1-t_n^2)^{-1/2}}^3$ satisfies, for all n large enough for which $2^{2N_n} \sin^2(\gamma_n) \geq 8$ and $\gamma_n \leq \pi/3$,

$$\bar{\mu}_{\frac{s_n}{\sqrt{1-t_n^2}}}(\tilde{C}_n) \geq \frac{\pi}{2} \quad \text{and} \quad \tilde{C}_n \subseteq [\frac{7}{16}, \frac{33}{16}] \times \mathbb{S}^2.$$

Set $a_n = s_n(1 - t_n^2)^{-1/2} = (2^{N_n} \sin \gamma_n)^{-1} \rightarrow 0$, as $n \rightarrow \infty$. Let $\tilde{f}_n = (R_n L^{t_n})^* f_n$ so that $\{\tilde{f}_n\}_n \subset L^2(\mathcal{H}^3)$ is also an L^2 -normalized extremizing sequence which satisfies, for some constant $c > 0$,

$$\int_{a_n^{-1}\tilde{\mathcal{C}}_n} |\tilde{f}_n(y)| d\tilde{\mu}(y) \geq c\tilde{\mu}(a_n^{-1}\tilde{\mathcal{C}}_n)^{1/2}, \quad \int_{a_n^{-1}\tilde{\mathcal{C}}_n} |\tilde{f}_n(y)|^2 d\tilde{\mu}(y) \geq c^2,$$

and $a_n^{-1}\tilde{\mathcal{C}}_n \subseteq [\frac{7}{16a_n}, \frac{33}{16a_n}] \times \mathbb{S}^2$.

Define the rescale $\tilde{g}_n(\cdot) := a_n^{-1}\tilde{f}_n(a_n^{-1}\cdot)$, so that for each n we have $\tilde{g}_n \in L^2(\tilde{\mathcal{H}}_{a_n}^3)$, $\|\tilde{g}_n\|_{L^2(\tilde{\mathcal{H}}_{a_n}^3)} = 1$ and there is a constant $c' > 0$ such that

$$\int_{\tilde{\mathcal{C}}_n} |\tilde{g}_n(y)| d\tilde{\mu}_{a_n}(y) \geq c\tilde{\mu}_{a_n}(\tilde{\mathcal{C}}_n)^{1/2} > c' > 0.$$

We are almost in the same situation as in Case 1, but we need the analog of (13.4) for the sequence $\{\tilde{g}_n\}_n$. After passing to a subsequence if necessary, $\{\tilde{f}_n\}_n$ satisfies the compactness alternative in Lemma 9.1. Denoting $\{\tilde{\ell}_n\}_n$ the corresponding sequence associated to $\{\tilde{f}_n\}_n$ as in (13.1) we then necessarily have that $\{\tilde{\ell}_n - \log_2(a_n^{-1})\}_n$ is bounded. This implies the desired analog of (13.4) for $\{\tilde{g}_n\}_n$. Therefore the argument in Case 1 applies showing that this subcase does not arise.

As a result, only Subcase 2a of Case 2 is possible, proving the theorem. □

14 Scaling

Here we record scaling properties of the family of operators $\{T_s\}_{s>0}$. Recall from Sect. 3 that for $s > 0$, $\mathcal{H}_s^3 = \{(y, \sqrt{|y|^2 - s^2}) : y \in \mathbb{R}^3\}$, equipped with the measure μ_s with density $d\mu_s(y, t) = \mathbb{1}_{\{|y|>s\}}\delta(t - \sqrt{|y|^2 - s^2})\frac{dy dt}{\sqrt{|y|^2 - s^2}}$.

The operator T_s , defined on $\mathcal{S}(\mathbb{R}^3)$, is given by

$$T_s f(x, t) = \widehat{f\mu_s}(-x, -t) = \int_{\{y \in \mathbb{R}^3, |y| \geq s\}} e^{ix \cdot y} e^{it\sqrt{|y|^2 - s^2}} f(y) \frac{dy}{\sqrt{|y|^2 - s^2}}.$$

We want to study the scaling of the quantity $\mathbf{H}_{p,s}$ defined by

$$\mathbf{H}_{p,s} := \sup_{0 \neq f \in L^2(\mathcal{H}_s^3)} \frac{\|T_s f\|_{L^p(\mathbb{R}^4)}}{\|f\|_{L^2(\mathcal{H}_s^3)}}.$$

Changing variables $y \rightsquigarrow sy$ in the expression defining $Tf(x, t) = T_1 f(x, t)$ we obtain

$$\begin{aligned} Tf(x, t) &= \int_{\{y \in \mathbb{R}^3, |y| \geq 1\}} e^{ix \cdot y} e^{it\sqrt{|y|^2-1}} f(y) \frac{dy}{\sqrt{|y|^2-1}} \\ &= s^{-1} \int_{\{y \in \mathbb{R}^3, |y| \geq s\}} e^{is^{-1}x \cdot y} e^{is^{-1}t\sqrt{|y|^2-s^2}} s^{-1} f(s^{-1}y) \frac{dy}{\sqrt{|y|^2-s^2}}, \end{aligned}$$

from where $sTf(sx, st) = T_s(s^{-1}f(s^{-1}\cdot))(x, t)$ and it follows that

$$s^{1-4/p} \|Tf\|_{L^p(\mathbb{R}^4)} = \|T_s s^{-1} f(s^{-1}\cdot)\|_{L^p(\mathbb{R}^4)}.$$

On the other hand

$$\int_{\{y \in \mathbb{R}^3, |y| \geq 1\}} |f(y)|^q \frac{dy}{\sqrt{|y|^2-1}} = \int_{\{y \in \mathbb{R}^3, |y| \geq s\}} |s^{-2/q} f(s^{-1}y)|^q \frac{dy}{\sqrt{|y|^2-s^2}},$$

that is $\|f\|_{L^q(\mu)} = \|s^{-2/q} f(s^{-1}\cdot)\|_{L^q(\mu_s)}$. Thus

$$s^{1-4/p} \|Tf\|_{L^p(\mathbb{R}^4)} \|f\|_{L^2(\mu)}^{-1} = \|T_s s^{-1} f(s^{-1}\cdot)\|_{L^p(\mathbb{R}^4)} \|s^{-1} f(s^{-1}\cdot)\|_{L^2(\mu_s)}^{-1},$$

and it follows that for all $s > 0$

$$\mathbf{H}_{p,s} = s^{1-4/p} \mathbf{H}_p.$$

In particular, if $p = 4$,

$$\mathbf{H}_{4,s} = \mathbf{H}_4,$$

for all $s > 0$.

Appendix A: Computation of a Limit

Let

$$\begin{aligned} I(a) &= 16\pi^3 \int_0^\infty e^{-a\tau} \left(\tau^2 \sqrt{\tau^2+4} - \frac{2}{3}(\tau^2+4)\sqrt{\tau^2+1} + \frac{8}{3} \right. \\ &\quad \left. + 2\tau \log(\tau + \sqrt{\tau^2+1}) \right) d\tau, \end{aligned}$$

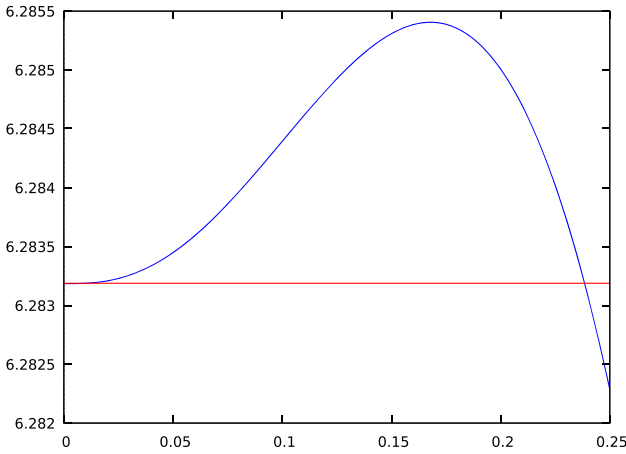


Fig. 1 Graph of the ratio $I(a)/II(a)$ and the constant 2π for $0 < a < 0.25$, illustrating the content of Lemma A.1

and

$$II(a) = 16\pi^2 \left(\int_0^\infty e^{-a\tau} \sqrt{\tau^2 + 1} \, d\tau \right)^2.$$

The ratio $I(a)/II(a)$ appeared in the proof of Proposition 4.1 while establishing that the best constant for the hyperboloid \mathcal{H}^3 is strictly greater than the best constant for the cone Γ^3 in their respective $L^2 \rightarrow L^4(\mathbb{R}^4)$ adjoint Fourier restriction inequalities. The purpose of this appendix is to prove the following lemma (Fig. 1).

Lemma A.1

$$\lim_{a \rightarrow 0^+} \frac{I(a)}{II(a)} = 2\pi, \quad \lim_{a \rightarrow 0^+} \frac{d}{da} \frac{I(a)}{II(a)} = 0, \quad \lim_{a \rightarrow 0^+} \frac{d^2}{da^2} \frac{I(a)}{II(a)} = 0,$$

and

$$\lim_{a \rightarrow 0^+} \frac{d^3}{da^3} \frac{I(a)}{II(a)} = 8\pi.$$

Therefore there exists $a_0 > 0$ such that

$$\frac{I(a)}{II(a)} > 2\pi,$$

for all $0 < a < a_0$.

Throughout this section we use the asymptotic notation $o_a(f(a))$ and $O_a(f(a))$ as $a \rightarrow 0^+$ in the usual way, namely $g(a) = o_a(f(a))$ if $g(a)/f(a) \rightarrow 0$ as $a \rightarrow$

0^+ , and $g(a) = O_a(f(a))$ if there exists a constant C , independent of a , such that $|g(a)| \leq C|f(a)|$ for all $a > 0$ small enough.

Changing variable $u = a\tau$ we obtain

$$I(a) = \frac{16\pi^3}{a^4} \left(\int_0^\infty e^{-u} \left(u^2 \sqrt{u^2 + 4a^2} - \frac{2}{3}(u^2 + 4a^2)\sqrt{u^2 + a^2} + 2a^2 u \log(u + \sqrt{u^2 + a^2}) \right) d\tau + \frac{8a^3}{3} - 2a^2 \log(a) \right),$$

and

$$II(a) = \frac{16\pi^2}{a^4} \left(\int_0^\infty e^{-u} \sqrt{u^2 + a^2} du \right)^2.$$

Using the Dominated Convergence Theorem it is direct to check that

$$\lim_{a \rightarrow 0^+} a^4 I(a) = 32\pi^3 \quad \text{and} \quad \lim_{a \rightarrow 0^+} a^4 II(a) = 16\pi^2,$$

so that

$$\lim_{a \rightarrow 0^+} \frac{I(a)}{II(a)} = 2\pi.$$

To address the limit of the derivatives of the ratio $I(a)/II(a)$ it will be convenient to introduce a rescaling. Let

$$N(a) := a^{4/3} I(a^{1/3}) = 16\pi^3 \left(\int_0^\infty e^{-u} \left(u^2 \sqrt{u^2 + 4a^{2/3}} - \frac{2}{3}(u^2 + 4a^{2/3})\sqrt{u^2 + a^{2/3}} + 2a^{2/3} u \log(u + \sqrt{u^2 + a^{2/3}}) \right) d\tau + \frac{8a}{3} - \frac{2}{3}a^{2/3} \log(a) \right),$$

and

$$D(a) := a^{4/3} II(a^{1/3}) = 16\pi^2 \left(\int_0^\infty e^{-u} \sqrt{u^2 + a^{2/3}} du \right)^2.$$

As we already know, and can readily check, $N(a) \rightarrow 32\pi^3$, $D(a) \rightarrow 16\pi^2$ and $N(a)/D(a) \rightarrow 2\pi$ as $a \rightarrow 0^+$. The remaining properties of the derivatives of $I(a)/II(a)$ in Lemma A.1 will follow if we show that $\frac{d}{da}(N(a)/D(a)) \rightarrow \frac{4\pi}{3}$ as $a \rightarrow 0^+$.

In what follows we write $(\cdot)'$ as a short for the derivative with respect to a . Given that both $N'(a)$ and $D'(a)$ diverge to $+\infty$ as $a \rightarrow 0^+$ it will be convenient to write the derivative of $N(a)/D(a)$ in the following way

$$\begin{aligned} \frac{d}{da} \frac{N(a)}{D(a)} &= \frac{16\pi^2 N'(a) - 32\pi^3 D'(a)}{D(a)^2} \\ &+ \frac{(D(a) - 16\pi^2) N'(a) - (N(a) - 32\pi^3) D'(a)}{D(a)^2}. \end{aligned} \tag{A.1}$$

We have the following lemma.

Lemma A.2 (i) $\lim_{a \rightarrow 0^+} \frac{d}{da} \frac{N(a)}{D(a)} = \frac{4\pi}{3}$.

(ii) As $a \rightarrow 0^+$,

$$N'(a) = O_a\left(\frac{\log a}{a^{1/3}}\right) \text{ and } D'(a) = O_a\left(\frac{\log a}{a^{1/3}}\right).$$

(iii) $\lim_{a \rightarrow 0^+} (N(a) - 32\pi^3) D'(a) = 0$ and $\lim_{a \rightarrow 0^+} (D(a) - 16\pi^2) N'(a) = 0$.

Proof In the course of the proof of this lemma we will make repeated use of the asymptotic behavior of some integrals as contained in Lemma A.3 below. We start with property (ii). For $a > 0$ the derivative of N is as follows,

$$\begin{aligned} N'(a) &= 16\pi^3 \left(\int_0^\infty e^{-u} \left(u^2 \frac{4}{3a^{1/3} \sqrt{u^2 + 4a^{2/3}}} - \frac{16}{9a^{1/3}} \sqrt{u^2 + a^{2/3}} \right. \right. \\ &\quad \left. \left. - \frac{2}{9} (u^2 + 4a^{2/3}) \frac{1}{a^{1/3} \sqrt{u^2 + a^{2/3}}} + \frac{4}{3a^{1/3}} u \log(u + \sqrt{u^2 + a^{2/3}}) \right. \right. \\ &\quad \left. \left. + \frac{2}{3} a^{1/3} u \frac{1}{(u + \sqrt{u^2 + a^{2/3}}) \sqrt{u^2 + a^{2/3}}} \right) du + \frac{8}{3} - \frac{4}{9a^{1/3}} \log(a) - \frac{2}{3a^{1/3}} \right) \\ &= 16\pi^3 \left(\frac{8}{3} - \frac{4}{3a^{1/3}} - \frac{4}{9a^{1/3}} \log(a) + \frac{4}{3a^{1/3}} \int_0^\infty e^{-u} u \log(u + \sqrt{u^2 + a^{2/3}}) du \right) \\ &\quad + o_a(1) \\ &= O_a\left(\frac{\log a}{a^{1/3}}\right), \end{aligned} \tag{A.2}$$

where we used (A.5), (A.8), (A.7), (A.10) and (A.11). The derivative of the function D is as follows

$$D'(a) = \frac{32\pi^2}{3} \int_0^\infty e^{-u} \sqrt{u^2 + a^{2/3}} du \cdot \int_0^\infty e^{-u} \frac{1}{a^{1/3} \sqrt{u^2 + a^{2/3}}} du,$$

so that (A.4) and (A.5) imply

$$D'(a) = O_a\left(\frac{1}{a^{1/3}}\right) O_a(\log a) = O_a\left(\frac{\log a}{a^{1/3}}\right),$$

and more explicitly using (A.13), as we will need later,

$$D'(a) = \frac{32\pi^2}{3a^{1/3}} \left(\int_0^\infty e^{-u} \log(u + \sqrt{u^2 + a^{2/3}}) du - \frac{1}{3} \log a \right) + o_a(1). \quad (\text{A.3})$$

We now turn to the proof of part (iii). Using that $\int_0^\infty e^{-u} u^3 du = 6$ we can write

$$\begin{aligned} N(a) - 32\pi^3 &= 16\pi^3 \left(\int_0^\infty e^{-u} \left(u^2(\sqrt{u^2 + 4a^{2/3}} - u) - \frac{2}{3}u^2(\sqrt{u^2 + a^{2/3}} - u) \right. \right. \\ &\quad \left. \left. - \frac{8}{3}a^{2/3}\sqrt{u^2 + a^{2/3}} + 2a^{2/3}u \log(u + \sqrt{u^2 + a^{2/3}}) \right) du \right. \\ &\quad \left. + \frac{8a}{3} - \frac{2}{3}a^{2/3} \log(a) \right) \\ &= 16\pi^3 a^{1/3} \left(\int_0^\infty e^{-u} \left(u^2 \frac{4a^{1/3}}{\sqrt{u^2 + 4a^{2/3}} + u} - \frac{2}{3}u^2 \frac{a^{1/3}}{\sqrt{u^2 + a^{2/3}} + u} \right. \right. \\ &\quad \left. \left. - \frac{8}{3}a^{1/3}\sqrt{u^2 + a^{2/3}} + 2a^{2/3}u \log(u + \sqrt{u^2 + a^{2/3}}) \right) du \right. \\ &\quad \left. + \frac{8a^{2/3}}{3} - \frac{2}{3}a^{1/3} \log(a) \right) \\ &= O_a(a^{2/3} \log a). \end{aligned}$$

Then

$$(N(a) - 32\pi^3) \cdot D'(a) = O_a(a^{2/3} \log a) O_a\left(\frac{\log a}{a^{1/3}}\right) = O_a(a^{1/3} \log^2 a) = o_a(1).$$

On the other hand

$$\begin{aligned} D(a) - 16\pi^2 &= 16\pi^2 \left(\int_0^\infty e^{-u} \sqrt{u^2 + a^{2/3}} du + 1 \right) \left(\int_0^\infty e^{-u} \sqrt{u^2 + a^{2/3}} du - 1 \right) \\ &= O_a(1) \left(\int_0^\infty e^{-u} (\sqrt{u^2 + a^{2/3}} - u) du \right) \end{aligned}$$

$$\begin{aligned}
 &= O_a(1) \left(\int_0^\infty e^{-u} \frac{a^{2/3}}{\sqrt{u^2 + a^{2/3}} + u} du \right) \\
 &= O_a(a^{2/3} \log a),
 \end{aligned}$$

where in the last line we used (A.9). Then

$$(D(a) - 16\pi^2) \cdot N'(a) = O_a(a^{2/3} \log a) O_a\left(\frac{\log a}{a^{1/3}}\right) = O_a(a^{1/3} \log^2 a) = o_a(1).$$

We now turn to the proof of (i). By (iii), the limit as $a \rightarrow 0^+$ of the second summand on the right hand side of (A.1) equals zero. We proceed to calculate the limit of the first summand. Combining (A.2) and (A.3) we obtain

$$\begin{aligned}
 &16\pi^2 N'(a) - 32\pi^3 D'(a) \\
 &= \frac{8(16)^2\pi^5}{3} - \frac{4(16)^2\pi^5}{3a^{1/3}} \\
 &\quad + \frac{(32)^2\pi^5}{3a^{1/3}} \int_0^\infty e^{-u} (u - 1) \log(u + \sqrt{u^2 + a^{2/3}}) du + o_a(1) \\
 &= \frac{2(32)^2\pi^5}{3} + \frac{(32)^2\pi^5}{3a^{1/3}} \int_0^\infty e^{-u} \left((u - 1) \log(u + \sqrt{u^2 + a^{2/3}}) - 1 \right) du \\
 &\quad + o_a(1).
 \end{aligned}$$

Using (A.12) to treat the integral in the previous expression we obtain

$$\lim_{a \rightarrow 0^+} (16\pi^2 N'(a) - 32\pi^3 D'(a)) = \frac{(32)^2\pi^5}{3},$$

therefore

$$\lim_{a \rightarrow 0^+} \frac{d}{da} \frac{N(a)}{D(a)} = \frac{(32)^2\pi^5}{3(16\pi^2)^2} = \frac{4\pi}{3}.$$

□

Finally, we state the asymptotic behavior of the many integrals used during the proof of the previous lemma.

Lemma A.3 *We have the following identities as $a \rightarrow 0^+$*

$$\int_0^\infty e^{-u} \frac{1}{\sqrt{u^2 + a^{2/3}}} du = O_a(\log a), \tag{A.4}$$

$$\int_0^\infty e^{-u} \frac{\sqrt{u^2 + a^{2/3}}}{a^{1/3}} du = \frac{1}{a^{1/3}} + O_a(a^{1/3} \log a), \tag{A.5}$$

$$\int_0^\infty e^{-u} \frac{u\sqrt{u^2 + a^{2/3}}}{a^{1/3}} du = \frac{2}{a^{1/3}} + O_a(a^{1/3}), \tag{A.6}$$

$$\int_0^\infty e^{-u} \frac{u^2}{a^{1/3}\sqrt{u^2 + 4a^{2/3}}} du = \frac{1}{a^{1/3}} + O_a(a^{1/3} \log a), \tag{A.7}$$

$$\int_0^\infty e^{-u} \frac{u^2 + 4a^{2/3}}{a^{1/3}\sqrt{u^2 + a^{2/3}}} du = \frac{1}{a^{1/3}} + O_a(a^{1/3} \log a), \tag{A.8}$$

$$\int_0^\infty e^{-u} \frac{a^{2/3}}{u + \sqrt{u^2 + a^{2/3}}} du = O_a(a^{2/3} \log a), \tag{A.9}$$

$$\int_0^\infty e^{-u} \frac{a^{1/3}u}{(u + \sqrt{u^2 + 4a^{2/3}})\sqrt{u^2 + 4a^{2/3}}} du = O_a(a^{1/3} \log a), \tag{A.10}$$

$$\int_0^\infty e^{-u} \frac{u}{a^{1/3}} \log(u + \sqrt{u^2 + a^{2/3}}) du = O_a\left(\frac{1}{a^{1/3}}\right), \tag{A.11}$$

$$\int_0^\infty \frac{e^{-u}}{a^{1/3}} ((u - 1) \log(u + \sqrt{u^2 + a^{2/3}}) - 1) du = -1 + o_a(1). \tag{A.12}$$

Proof The identities are elementary but we choose to give details for the sake of completeness.

Verification of (A.4) and (A.5) Integration by parts shows that

$$\begin{aligned} \int_0^\infty e^{-u} \frac{1}{\sqrt{u^2 + a^{2/3}}} du &= \int_0^\infty e^{-u} \log(u + \sqrt{u^2 + a^{2/3}}) du - \frac{1}{3} \log a \\ &= O_a(1) + O_a(\log a), \end{aligned} \tag{A.13}$$

and

$$\begin{aligned} \frac{1}{a^{1/3}} \int_0^\infty e^{-u} \sqrt{u^2 + a^{2/3}} du &= \frac{1}{2a^{1/3}} \int_0^\infty e^{-u} (a^{2/3} \log(u + \sqrt{u^2 + a^{2/3}}) \\ &\quad + u\sqrt{u^2 + a^{2/3}} - \frac{1}{3}a^{2/3} \log a) du \\ &= O_a(a^{1/3}) + O_a(a^{1/3} \log a) + \frac{1}{a^{1/3}} \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1}{2} \int_0^\infty e^{-u} \frac{u}{a^{1/3}} (\sqrt{u^2 + a^{2/3}} - u) \, du \\
 &= \frac{1}{a^{1/3}} + O_a(a^{1/3} \log a) + \frac{a^{1/3}}{2} \int_0^\infty e^{-u} \frac{u}{\sqrt{u^2 + a^{2/3}} + u} \, du \\
 &= \frac{1}{a^{1/3}} + O_a(a^{1/3} \log a) + O_a(a^{1/3}).
 \end{aligned}$$

Verification of (A.6) Using that $\int_0^\infty e^{-u} u^2 \, du = 2$ we have

$$\begin{aligned}
 \int_0^\infty e^{-u} \frac{u \sqrt{u^2 + a^{2/3}}}{a^{1/3}} \, du &= \frac{2}{a^{1/3}} + \frac{1}{a^{1/3}} \int_0^\infty e^{-u} u (\sqrt{u^2 + a^{2/3}} - u) \, du \\
 &= \frac{2}{a^{1/3}} + a^{1/3} \int_0^\infty e^{-u} \frac{u}{\sqrt{u^2 + a^{2/3}} + u} \, du \\
 &= \frac{2}{a^{1/3}} + O_a(a^{1/3}).
 \end{aligned}$$

Verification of (A.7)

$$\begin{aligned}
 \int_0^\infty e^{-u} \frac{u^2}{a^{1/3} \sqrt{u^2 + 4a^{2/3}}} \, du &= \int_0^\infty e^{-u} \frac{\sqrt{u^2 + 4a^{2/3}}}{a^{1/3}} \, du - 4a^{1/3} \int_0^\infty e^{-u} \frac{1}{\sqrt{u^2 + 4a^{2/3}}} \, du \\
 &= \frac{1}{a^{1/3}} + O_a(a^{1/3} \log a) + a^{1/3} O_a(\log a),
 \end{aligned}$$

where we used (A.4) and (A.5).

Verification of (A.8)

$$\begin{aligned}
 \int_0^\infty e^{-u} \frac{u^2 + 4a^{2/3}}{a^{1/3} \sqrt{u^2 + a^{2/3}}} \, du &= \frac{1}{a^{1/3}} \int_0^\infty e^{-u} \sqrt{u^2 + a^{2/3}} \, du + 3a^{1/3} \int_0^\infty e^{-u} \frac{1}{\sqrt{u^2 + a^{2/3}}} \, du \\
 &= \frac{1}{a^{1/3}} + O_a(a^{1/3} \log a) + O_a(a^{1/3}),
 \end{aligned}$$

where in the last line we used (A.4) and (A.5).

Verification of (A.9)

$$\begin{aligned}
 \int_0^\infty e^{-u} \frac{a^{2/3}}{u + \sqrt{u^2 + a^{2/3}}} \, du &= \int_0^\infty e^{-u} (\sqrt{u^2 + a^{2/3}} - u) \, du \\
 &= 1 + a^{1/3} O_a(a^{1/3} \log a) - 1 \\
 &= O_a(a^{2/3} \log a),
 \end{aligned}$$

where we used (A.5).

Verification of (A.10)

$$\begin{aligned} \int_0^\infty e^{-u} \frac{a^{1/3}u}{(u + \sqrt{u^2 + 4a^{2/3}})\sqrt{u^2 + 4a^{2/3}}} du &= \int_0^\infty e^{-u} \frac{a^{1/3}}{\sqrt{u^2 + 4a^{2/3}}} du \\ &\quad - \int_0^\infty e^{-u} \frac{a^{1/3}}{u + \sqrt{u^2 + 4a^{2/3}}} du \\ &= O_a(a^{1/3} \log a), \end{aligned}$$

where we used (A.4) and (A.9).

Verification of (A.11) The identity is immediate since $e^{-u}u \log(u) \in L^p([0, \infty))$ for all $p \in [1, \infty]$.

Verification of (A.12) For $a > 0$, integration by parts shows

$$\int_0^\infty e^{-u}(u - 1) \log(u + \sqrt{u^2 + a^{2/3}}) du = \int_0^\infty e^{-u} \frac{u}{\sqrt{u^2 + a^{2/3}}} du,$$

so that to prove the last identity we need to show

$$\lim_{a \rightarrow 0^+} \frac{1}{a} \int_0^\infty e^{-u} \left(1 - \frac{u}{\sqrt{u^2 + a^2}} \right) du = 1.$$

Changing variable $u \rightsquigarrow au$ gives

$$\begin{aligned} \frac{1}{a} \int_0^\infty e^{-u} \left(1 - \frac{u}{\sqrt{u^2 + a^2}} \right) du &= \int_0^\infty e^{-au} \left(1 - \frac{u}{\sqrt{u^2 + 1}} \right) du \\ &= \int_0^\infty e^{-au} \frac{1}{(u + \sqrt{u^2 + 1})\sqrt{u^2 + 1}} du, \end{aligned}$$

hence

$$\lim_{a \rightarrow 0^+} \frac{1}{a} \int_0^\infty e^{-u} \left(1 - \frac{u}{\sqrt{u^2 + a^2}} \right) du = \int_0^\infty \frac{1}{(u + \sqrt{u^2 + 1})\sqrt{u^2 + 1}} du.$$

Changing variable $u = \sinh t$ we obtain

$$\int_0^\infty \frac{1}{(u + \sqrt{u^2 + 1})\sqrt{u^2 + 1}} du = \int_0^\infty \frac{1}{\sinh t + \cosh t} dt = \int_0^\infty e^{-t} dt = 1.$$

□

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