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Injectivity of Spherical Means on *H*-Type Groups

E. K. Narayanan[1] · P. K. Sanjay[2] · K. T. Yasser[2]

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Abstract

We establish injectivity results for three different spherical means on an H-type group, G. The first one is the standard spherical mean operator, which is defined as the average of a function over the spheres in the complement of the center, the second one is the average over the product of spheres in the center and its complement, and the third one is the average over the spheres defined by a homogeneous norm on G. If m is the dimension of the center of G, injectivity of these spherical means is proved for the range $1 \leq p \leq \frac{2m}{m-1}$. Examples are provided to show the sharpness of our results in the first two cases.

Keywords Spherical means · Injectivity · H-type groups · Spectral projections · Singular Integrals · Spherical functions

Mathematics Subject Classification Primary 43A80 · Secondary 22E25 · 43A90 · 44A35 · 42C10

1 Introduction

One of the problems in Integral Geometry is to find out whether a function can be determined from its averages on spheres of a fixed radius $r > 0$. This leads to the question of injectivity of the so called spherical mean operator. Let μ_r^n be the normalized surface measure on the sphere $\{x \in \mathbb{R}^n : |x| = r\}$ in $\mathbb{R}^n$. Here (as elsewhere

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✉ E. K. Narayanan
naru@iisc.ac.in

P. K. Sanjay
sanjaypk@nitc.ac.in

K. T. Yasser
yasser_p160082ma@nitc.ac.in

1 Department of Mathematics, Indian Institute of Science, Bangalore 560012, India

2 Department of Mathematics, National Institute of Technology, Calicut, India

in this paper), normalized means the total mass is one. We use the superscript n to denote the dimension of the ambient space. The spherical means of a function f are then defined to be the convolution $f * \mu_r^n$:

$$f * \mu_r^n(x) = \int_{|y|=r} f(x-y)d\mu_r^n(y).$$

The above is nothing but the average of the function f over the sphere of radius r centered at the point x. The injectivity question is the following:

Suppose that, for a fixed $r > 0$, $f * \mu_r^n(x) = 0$ for all $x \in \mathbb{R}^n$. Does it follow that f is identically zero?

In general, the answer to this question is no. For $\lambda > 0$, let

$$\varphi_\lambda(x) = c\,\frac{J_{\frac{n}{2}-1}(\lambda|x|)}{(\lambda|x|)^{\frac{n}{2}-1}},\ x \in \mathbb{R}^n, \tag{1.1}$$

where J_α denotes the Bessel function of order α and c is a constant that makes $\varphi_\lambda(0) = 1$. Then it is well known that

$$\varphi_\lambda * \mu_r^n(x) = \varphi_\lambda(r)\varphi_\lambda(x),\ \forall r > 0,\ x \in \mathbb{R}^n.$$

Hence, if $r > 0$ is a zero of the function $s \to J_{\frac{n}{2}-1}(\lambda s)$ (which exists) then $\varphi_\lambda * \mu_r^n$ is identically zero. On the other hand, Zalcman [21] proved that, if we consider averages over spheres of two different radii $r, s > 0$, then a two radius theorem is true, provided r/s is not a quotient of the zeroes of the Bessel function $J_{\frac{n}{2}-1}(t)$. That is if both the convolutions $f * \mu_r^n$ and $f * \mu_s^n$ vanish identically, then f too vanishes identically provided r/s is not a quotient of the zeroes of the Bessel function $J_{\frac{n}{2}-1}(t)$ (see [21] for the proof).

It is known that the function φ_λ (see 1.1) is in $L^p(\mathbb{R}^n)$ if and only if $p > \frac{2n}{n-1}$. It follows that injectivity question raised above fails for $L^p(\mathbb{R}^n)$, $\frac{2n}{n-1} < p \leq \infty$. In [20], a one radius theorem is proved for $L^p(\mathbb{R}^n)$, which establishes the injectivity for the range $1 \leq p \leq \frac{2n}{n-1}$. In other words, if $f \in L^p(\mathbb{R}^n)$ and $f * \mu_r^n$ is identically zero for a fixed radius $r > 0$, then f vanishes identically, provided $1 \leq p \leq \frac{2n}{n-1}$.

1.1 Spherical Means on the Heisenberg Group

Consider the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z,t)(w,s) = \left(z + w, t + s + \frac{1}{2}\Im(z \cdot \overline{w})\right),$$

which makes $\mathbb{H}^n$ into a step two nilpotent Lie group. Consider μ_r^{2n}, the normalized surface measure on the sphere $\{z \in \mathbb{C}^n : |z| = r\}$ as a measure on $\mathbb{H}^n$. The spherical means of a function f on $\mathbb{H}^n$ is then defined to be $f * \mu_r^{2n}(z,t)$:

$$f * \mu_r^{2n}(z,t) = \int_{|w|=r} f\left(z - w, t - \frac{1}{2}\Im(z \cdot \overline{w})\right) d\mu_r^{2n}(w).$$

In [20], Thangavelu investigated the injectivity question for the above spherical means on $\mathbb{H}^n$ and established the following theorem:

Theorem 1.1 *If $f \in L^p(\mathbb{H}^n)$, $1 \leq p < \infty$ and for a fixed $r > 0$, $f * \mu_r^{2n}(z,t) = 0$ for all $(z,t) \in \mathbb{H}^n$, then f vanishes identically.*

To prove the above result, Thangavelu used the spectral decomposition of the sublaplacian on $\mathbb{H}^n$, and summability results proved by Strichartz in [18]. Below, we briefly describe the method used to prove the above theorem.

Let $\mathcal{L}$ be the sublaplacian on the Heisenberg group. Let $L_k^{n-1}(t)$ be the Laguerre polynomial of type $(n-1)$. For $\lambda \neq 0$, let

$$\varphi_k^\lambda(z) = L_k^{n-1}\left(\frac{1}{2}|\lambda||z|^2\right) e^{-\frac{1}{4}|\lambda||z|^2}, z \in \mathbb{C}^n, \tag{1.2}$$

and define

$$e_k^\lambda(z,t) = e^{-i\lambda t}\varphi_k^\lambda(z).$$

These functions are joint eigenfunctions of $\mathcal{L}$ and $T = i\frac{\partial}{\partial t}$:

$$\mathcal{L}e_k^\lambda = (2k+n)|\lambda|e_k^\lambda, \quad Te_k^\lambda = \lambda e_k^\lambda.$$

Given a function f on $\mathbb{H}^n$, we can decompose f into the joint eigenfunctions of $\mathcal{L}$ and T as

$$f(z,t) = (2\pi)^{-n-1} \sum_{k=0}^{\infty} \int_{\mathbb{R}} f * e_k^\lambda(z,t)\, |\lambda|^n\, d\lambda. \tag{1.3}$$

The above was studied in detail by Strichartz [18]. Among the many results established by Strichartz, we mention the following Abel summability result, which played a crucial role in the injectivity proof by Thangavelu [20].

Theorem 1.2 *For any $f \in L^p(\mathbb{H}^n)$, $1 < p < \infty$, the modified Abel means*

$$(2\pi)^{-n-1} \sum_{k=0}^{N^2} \left(1 - \frac{1}{N}\right)^k \int_{-N}^{N} f * e_k^\lambda(z,t)\, |\lambda|^n\, d\lambda$$

converges to f in the L^p norm as $N \to \infty$.

Now, we highlight the key ingredients in the proof in [20] as we will be closely following these in our proofs.

(A) The functions $e_k^\lambda(z,t)$ are eigenfunctions for the spherical mean operator. Indeed,

$$e_k^\lambda * \mu_r^{2n}(z,t) = \frac{k!(n-1)!}{(k+n-1)!}\,\varphi_k^\lambda(r) e_k^\lambda(z,t).$$

(B) L^p-boundedness of the spectral projection operator: For each k, define the spectral projection P_k by

$$P_k f = \int_{\mathbb{R}} f * e_k^\lambda(z,t)\, |\lambda|^n \, d\lambda.$$

Then $f \to P_k f$ is a bounded operator on $L^p(\mathbb{H}^n)$, for $1 < p < \infty$.
(C) Applying Theorem 1.2 to $f * \mu_r^{2n}$ and using (A) and (B) above, it can be shown that the Fourier transform of $P_k f$ in the t-variable is supported on a discrete subset of $\mathbb{R}$ which implies that $P_k f = 0$ for every k, as $p < \infty$.

For $p = \infty$, one has a two radius theorem for $\mathbb{H}^n$ which is proved using a Wiener-Tauberian theorem for the radial functions on the Heisenberg group (see [4]). We refer the reader to [2, 3] for related results. See also [19] for a generalisation in the context of Gelfand pairs associated to $\mathbb{H}^n$.

Extending and generalising the result in [20], we establish injectivity results for three different spherical means on an H-type group. In the remaining of this section we define these spherical means and state the injectivity results obtained.

1.2 Spherical Means on *H*-Type Groups

Let G be an H-type group, identified with its Lie algebra $\mathfrak{g}$ via the exponential map. Then $\mathfrak{g}$ admits an orthogonal decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center and $\mathfrak{v}$ its orthogonal complement. It is known that $\dim \mathfrak{v}$ has to be even, say $\dim \mathfrak{v} = 2n$, and let m denote $\dim \mathfrak{z}$. We will identify $\mathfrak{v}$ with $\mathbb{C}^n$ and $\mathfrak{z}$ with $\mathbb{R}^m$. This requires fixing an orthonormal basis on $\mathfrak{v}$ and $\mathfrak{z}$. For most of our purposes, this can be an arbitrary chosen orthonormal basis, however for certain computations we will choose a basis with some properties (see (2.1), (2.2), (2.3)). We will write (z,t) for points in G, where $z \in \mathbb{C}^n$ (identified with $\mathfrak{v}$) and $t \in \mathbb{R}^m$ (identified with $\mathfrak{z}$). The group law then is given by

$$(z,t)(w,s) = \left(z+w, t+s+\frac{1}{2}[z,w]\right),$$

where $[\ ,\ ]$ denotes the Lie bracket. The Haar measure on G is given by the Lebesgue measure on $\mathfrak{g}$ and will be denoted by $dzdt$. Denote by $Q = 2n+2m$ the homogeneous dimension of G.

Next, we define three different spherical means and state the injectivity results. Since $m = 1$ corresponds to the Heisenberg group, we will always assume that $m \geq 2$ unless explicitly stated. As earlier, let μ_r^{2n} denote the normalized surface measure on the sphere $\{z \in \mathfrak{v} : |z| = r\}$ and consider the spherical means of a function f,

$$f * \mu_r^{2n}(z,t) = \int_{|w|=r} f\left(z - w, t - \frac{1}{2}[z,w]\right) d\mu_r^{2n}(w).$$

Theorem 1.3 *Let $f \in L^p(G)$, $1 \leq p \leq \frac{2m}{m-1}$. If for a fixed $r > 0$,*

$$f * \mu_r^{2n}(z,t) = 0 \text{ for all } (z,t) \in G,$$

then f vanishes identically. Moreover, for any $p > \frac{2m}{m-1}$, the injectivity fails.

Next, let μ_s^m, $s > 0$ be the normalized surface measure on the sphere $\{y \in \mathfrak{z} : |y| = s\}$. Consider the measure $\mu_{r,s} = \mu_r^{2n} \times \mu_s^m$. That is,

$$\int_G f(z,t)\, d\mu_{r,s}(z,t) = \int_G f(z,t)\, d\mu_r^{2n}(z) d\mu_s^m(t).$$

Then, we define the *bi-spherical means* of f by

$$f * \mu_{r,s}(z,t) = \int_{|w|=r} \int_{|u|=s} f\left(z - w, t - u - \frac{1}{2}[z,w]\right) d\mu_r^{2n}(w) d\mu_s^m(u).$$

We have the following theorem.

Theorem 1.4 *Let $f \in L^p(G)$, $1 \leq p \leq \frac{2m}{m-1}$. If for a fixed $r > 0$,*

$$f * \mu_{r,s}(z,t) = 0 \text{ for all } (z,t) \in G,$$

then $f \equiv 0$. Moreover, for any $p > \frac{2m}{m-1}$, the injectivity fails.

Finally, we define the *homogeneous spherical means*. Let $|(z,t)|$ denote a homogeneous norm on G (see the next section for definition). There exists a unique Radon measure σ on the unit sphere $\Sigma = \{(z,t) : |(z,t)| = 1\}$ such that for all $f \in L^1(G)$

$$\int_G f(z,t)\, dz\, dt = \int_0^\infty \int_\Sigma f(\delta_r(z,t))\, d\sigma(z,t)\, r^{Q-1} dr$$

where δ_r denote the dilations that act as automorphisms of G (see the next section for the definition). Dilating the measure σ using δ_r, for $r > 0$ we can define σ_r by

$$\sigma_r(f) = \sigma(\delta_r f) = \int_\Sigma f(\delta_r(z,t))\, d\sigma(z,t).$$

The homogeneous spherical mean of a function f is defined as the convolution $f * \sigma_r$, of f with σ_r. For the homogeneous spherical means we have the following theorem.

Theorem 1.5 (1) *Let* $m \geq 2$ *and let* $r > 0$. *If* $f \in L^p(G)$, $1 \leq p \leq \frac{2m}{m-1}$ *and* $f * \sigma_r(z, t) = 0$ *for all* $(z, t) \in G$ *then* f *vanishes identically.*
(2) *Let* $G = \mathbb{H}^n$, *that is* $m = 1$, *then the above injectivity holds for the range* $1 \leq p < \infty$.

The plan of the paper is as follows: In the next section we recall all the required definitions and also state some known results that will be used later. In the third section we study the spectral decomposition of the sublaplacian of G and prove the Abel summability. Using this, we prove the injectivity results in the final section.

2 Preliminaries

In this section, we recall some definitions and properties of H-type groups introduced by Kaplan [14]. Let $\mathfrak{g}$ be a finite dimensional real inner product space endowed with a Lie bracket that makes it into a two step nilpotent Lie algebra. Let $\mathfrak{z}$ be its centre and $\mathfrak{v}$ be the orthogonal complement of $\mathfrak{z}$. For each $v \in \mathfrak{v}$, consider the map $ad_v : \mathfrak{v} \to \mathfrak{z}$ defined by

$$ad_v(v') = [v, v'].$$

Let $\mathfrak{k}_v$ be the kernel of this map and $\mathfrak{b}_v$ its orthogonal complement so that

$$\mathfrak{v} = \mathfrak{k}_v \oplus \mathfrak{b}_v.$$

We shall say that $\mathfrak{g}$ is Heisenberg type or H-type if the map ad_v is a surjective isometry for every unit vector $v \in \mathfrak{v}$. A connected and simply connected Lie group G is of Heisenberg type if its Lie algebra is H-type. For each non-zero $z \in \mathfrak{z}$ we can define the linear operator $J_z : \mathfrak{v} \to \mathfrak{v}$ by

$$\langle J_z(v), v' \rangle = \langle z, [v, v'] \rangle \qquad \text{for all } v, v' \in \mathfrak{v}.$$

Then J_z is a skew-symmetric linear isomorphism. Then $\mathfrak{g}$ is H-type if and only if

$$J_z^2 = -|z|^2 I.$$

This means that J_z defines a complex structure on $\mathfrak{v}$ when $|z| = 1$ and therefore the dimension of $\mathfrak{v}$ is even. Hence, we identify $\mathfrak{v}$ with $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ and $\mathfrak{z}$ with $\mathbb{R}^m$ for $n, m \in \mathbb{N}$. As mentioned in the introduction this requires fixing an orthonormal basis in $\mathfrak{v}$ and $\mathfrak{z}$.

The exponential map from $\mathfrak{g}$ to G is a diffeomorphism. We can therefore parametrise the elements of $G = \exp \mathfrak{g}$ by (z, t), for z in $\mathfrak{v} \equiv \mathbb{C}^n$ and t in $\mathfrak{z} \equiv \mathbb{R}^m$. By the Baker–Campbell–Hausdorff formula, it follows that the group law in G is

$$(z, t)(z', t') = \left(z + z', t + t' + \frac{1}{2}[z, z']\right) \quad \forall (z, t), (z', t') \in G.$$

Since $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$, the Lie bracket on $\mathfrak{v}$ can be written as (see [6])

$$[z, z']_j = \langle z, U^j z' \rangle$$

in terms of $2n \times 2n$ skew-symmetric matrices U^j, $j = 1, 2, \ldots, m$. Since $J_z^2 = -|z|^2 I$, U^j are orthogonal and satisfy

$$U^i U^j + U^j U^i = 0, \ i \neq j.$$

The left invariant vector fields on G which agree respectively with $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{m} \left(\sum_{l=1}^{2n} z_l U_{l,j}^k \right) \frac{\partial}{\partial t_k},$$
$$Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} \sum_{k=1}^{m} \left(\sum_{l=1}^{2n} z_l U_{l,j+n}^k \right) \frac{\partial}{\partial t_k},$$

where $z_l = x_l, z_{l+n} = y_l, l = 1, 2, \ldots, n$. The vector fields $T_k = \frac{\partial}{\partial t_k}, k = 1, 2, \ldots, m$ correspond to the centre of $\mathfrak{g}$. Then the sublaplacian $\mathcal{L}_G = -\sum_j (X_j^2 + Y_j^2)$ is given by

$$\mathcal{L}_G = -\sum_{j=1}^{n} (X_j^2 + Y_j^2) = -\Delta_z + \frac{1}{4} |z|^2 T - \sum_{k=1}^{m} \langle z, U^k \nabla_z \rangle T_k,$$

where

$$\Delta_z = \sum_{j=1}^{2n} \frac{\partial^2}{\partial z_j \partial \overline{z_j}}, \quad T = -\sum_{k=1}^{m} \frac{\partial^2}{\partial t_k^2}, \quad \nabla_z = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_{2n}} \right)^T.$$

For $a \in \mathbb{R}^m$ (identified with $\mathfrak{z}^*$) let $f^a(z)$ stand for the inverse Fourier transform of the function $f(z, t)$ in the central variable. That is

$$f^a(z) = \int_{\mathbb{R}^m} f(z, t) \, e^{i\langle a, t \rangle} \, dt.$$

For $a \neq 0$, let J_a be the linear mapping on $\mathfrak{z}^\perp$ defined earlier by

$$\langle J_a u, w \rangle = a([u, w]), \quad \text{for any } u, w \in \mathfrak{z}^\perp.$$

Choose an orthonormal basis

$$\left\{ E_1(a), E_2(a), \ldots, E_n(a), \bar{E}_1(a), \bar{E}_2(a), \ldots, \bar{E}_n(a) \right\} \tag{2.1}$$

of $\mathfrak{z}^\perp$ such that

$$J_a E_i(a) = -|a|\bar{E}_i(a),\ J_a\bar{E}_i(a) = |a|E_i(a)$$

and an orthonormal basis

$$\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\} \tag{2.2}$$

for $\mathfrak{z}$, such that $\langle a, \epsilon_1\rangle = |a|$ and $\langle a, \epsilon_j\rangle = 0$ for $j = 2, 3, \ldots, m$. If $\mathfrak{g}$ is identified with $\mathbb{C}^n \times \mathbb{R}^m$ via this orthonormal basis, the first coordinate of the Lie bracket takes the form (see [17])

$$[z, z']_1 = \langle z, U^1 z'\rangle = \sum_{i=0}^{n}(x_i' y_i - y_i' x_i) = \Im(z \cdot \bar{z'}).$$

Hence the convolution with functions of the form $g(z, t) = e^{-i\langle a,t\rangle}\varphi(z)$ can be written as

$$\begin{aligned} f * g(z, t) &= \int_{\mathbb{C}^n}\int_{\mathbb{R}^m} f\left(z - w, t - s - \frac{1}{2}[z, w]\right)\varphi(w)e^{-i\langle a,s\rangle}\, dwds \\ &= \int_{\mathbb{C}^n} f^a(z - w)\varphi(w)e^{-i\langle a,t\rangle}e^{\frac{i}{2}\langle a,[z,w]\rangle}\, dw \\ &= e^{-i\langle a,t\rangle} f^a \times_{|a|} \varphi(z), \end{aligned} \tag{2.3}$$

where the twisted convolution $\times_{|a|}$ of two suitable functions f_1 and f_2 on $\mathbb{C}^n$ is defined by

$$f_1 \times_{|a|} f_2(z) = \int_{\mathbb{C}^n} f_1(z - w)\ f_2(w)\ e^{\frac{i}{2}|a|\Im z\cdot\overline{w}}\, dw.$$

Also, one obtains the following result regarding the action of the sublaplacian $\mathcal{L}_G$ on functions of the form $e^{-i\langle a,t\rangle}\varphi(z)$.

Lemma 2.1 *Let $0 \neq a \in \mathfrak{z}^*$. If $f(z, t) = e^{-i\langle a,t\rangle}\varphi(z)$, then*

$$\mathcal{L}_G f(z, t) = e^{-i\langle a,t\rangle} L_{|a|}\varphi(z)$$

where, for $\lambda > 0$

$$L_\lambda = -\Delta_z + \frac{\lambda^2|z|^2}{4} - i\lambda\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right)$$

is the twisted Laplacian on $\mathbb{C}^n$.

For a proof, see Lemma 1 in [17]. Define, for $0 \neq a \in \mathfrak{z}$,

$$e_k^a(z, t) = e^{-i\langle a,t\rangle}\varphi_k^{|a|}(z) \tag{2.4}$$

where $\varphi_k^{|a|}$ is defined in (1.2). Then, from the above lemma it follows that

$$\mathcal{L}_G e_k^a = (2k+n)|a|e_k^a. \tag{2.5}$$

An H-type group admits a family of dilations which act as automorphisms of G by

$$\delta_r(z,t) = (rz, r^2 t), r > 0.$$

It is easy to see that G with this family of dilations is a homogeneous Lie group whose homogeneous dimension is $2n + 2m$ which we denote by Q (see [13]). The Korányi norm on G is defined as

$$|(z,t)| = \left(|z|^4 + |t|^2\right)^{1/4}.$$

It is clear that $|\delta_r(z,t)| = r|(z,t)|$.

A smooth kernel K on $G \setminus \{0\}$ is said to be homogeneous of degree $-Q$ if

$$K(\delta_r(z,t)) = r^{-Q} K(z,t), \forall (z,t) \in G \setminus \{0\}.$$

Smooth (away from identity) homogeneous kernels K which satisfy a cancellation condition (see below) define singular integral operators on G via principal value integrals. We will denote such an operator by $f \mapsto \text{P. V. } f * K$. The cancellation condition is given by

$$\int_{a<|(z,t)|<b} K(z,t)\, dzdt = 0, \forall\, 0 < a < b < \infty. \tag{2.6}$$

Notice that, since $\{(z,t) : a < |(z,t)| < b\}$ is relatively compact, the above integral is well defined. For more details on such operators, we refer to [13]. Now we collect some of the results about singular integral operators on G which will be used later.

Theorem 2.1 *Let G be an H-type group and let $K \in C^\infty(G \setminus \{0\})$ be a kernel which is homogeneous of degree $-Q$. Assume that K satisfies the cancellation condition*

$$\int_{a<|(z,t)|<b} K(z,t)\, dzdt = 0, \forall\, 0 < a < b < \infty.$$

Then the singular integral operator, defined by

$$f \mapsto \text{P. V. } f * K$$

is bounded on $L^2(G)$.

Proof This is a special case of Theorem 1 in [15, p. 494]. □

The next theorem says that for the above operators, the L^2-boundedness imply the L^p-boundedness.

Theorem 2.2 *Let G be an H-type group and $K \in C^\infty(G \setminus \{0\})$ be a kernel that satisfy the cancellation condition and is homogeneous of degree $-Q$. If the operator*

$$f \mapsto \mathrm{P.V.}\, f * K$$

is bounded on $L^2(G)$, then it is bounded on $L^p(G)$ for $1 < p < \infty$.

Proof Follows from Theorem 5.1 of [16]. □

We end this section by restating the cancellation condition.

Lemma 2.2 *Let $K \in C^\infty(G \setminus \{0\})$ be homogeneous of degree $-Q$. Then the cancellation condition in (2.6) is equivalent to the condition*

$$\int_{\mathbb{C}^n} \int_{S^{m-1}} K(z,u)\, dz d\mu_1^m(u) = 0$$

where μ_1^m is the normalised surface measure on the unit sphere in $\mathfrak{z}$. In particular, if K is radial in the t-variable, the cancellation condition is equivalent to

$$\int_{\mathbb{C}^n} K(z,1)\, dz = 0.$$

Proof Since K is homogeneous of degree $-Q$, one has

$$\begin{aligned}
\int_{a<|(z,t)|<b} K(z,t)\, dzdt &= \int_{\mathbb{C}^n} \int_{S^{m-1}} \int_{a^4<|z|^4+s^2<b^4} K(z,su) s^{m-1} ds d\mu_1^m(u) dz \\
&= \int_{\mathbb{C}^n} \int_{S^{m-1}} \int_{a^4<|z|^4+s^2<b^4} K\left(\frac{z}{\sqrt{s}}, u\right) s^{-n-1} ds d\mu_1^m(u) dz \\
&= \int_{\mathbb{C}^n} \int_{S^{m-1}} \int_{a^4<s^2(1+|w|^4)<b^4} \frac{ds}{s} K(w,u) d\mu_1^m(u) dz.
\end{aligned}$$

Now the result follows from the fact that

$$\int_{a^4<s^2(1+|w|^4)<b^4} \frac{ds}{s} = \int_{\frac{a^2}{\sqrt{1+|w|^4}}}^{\frac{b^2}{\sqrt{1+|w|^4}}} \frac{ds}{s} = \log\left(\frac{b^2}{a^2}\right)$$

is independent of w. □

We need the following result which is a special case of a result due to Christ (see [7, p. 575]).

Theorem 2.3 *Let G be an H-type group, with dilations $\{\delta_t : t > 0\}$. Let $\gamma : \mathbb{R} \to G$ be an odd homogeneous curve, that is $\gamma(t) = \exp(\delta_t(Y_+))$ for $t > 0$ and $\gamma(t) =$*

$\exp(\delta_{-t}(Y_-))$ *where* $Y_+ = -Y_- \in \mathfrak{g}$, *so that* $\gamma(t) = -\gamma(-t) = \gamma(t^{-1})$. *Then, the operator*

$$H_\gamma f(x) = \mathrm{P.\,V.} \int_{\mathbb{R}} f(x \cdot \gamma(t)^{-1}) \, \frac{dt}{t},$$

is bounded on $L^p(G)$ *for* $1 < p < \infty$ *with norm independent of the curve* γ.

We shall also need the following result connecting the L^p membership of a function on $\mathbb{R}^m$ with the dimension of the support of the Fourier transform of the function.

Theorem 2.4 *Let* $f \in L^p(\mathbb{R}^m)$ *and support of* $\widehat{f}$ *(distributional Fourier transform of* f*) is contained in a* C^1*-manifold of dimension* $0 < d < m$. *Then* f *vanishes identically provided* $1 \leq p \leq \frac{2m}{d}$. *If* $d = 0$, f *vanishes identically provided* $1 \leq p < \infty$.

Proof When the support is a sphere, this follows from [20] (see Lemma 2.2 and Theorem 2.2 there). For the general case see [1] (Theorem 1). □

3 Spectral Projections and Abel Summability

In this section, we prove a summability result for the spectral decomposition of the sublaplacian on L^p for $2 \leq p < \infty$. We follow the methods in [18]. For $a \in \mathbb{R}^m$ and $(z, t) \in G \equiv \mathbb{C}^n \times \mathbb{R}^m$, recall that (see 2.4)

$$e_k^a(z, t) = e^{-i\langle a, t\rangle} \varphi_k^{|a|}(z),$$

where the scaled Laguerre functions φ_k^λ for $\lambda > 0$, defined by

$$\varphi_k^\lambda(z) = L_k^{n-1}\left(\frac{\lambda|z|^2}{2}\right) e^{-\frac{1}{4}\lambda|z|^2}, k = 0, 1, 2, \dots$$

in terms of the Laguerre polynomials L_k^{n-1}, are the eigenfunctions of the twisted Laplacian L_λ with eigenvalue $(2k + n)|\lambda|$. Hence,

$$\mathcal{L}_G e_k^a(z, t) = e^{-i\langle a, t\rangle} L_{|a|} \, \varphi_k^{|a|}(z) = (2k + n)|a| e_k^a(z, t).$$

Next, we explain the L^2 spectral decomposition. Applying the Fourier inversion formula in the central variable and using the special Hermite expansion of a function on $\mathbb{C}^n$, we obtain

$$
\begin{aligned}
f(z,t) &= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} f^a(z)e^{-i\langle a,t\rangle}\,da \\
&= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} \frac{|a|^n}{(2\pi)^n}\sum_{k=0}^{\infty}(f^a \times_{|a|} \varphi_k^{|a|}(z))e^{-i\langle a,t\rangle}\,da. \\
&= \frac{1}{(2\pi)^{n+m}}\int_{\mathbb{R}^m}\sum_{k=0}^{\infty} f * e_k^a(z,t))\,|a|^n\,da \\
&= \frac{1}{(2\pi)^{n+m}}\sum_{k=0}^{\infty}\int_{\mathbb{R}^m} f * e_k^a(z,t)\,|a|^n\,da.
\end{aligned}
$$

Since $\mathcal{L}_G$ is left invariant, $\mathcal{L}_G(f*g) = f*\mathcal{L}_G g$. Hence $f*e_k^a$ are eigenfunctions of the sub-Laplacian $\mathcal{L}_G$ with eigenvalues $(2k+n)|a|$. Therefore the above expansion is in fact the L^2 spectral decomposition of f. We also have, by the Plancherel formula (see [17, p. 2717]),

$$
\|f\|_{L^2(G)} = \frac{1}{(2\pi)^{n+m}}\sum_{k=0}^{\infty}\int_{\mathbb{R}^m}|a|^{2n}\int_{\mathbb{C}^n}|f*e_k^a(z,0)|^2\,dz\,da.
$$

Let $\mathcal{A}_k$ denote the spectral projection operator on L^2 defined by

$$
\mathcal{A}_k f(z,t) = \int_{\mathbb{R}^m} f*e_k^a(z,t)\,|a|^n\,da. \tag{3.1}
$$

Our aim is to extend this spectral projection operator $\mathcal{A}_k$ to $L^p(G)$ and prove its L^p boundedness. We will achieve this by showing that each $\mathcal{A}_k$ is a singular integral operator whose kernel satisfies the requirements of Theorem 2.2. Notice that if $f(z,t)$ is a Schwartz class function on G, whose Fourier transform in the t-variable is compactly supported, then following the proof given in [20] (see pp. 269–270) we can show that

$$
\int_{\mathbb{R}^m} f*e_k^a(z,t)\,|a|^n\,da = f*A_k(z,t), \tag{3.2}
$$

where A_k is given by

$$
\begin{aligned}
A_k(z,t) &= \int_{\mathbb{R}^m} e_k^a(z,t)\,|a|^n\,da \\
&= \int_{\mathbb{R}^m} e^{-i\langle a,t\rangle}\,\varphi_k^{|a|}(z)|a|^n\,da.
\end{aligned}
$$

Due to the presence of the Gaussian in the integral defining A_k, it is easy to show that A_k is smooth away from identity. Since $\varphi_k^{|a|}(z) = L_k^{n-1}\left(\frac{|a||z|^2}{2}\right)e^{-\frac{|a|}{4}|z|^2}$, the kernel

$A_k(z,t)$ is a linear combination of functions of the form

$$A_k^j(z,t) = |z|^{2j} \int_{\mathbb{R}^m} e^{-i\langle a,t\rangle}\, e^{-\frac{|a|}{4}|z|^2}\, |a|^{n+j}\, da, \;\; j = 0, 1, \ldots, k,$$

which too are smooth away from the identity. A simple change of variables shows that

$$A_k^j(sz, s^2 t) = s^{-Q} A_k^j(z,t),$$

which is the required homogeneity for singular integral operators on G.

Using polar coordinates, we obtain

$$A_k^j(z,t) = c_m |z|^{2j} \int_0^\infty \frac{J_{\frac{m}{2}-1}(\lambda|t|)}{(\lambda|t|)^{\frac{m}{2}-1}} e^{-\frac{\lambda}{4}|z|^2} \lambda^{n+m+j-1}\, d\lambda,$$

where c_m is a constant that depends only on m. We prove that $A_k(z,t)$ is a Calderón–Zygmund kernel by showing that each $A_k^j(z,t)$ is. Since $A_k^j(z,t)$ is homogeneous of degree $-Q$ and belongs to $C^\infty(G \setminus \{0\})$, we need to show that these kernels satisfy the cancellation condition as in Lemma 2.2. Since $A_k^j(z,t)$ is radial in t, it suffices to show the following:

Lemma 3.1

$$\int_{\mathbb{C}^n} A_k^j(z,1)\, dz = 0, \;\; j = 0, 1, 2, \ldots, k.$$

Proof We start with the integral

$$I_m(\tau) = \int_0^\infty \frac{J_{\frac{m}{2}-1}(\lambda)}{\lambda^{\frac{m}{2}-1}}\, e^{-\tau\lambda}\, \lambda^{m-1}\, d\lambda, \;\; \tau > 0. \tag{3.3}$$

Then for any $t \in \mathbb{R}^m$ such that $|t| = 1$, it is easy to see that the above (up to a constant) equals

$$\int_{\mathbb{R}^m} e^{-i\langle x,t\rangle}\, e^{-\tau|x|}\, dx,$$

which equals the Poisson kernel,

$$c_m \frac{\tau}{(1+\tau^2)^{\frac{m+1}{2}}}$$

for some constant c_m. Now,

$$\int_0^\infty \frac{J_{\frac{m}{2}-1}(\lambda)}{\lambda^{\frac{m}{2}-1}}\, e^{-\tau\lambda}\, \lambda^{n+m+j-1}\, d\lambda = \frac{d^{n+j}}{d\tau^{n+j}}\,(I_m(\tau))$$

$$= I_m^{(n+j)}(\tau).$$

Hence, to prove the lemma, we need to show that

$$\int_{\mathbb{C}^n} |z|^{2j} \; I_m^{(n+j)}\left(\frac{|z|^2}{4}\right) \, dz = 0, \;\; j = 0, 1, 2, \dots, k.$$

Since the integrand is radial, this reduces to showing that

$$\int_0^\infty I_m^{(n+j)}\left(\frac{r^2}{4}\right) r^{2n+2j-1} \, dr = 2^{2n+2j-1} \int_0^\infty I_m^{(n+j)}(b) \, b^{n+j-1} \, db = 0.$$

Now, writing

$$\Psi(b) = \frac{1}{(1+b^2)^{\frac{m+1}{2}}},$$

we get,

$$I_m^{(n+j)}(b) = b\Psi^{(n+j)}(b) + (n+j)\Psi^{(n+j-1)}(b).$$

Hence

$$\begin{aligned}\int_0^\infty I_m^{(n+j)}(b) \, b^{n+j-1} \, db &= \int_0^\infty \Psi^{(n+j)}(b) \, b^{n+j} \, db \\ &\quad + (n+j) \int_0^\infty \Psi^{(n+j-1)}(b) \, b^{n+j-1} \, db \\ &= \lim_{b\to\infty} b^{n+j} \Psi^{(n+j-1)}(b)\end{aligned}$$

which is easily verified to be zero as $m \geq 2$. This proves the lemma. □

From Theorem 2.1, it follows that the operator

$$f \mapsto \mathrm{P.V.}\, f * A_k$$

is a bounded operator on $L^2(G)$ and therefore (by Theorem 2.2) bounded on $L^p(G)$, $1 < p < \infty$ as well.

Hence, we have proved the following theorem.

Theorem 3.1 *The spectral projection operator $\mathcal{A}_k$ is the convolution operator $f \mapsto$ P. V. $f * A_k$ and is bounded on $L^p(G)$, $1 < p < \infty$.*

Next we show the Abel summability of the spectral decomposition for $f \in L^p(G)$.

Theorem 3.2 *Let* $2 \leq p < \infty$ *and* $f \in L^p(G)$*. Then*

$$\lim_{r \to 1} \sum_{k=0}^{\infty} r^k \int_{\mathbb{R}^m} f * e_k^a(z,t)\, |a|^n\, da = f(z,t)$$

in the L^p *norm.*

As in [18] (see Theorem 3.3 and Corollary 3.4 there, also the first paragraph in p. 375), it is enough to show that the operators

$$T_r f(z,t) = \sum_{k=0}^{\infty} r^k \int_{\mathbb{R}^m} f * e_k^a(z,t)\, |a|^n\, da, \tag{3.4}$$

are uniformly bounded on $L^p(G)$, $2 \leq p < \infty$. To prove this, we need the following lemma.

Lemma 3.2 *Let* $K(z,t)$ *be an odd kernel that is smooth away from the identity and homogeneous of degree* $-Q$*. Then the operator norm of* $f \mapsto \mathrm{P.V.}\, f * K$ *on* $L^p(G)$, $1 < p < \infty$ *is bounded by*

$$C_p \int_{\mathbb{C}^n} \int_{S^{m-1}} |K(w,u)|\, dw du$$

for some constant C_p *depending only on* p*.*

Proof Using the homogeneity of the kernel K, we can write $f * K$ as,

$$\begin{aligned} f * K(z,t) &= \int_{\mathbb{C}^n} \int_{\mathbb{R}^m} f\left(z - w, t - s - \frac{1}{2}[z,w]\right) K(w,s)\, dw ds \\ &= \int_{\mathbb{C}^n} \int_0^{\infty} \int_{S^{m-1}} f\left(z - w, t - ru - \frac{1}{2}[z,w]\right) r^{-n-m} K\left(\frac{w}{\sqrt{r}}, u\right) \\ &\quad r^{m-1}\, dr du dw \\ &= \int_0^{\infty} \int_{S^{m-1}} \int_{\mathbb{C}^n} f\left(z - \sqrt{r} w, t - ru - \frac{\sqrt{r}}{2}[z,w]\right) K(w,u) r^{-1}\, dw dr du \\ &= 2 \int_{\mathbb{C}^n} \int_{S^{m-1}} \int_0^{\infty} f\left(z - rw, t - r^2 u - \frac{r}{2}[z,w]\right) K(w,u)\, \frac{dr}{r} du dw. \end{aligned}$$

Since K is odd, the above integral becomes

$$\begin{aligned} &2 \int_{\mathbb{C}^n} \int_{S^{m-1}} \left(\int_0^{\infty} f(z - rw, t - r^2 u - \frac{r}{2}[z,w])\, \frac{dr}{r} \right. \\ &\quad \left. - \int_0^{\infty} f(z + rw, t + r^2 u + \frac{r}{2}[z,w])\, \frac{dr}{r} \right) K(w,u)\, du dw \end{aligned}$$

Now, the inner integral,

$$\int_0^\infty f\left(z - rw, t - r^2u - \frac{r}{2}[z, w]\right) \frac{dr}{r} - \int_0^\infty f\left(z + rw, t + r^2u + \frac{r}{2}[z, w]\right) \frac{dr}{r}$$

equals

$$\int_0^\infty f((z,t)(\delta_r(w,u))^{-1}) \frac{dr}{r} - \int_0^\infty f((z,t)(\delta_r(w,u))) \frac{dr}{r} \tag{3.5}$$

since

$$\begin{aligned}\int_{-\infty}^0 f((z,t)(\delta_{-r}(-w,-s))^{-1}) \frac{dr}{r} &= \int_{-\infty}^0 f((z,t)\delta_{-r}(w,s)) \frac{dr}{r} \\ &= -\int_0^\infty f((z,t)\delta_r(w,s)) \frac{dr}{r}.\end{aligned}$$

The expression (3.5) is the Hilbert transform $H_{\gamma_{(w,s)}} f$ of f along the curve $\gamma_{(w,s)}$ in G, given by,

$$\gamma_{(w,s)}(r) = \begin{cases} \delta_r(w,s) & \text{for } r > 0 \\ \delta_{-r}(-w,-s) & \text{for } r \leq 0 \end{cases}$$

Hence

$$f * K(z,t) = \int_{\mathbb{C}^n} \int_{S^{m-1}} H_{\gamma_{(w,s)}} f(z,t) K(w,s)\, dw ds.$$

Therefore,

$$\begin{aligned}\|f * K\|_p &\leq \int_{\mathbb{C}^n} \int_{S^{m-1}} \|H_{\gamma_{(w,s)}} f\|_p |K(w,s)|\, dw ds \\ &\leq C_p \|f\|_p \int_{\mathbb{C}^n} \int_{S^{m-1}} |K(w,s)|\, dw ds\end{aligned}$$

where C_p is a constant that depends only on p (by Theorem 2.3). □

Now we are in a position to prove the uniform boundedness of $\|T_r\|_p$ using the previous lemma. We note that T_r is a convolution operator with kernel

$$\sum_{k=0}^\infty r^k \int_{\mathbb{R}^m} e_k^a(z,t) |a|^n \, da,$$

which we compute using the following generating function identity of Laguerre polynomials:

$$\sum_{k=0}^\infty r^k L_k^\alpha(x) = (1-r)^{-\alpha-1} e^{-\frac{rx}{1-r}}, \ |r| < 1.$$

It then follows that

$$\sum_{k=0}^{\infty} r^k \int_{\mathbb{R}^m} e_k^a(z,t)|a|^n\, da = (1-r)^{-n} \int_{\mathbb{R}^m} e^{-i\langle a,t\rangle} e^{-\frac{1}{4}\frac{1+r}{1-r}|a||z|^2} |a|^n\, da.$$

Since this is not an odd kernel, we bring in the Riesz transform in the t- variable. Define the operator $\mathcal{R}_j$ by

$$(\mathcal{R}_j f)^a(z) = \frac{a_j}{|a|}\, f^a(z),$$

which is just the j-th Riesz transform in the central variable. Clearly, $\mathcal{R}_j$ is bounded on $L^p(G)$ for $1 < p < \infty$. Now, define the operator

$$\mathcal{R}_j \mathcal{A}_k f(z,t) = \int_{\mathbb{R}^m} f * e_k^a(z,t) \frac{a_j}{|a|} |a|^n\, da.$$

Since $\sum_{j=1}^m \mathcal{R}_j^2 = I$, it suffices to prove that the operator norm of $\sum_{k=0}^{\infty} r^k \mathcal{R}_j \mathcal{A}_k$ is indepedent of r. Now the kernel of the above operator is

$$\sum_{k=0}^{\infty} r^k \int_{\mathbb{R}^m} e_k^a(z,t) \frac{a_j}{|a|} |a|^n\, da = (1-r)^{-n} \int_{\mathbb{R}^m} e^{-i\langle a,t\rangle}\, e^{-\frac{1}{4}\frac{1+r}{1-r}|a||z|^2}\, \frac{a_j}{|a|}\, |a|^n\, da.$$

Writing in terms of the polar coordinates and using the Hecke-Bochner identity, we obtain that the above integral is a constant multiple of

$$(1-r)^{-n}\, t_j \int_0^{\infty} \frac{J_{\frac{m}{2}}(\lambda|t|)}{(\lambda|t|)^{\frac{m}{2}}}\, e^{-\frac{1}{4}\frac{1+r}{1-r}\lambda|z|^2}\, \lambda^{n+m-1}\, d\lambda.$$

When $t \in S^{m-1}$, we can write the above expression using the function I_m (see (3.3)) as

$$(1-r)^{-n} t_j I_{m+2}^{(n-2)} \left(-\frac{1}{4}\frac{1+r}{1-r}|z|^2\right).$$

Since

$$\int_{\mathbb{C}^n} \int_{S^{m-1}} \left| (1-r)^{-n} t_j I_{m+2}^{(n-2)} \left(-\frac{1}{4}\frac{1+r}{1-r}|z|^2\right) \right| dz dt$$
$$\leq C \frac{1}{(1+r)^n} \int_0^{\infty} |I_{m+2}^{(n-2)}(a)| a^{n-1}\, da,$$

the proof is complete as it can easily be verified that

$$\int_0^{\infty} |I_{m+2}^{(n-2)}(a)| a^{n-1}\, da \leq C.$$

Remark 3.1 We comment on the difference in the proofs in the case of Heisenberg group and Heisenberg type groups. In [18], Strichartz obtains explicit expressions for the kernels of the spectral projections (see pp. 361–362 in [18]). From the expressions, the necessary properties of the kernel can be deduced. However, in the present case, due to the higher dimension of the center, this does not seem to be possible. Nevertheless, we obtain an integral expression for the kernel from which we are able to deduce the properties of the kernel. Notice that, using the generating function for the Laguerre polynomials, it is possible to obtain an expression (not explicit) for the kernel of the spectral projections $\mathcal{A}_k$. Indeed,

$$\sum_{k=0}^{\infty} r^k \int_{\mathbb{R}^m} e_k^a(z,t)\,|a|^n\,da = (1-r)^{-n} \int_{\mathbb{R}^m} e^{-i\langle a,t\rangle}\, e^{-\frac{1}{4}\frac{1+r}{1-r}|a||z|^2}\,|a|^n\,da.$$

Using polar coordinates in the above leads to the expression (up to a constant)

$$(1-r)^{-n} \int_0^{\infty} \frac{J_{\frac{m}{2}-1}(\lambda|t|)}{(\lambda|t|)^{\frac{m}{2}-1}}\, e^{-\frac{1}{4}\frac{1+r}{1-r}\lambda|z|^2}\,\lambda^{n+m-1}\,d\lambda.$$

Substituting the well known integral formula for the Bessel function in the above, we get (again up to a constant)

$$\int_{-1}^{1} (1-s^2)^{\frac{m-3}{2}} \left((1-r)^{-n} \int_0^{\infty} e^{is\lambda|t|}\, e^{-\frac{1}{4}\frac{1+r}{1-r}\lambda|z|^2}\,\lambda^{n+m-1}\,d\lambda\right) ds.$$

Now, the kernel A_k is the k^{th}-derivative of the above with respect to r, evaluated at $r=0$. However, the inner integral can be computed as in [18, p. 362]. We obtain that the expression

$$(1-r)^{-n} \int_0^{\infty} e^{is\lambda|t|}\, e^{-\frac{1}{4}\frac{1+r}{1-r}\lambda|z|^2}\,\lambda^{n+m-1}\,d\lambda$$

equals (ignoring some constants that depend only on n and m)

$$(1-r)^m \left[(|z|^2 - 4is|t|) + r(|z|^2 + 4is|t|)\right]^{-n-m}.$$

Differentiating the above k times and evaluating at $r=0$, we obtain that

$$A_k(z,t) = (-1)^k c_{n,m} \int_{-1}^{1} (1-s^2)^{\frac{m-3}{2}} P_k(z, s|t|)\,ds \tag{3.6}$$

where $c_{n,m}$ is a constant depending only on n and m and

$$P_k(z,t) = \left[\sum_{j=0}^{\min(k,m)} \binom{k}{j} \frac{m!}{(m-j)!} \frac{(n+m+k-j-1)!}{k!(n+m-1)!} \frac{(|z|^2-4it)^j}{(|z|^2+4it)^j} \right] \frac{(|z|^2+4it)^k}{(|z|^2-4it)^{n+m+k}}.$$

When m is odd, $p = \frac{m-3}{2}$ is a non-negative integer and one can expand the term $(1-s^2)^p$ in (3.6) and prove the cancellation condition for the kernel A_k by a somewhat long induction argument. However, this does not seem to work when m is even.

4 Injectivity of Spherical Means

In this section we prove the theorems stated in the introduction. We follow the proofs given in [20] closely. The important point is that the functions $e_k^a(z,t)$ are eigenfunctions for the three spherical mean operators we have considered.

4.1 Proof of Theorem 1.3

First we look at the spherical means with respect to the normalized surface measure μ_r^{2n} on the sphere $\{z \in \mathfrak{v} : |z| = r\}$. As in (2.3), we can see that

$$e_k^a * \mu_r^{2n}(z,t) = e^{-i\langle a,t\rangle} \varphi_k^{|a|} \times_{|a|} \mu_r^{2n}(z).$$

Since (see [20])

$$\varphi_k^{|a|} \times_{|a|} \mu_r^{2n}(z) = \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^{|a|}(r) \varphi_k^{|a|}(z)$$

we obtain,

$$e_k^a * \mu_r^{2n}(z,t) = \ c_{k,n}\ \varphi_k^{|a|}(r)\ e_k^a(z,t),\ \ \forall\ (z,t) \in G, \tag{4.1}$$

where $c_{k,n} = \frac{k!(n-1)!}{(k+n-1)!}$. Now, let $f \in L^p(G)$, $1 \leq p \leq \frac{2m}{m-1}$ and assume that $f * \mu_r^{2n}$ vanishes identically. Convolving f with a smooth approximate identity, we may assume that $f \in L^p$ for $2 \leq p \leq \frac{2m}{m-1}$. From the above identity (4.1), the spectral decomposition of $f * \mu_r^{2n}$ is given by

$$f * \mu_r^{2n}(z,t) = \sum_{k=0}^{\infty} \int_{\mathbb{R}^m} c_{k,n}\ \varphi_k^{|a|}(r)\ f * e_k^a(z,t)\ |a|^n\, da.$$

If $f * \mu_r^{2n}(z,t) = 0$ for all (z,t), by Theorem 3.2,

$$\lim_{s\to 1} \sum_{k=0}^{\infty} c_{k,n}\, s^k \int_{\mathbb{R}^m} \varphi_k^{|a|}(r)\ f * e_k^a(z,t)\ |a|^n\, da = 0$$

where the convergence is in $L^p(G)$. Applying the k^{th} spectral projection operator $\mathcal{A}_k$ and using Theorem 3.1 we obtain that

$$\int_{\mathbb{R}^m} \varphi_k^{|a|}(r) \left(f^a \times_{|a|} \varphi_k^{|a|}\right)(z)\, e^{-i\langle a,t\rangle} |a|^n\, da = 0, \quad \forall (z,t), \forall k = 0, 1, 2, \dots$$

Arguing as in [20, p.276] (also see [19, pp. 257–258]), we obtain that, for almost all $z \in \mathbb{C}^n$, the support of $f^a \times_{|a|} \varphi_k^{|a|}(z)$, the distributional Fourier transform of $\mathcal{A}_k f(z,\cdot)$, is contained in the zero set of $L_k^{n-1}(\frac{1}{2}|a|r^2)$, which is a finite union of spheres in $\mathbb{R}^m$. But this implies, by Theorem 2.4, that $\mathcal{A}_k f(z,t)$ is zero as $\mathcal{A}_k f \in L^p$ for $1 < p \le \frac{2m}{m-1}$. This finishes the proof of Theorem 1.3.

Next, we show that the above range is optimal by an example. For a fixed $k \ge 1$ and $s > 0$, let

$$\begin{aligned} F(z,t) &= \frac{J_{\frac{m}{2}-1}(s|t|)}{(s|t|)^{\frac{m}{2}-1}}\ \varphi_k^s(z) \\ &= \int_{|a|=s} e^{-i\langle a,t\rangle}\ \varphi_k^{|a|}(z)\, d\mu_s^m(a) \\ &= \int_{|a|=s} e_k^a(z,t)\, d\sigma_s(a), \end{aligned}$$

where μ_s^m as earlier, is the normalized surface measure on the sphere $\{a \in \mathbb{R}^m : |a| = s\}$. An easy computation using (4.1) shows that,

$$F * \mu_r^{2n}(z,t) = c_{k,n}\, \varphi_k^s(r)\ F(z,t),$$

for all $(z,t) \in G$. Choosing s suitably, we can make sure that $\varphi_k^s(r) = 0$. From the asymptotics of the Bessel function it is clear that $F \in L^p(G)$ if and only if $p > \frac{2m}{m-1}$, which proves our claim.

4.2 Proof of Theorem 1.4

Now we look at the bi-spherical means defined using the measures $\mu_{r,s} = \mu_r^{2n} \times \mu_s^m$. Recall that the measure $\mu_{r,s}$ for $r > 0$, $s > 0$ was defined by

$$\mu_{r,s}(f) = \int_{|z|=r} \int_{|t|=s} f(z,t)\, d\mu_r^{2n}(z)\, d\mu_s^m(t),$$

where $d\mu_r^{2n}$ and $d\mu_s^m$ are the normalized surface measures on the spheres $\{z : |z| = r\}$ and $\{t : |t| = s\}$ respectively. Assume that $f \in L^p(G)$ for $2 \leq p \leq \frac{2m}{m-1}$ and $f * \mu_{r,s}$ vanishes identically. Proceeding as in the earlier proof, using the identity (4.1), we get

$$\begin{aligned} e_k^a * \mu_{r,s}(z,t) &= c_{k,n}\, e^{-i\langle a,t\rangle} \frac{J_{\frac{m}{2}-1}(s|a|)}{(s|a|)^{\frac{m}{2}-1}} \varphi_k^{|a|}(r)\, \varphi_k^{|a|}(z) \\ &= c_{k,n} \frac{J_{\frac{m}{2}-1}(s|a|)}{(s|a|)^{\frac{m}{2}-1}} \varphi_k^{|a|}(r)\, e_k^a(z,t). \end{aligned}$$

Continuing exactly as above we get that the distributional Fourier transform of $\mathcal{A}_k f(z,t)$ in the t variable is supported in the zero set of (as a function of a)

$$\frac{J_{\frac{m}{2}-1}(s|a|)}{(s|a|)^{\frac{m}{2}-1}} \varphi_k^{|a|}(r),$$

which is a union of infinitely many spheres in $\mathbb{R}^m$. It then follows that $\mathcal{A}_k f = 0$ from Theorem 2.4, if $1 \leq p \leq \frac{2m}{m-1}$. This completes the proof of Theorem 1.4.

Next we show that the above range is the best possible. To this end, we need to recall some results on bi-radial functions on an H-type group G. Define the averaging operator (see [5, p. 221]) Π on integrable functions on G by

$$\Pi(f)(z,t) = \int_{S^{m-1}} \int_{S^{2n-1}} f(|z|u, |t|v)\, d\mu_1^{2n}(u)\, d\mu_1^m(v),$$

where $d\mu_1^{2n}$ and $d\mu_1^m$ are the normalized surface measures on the unit spheres $\{z : |z| = 1\}$ and $\{t : |t| = 1\}$ respectively. The operator Π is then an averaging projector satisfying several properties (see [5, p. 220]).

A bi-radial function on G is a function f that satisfies $\Pi(f) = f$. Clearly, f is bi-radial if and only if f is radial in both the z and t variables. For $k = 0, 1, 2, \ldots$ and $\lambda > 0$, define the functions $\Phi_k^\lambda(z,t)$ by

$$\Phi_k^\lambda(z,t) = C(k,n,m)\, \varphi_k^\lambda(z) \frac{J_{\frac{m}{2}-1}(s|t|)}{(s|t|)^{\frac{m}{2}-1}},$$

where $C(k,n,m)$ is a constant so that $\Phi_k^\lambda(0,0) = 1$. We have the following result about the class of integrable bi-radial functions, denoted by $L^1(G)^\#$.

Theorem 4.1 (1) *The space $L^1(G)^\#$ is a commutative Banach algebra under convolution.*

(2) *The space of multiplicative linear functionals on $L^1(G)^\#$ coincides with the collection $\{\Phi_k^\lambda : \lambda > 0,\ k = 0, 1, 2, \ldots\}$.*

For the proof of above see [5, Proposition 5.3]. We also need the product formula satisfied by the functions Φ_k^λ.

Proposition 4.1 *Let $\Phi = \Phi_k^\lambda$ for some k and λ. Let ${}_{(z,t)}\Phi$ denote the left translate of the function Φ by the point (z, t). Then,*

$$\Pi({}_{(z,t)}\Phi)(w, s) = \Phi(z, t)\ \Phi(w, s).$$

For a proof, see Proposition 2.3 in [9]. Now, a simple computation shows that the identity in Proposition 4.1 reduces to

$$\Phi_k^\lambda * \mu_{r,s}(z, t) = \Phi_k^\lambda(r, s)\ \Phi_k^\lambda(z, t).$$

Choosing λ and $k > 0$ such that $\Phi_k^\lambda(r, s) = 0$, we get

$$\Phi_k^\lambda * \mu_{r,s}(z, t) = 0\ \forall\ (z, t) \in G,$$

which proves our claim as $\Phi_k^\lambda(z, t) \in L^p$ if and only if $p > \frac{2m}{m-1}$.

4.3 Proof of Theorem 1.5

Finally, we look at the homogeneous spherical means defined using the measure σ_r. First we deal with the case $m \geq 2$. Recall the homogeneous norm on G, given by

$$|(z, t)| = (|z|^4 + |t|^2)^{\frac{1}{4}}.$$

Also, recall that there exists a unique Radon measure σ on the unit sphere $\Sigma = \{(z, t) : |(z, t)| = 1\}$ such that for all $f \in L^1(G)$

$$\int_G f(g)\, dg = \int_0^\infty \int_\Sigma f(\delta_r(z, t))\, d\sigma(z, t)\, r^{Q-1} dr$$

where δ_r denote the dilations that act as automorphisms of G. The measures σ_r, for $r > 0$ are defined by

$$\sigma_r(f) = \sigma(\delta_r f) = \int_\Sigma f(\delta_r(z, t))\, d\sigma(z, t).$$

The homogeneous spherical means of a function f is then defined as the convolution $f * \sigma_r$, of f with σ_r.

We have the formula for the measure σ_s, $s > 0$ given by

$$\begin{aligned}\sigma_s(f) &= \int f(z, t)\, d\sigma_s(z, t)\\ &= 2\int_0^1 \int_{|z|=1} \int_{|t|=1} f\left(srz, s^2\sqrt{1-r^4}t\right)\, d\mu_1^{2n}(z) d\mu_1^m(t)\, r^{2n-1}\, (1-r^4)^{\frac{m-2}{2}}\, dr.\end{aligned}$$

See [12, Proposition 2.7] or [11, p. 102] for the proof of this formula.

It follows that

$$f * \sigma_s = 2 \int_0^1 f * \mu_{sr, s^2\sqrt{1-r^4}} \, r^{2n-1} \, (1-r^4)^{\frac{m-2}{2}} \, dr$$

where $f * \mu_{sr,s^2\sqrt{1-r^4}}$ are the bi-spherical means defined earlier.

As above we compute

$$\begin{aligned} e_k^a * \sigma_s(z,t) &= 2 \int_0^1 e_k^a * \mu_{sr,s^2\sqrt{1-r^4}} \, (z,t) \, r^{2n-1} \, (1-r^4)^{\frac{m-2}{2}} \, dr \\ &= c_{k,n} \left(2 \int_0^1 \frac{J_{\frac{m}{2}-1}(s^2\sqrt{1-r^4}|a|)}{(s^2\sqrt{1-r^4}|a|)^{\frac{m}{2}-1}} \, \varphi_k^{|a|}(r) \, r^{2n-1} \, (1-r^4)^{\frac{m-2}{2}} \, dr \right) e_k^a(z,t). \end{aligned}$$

Write $|a| = \lambda$, and notice that the function

$$\lambda \mapsto \int_0^1 \frac{J_{\frac{m}{2}-1}(s^2\sqrt{1-r^4}\lambda)}{(s^2\sqrt{1-r^4}\lambda)^{\frac{m}{2}-1}} \, \varphi_k^{\lambda}(r) \, r^{2n-1} \, (1-r^4)^{\frac{m-2}{2}} dr \tag{4.2}$$

is holomorphic for $\Re\lambda > 0$ and so the above function has at most countably many zeros $\lambda \in (0, \infty)$. Now the proof can be completed as above for the range $1 \leq p \leq \frac{2m}{m-1}$, if $m \geq 2$. We believe that the range obtained is optimal. This will be true if the function in (4.2) has a zero in $(0, \infty)$.

When $m = 1$, $G = \mathbb{H}^n$, the Heisenberg group. The formula for the measure σ_s takes the following form (see [10, p. 95]):

$$\sigma_s = c_n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mu_{s\sqrt{\cos\theta}, \frac{1}{2}s^2 \sin\theta} \, (\cos\theta)^{n-1} \, d\theta,$$

where the measure $\mu_{r,s}$ is the normalized surface measure on the sphere $\{(z,s) \in \mathbb{H}^n : |z| = r\}$. Now the proof can be completed as earlier. We omit the details. This completes the proof of Theorem 1.5.

Remark 4.1 The Abel summability result for the spectral decomposition will be true for all $1 < p < \infty$, if we can estimate the operator norm of $\mathcal{A}_k$. It is a natural question whether a two radius theorem is true for functions in $L^p(G)$ for $\frac{2m}{m-1} < p \leq \infty$ and whether our results can be proved for averages over K-orbits where $(G \rtimes K, K)$ is a Gelfand pair as in the case of the Heisenberg group. We hope to return to these questions and some others in the near future.

Remark 4.2 When $1 \leq p \leq 2$, it is possible to take the Fourier transform in the central variable and prove the injectivity results for the spherical means with weaker conditions of growth on the function. See [8].

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