Journal of Fourier Analysis and Applications (2024) 30:31
https://doi.org/10.1007/s00041-024-10088-w

Existence of Exponential Orthonormal Bases for Infinite Convolutions on $\mathbb{R}^n$

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Received: 14 October 2022 / Revised: 23 March 2024 / Accepted: 10 April 2024 /
Published online: 7 May 2024

Abstract

In this paper we investigate the harmonic analysis of infinite convolutions generated by admissible pairs on Euclidean space $\mathbb{R}^n$. Our main results give several sufficient conditions so that the infinite convolution μ to be a spectral measure, that is, its Hilbert space $L^2(\mu)$ admits a family of orthonormal basis of exponentials. As a concrete application, we give a complete characterization on the spectral property for certain infinite convolution on the plane $\mathbb{R}^2$ in terms of admissible pairs.

Keywords Spectral measures · Infinite convolutions · Spectra · Exponential orthonormal bases

Mathematics Subject Classification 42A85 · 28A80

Communicated by Yurii Lyubarskii.

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1 Introduction

Let μ be a Borel probability measure on Euclidean space $\mathbb{R}^n$, and let $\widehat{\mu}$ be the *Fourier-Stieltjes transform* of μ on $\mathbb{R}^n$:

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \langle \xi, x\rangle} d\mu(x) \quad (\, \xi \in \mathbb{R}^n).$$

A fundamental problem in harmonic analysis for $L^2(\mu)$ is whether there exists a discrete set $\Lambda \subseteq \mathbb{R}^n$ such that the collection of exponential functions

$$E(\Lambda) := \{e_\lambda(x) := e^{2\pi i \langle \lambda, x\rangle} : \lambda \in \Lambda\}$$

forms an orthonormal basis for $L^2(\mu)$. If this holds, μ is called a *spectral measure* and Λ is a *spectrum* for μ, and (μ, Λ) is called a *spectral pair*. Thus, for a discrete set Λ, there are two crucial ingredients for Λ being a spectrum of μ: one is that the system $E(\Lambda)$ forms an *orthogonal set* for $L^2(\mu)$, that is,

$$\widehat{\mu}(\lambda' - \lambda) = \langle e_\lambda, e_{\lambda'}\rangle_{L^2(\mu)} = 0 \qquad \text{for } \lambda' \neq \lambda \in \Lambda;$$

the other is that $E(\Lambda)$ is *total* in $L^2(\mu)$, that is, if $\langle f, e_\lambda\rangle_{L^2(\mu)} = 0$ for all $\lambda \in \Lambda$, then $f = 0$ holds μ-almost everywhere.

It is known that the study of spectral measures has a long history, and the wide research of spectral measures dates back to the famous Fuglede conjecture in 1974, which asserted that the normalized Lebesgue measure restricted on a Borel set Ω is a spectral measure if and only if Ω is a translational tile in $\mathbb{R}^n$, the interested readers are referred to [16, 25, 36] and references therein. In the year 1998, Jorgensen and Pedersen [22] announced the first class of singular continuous spectral measures, in which they proved that Bernoulli convolutions $\mu_{R,2}$ is a spectral measure if $R = 2k$, and it is not spectral if $R = 2k + 1$. Later on, after the pioneering works of Strichartz [34] and Łaba-Wang [26], many significant progresses have been made in constructing new spectral measures (e.g., see [1, 2, 5, 10]), classifying the structures of spectra for some singular spectral measures (e.g., see [6, 8, 11, 17]) and investigating the convergence or divergence of mock Fourier series (e.g., see [12, 20, 34, 35]). Among them, a surprising and interesting phenomenon was that there are uncountably many spectra such that the associated mock Fourier series of continuous functions converge uniformly, and the mock Fourier series of L^p-functions converge pointwise almost everywhere [20, 35].

The present paper is devoted to investigating the question of spectrality of infinite convolutions on $\mathbb{R}^n$. More precisely, here and below, we use the symbol $*$ to denote the convolution of two measures, and for a discrete set D in $\mathbb{R}^n$, we define a discrete probability measure δ_D as follows

$$\delta_D = \frac{1}{\#D} \sum_{d \in D} \delta_d, \tag{1.1}$$

where δ_d is the Dirac point mass measure at the point d, and $\#D$ denotes the cardinality of D. The interest of this paper is concentrated on the following well-studied problem in spectral theory of measures.

Question 1.1 Given a sequence of matrices $R_k \in GL(n, \mathbb{R})$ and a sequence of digit sets D_k in $\mathbb{R}^n$, under what conditions is the infinite convolution

$$\mu(R_k, D_k) := \delta_{R_1^{-1}D_1} * \delta_{R_1^{-1}R_2^{-1}D_2} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}D_3} * \cdots \tag{1.2}$$

(if it exists in the weak-star topology) a spectral measure?

The investigation of measure $\mu(R_k, D_k)$ and similar measures dates back to the 1930s, and it was shown [23] that it is either absolutely continuous or singular continuous with respect to Lebesgue measure. Also, several sufficient and necessary conditions for the the existence of $\mu(R_k, D_k)$ were given in [23] and [29]. In particular, if $R_k = R$ and $D_k = D$ for all $k \in \mathbb{N}$, they are *self-affine measures* $\mu_{R,D} := \mu(R_k, D_k)$ in fractal geometry (cf., [14, 21]), and Bernoulli convolution $\mu_{R,2}$ mentioned above is obtained by taking $R_k = R \in \mathbb{R}$ and $D_k = \{-1, 1\}$ for all $k \in \mathbb{N}$.

In the published literatures, in order to settle down Question 1.1, one basic but most important condition is the concept of admissible pairs (cf., [10, 22, 26, 34]):

Definition 1.2 Let $R \in GL(n, \mathbb{Z})$ be an $n \times n$ integer matrix, and let $D \subseteq \mathbb{Z}^n$ be a finite subset. The pair (R, D) is called an *admissible pair* if there is a set $C \subseteq \mathbb{Z}^n$ with the same cardinality as D such that the matrix

$$H_{R^{-1}D,C} := \frac{1}{\sqrt{\#D}} \left[e^{2\pi i \langle R^{-1}d, c\rangle} \right]_{d\in D, c\in C}$$

is unitary, i.e., $H^*_{R^{-1}D,C} H_{R^{-1}D,C} = I$. Following [10, 26, 34], the system $(R^{-1}D, C)$ is called a *compatible pair*, and (R, D, C) is called a *Hadamard triple*.

Based on admissible pairs, a lot of one-dimensional spectral infinite convolutions $\mu(R_k, D_k)$ as in (1.2) were obtained in [26, 34] and the references given there, especially see the recent works [19, 28–30, 32]. However, to the best of our knowledge, there are only a few classes of higher dimensional spectral measures $\mu(R_k, D_k)$ on $\mathbb{R}^n$ are known. Let us describe some of the previous work on higher dimensional infinite convolutions being spectral. Strichartz first showed [34, Theorem 2.8] the following theorem:

Theorem A [34] *If the measure $\mu(R_k, D_k)$ on $\mathbb{R}^n$ as in (1.2) satisfies that*

(i) *there are digit sets C_k with $\mathbf{0} \in C_k$ such that $\{(R_k^{-1}D_k, C_k)\}_{k=1}^{\infty}$ forms a sequence of compatible pairs which are chosen from a finite set of compatible pairs, and each matrix R_k is expanding[1];*

[1] Here, the notation "expanding" denotes that, for the sequence $\{R_k\}$ with finitely many distinct matrices, there exists $r > 1$ such that $\|R_k x\|_2 \geq r\|x\|_2$ for all k, where $\|\cdot\|_2$ denotes the Euclidean 2-norm on $\mathbb{R}^n$, see [34, pp. 216, line 1–2]

(ii) *the zero set Z_n of the Fourier transform of $\delta_{R_n^{-1}D_n}$ is uniformly disjoint from the sets $(R_1^* \cdots R_n^*)^{-1}C_1 + (R_2^* \cdots R_n^*)^{-1}C_2 \cdots + R_n^{*-1}C_n$ for all $n \in \mathbb{N}$. That is, there is a common positive number $\delta > 0$ (independent on n) such that their distance satisfies that*

$$d(Z_n,\ (R_1^* \cdots R_n^*)^{-1}C_1 + (R_2^* \cdots R_n^*)^{-1}C_2 \cdots + R_n^{*-1}C_n) > \delta \quad \textit{for all}\ \ n \in \mathbb{N}.$$

Then the measure $\mu(R_k, D_k)$ in (1.2) is a spectral measure with a spectrum

$$\Lambda = \bigcup_{n=1}^{\infty} \big(C_1 + R_1^* C_2 + \cdots + R_1^* \cdots R_{n-1}^* C_n\big).$$

Unfortunately, the condition in Theorem A(ii) is not a necessary condition, and it might be very difficult to check, even for self-affine measures. Fortunately, Dutkay, Haussermann and Lai [10] completely showed that one admissible pair automatically yields a self-affine spectral measure on $\mathbb{R}^n$, i.e., condition (ii) can be removed in this case. Nevertheless, there are non-spectral measures of infinite convolutions generated by more than one admissible pair, cf., [4, Example 5.2]. To some extent, this means that it is a very challenging question to give a complete answer to Question 1.1 for general convolutions. At last, we point out that Dutkay and Lai [13] (also see [18, Section 3]) investigated the spectrality of a class of infinite convolutions on $\mathbb{R}^n$, where $R_k = R$ for all k and there are only finitely many distinct D_k such that $(R^{-1}D_k, C)$ forms a compatible pair for some *common digit set* $C \subseteq \mathbb{Z}^n$, and some additional conditions were given to guarantee infinite convolutions to be spectral (e.g., see [13, Theorem 1.5]).

Continuing the line of the research above, the purpose of this paper is to further investigate the spectrality of higher dimensional measures $\mu(R_k, D_k)$ as in (1.2) on $\mathbb{R}^n$ under the condition of admissible pairs or compatible pairs, and we will settle down Question 1.1 partially. More explicitly, given a sequence of admissible pairs $\{(R_k, D_k) : k \in \mathbb{N}\}$ on $\mathbb{R}^n$, we define

$$\mathbf{R}_k = R_k R_{k-1} \ldots R_1 \tag{1.3}$$

and assume that the following infinite convolution as in (1.2)

$$\mu := \mu(R_k, D_k) = \delta_{\mathbf{R}_1^{-1}D_1} * \delta_{\mathbf{R}_2^{-1}D_2} * \cdots * \delta_{\mathbf{R}_k^{-1}D_k} * \cdots \tag{1.4}$$

converges to a Borel probability measure μ in the weak-star topology. For each $k \in \mathbb{N}$, the measure μ in (1.4) can be decomposed as

$$\mu = \mu_k * (\nu_k \circ \mathbf{R}_k),$$

where $\nu_k \circ \mathbf{R}_k(E) = \nu_k(\mathbf{R}_k(E))$ for all Borel set $E \subseteq \mathbb{R}^n$, and the Borel measures μ_k, ν_k are respectively represented as (sometimes, we also write $\nu_0 = \mu(R_k, D_k)$)

$$\mu_k = \delta_{\mathbf{R}_1^{-1} D_1} * \delta_{\mathbf{R}_2^{-1} D_2} * \cdots * \delta_{\mathbf{R}_k^{-1} D_k}$$

and

$$\nu_k = \delta_{R_{k+1}^{-1} D_{k+1}} * \delta_{R_{k+1}^{-1} R_{k+2}^{-1} D_{k+2}} * \delta_{R_{k+1}^{-1} R_{k+2}^{-1} R_{k+3}^{-1} D_{k+3}} * \cdots . \tag{1.5}$$

As in [2, 10, 13], we know that the sequence of measures ν_k plays a critical role in determining the spectral property of $\mu(R_k, D_k)$. Here, based on [10, Lemma 4.7], it is reasonable to define the concept of *equi-positive family on some compact set*.

Definition 1.3 Let $M(\mathbb{R}^n)$ be the convolutional algebra of Borel probability measures on $\mathbb{R}^n$ and let $X \subseteq \mathbb{R}^n$ be a compact set. A subset Φ of $M(\mathbb{R}^n)$ is said to be an *equi-positive family on* X if there are positive numbers ϵ, δ such that for all $x \in X$ and for all $\nu \in \Phi$, there is an $h(x, \nu) \in \mathbb{Z}^n$ such that

$$|\widehat{\nu}(x + h(x, \nu) + y)| > \epsilon$$

for all $y \in \mathbb{R}^n$ with $\|y\|_2 < \delta$, and $h(x, \nu) = \mathbf{0}$ if $x = \mathbf{0}$.

Principal Assumption. Throughout this paper, by $\sigma(R_k)$ we denote the set of all singular values of $R_k \in GL(n, \mathbb{Z})$, i.e., the set of all nonnegative square roots of the eigenvalues of the positive semidefinite matrix $R_k^* R_k$. We always assume that the singular values of R_k, appeared in (1.2) or (1.4), satisfy that

$$\kappa := \inf_{k \geq 1} \min \sigma(R_k) > 1, \tag{1.6}$$

(this ensures that all matrices R_k are expanding in the sense of Strichartz [34]) and the compact set X in Definition 1.3 is chosen to be $X = B(\kappa)$, where

$$B(\kappa) := \left\{ x \in \mathbb{R}^n : \|x\|_2 \leq \frac{\sqrt{n}\kappa}{2(\kappa - 1)} \right\} \tag{1.7}$$

(this requirement will become clearer in the proof of (3.4)) is the 2-norm closed ball in $\mathbb{R}^n$, with the interpretation that

$$B(\infty) := \left\{ x \in \mathbb{R}^n : \|x\|_2 \leq \sqrt{n}/2 \right\}.$$

Here, $\|x\|_2$ denotes the Euclidean 2-norm of a vector $x \in \mathbb{R}^n$. Clearly, $\left[-\frac{1}{2}, \frac{1}{2}\right]^n \subseteq B(\infty) \subseteq B(\kappa)$ for all $\kappa > 1$.

The following is our first main result, which gives a sufficient condition to guarantee the spectrality of arbitrary infinite convolutions.

Theorem 1.4 *Suppose $\{(R_k, D_k) : k \in \mathbb{N}\}$ is a sequence of admissible pairs on $\mathbb{R}^n$ such that the measure μ as in (1.4) exists and (1.6) holds (i.e., $\kappa > 1$). If there is a strictly increasing sequence of positive integers $\{k_j\}$ such that the family $\{\nu_{k_j}\}$ written as in (1.5) is equi-positive on $B(\kappa)$ as in (1.7), then the measure μ in (1.4) is a spectral measure.*

Remark 1.5 (1) The similar results for one-dimensional spectral measures have been obtained in [2, Theorem 3.2], [32, Theorem 2.4] and [28, Theorem 1.4], in which $R_k \geq 2$, $D_k \subseteq \mathbb{Z}$ and the compact set $B(\kappa)$ in Theorem 1.4 is chosen to be the closed interval $[0, 1]$. After the submission of this paper to JFAA, we noted that an analogous result of Theorem 1.4 was also obtained in [31, Theorem 1.1]. (2) If we take $n = 1$ in Theorem 1.4, then $R_k \in GL(n, \mathbb{Z})$ and $D_k \subseteq \mathbb{Z}^n$ are just respectively reduced to that $0 \neq R_k \in \mathbb{Z}$ and $D_k \subseteq \mathbb{Z}$, and $\kappa > 1$ furthermore implies that $\kappa \geq 2$ or $R_k \geq 2$ for all $k \in \mathbb{N}$ and hence the compact set $B(\kappa)$ is always contained in the closed interval $[-1, 1]$. Therefore, Theorem 1.4 gives an effective supplement to the one-dimensional case compared with the previous research mentioned in (1) for $\mathbb{R}$. (3) The spectral measures given in Theorem 1.4 may have no compact support, one can find more one-dimensional such examples in [29]. Also, it should be pointed out that infinite convolutions appeared in other results in this paper all have compact support. (4) By the definition of equi-positivity in Definition 1.3 and the fact that $\left[-\frac{1}{2}, \frac{1}{2}\right]^n \subseteq B(\kappa)$ for all $\kappa > 1$, it follows that a subset Φ of $M(\mathbb{R}^n)$ is equi-positive on $B(\kappa)$ if and only if Φ is equi-positive on $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$. Therefore, the equi-positive condition in Theorem 1.4 can be changed to that $\{\nu_{k_j}\}$ *in* (1.5) *is equi-positive on* $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$.

Our second main result Theorem 1.6 is concentrated on the spectrality of infinite convolution μ generated by finitely many admissible pairs firstly studied by Strichartz [34]. Here we provide some easier verifiable sufficient criteria (compared with Theorem A) so that μ is spectral. In the following, for a digit set $F \subseteq \mathbb{R}^n$, we denote by $\mathbb{Z}[F]$ the additive group generated by the elements of F, that is,

$$\mathbb{Z}[F] := \{k_1 f_1 + \cdots + k_n f_n : k_i \in \mathbb{Z},\ \ f_i \in F \ \text{ for } i = 1, 2, \ldots, n\},$$

and $F \subseteq \mathbb{R}^n$ is called *uniformly discrete* if there is a positive number $\delta > 0$ such that $\|f_1 - f_2\|_2 > \delta$ for all distinct $f_1, f_2 \in F$, and $\widehat{\delta_F}$ denotes the Fourier-Stieltjes transform of the measure δ_F as in (1.1):

$$\widehat{\delta_F}(\xi) = \int e^{-2\pi i \langle \xi, x\rangle} d\delta_F(x) = \frac{1}{\# F} \sum_{f \in F} e^{-2\pi i \langle \xi, f\rangle} \quad (\xi \in \mathbb{R}^n).$$

Moreover, we use the symbol $\mathcal{Z}(f)$ to stand for the zero set of the function f on $\mathbb{R}^n$.

Theorem 1.6 *Given a sequence of admissible pairs $\{(R_k, D_k)\}$ which are chosen from a finite set of admissible pairs such that (1.6) (i.e., $\kappa > 1$) holds. Assume that, for each $k \in \mathbb{N}$, one has*

(i) $\mathbb{Z}[D_k - d_k] = \mathbb{Z}^n$ *for some $d_k \in D_k$.*

(ii) *the set* $\mathcal{Z}(\widehat{\delta_{D_k}}) \cap \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ *is a uniformly discrete set.*

Then the associated measure μ *in* (1.4) *exists and it is a spectral measure.*

Our third main result Theorem 1.7 gives another sufficient condition for infinite convolutions to be spectral, which can be used to deal with the case that the zero set $\mathcal{Z}(\widehat{\delta_D})$ is the union of linear manifolds, see Example 1.10. An obvious difference to Theorem 1.6 is that it allows the matrices R_k and digit set D_k to be chosen arbitrarily.

Theorem 1.7 *Let* $\{(R_k, D_k)\}_{k=1}^{\infty}$ *be a sequence of admissible pairs, where the matrices* $\{R_k\}_{k=1}^{\infty} \subseteq GL(n, \mathbb{Z})$ *satisfies* (1.6) *(i.e.,* $\kappa > 1$*) and the digit sets* $\{D_k\}_{k=1}^{\infty}$ *is contained in* $\mathbb{Z}^n$*, and let*

$$B\left(\boldsymbol{0}, \frac{\sqrt{n}}{2\kappa}\right) := \left\{\xi \in \mathbb{R}^n : \|\xi\|_2 \leq \frac{\sqrt{n}}{2\kappa}\right\} \tag{1.8}$$

be the 2*-normed closed ball. Assume that* $\sup_{k\in\mathbb{N}} \sup_{\boldsymbol{d}_k \in D_k} \{\| R_k^{-1} \boldsymbol{d}_k \|_2\} < \infty$*, and*

$$\eta := \inf_{k\in\mathbb{N}} \inf_{\xi \in B\left(\boldsymbol{0}, \frac{\sqrt{n}}{2\kappa}\right)} |\widehat{\delta_{D_k}}(\xi)| > 0. \tag{1.9}$$

Then the associated measure μ *in* (1.4) *exists, and it is a spectral measure.*

The following Corollary is an immediate consequence of Theorem 1.7.

Corollary 1.8 *Given a sequence of admissible pairs* $\{(R_k, D_k)\}_{k=1}^{\infty}$*, where the matrices* $\{R_k\}_{k=1}^{\infty} \subseteq GL(n, \mathbb{Z})$ *satisfies* (1.6) *(i.e.,* $\kappa > 1$*) and* $\{D_k\}_{k=1}^{\infty} \subseteq \mathbb{Z}^n$ *is a sequence of digit sets chosen from a finite set, say* $D(1), \ldots, D(N)$ *for some* $N \in \mathbb{N}$*. Assume that the union of the zero set* $\bigcup_{k=1}^{N} \mathcal{Z}(\widehat{\delta_{D(k)}})$ *is separated from the* 2*-normed closed ball* $B(\boldsymbol{0}, \frac{\sqrt{n}}{2\kappa})$ *as in* (1.8)*. Then the associated measure* μ *in* (1.4) *exists and it is a spectral measure.*

Based on the research mentioned above, we are very surprised to find that admissible pairs might be a necessary and sufficient condition for some special infinite convolutions to be spectral.

Theorem 1.9 *Let* $\{R_k\}_{k=1}^{\infty} \subseteq GL(n, \mathbb{Z})$ *be a sequence of matrices satisfies* (1.6) *(i.e.,* $\kappa > 1$*), and let* $\{D_k\}$ *be a sequence of digit sets chosen from* $\{D^{(3)}, D^{(4)}\}$*, where*

$$D^{(3)} = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \text{ and } D^{(4)} = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}.$$

If the associated measure μ *in* (1.4) *is a spectral measure, then* (R_k, D_k) *is an admissible pair for each* $k \geq 2$*. Furthermore, the converse of this statement hold in the following two cases:*

(i) *If* $\{R_k\}_{k=1}^{\infty}$ *is a finite sequence, then the associated measure* μ *in* (1.4) *is a spectral measure if and only if* (R_k, D_k) *is an admissible pair for each* $k \geq 2$*.*

(ii) *If* $\kappa := \inf_{k\geq 1} \min \sigma(R_k) > \frac{3}{2}$, *then the associated measure* μ *in* (1.4) *is a spectral measure if and only if* (R_k, D_k) *is an admissible pair for each* $k \geq 2$.

It is remarked here that the part "*admissible pairs* $\Rightarrow$ *spectral*" of Theorem 1.9 is a direct result of Theorem 1.6 or Corollary 1.8. However, for the converse part "*spectral* $\Rightarrow$ *admissible pairs*" of Theorem 1.9, it needs us to do much research on the structure of spectra for measures ν_k written as in (1.5) (see Theorem 5.2 for more details). The ideas adopted here are totally different from the previous studies, which is of independent interest and might shed some new light on the characterization of necessary condition for general infinite convolutions being spectral measures.

Finally, we provide a simple and concrete example of spectral infinite convolution to illustrate Theorem 1.7 or Corollary 1.8.

Example 1.10 Suppose that $\{R_k\}_{k=1}^{\infty} \subseteq GL(n, \mathbb{Z})$ is a sequence of matrices such that $\kappa := \inf_{k\geq 1} \min \sigma(R_k) > \sqrt{2}$ and (R_k, D) is an admissible pair for each $k \in \mathbb{N}$, where

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Then the associated measure

$$\mu(R_k, D) := \delta_{R_1^{-1}D} * \delta_{R_1^{-1}R_2^{-1}D} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}D} * \cdots$$

exists, and it is a spectral measure. As a consequence, if

$$R(1) = \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix}, \quad R(2) = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}, \quad R(3) = \begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}, \quad R(4) = \begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix},$$

then for any $X : \mathbb{N} \to \{1, 2, 3, 4\}$, the associated infinite convolution measure

$$\mu(X, D) := \delta_{R_1^{-1}D} * \delta_{R_1^{-1}R_2^{-1}D} * \delta_{R_1^{-1}R_2^{-1}R_3^{-1}D} * \cdots \qquad \text{with } R_k = R(X(k))$$

is a spectral measure.

The rest of the paper is organized as follows. In Sect. 2 we review some basic facts about admissible pairs and spectral measures. In Sect. 3 we prove Theorem 1.4, in Sect. 4 we prove Theorems 1.6, 1.7, Corollary 1.8 and Example 1.10, and in the last Sect. 5 we prove Theorem 1.9.

2 Admissible Pairs and Spectral Measures

In this section we collect some basic properties of admissible pairs or compatible pairs on $\mathbb{R}^n$, and state the classical criterion for the completeness of exponential functions due to Jorgensen and Pedersen [22].

Lemma 2.1 [26] *Let $R \in GL(n, \mathbb{Z})$ be an $n \times n$ matrix, and let $D, C \subseteq \mathbb{Z}^n$ be two finite subsets of $\mathbb{Z}^n$ with the same cardinality. Then the following statements are equivalent:*

(i) *$(R^{-1}D, C)$ is a compatible pair.*
(ii) *C is a spectrum of the measure $\delta_{R^{-1}D}$.*
(iii) *$\widehat{\delta_{R^{-1}D}}(c_1 - c_2) = 0$ for any distinct $c_1, c_2 \in C$.*
(iv) *$\sum_{c \in C} |\widehat{\delta_{R^{-1}D}}(\xi + c)|^2 = 1 \qquad (\forall \xi \in \mathbb{R}^n)$.*

Lemma 2.2 [26, 34] *Let $R \in GL(n, \mathbb{Z})$ be an $n \times n$ matrix, and let $D, C \subseteq \mathbb{Z}^n$ be two finite subsets of $\mathbb{Z}^n$ with the same cardinality such that $(R^{-1}D, C)$ forms a compatible pair. Then the following statements hold.*

(i) *$(R^{-1}D + a, C + b)$ is a compatible pair for any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$.*
(ii) *No two elements in D are congruent modulo R (i.e., $d_i - d_j \notin R\mathbb{Z}^n$ for distinct elements $d_i, d_j \in D$), and no two elements in C are congruent modulo R^*.*
(iii) *If $\widetilde{C} \subseteq \mathbb{Z}^n$ such that $\widetilde{C} \equiv C \pmod{R^*}$, then $(R^{-1}D, \widetilde{C})$ is a compatible pair;*
(iv) *If the matrices $R_j \in GL(n, \mathbb{Z})$ and the digit sets $D_j, C_j \subseteq \mathbb{Z}^n$ satisfy that $(R_j^{-1}D_j, C_j)$ is a compatible pair and $\mathbf{0} \in C_j$ for each $j \in \mathbb{N}$, then*

$$\begin{aligned}&(R_1^{-1}D_1 + R_1^{-1}R_2^{-1}D_2 + \cdots + R_1^{-1}R_2^{-1}\ldots R_{k-1}D_k,\\&\ C_1 + R_1^*C_2 + \cdots + R_1^*\ldots R_{k-1}^*C_k)\end{aligned}$$

is a compatible pair for each $k \in \mathbb{N}$.

Associated to a Borel probability measure μ and a discrete set $\Lambda \subseteq \mathbb{R}^n$, we set

$$Q_{\mu,\Lambda}(\xi) = \sum_{\lambda \in \Lambda} |\widehat{\mu}(\xi + \lambda)|^2 \qquad (\xi \in \mathbb{R}^n).$$

Lemma 2.3 [22] *Let μ be a Borel probability measure on $\mathbb{R}^n$, and let Λ be a discrete set in $\mathbb{R}^n$. Then the following three statements holds.*

(i) *Λ is an orthogonal set for μ if and only if $Q_{\mu,\Lambda}(\xi) \le 1$ for all $\xi \in \mathbb{R}^n$.*
(ii) *Λ is a spectrum for μ if and only if $Q_{\mu,\Lambda}(\xi) = 1$ for all $\xi \in \mathbb{R}^n$.*
(iii) *$Q_{\mu,\Lambda}$ has an entire analytic extension to $\mathbb{C}^n$ if Λ is an orthogonal set for the measure μ with compact support.*

It is remarked here that Lemma 2.3(i) is an immediate result of Bessel's inequality, and Lemma 2.3(iii) follows from [22, Lemma 4.2], while Lemma 2.3(ii) was only proved in [22, Lemma 3.3] for compactly supported Borel probability measure by using Stone-Weierstrass theorem. Recently, Li et al. [28] proved that Lemma 2.3(ii) actually holds for probability measures without compact support.

3 Proof of Theorem 1.4

In this section we give the proof of Theorem 1.4, and it is divided into two steps.

- The first step is to construct an orthogonal set Γ (see (3.8)) for the infinite convolution measure $\mu := \mu(R_k, D_k)$ in (1.4) by using the properties of admissible pairs stated in Lemmas 2.1 and 2.2. It is worthy noting that this step involves adjusting the method of [10, Section 4] for self-affine measures to the present context.
- The second step is to show that $E(\Gamma)$ constructed in the first step is *total* in $L^2(\mu)$ by applying Jorgensen–Pedersen's completeness criterion (Lemma 2.3(ii)). This step depends on developing Strichartz's method in the proof of Theorem A (also see [34, Theorem 2.8]) for infinite convolutions generated by finitely many admissible pairs.

Proof of Theorem 1.4 First, since (R_k, D_k) is an admissible pair, by the statements (i) and (iii) of Lemma 2.2, one might without loss of generality assume that

$$\mathbf{0} \in C_k \subseteq R_k^* \left[-\frac{1}{2}, \frac{1}{2}\right]^n \cap \mathbb{Z}^n \tag{3.1}$$

such that $(R_k^{-1} D_k, C_k)$ forms a compatible pair for each $k \in \mathbb{N}$. Then, it is not hard to check (e.g., see [34, Theorem 2.7]) that the discrete set $E(\Lambda) = \{e^{2\pi i \langle \lambda, \cdot \rangle} : \lambda \in \Lambda\}$, where

$$\Lambda = C_1 + \mathbf{R}_1^* C_2 + \mathbf{R}_2^* C_3 + \cdots + \mathbf{R}_k^* C_k + \cdots , \tag{3.2}$$

forms an infinite orthogonal set for $L^2(\mu)$, where $\mathbf{R}_k$ is defined as in (1.3).

For the convenience of discussions, we introduce the following notations: for any two non-negative integers $s > t \geq 0$, we define that

$$\begin{aligned}
\mathbf{R}_{s,t} &= R_s R_{s-1} \ldots R_{t+1} \qquad \text{where} \qquad \mathbf{R}_{s,s-1} = R_s, \quad \mathbf{R}_{s,0} = \mathbf{R}_s \\
\mathbf{D}_{s,t} &= \mathbf{R}_{s,t} \left(\mathbf{R}_{t+1,t}^{-1} D_{t+1} + (\mathbf{R}_{t+2,t})^{-1} D_{t+2} + \cdots + (\mathbf{R}_{s,t})^{-1} D_s \right), \\
\mathbf{C}_{s,t} &= C_{t+1} + \mathbf{R}_{t+1,t}^* C_{t+2} + \mathbf{R}_{t+2,t}^* C_{t+3} + \cdots + \mathbf{R}_{s-1,t}^* C_s .
\end{aligned} \tag{3.3}$$

Obviously, from Lemma 2.2(iv), one concludes that, for each $s > t \geq 0$, $(\mathbf{R}_{s,t}, \mathbf{D}_{s,t})$ is an admissible pair, or $(\mathbf{R}_{s,t}^{-1} \mathbf{D}_{s,t}, \mathbf{C}_{s,t})$ is a compatible pair, or $(\mathbf{R}_{s,t}, \mathbf{D}_{s,t}, \mathbf{C}_{s,t})$ is a Hadamard triple.

With (3.3), for any strictly increasing sequence of positive integers $\{\ell_k\}_{k=0}^{\infty}$ with $\ell_0 = 0$ and $\ell_1 \geq 1$, the measure μ in (1.4) and its orthogonal set Λ in (3.2) are respectively rewritten as

$$\mu = \mu_{\ell_s} * \nu_{\ell_s} \circ \mathbf{R}_{\ell_s} \qquad \text{and} \qquad \Lambda = \bigcup_{s=1}^{\infty} \Lambda_{\ell_s},$$

where

$$\begin{aligned}
\mu_{\ell_s} &= \delta_{\mathbf{R}_{\ell_1}^{-1} \mathbf{D}_{1,\ell_0}} * \delta_{\mathbf{R}_{\ell_1}^{-1} \mathbf{R}_{\ell_2,\ell_1}^{-1} \mathbf{D}_{2,\ell_1}} * \delta_{\mathbf{R}_{\ell_2}^{-1} \mathbf{R}_{\ell_3,\ell_2}^{-1} \mathbf{D}_{3,\ell_2}} * \cdots * \delta_{\mathbf{R}_{\ell_{s-1}}^{-1} \mathbf{R}_{\ell_s,\ell_{s-1}}^{-1} \mathbf{D}_{\ell_s,\ell_{s-1}}}, \\
\nu_{\ell_s} &= \delta_{R_{\ell_s+1}^{-1} D_{\ell_s+1}} * \delta_{R_{\ell_s+1}^{-1} R_{\ell_s+2}^{-1} D_{\ell_s+2}} * \delta_{R_{\ell_s+1}^{-1} R_{\ell_s+2}^{-1} R_{\ell_s+3}^{-1} D_{\ell_s+3}} * \cdots ,
\end{aligned}$$

and

$$\Lambda_{\ell_s} = \mathbf{C}_{\ell_1,\ell_0} + \mathbf{R}^*_{\ell_1}\mathbf{C}_{\ell_2,\ell_1} + \mathbf{R}^*_{\ell_2}\mathbf{C}_{\ell_3,\ell_2} + \cdots + \mathbf{R}^*_{\ell_{s-1}}\mathbf{C}_{\ell_s,\ell_{s-1}}.$$

Step I. *To construct a new orthogonal set* Γ *(see (3.8)) for* μ, *instead of* Λ *in* (3.2).
To do this, we first claim that for each $s > t \geq 0$, one has

$$(\mathbf{R}^*_{s,t})^{-1}\mathbf{C}_{s,t} \subseteq B(\kappa),$$

where $B(\kappa)$ is written as in (1.7). This is equivalent to that for any $\mathbf{c}_{s,t} \in \mathbf{C}_{s,t}$, one has

$$\|(\mathbf{R}^*_{s,t})^{-1}\mathbf{c}_{s,t}\|_2 \leq \frac{\sqrt{n}\kappa}{2(\kappa-1)}. \tag{3.4}$$

For the proof of (3.4), we first notice that each $\mathbf{c}_{s,t} \in \mathbf{C}_{s,t}$ can be written as

$$\mathbf{c}_{s,t} = c_{t+1} + R^*_{t+1}c_{t+2} + R^*_{t+1}R^*_{t+2}c_{t+3} + \cdots + R^*_{t+1}R^*_{t+2}\ldots R^*_{s-1}c_s,$$

where $c_j \in C_j$ for each $j = t+1, t+2, \ldots, s$. With an easy calculation, we get, from (3.1), that

$$\begin{aligned}(\mathbf{R}^*_{s,t})^{-1}\mathbf{c}_{s,t} &= (\mathbf{R}^*_{s,t+1})^{-1}(R^*_{t+1})^{-1}c_{t+1} + (\mathbf{R}^*_{s,t+2})^{-1}(R^*_{t+2})^{-1}c_{t+2} + \cdots + (\mathbf{R}^*_{s,s-1})^{-1}c_s \\ &\in \left((\mathbf{R}^*_{s,t+1})^{-1} + (\mathbf{R}^*_{s,t+2})^{-1} + \cdots + I_n\right)\left[-\frac{1}{2}, \frac{1}{2}\right]^n.\end{aligned} \tag{3.5}$$

Remembering that (e.g., see [33, pp. 414]) the usual Euclidean norm or matrix norm of a nonsingular square matrix $R^{-1} \in GL(n, \mathbb{R})$ satisfies that

$$\|R^{-1}\|_2 := \sup_{\|x\|_2=1} \|R^{-1}x\|_2 = \frac{1}{\min \sigma(R)}.$$

Thus, the assumption (1.6) implies that

$$\sup_{k\geq 1} \|(R^*_k)^{-1}\|_2 = \sup_{k\geq 1} \|(R_k^{-1})^*\|_2 = \sup_{k\geq 1} \|R_k^{-1}\|_2 = \kappa^{-1} < 1, \tag{3.6}$$

and hence it follows from (3.5) and (3.6), and $\|A \circ B\|_2 \leq \|A\|_2\|B\|_2$ $(A, B \in GL(n, \mathbb{Z}))$ that

$$\|(\mathbf{R}^*_{s,t})^{-1}\mathbf{c}_{s,t}\|_2 \leq (\kappa^{-(s-t-1)} + \kappa^{-(s-t-2)} + \cdots + 1)\frac{\sqrt{n}}{2} \leq \frac{\sqrt{n}\kappa}{2(\kappa-1)},$$

the desired result (3.4) holds.

Next, we are going to construct a discrete set Γ (see (3.8) below) by using (3.4) and the "*equi-positivity*" of the family $\{\nu_{k_j}\}$. More precisely, since the family $\{\nu_{k_j}\}$ is equi-positive on the 2-norm closed ball $B(\kappa)$, there are positive numbers ϵ, δ such that for each $x \in B(\kappa)$ there is an $h(x, \nu_{k_j}) \in \mathbb{Z}^n$ such that

$$|\widehat{\nu_{k_j}}(x + h(x, \nu_{k_j}) + y)| > \epsilon \tag{3.7}$$

for all $y \in \mathbb{R}^n$ with $\|y\|_2 < \delta$, and $h(x, \nu_{k_j}) = \mathbf{0}$ if $x = \mathbf{0}$.

By applying (3.4) and (3.7) recursively, we can choose a suitable *special* strictly increasing subsequence $\{\ell_s\}$ of $\{k_j\}$ and construct a sequence of digit sets $\widetilde{\mathbf{C}_{\ell_s,\ell_{s-1}}}$, where $\ell_0 = 0$ and $\ell_1 \geq 1$, such that the discrete sets

$$\begin{aligned} \Gamma_{\ell_s} &= \widetilde{\mathbf{C}_{\ell_1,\ell_0}} + \mathbf{R}^*_{\ell_1}\widetilde{\mathbf{C}_{\ell_2,\ell_1}} + \mathbf{R}^*_{\ell_2}\widetilde{\mathbf{C}_{\ell_3,\ell_2}} + \cdots + \mathbf{R}^*_{\ell_{s-1}}\widetilde{\mathbf{C}_{\ell_s,\ell_{s-1}}} \\ \Gamma &= \bigcup_{s=1}^{\infty} \Gamma_{\ell_s}, \qquad \text{where} \quad \mathbf{0} \in \Gamma_{\ell_s} \subseteq \Gamma_{\ell_{s+1}} \quad \text{for all } s \geq 1 \end{aligned} \tag{3.8}$$

satisfying the following properties:

(i) The integer ℓ_1 is an arbitrary positive integer in the sequence $\{k_j\}$ such that

$$\widetilde{\mathbf{C}_{\ell_1,\ell_0}} = \left\{\mathbf{c}_{\ell_1,\ell_0} + \mathbf{R}^*_{\ell_1,\ell_0} h(\mathbf{c}_{\ell_1,\ell_0}, \nu_{\ell_1}) : \ \mathbf{c}_{\ell_1,\ell_0} \in \mathbf{C}_{\ell_1,\ell_0},\ h(\mathbf{c}_{\ell_1,\ell_0}, \nu_{\ell_1}) \in \mathbb{Z}^n\right\},$$

where the integer vector $h(\mathbf{c}_{\ell_1,\ell_0}, \nu_{\ell_1}) \in \mathbb{Z}^n$ is chosen such that

$$\left|\widehat{\nu_{\ell_1}}\left(\left(\mathbf{R}^*_{\ell_1,\ell_0}\right)^{-1} \mathbf{c}_{\ell_1,\ell_0} + h(\mathbf{c}_{\ell_1,\ell_0}, \nu_{\ell_1})\right)\right| > \epsilon$$

for all $\mathbf{c}_{\ell_1,\ell_0} \in \mathbf{C}_{\ell_1,\ell_0}$. In particular, we choose $h(\mathbf{c}_{\ell_1,\ell_0}, \nu_{\ell_1}) = \mathbf{0}$ if $\mathbf{c}_{\ell_1,\ell_0} = \mathbf{0} \in \mathbf{C}_{\ell_1,\ell_0}$. This is guaranteed by the "*equi-positivity*" of the family $\{\nu_{k_j}\}$ and the fact that $\left(\mathbf{R}^*_{\ell_1,\ell_0}\right)^{-1} \mathbf{c}_{\ell_1,\ell_0} \in B(\kappa)$ by (3.4).

(ii) For each $s \geq 1$ and the constructed ℓ_s, one can choose $\ell_{s+1} > \ell_s$ and define

$$\begin{aligned} \widetilde{\mathbf{C}_{\ell_{s+1},\ell_s}} = \Big\{&\mathbf{c}_{\ell_{s+1},\ell_s} + \mathbf{R}^*_{\ell_{s+1},\ell_s} h(\mathbf{c}_{\ell_{s+1},\ell_s}, \\ &\nu_{\ell_{s+1}}) : \ \mathbf{c}_{\ell_{s+1},\ell_s} \in \mathbf{C}_{\ell_{s+1},\ell_s},\ h(\mathbf{c}_{\ell_{s+1},\ell_s}, \nu_{\ell_{s+1}}) \in \mathbb{Z}^n\Big\}, \end{aligned} \tag{3.9}$$

where the integer vector $h(\mathbf{c}_{\ell_{s+1},\ell_s}, \nu_{\ell_{s+1}}) \in \mathbb{Z}^n$ is chosen so that

$$\left|\widehat{\nu_{\ell_{s+1}}}\left(\left(\mathbf{R}^*_{\ell_{s+1}}\right)^{-1} \gamma_s + \left(\mathbf{R}^*_{\ell_{s+1},\ell_s}\right)^{-1} \mathbf{c}_{\ell_{s+1},\ell_s} + h(\mathbf{c}_{\ell_{s+1},\ell_s}, \nu_{\ell_{s+1}})\right)\right| > \epsilon, \tag{3.10}$$

and

$$\left\|\left(\mathbf{R}^*_{\ell_{s+1}}\right)^{-1} \gamma_s\right\|_2 < \frac{\delta}{2}, \tag{3.11}$$

for all $\gamma_s \in \Gamma_{\ell_s}$. In particular, we choose $h(\mathbf{c}_{\ell_{s+1},\ell_s}, \nu_{\ell_{s+1}}) = \mathbf{0}$ if $\mathbf{c}_{\ell_{s+1},\ell_s} = \mathbf{0} \in \mathbf{C}_{\ell_{s+1},\ell_s}$.

The above constructions are reasonable because the finiteness of the cardinality of the set Γ_{ℓ_s} means that we can choose $\ell_{s+1} > \ell_s$ large enough such that (3.11) holds, and then the claim

$$\left(\mathbf{R}^*_{\ell_{s+1},\ell_s}\right)^{-1} \mathbf{c}_{\ell_{s+1},\ell_s} \in B(\kappa)$$

as in (3.4) and the equi-positivity of $\{\nu_{k_j}\}$ guarantees (3.10) holds.

Finally, we show that the set Γ in (3.8) forms an orthogonal set of μ as in (1.4). In fact, since (3.9) means that $\mathbf{C}_{s,t} \equiv \widetilde{\mathbf{C}_{\ell_s,\ell_{s-1}}} \pmod{\mathbf{R}^*_{\ell_s,\ell_{s-1}}}$, it follows from Lemma 2.2 (iii), (iv) that all $(\mathbf{R}^{-1}_{\ell_s,\ell_{s-1}}\mathbf{D}_{\ell_s,\ell_{s-1}}, \widetilde{\mathbf{C}_{\ell_s,\ell_{s-1}}})$ are compatible pairs. By Lemma 2.2(iv), $(\mu_{\ell_s}, \Gamma_{\ell_s})$ is a spectral pair, it follows from Lemma 2.1 that

$$\sum_{\gamma\in\Gamma_{\ell_s}} |\widehat{\mu_{\ell_s}}(\xi + \gamma_{\ell_s})|^2 = 1 \qquad \text{for all } \xi \in \mathbb{R}^n. \tag{3.12}$$

Thus, the orthogonality of exponential functions $E(\Gamma_{\ell_s})$ in the space $L^2(\mu_{\ell_s})$ means that

$$\Gamma_{\ell_s} - \Gamma_{\ell_s} \subseteq \mathcal{Z}(\widehat{\mu_{\ell_s}}) \cup \{0\} \qquad \text{for all } s \geq 1.$$

Notice that for each $s \geq 1$, we get that $\mu = \mu_{\ell_s} * \nu$ for some probability measure ν. It follows that $\widehat{\mu} = \widehat{\mu_{\ell_s}} \cdot \widehat{\nu}$, and hence $\mathcal{Z}(\widehat{\mu_{\ell_s}}) \subseteq \mathcal{Z}(\widehat{\mu})$. Since Γ_{ℓ_s} is an increasing set as in s, it follows from (3.8) that

$$\Gamma - \Gamma \subseteq \mathcal{Z}(\widehat{\mu}) \cup \{0\},$$

i.e., Γ is an orthogonal set of μ. By Bessel's inequality,

$$Q_{\mu,\Gamma}(\xi) := \sum_{\gamma\in\Gamma} |\widehat{\mu}(\xi + \gamma)|^2 \leq 1 \qquad \text{for all } \xi \in \mathbb{R}^n. \tag{3.13}$$

Step II. *To show that* Γ *in* (3.8) *is a spectrum of* μ *as in* (1.4).

By Lemma 2.3, it suffices to show that $Q_{\mu,\Gamma}(\xi) = 1$ for all $\xi \in \mathbb{R}^n$. For this, we fix $\xi \in \mathbb{R}^n$ and define

$$f_{\ell_s}(\gamma) = \begin{cases} |\widehat{\mu_{\ell_s}}(\xi + \gamma)|^2, & \gamma \in \Gamma_{\ell_s}; \\ 0, & \text{others}, \end{cases} \quad \text{and} \quad f(\gamma) = \begin{cases} |\widehat{\mu}(\xi + \gamma)|^2, & \gamma \in \Gamma; \\ 0, & \text{others}. \end{cases} \tag{3.14}$$

Since $\mu = \mu_{\ell_s} * (\nu_{\ell_s} \circ \mathbf{R}_{\ell_s})$ for all $s \geq 1$, it follows that

$$\widehat{\mu}(\cdot) = \widehat{\mu_{\ell_s}}(\cdot)\widehat{\nu_{\ell_s}}\left((\mathbf{R}^*_{\ell_s})^{-1}\cdot\right), \tag{3.15}$$

which yields, from (3.14) and $\widehat{\nu_{\ell_s}}(\mathbf{0}) = 1$ $(s \geq 1)$, that

$$\lim_{s\to\infty} f_{\ell_s}(\gamma) = f(\gamma). \tag{3.16}$$

On the other hand, by (3.8) and (3.9), each $\gamma_{s+1} \in \Gamma_{\ell_{s+1}}$ is in the form

$$\gamma_{s+1} = \gamma_s + \mathbf{R}^*_{\ell_s}\left(\mathbf{c}_{\ell_{s+1},\ell_s} + \mathbf{R}^*_{\ell_{s+1},\ell_s} h(\mathbf{c}_{\ell_{s+1},\ell_s}, \nu_{\ell_{s+1}})\right)$$

for some $\gamma_s \in \Gamma_{\ell_s}$, $\mathbf{c}_{\ell_{s+1},\ell_s} \in \mathbf{C}_{\ell_{s+1},\ell_s}$ and $h(\mathbf{c}_{\ell_{s+1},\ell_s}, \nu_{\ell_{s+1}}) \in \mathbb{Z}^n$, By (3.15), one has that

$$\begin{aligned}
\widehat{\mu}(\xi + \gamma_{s+1}) &= \widehat{\mu_{\ell_{s+1}}}(\xi + \gamma_{s+1})\widehat{\nu_{\ell_{s+1}}}\left(\left(\mathbf{R}^*_{\ell_{s+1}}\right)^{-1}(\xi + \gamma_{s+1})\right) \\
&= \widehat{\mu_{\ell_{s+1}}}(\xi + \gamma_{s+1})\widehat{\nu_{\ell_{s+1}}}\left(\left(\mathbf{R}^*_{\ell_{s+1}}\right)^{-1}(\xi + \gamma_s) + \left(\mathbf{R}^*_{\ell_{s+1},\ell_s}\right)^{-1}\mathbf{c}_{\ell_{s+1},\ell_s}\right. \\
&\quad \left. + h(\mathbf{c}_{\ell_{s+1},\ell_s}, \nu_{\ell_{s+1}})\right).
\end{aligned} \tag{3.17}$$

By (1.6), one knows that there is an $s_0 \in \mathbb{N}$ (depending on ξ) such that $s \geq s_0$ implies that

$$\left\|\left(\mathbf{R}^*_{\ell_{s+1}}\right)^{-1}(\xi)\right\|_2 < \frac{\delta}{2}.$$

Combining this with (3.10) and (3.11) and $\left(\mathbf{R}^*_{\ell_{s+1},\ell_s}\right)^{-1}\mathbf{c}_{\ell_{s+1},\ell_s} \in B(\kappa)$, the equi-positivity of measures $\{\nu_k\}$ on the compact set $B(\kappa)$ (see (3.10)) implies that the equality (3.17) becomes

$$|\widehat{\mu}(\xi + \gamma_{s+1})| \geq \epsilon \left|\widehat{\mu_{\ell_{s+1}}}(\xi + \gamma_{s+1})\right| \qquad \text{for all } s \geq s_0,$$

which, together with (3.14), yields that

$$f(\gamma) \geq \epsilon^2 f_s(\gamma) \qquad \text{for all } \gamma \in \Gamma.$$

Notice that $\sum_{\gamma\in\Gamma} f(\gamma) \leq 1$ by (3.13). Then $\sum_{\gamma\in\Gamma} f_s(\gamma) \leq \epsilon^{-2}$. Now, applying Lebesgue's dominated convergence theorem, we get, from (3.12) and (3.16), that

$$Q_{\mu,\Gamma}(\xi) = \sum_{\gamma\in\Gamma} |\widehat{\mu}(\xi + \gamma)|^2 = 1 \qquad \text{for all } \xi \in \mathbb{R}^n.$$

By Lemma 2.3(ii), the measure μ in (1.4) is a spectral measure.

This finishes the proof of Theorem 1.4. □

4 Proof of Theorems 1.6 and 1.7

Recall that (see [10]) the *integral periodic zero set* of a Borel probability measure μ on $\mathbb{R}^n$ is defined by

$$\mathbf{Z}(\mu) = \{\xi \in \mathbb{R}^n : \widehat{\mu}(\xi + k) = 0 \text{ for all } k \in \mathbb{Z}^n\}.$$

4.1 Proof of Theorem 1.6

The proof of Theorem 1.6 relies on Theorem 1.4 and Lemmas 4.1 and 4.2.

The following lemma does not require the finiteness of the choices of admissible pairs (R_k, D_k).

Lemma 4.1 *Suppose $\{(R_k, D_k) : k \in \mathbb{N}\}$ is a sequence of admissible pairs on $\mathbb{R}^n$ such that the measure μ as in (1.4) exists, and the measures $\nu_k (k \geq 1)$ as in (1.5) exist, and (1.6) holds (i.e., $\kappa > 1$). Assume that, for each $k \in \mathbb{N}$, one has*

(i) $\mathbb{Z}[D_k - d_k] = \mathbb{Z}^n$ *for some* $d_k \in D_k$.
(ii) *the set* $\mathcal{Z}(\widehat{\delta_{D_k}}) \cap \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ *is a uniformly discrete set.*

Then the associated measure $\mu = \mu(R_k, D_k)$ in (1.4) satisfies $\mathbf{Z}(\mu) = \emptyset$. Moreover, $\mathbf{Z}(\nu_k) = \emptyset$ for all $k \in \mathbb{N}$.

Proof Suppose on the contrary that $\mathbf{Z}(\mu) \neq \emptyset$. Thus, there is a $\xi_0 \in \mathbb{R}^n \setminus \mathbb{Z}^n$ such that

$$\widehat{\mu}(\xi_0 + \mathbf{m}) = 0 \qquad \text{for all } \mathbf{m} \in \mathbb{Z}^n. \tag{4.1}$$

As we do in the proof of Theorem 1.4, in what follows, we still without loss of generality assume that

$$\mathbf{0} \in C_k \subseteq R_k^* \left[-\frac{1}{2}, \frac{1}{2}\right]^n \cap \mathbb{Z}^n$$

such that $(R_k^{-1} D_k, C_k)$ forms a compatible pair for each $k \geq 1$. By Lemma 2.1,

$$\sum_{c_k \in C_k} \left|\widehat{\delta_{D_k}}(\tau_{k,c_k}(\xi))\right|^2 \equiv 1 \qquad \text{for all } \xi \in \mathbb{R}^n, \tag{4.2}$$

where $\{\tau_{k,c_k} : c_k \in C_k\}$ denotes the functions system:

$$\tau_{k,c_k}(\xi) = (R_k^*)^{-1}(\xi + c_k). \tag{4.3}$$

Fix $\xi_0 \in \mathbb{R}^n \setminus \mathbb{Z}^n$. We will yield contradictions by proving the following three claims.
Claim 1. We show that $\mathbf{Z}(\nu_k) \neq \emptyset$ for all $k \geq 1$ if $\mathbf{Z}(\mu) \neq \emptyset$.

In fact, since the Fourier-Stieltjes transform $\widehat{\mu}$ of the measure $\mu := \mu(R_k, D_k)$ in (1.4) is

$$\widehat{\mu}(\xi) = \int e^{-2\pi i\langle \xi, x\rangle} d\mu(x) = \prod_{k=1}^{\infty} \widehat{\delta}_{D_k}((\mathbf{R}_k^*)^{-1}\xi) \qquad (\xi \in \mathbb{R}^n),$$

where $\mathbf{R}_k$ is defined as in (1.3), it follows from (4.1) and the $\mathbb{Z}^n$-periodicity of $\widehat{\delta_{D_j}}$ that, for each $k \in \mathbb{N}$,

$$\begin{aligned} 0 &= \widehat{\mu}\left(\xi_0 + c_1 + R_1^* c_2 + \mathbf{R}_2^* c_3 + \cdots + \mathbf{R}_{k-1}^* c_k + \mathbf{R}_k^* \mathbf{m}\right) \\ &= \widehat{\delta_{D_1}}\left((R_1^*)^{-1}(\xi_0 + c_1)\right) \cdot \widehat{\delta_{D_2}}\left((\mathbf{R}_2^*)^{-1}(\xi_0 + c_1 + R_2^* c_2)\right) \\ &\quad \cdots \widehat{\delta_{D_k}}\left((\mathbf{R}_k^*)^{-1}(\xi_0 + c_1 + R_1^* c_2 + \cdots + R_{k-1}^* c_k) + \mathbf{m}\right) \\ &\quad \widehat{\nu_k}\left((\mathbf{R}_k^*)^{-1}(\xi_0 + c_1 + R_1^* c_2 + \cdots + R_{k-1}^* c_k) + \mathbf{m}\right) \\ &= \widehat{\delta_{D_1}}\left(\tau_{1,c_1}(\xi_0)\right) \cdot \widehat{\delta_{D_2}}\left(\tau_{2,c_2} \circ \tau_{1,c_1}(\xi_0)\right) \cdots \widehat{\delta_{D_k}}\left(\tau_{k,c_k} \circ \cdots \circ \tau_{2,c_2} \circ \tau_{1,c_1}(\xi_0)\right) \\ &\quad \cdot \widehat{\nu_k}\left(\tau_{k,c_k} \circ \cdots \circ \tau_{2,c_2} \circ \tau_{1,c_1}(\xi_0) + \mathbf{m}\right) \qquad \text{(by (4.3))} \end{aligned} \tag{4.4}$$

for each $c_j \in C_j \subseteq \mathbb{Z}^n$, where $j = 1, 2, \ldots, k$.

Note that, by applying (4.2) several times, we get that, for each $j = 1, \ldots, k$, there is at least one $c_j \in C_j$ such that

$$\begin{aligned} &\widehat{\delta_{D_1}}\left(\tau_{1,c_1}(\xi_0)\right) \neq 0,\ \widehat{\delta_{D_2}}\left(\tau_{2,c_2} \circ \tau_{1,c_1}(\xi_0)\right) \\ &\quad \neq 0,\ \ldots,\ \widehat{\delta_{D_k}}\left(\tau_{k,c_k} \circ \cdots \circ \tau_{2,c_2} \circ \tau_{1,c_1}(\xi_0)\right) \neq 0, \end{aligned}$$

which means, together with (4.4), that

$$\widehat{\nu_k}\left(\tau_{k,c_k} \circ \cdots \circ \tau_{2,c_2} \circ \tau_{1,c_1}(\xi_0) + \mathbf{m}\right) = 0 \qquad \text{for all } \mathbf{m} \in \mathbb{Z}^n,$$

i.e., $\tau_{k,c_k} \circ \cdots \circ \tau_{2,c_2} \circ \tau_{1,c_1}(\xi_0) \in \mathbf{Z}(\nu_k)$ for each $k \in \mathbb{N}$, and hence the Claim 1 is proved.

From the arguments in Claim 1, it makes sense to put $Y_0 = \{\xi_0\}$, and for each $k \in \mathbb{N}$ we put

$$Y_k = \left\{\tau_{k,c_k} \circ \cdots \circ \tau_{2,c_2} \circ \tau_{1,c_1}(\xi_0) \in \mathbf{Z}(\nu_k) : \widehat{\delta_{D_j}}\left(\tau_{j,c_j} \circ \cdots \circ \tau_{1,c_1}(\xi_0)\right) \neq 0,\ j \geq 1\right\}. \tag{4.5}$$

Claim 2. We show that $\#Y_k \leq \#Y_{k+1}$ for each $k \in \mathbb{N}$.

In fact, this can be proved by showing that all the elements of Y_k are distinct. Precisely, if there are two distinct sequences $(c_1, c_2, \ldots, c_k)$ and $(c_1', c_2', \ldots, c_k')$ in

$\prod_{j=1}^{k} C_j$ such that

$$\tau_{k,c_k} \circ \tau_{k-1,c_{k-1}} \circ \cdots \circ \tau_{1,c_1}(\xi_0) = \tau_{k,c'_k} \circ \tau_{k-1,c'_{k-1}} \circ \cdots \circ \tau_{1,c'_1}(\xi_0),$$

it follows from (4.3) that

$$c_1 + R_1^* c_2 + \cdots + R_1^* \cdots R_{k-1}^* c_k = c'_1 + R_1^* c'_2 + \cdots + R_1^* \cdots R_{k-1}^* c'_k.$$

Setting $i_0 = \min\{i : c_i \neq c'_i\}$, we get that $c_{i_0} \equiv c'_{i_0} \pmod{R_{i_0}^*}$, where $c_{i_0}, c'_{i_0} \in C_{i_0}$, which is a contradiction to Lemma 2.2(ii) since $(R_{i_0}^{-1} D_{i_0}, C_{i_0})$ is a compatible pair. Therefore, the Claim 2 is proved.

Claim 3. We finally show that $\#Y_k = \#Y_{k+1}$ for k large enough.

In fact, each element $\tau_{k,c_k} \circ \cdots \circ \tau_{2,c_2} \circ \tau_{1,c_1}(\xi_0)$ of Y_k in (4.5) satisfies that

$$\begin{aligned}
&\|\tau_{k,c_k} \circ \cdots \circ \tau_{2,c_2} \circ \tau_{1,c_1}(\xi_0)\|_2 \\
&= \left\| (\mathbf{R}_k^*)^{-1}(\xi_0 + c_1 + R_1^* c_2 + \cdots + R_{k-1}^* c_k) \right\|_2 \\
&\le \left\| (\mathbf{R}_k^*)^{-1} \xi_0 \right\|_2 + \left\| (\mathbf{R}_k^*)^{-1}(c_1 + R_1^* c_2 + \cdots + R_{k-1}^* c_k) \right\|_2 \\
&\le \left\| (\mathbf{R}_k^*)^{-1} \right\| \cdot \|\xi_0\|_2 + \frac{\sqrt{n}\kappa}{2(\kappa - 1)} \qquad \text{(by (3.4))} \\
&\le \kappa^{-k} \|\xi_0\|_2 + \frac{\sqrt{n}\kappa}{2(\kappa - 1)} \qquad \text{(by (3.6))}.
\end{aligned}$$

As $\kappa > 1$ in (1.6), there is a large $k_0 \in \mathbb{N}$ such that $k \ge k_0$ implies that $\kappa^{-k} \|\xi_0\|_2 < 1$, therefore

$$Y_k \subseteq B_0^1(\kappa) := \left\{ x \in \mathbb{R}^n : \|x\|_2 \le \frac{\sqrt{n}\kappa}{2(\kappa - 1)} + 1 \right\} \qquad \text{for all } k \ge k_0. \tag{4.6}$$

Let

$$N := \max \# \left\{ \mathcal{Z}(\widehat{\delta_{D_k}}) \cap B_0^1(\kappa) : k \ge 1 \right\} \tag{4.7}$$

and

$$\delta_0 = \min \left\{ \|x\|_2 : x \in \mathcal{Z}(\widehat{\delta_{D_k}}) \cap B_0^1(\kappa), k \ge 1 \right\}. \tag{4.8}$$

Notice that each function $\mathcal{Z}(\widehat{\delta_{D_k}})$ is $\mathbb{Z}^n$-periodic and $\widehat{\delta_{D_k}}(\mathbf{0}) = 1$, and

$$\left[-\frac{1}{2}, \frac{1}{2} \right]^n \subseteq B(\kappa) \subseteq B_0^1(\kappa).$$

Thus the assumption (ii) implies that N is a finite positive integer and $\delta_0 > 0$.

As $\delta_0 > 0$, there is a smallest positive integer $j_0 \in \mathbb{N}$ such that $j > j_0$ implies that

$$\kappa^j \delta_0 > \frac{\sqrt{n}\kappa}{2(\kappa - 1)} + 1$$

and hence the requirement $\kappa > 1$ in (1.6) and (4.6), (4.8) yield that, for all $j > j_0$,

$$\mathcal{Z}\left(\widehat{\delta}_{\mathbf{R}_{k+j,k}^{-1} D_{k+j}}\right) \cap B_0^1(\kappa) = \mathbf{R}_{k+j,k}^* \mathcal{Z}(\widehat{\delta}_{D_{k+j}}) \cap B_0^1(\kappa) = \emptyset \qquad (\forall k \geq 1).$$

Consequently,

$$\mathcal{Z}(\widehat{\nu_k}) \cap B_0^1(\kappa) \subseteq \bigcup_{j=1}^{j_0} \left(\mathbf{R}_{k+j,k}^* \mathcal{Z}(\widehat{\delta}_{D_{k+j}}) \cap B_0^1(\kappa)\right),$$

and whence, for each $k \in \mathbb{N}$, one has that

$$\begin{aligned} &\#\left(\mathcal{Z}(\widehat{\nu_k}) \cap B_0^1(\kappa)\right) \\ &\leq \sum_{j=1}^{j_0} \#\left(\mathbf{R}_{k+j,k}^* \mathcal{Z}(\widehat{\delta}_{D_{k+j}}) \cap B_0^1(\kappa)\right) \leq \sum_{j=1}^{j_0} \#\left(\mathcal{Z}(\widehat{\delta}_{D_{k+j}}) \cap B_0^1(\kappa)\right) \leq N j_0, \end{aligned}$$

where the last inequality follows from (4.7). Furthermore, since $Y_k \subseteq \mathbf{Z}(\nu_k) \subseteq \mathcal{Z}(\widehat{\nu_k})$ for all $k \in \mathbb{N}$, it follows from (4.6) that

$$\#Y_k = \#\left(\mathbf{Z}(\nu_k) \cap Y_k\right) \leq \#\left(\mathcal{Z}(\widehat{\nu_k}) \cap B_0^1(\kappa)\right) \leq N j_0 \qquad \text{for all } k \geq k_0.$$

Combining this with Claim 2, we get the desired result of Claim 3.

By Claim 3, we might assume that there is a $k_0 \in \mathbb{N}$ large enough such that

$$\#Y_k = \#Y_{k_0} \qquad \text{for } k \geq k_0.$$

Fixing $k \geq k_0$. By the definition of Y_k in (4.5), each $\xi \in Y_k$ corresponds to a unique $c_{k+1} \in C_{k+1}$ such that

$$\widehat{\delta}_{D_{k+1}}(\tau_{k+1,c_{k+1}}(\xi)) \neq 0,$$

where $\tau_{k+1,c_{k+1}}(\xi) \in Y_{k+1} \subseteq \mathbf{Z}(\nu_{k+1})$, and therefore it follows from (4.2) that $|\widehat{\delta}_{D_{k+1}}(\tau_{k+1,c_{k+1}}(\xi))| = 1$, i.e.,

$$1 = \left|\frac{1}{\#D_{k+1}} \sum_{d \in D_{k+1}} e^{2\pi i \langle d, \tau_{k+1,c_{k+1}}(\xi)\rangle}\right| = \frac{1}{\#D_{k+1}} \left|\sum_{d \in D_{k+1}} e^{2\pi i \langle d - d_{k+1}, \tau_{k+1,c_{k+1}}(\xi)\rangle}\right|, \tag{4.9}$$

where $d_{k+1} \in D_{k+1}$ is chosen so that $\mathbb{Z}[D_{k+1} - d_{k+1}] = \mathbb{Z}^n$ satisfying the assumption (i). Since $\mathbf{0} = (0, \ldots, 0)^* \in D_{k+1} - d_{k+1}$, by using the triangle inequality, we get that all terms in the sum of (4.9) must be equal to 1. Thus it follows from (4.9) that

$$\langle d - d_{k+1}, \tau_{k+1,c_{k+1}}(\xi)\rangle \in \mathbb{Z} \text{ for all } d \in D_{k+1}. \tag{4.10}$$

Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^*$ be the standard basis of $\mathbb{R}^n$, where $i = 1, 2, \ldots, n$. Whence, it follows from the assumption $\mathbb{Z}[D_{k+1} - d_{k+1}] = \mathbb{Z}^n$ and (4.10) that

$$\langle e_i, \tau_{k+1,c_{k+1}}(\xi)\rangle \in \mathbb{Z} \qquad \text{for all } i = 1, 2, \ldots, n.$$

This implies that

$$\tau_{k+1,c_{k+1}}(\xi) \in \mathbb{Z}^n,$$

and therefore

$$\widehat{\nu_{k+1}}(\tau_{k+1,c_{k+1}}(\xi) - \tau_{k+1,c_{k+1}}(\xi)) = \widehat{\nu_{k+1}}(\mathbf{0}) = 1.$$

This yields that $\tau_{k+1,c_{k+1}}(\xi) \notin \mathbf{Z}(\nu_{k+1})$. It is a contradiction. Thus, we get that $\mathbf{Z}(\mu) = \emptyset$.

Next, by repeating the arguments above to each measure ν_k, we similarly get that $\mathbf{Z}(\nu_k) = \emptyset$. This completes the proof of Lemma 4.1. □

Lemma 4.2 *Given a sequence of admissible pairs $\{(R_k, D_k) : k \in \mathbb{N}\}$ which are chosen from a finite set of admissible pairs on $\mathbb{R}^n$ such that (1.6) and the assumptions (i) and (ii) of Lemma 4.1 hold. Let ν_k ($k \geq 1$) be as in (1.5). Then there is a subsequence $\{\nu_{k_j}\}_{j=1}^{\infty}$ of $\{\nu_k\}_{k=1}^{\infty}$ such that the measures $\{\nu_{k_j}\}$ converges to an infinite convolution measure*

$$\mu(\widetilde{R}_k, \widetilde{D}_k) = \delta_{\widetilde{R}_1^{-1}\widetilde{D}_1} * \delta_{(\widetilde{R}_2\widetilde{R}_1)^{-1}\widetilde{D}_2} * \cdots * \delta_{(\widetilde{R}_k \ldots \widetilde{R}_1)^{-1}\widetilde{D}_k} * \cdots$$

in the weak-star topology, where $\{(\widetilde{R}_k, \widetilde{D}_k)\}_{k=1}^{\infty}$ is a subsequence of $\{(R_k, D_k)\}_{k=1}^{\infty}$. As a consequence, one has $\mathbf{Z}(\mu(\widetilde{R}_k, \widetilde{D}_k)) = \emptyset$ by Lemma 4.1.

Proof For each $j \in \mathbb{N}$, we write $\nu_k = \nu_{k,j} * \omega_{k,j}$, where

$$\nu_{k,j} = \delta_{R_{k+1}^{-1} D_{k+1}} * \delta_{R_{k+1}^{-1} R_{k+2}^{-1} D_{k+2}} \cdots * \delta_{R_{k+1}^{-1} \cdots R_{k+j}^{-1} D_{k+j}},$$

and

$$\omega_{k,j} = \delta_{R_{k+1}^{-1} \cdots * R_{k+j+1}^{-1} D_{k+j+1}} * \delta_{R_{k+1}^{-1} \cdots * R_{k+j+2}^{-1} D_{k+j+2}} * \cdots . \tag{4.11}$$

By setting

$$M := \sup_{k \in \mathbb{N}} \max\{\|\mathbf{d}_k\|_2 : \mathbf{d}_k \in D_k\},$$

it follows from (1.6), (3.6) and (4.11) that the support of $\omega_{k,j}$ is contained in the 2-normed closed ball

$$B_j(\kappa) := \left\{ \xi \in \mathbb{R}^n : \|\xi\|_2 \le \frac{\kappa M}{\kappa^j(\kappa - 1)} \right\}. \tag{4.12}$$

Since the sequence $\{(R_k, D_k)\}_{k=1}^{\infty}$ contains only finitely many distinct admissible pairs, we can apply Bolzano-Weierstrass theorem to each level of infinite convolutions ν_k. By the famous Cantor's diagonal process arguments, we can choose a subsequence $\{\nu_{k_j}\}_{j=1}^{\infty}$ of the measures $\{\nu_k\}$ such that

$$\nu_{k_j,j} = \nu_{k_{j+1},j} = \nu_{k_{j+2},j} = \cdots \qquad \text{for all } j \in \mathbb{N}.$$

Let $(\widetilde{R}_j, \widetilde{D}_j) = (R_{k_j+j}, D_{k_j+j})$ $(j \in \mathbb{N})$ and define the measure

$$\mu(\widetilde{R}_k, \widetilde{D}_k) := \delta_{\widetilde{R}_1^{-1}\widetilde{D}_1} * \delta_{\widetilde{R}_1^{-1}\widetilde{R}_2^{-1}\widetilde{D}_2} * \cdots * \delta_{\widetilde{R}_1^{-1}\ldots\widetilde{R}_j^{-1}\widetilde{D}_j} * \cdots .$$

Notice that, for each $k_j \in \mathbb{N}$, the measure $\mu(\widetilde{R}_k, \widetilde{D}_k)$ can be decomposed as

$$\mu(\widetilde{R}_k, \widetilde{D}_k) = \nu_{k_j,j} * \widetilde{\omega_{k_j,j}} \tag{4.13}$$

for some infinite convolution measure $\widetilde{\omega_{k_j,j}}$, whose support is also contained in the 2-norm closed ball $B_j(\kappa)$ as in (4.12).

Therefore, for any $\xi \in \mathbb{R}^n$ and $j \ge 1$, we have

$$\begin{aligned}
\widehat{\nu_{k_j}}(\xi) &= \widehat{\nu_{k_j,j}}(\xi)\widehat{\omega_{k_j,j}}(\xi) = \widehat{\nu_{k_j,j}}(\xi) \int_{\mathbb{R}^n} e^{-2\pi i\langle \xi, x\rangle} d\omega_{k_j,j}(x) \\
&= \widehat{\nu_{k_j,j}}(\xi) \left(\int_{B_j(\kappa)} \cos(2\pi\langle \xi, x\rangle) d\omega_{k_j,j}(x) - i \int_{B_j(\kappa)} \sin(2\pi\langle \xi, x\rangle) d\omega_{k_j,j}(x) \right) \\
&= \widehat{\nu_{k_j,j}}(\xi)(\cos 2\pi\langle \xi, x_1\rangle - i \sin 2\pi\langle \xi, x_2\rangle)
\end{aligned} \tag{4.14}$$

for some $x_1, x_2 \in B_j(\kappa)$. Here, the last equality follows from the intermediate value property of continuous functions $\cos(2\pi\langle \xi, x\rangle)$ and $\sin(2\pi\langle \xi, x_2\rangle)$ on the compact set $B_j(\kappa)$.

Likewise (4.14), one can obtain from (4.13) that

$$\widehat{\mu}(\widetilde{R}_k, \widetilde{D}_k) = \widehat{\nu_{k_j,j}}(\xi)(\cos 2\pi\langle \xi, x_3\rangle - i \sin 2\pi\langle \xi, x_4\rangle) \tag{4.15}$$

for some $x_3, x_4 \in B_j(\kappa)$. Whence, we get from (4.14) and (4.15) that

$$\begin{aligned}
&|\widehat{\nu}_{k_j}(\xi) - \widehat{\mu}(\widetilde{R}_k, \widetilde{D}_k)| \\
&\quad \le |\cos 2\pi\langle \xi, x_1\rangle - \cos 2\pi\langle \xi, x_3\rangle| + |\sin 2\pi\langle \xi, x_2\rangle - \sin 2\pi\langle \xi, x_4\rangle| \\
&\quad \le |e^{2\pi\langle \xi, x_1\rangle} - e^{2\pi\langle \xi, x_2\rangle}| + |e^{2\pi\langle \xi, x_3\rangle} - e^{2\pi\langle \xi, x_4\rangle}| \\
&\quad \le 2 \sup_{x,y\in B_j(\kappa)} |1 - e^{2\pi\langle \xi, y-x\rangle}| \to 0 \qquad \text{as } j \to \infty.
\end{aligned}$$

Because $\widehat{\mu}(\widetilde{R}_k, \widetilde{D}_k)$ is a continuous function, by [24, Chapter VI, Lemma 2.3], we get that the measures ν_{k_j} converge to $\mu(\widetilde{R}_k, \widetilde{D}_k)$ in the weak-star topology. Furthermore, in the proof of Lemma 4.1, if we replace μ by $\mu(\widetilde{R}_k, \widetilde{D}_k)$, one similarly yield that $\mathbf{Z}(\mu(\widetilde{R}_k, \widetilde{D}_k)) = \emptyset$. This finishes the proof of Lemma 4.2. □

In fact, combining the the arguments of Lemmas 4.2 and 4.1, we actually get the following fact: "*Under the assumptions of Lemma 4.2, any subsequence* $\{\nu_{k_j}\}_{j=1}^{\infty}$ *of* $\{\nu_k\}_{k=1}^{\infty}$ *contains a subsequence converging to an infinite convolution*

$$\mu(\widetilde{\widetilde{R}}_k, \widetilde{\widetilde{D}}_k) := \delta_{\widetilde{\widetilde{R}}_1^{-1}\widetilde{\widetilde{D}}_1} * \delta_{(\widetilde{\widetilde{R}}_2\widetilde{\widetilde{R}}_1)^{-1}\widetilde{\widetilde{D}}_2} * \cdots * \delta_{(\widetilde{\widetilde{R}}_k\ldots\widetilde{\widetilde{R}}_1)^{-1}\widetilde{\widetilde{D}}_k} * \cdots$$

in the weak-star topology, where $\{(\widetilde{\widetilde{R}}_k, \widetilde{\widetilde{D}}_k)\}_{k=1}^{\infty}$ *is a subsequence of* $\{(R_k, D_k)\}_{k=1}^{\infty}$. *Moreover,* $\mathbf{Z}(\mu(\widetilde{\widetilde{R}}_k, \widetilde{\widetilde{D}}_k)) = \emptyset$.

Now we have all ingredients for the proof of Theorem 1.6.

Proof of Theorem 1.6 Let ν_k be as in (1.5). By Theorem 1.4, it suffices to show that the sequence of measures $\{\nu_k\}$ is equi-positive on the 2-norm closed ball $B(\kappa)$, that is, there exist constants $\epsilon > 0$, $\delta > 0$ such that for each $x \in B(\kappa)$, there exists an $h(x, \nu_k) \in \mathbb{Z}^n$ such that

$$\inf_{k\ge1} \inf_{\|y\|_2<\delta} |\widehat{\nu}_k(y + x + h(x, \nu_k))| > \epsilon. \tag{4.16}$$

In order to prove (4.16), we need to show the following two claims.

Claim 1. For any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{k\ge0} \sup_{\|x-y\|_2<\delta} |\widehat{\nu}_k(x) - \widehat{\nu}_k(y)| < \epsilon.$$

Indeed, recall that $M := \sup_{k\in\mathbb{N}} \max\{\|\mathbf{d}_k\|_2 : \mathbf{d}_k \in D_k\}$ by (4.7), it follows from (1.5) and (1.6) that

$$\bigcup_{k\ge1} \operatorname{supp}(\nu_k) \subset \left\{\xi \in \mathbb{R}^n : \|\xi\|_2 \le \frac{M}{\kappa - 1}\right\} =: B.$$

Therefore, for any $x, y \in \mathbb{R}^n$, we have

$$|\widehat{\nu_k}(x) - \widehat{\nu_k}(y)| \leq \int_B |e^{-2\pi i\langle x,t\rangle} - e^{-2\pi i\langle y,t\rangle}| d\nu_k(t) \\ \leq \sup_{t\in B} |1 - e^{-2\pi i\langle y-x,t\rangle}| \leq 2\pi \|y - x\|_2 \cdot \sup_{t\in B} \|t\|_2 \tag{4.17}$$

Thus, for any $\epsilon > 0$, it suffice to choose $\delta := \frac{\epsilon}{2\pi \sup_{t\in B} \|t\|_2}$. This completes the proof of claim 1.

Claim 2. For each $x \in B(\kappa)$, there exists a $\epsilon_x > 0$ and an integral vector $h(x, \nu_k) \in \mathbb{Z}^n$ such that

$$\inf_{k\geq 1} |\widehat{\nu_k}(x + h(x, \nu_k))| > \epsilon_x.$$

Indeed, suppose on the contrary that there is a $x_0 \in B(\kappa)$ and a strictly increasing sequence k_j such that

$$|\widehat{\nu_{k_j}}(x_0 + \mathbf{m})| \leq \frac{1}{j} \qquad \text{for all } \mathbf{m} \in \mathbb{Z}^n.$$

By the remark following Lemma 4.2, the sequence of measures $\{\nu_{k_j}\}_{j=1}^{\infty}$ contains a subsequence converging to an infinite convolution

$$\mu(\widetilde{\widetilde{R}}_k, \widetilde{\widetilde{D}}_k) := \delta_{\widetilde{\widetilde{R}}_1^{-1}\widetilde{\widetilde{D}}_1} * \delta_{(\widetilde{\widetilde{R}}_2\widetilde{\widetilde{R}}_1)^{-1}\widetilde{\widetilde{D}}_2} * \cdots * \delta_{(\widetilde{\widetilde{R}}_k\ldots\widetilde{\widetilde{R}}_1)^{-1}\widetilde{\widetilde{D}}_k} * \cdots$$

in the weak-star topology, where $\{(\widetilde{\widetilde{R}}_k, \widetilde{\widetilde{D}}_k)\}_{k=1}^{\infty}$ is a subsequence of $\{(R_k, D_k)\}_{k=1}^{\infty}$. Therefore,

$$\widehat{\mu}(\widetilde{\widetilde{R}}_k, \widetilde{\widetilde{D}}_k)(x_0 + \mathbf{m}) = \lim_{j\to\infty} \widehat{\nu_{k_j}}(x_0 + \mathbf{m}) = 0$$

holds for all $\mathbf{m} \in \mathbb{Z}^n$, that is, $x_0 \in \mathbf{Z}(\mu(\widetilde{\widetilde{R}}_k, \widetilde{\widetilde{D}}_k))$. However, the arguments of Lemma 4.1 implies that $\mathbf{Z}(\mu(\widetilde{\widetilde{R}}_k, \widetilde{\widetilde{D}}_k)) = \emptyset$. This contradiction yields claim 2.

By Claim 1 and Claim 2, for each $x \in B(\kappa)$, we choose a positive numbers $\epsilon_x > 0$, an integral vector $h(x, \nu_k) \in \mathbb{Z}^n$ and a δ_x small enough such that the following inequalities

$$\sup_{k\geq 1} \sup_{\|x-y\|_2<\delta_x} |\widehat{\nu_k}(x) - \widehat{\nu_k}(y)| < \tfrac{\epsilon_x}{3} \tag{4.18}$$

and

$$\inf_{k\geq 1} |\widehat{\nu_k}(\xi + h(x, \nu_k))| > \tfrac{2\epsilon_x}{3} \text{ for all } \xi \in B(x, \delta_x) \tag{4.19}$$

hold at the same time, where $B(x, \delta_x)$ denotes the 2-normed closed ball with center x and radius δ_x.

As $B(\kappa)$ is compact, we can find $x_1, x_2, \dots, x_m \in B(\kappa)$ so that

$$B(\kappa) \subset \bigcup_{i=1}^{m} B(x_i, \delta_{x_i}).$$

Letting

$$\epsilon := \min_i \left\{ \frac{\epsilon_{x_i}}{3} \right\} \qquad \text{and} \qquad \delta < \min\{\delta_{x_1}, \dots, \delta_{x_m}\}.$$

Thus, by (4.19), for any $x \in B(\kappa)$ there is a $i_0 \in \{1, 2, \cdots, m\}$ and an integral vector $h(x, \nu_k) := h(x_{i_0}, \nu_k)$ such that $x \in B(x_{i_0}, \delta_{x_{i_0}})$ and

$$|\widehat{\nu_k}(x + h(x, \nu_k))| \geq \frac{2\epsilon_{x_{i_0}}}{3}.$$

Consequently, from trigonometric inequality and (4.18), we get that, for any $||y|| < \delta$,

$$\begin{aligned} \inf_{k\geq 1} |\widehat{\nu_k}(y + x + h(x, \nu_k))| &\geq \inf_{k\geq 1} |\widehat{\nu_k}(x + h(x, \nu_k))| \\ &\quad - \sup_{k\geq 1} |\widehat{\nu_k}(y + x + h(x, \nu_k)) - \widehat{\nu_k}(x + h(x, \nu_k))| \\ &> \frac{\epsilon_{x_{i_0}}}{3} \geq \epsilon. \end{aligned}$$

We obtain the desired result (4.16) and the proof of Theorem 1.6 is complete. □

Remark 4.3 In the proof of Theorem 1.6, we actually establish a fact that if $\{\nu_k\}$ converges to ν in the weak-star topology with $\mathbf{Z}(\nu) = \emptyset$, then $\{\nu_k\}$ is equi-positive. As the referee mentioned to us, this fact in one-dimensional case was obtained in [30, Theorem 1.1], and was also generalized to higher dimensional case in [31, Theorem 1.2].

4.2 Proof of Theorem 1.7 and Corollary 1.8

The proof of Theorem 1.7 needs the following lemma.

Lemma 4.4 *Under the assumptions of Theorem 1.7, we get that*

$$\inf_{k\geq 1} \inf_{\xi\in\left[-\frac{1}{2},\frac{1}{2}\right]^n} |\widehat{\nu_k}(\xi)| \geq c \tag{4.20}$$

for some positive constant c, independent of k, where ν_k is written as in (1.5).

Consequently, $\mathbf{Z}(\nu_k) = \emptyset$ for all $k \in \mathbb{N}$. Moreover, if ν is a limit point (in weak-star topology) of the sequence $\{\nu_k\}$, one also has that $\mathbf{Z}(\nu) = \emptyset$.

Proof Fixing $\xi \in \left[-\frac{1}{2}, \frac{1}{2}\right]^n$. By the hypothesis, we set

$$M_1 := \sup_{k\in\mathbb{N}} \sup_{\mathbf{d}_k\in D_k} \{\|R_k^{-1}\mathbf{d}_k\|_2\} < \infty. \tag{4.21}$$

Notice that, for each $j > k$ and all $d_j \in D_j$, one has

$$\begin{aligned}\|\mathbf{R}_{j,k}^{-1}d_j\|_2 &= \|R_{k+1}^{-1}R_{k+2}^{-1}\dots R_{j-1}^{-1}R_j^{-1}d_j\|_2 \le \|R_{k+1}^{-1}R_{k+2}^{-1}\dots R_{j-1}^{-1}\| \\ &\cdot\|R_j^{-1}d_j\|_2 \le \kappa^{-(j-k-1)}M_1.\end{aligned}$$

Thus, if we choose the smallest positive integer $j_0(k) > k$ such that

$$\kappa^{-(j_0(k)-k-1)}M_1 < \frac{1}{4\sqrt{n}},$$

then $j > j_0(k)$ implies that

$$2\pi|\langle\mathbf{R}_{j,k}^{-1}d_j, \xi\rangle| \le 2\pi\|\mathbf{R}_{j,k}^{-1}d_j\|_2 \cdot \|\xi\|_2 \le \frac{\pi}{4\kappa^{j-j_0(k)}}. \tag{4.22}$$

Moreover, it follows from (1.6) that for all $j > k \ge 1$, one has

$$\|\mathbf{R}_{j,k}^{*-1}\xi\|_2 = \|R_j^{*-1}R_{j-1}^{*-1}\dots R_{k+1}^{*-1}\xi\|_2 \le \kappa^{-(j-k)}\sqrt{n}/2 \le \frac{\sqrt{n}}{2\kappa},$$

that is, $\mathbf{R}_{j,k}^{*-1}\xi \in B(\mathbf{0}, \frac{\sqrt{n}}{2\kappa})$ for all $j > k$. Note that $j_0(k)-k$ is a constant for all $k \in \mathbb{N}$. Whence,

$$\begin{aligned}|\widehat{\nu_k}(\xi)| &= \prod_{j=k+1}^{j_0(k)} |\widehat{\delta_{D_j}}(\mathbf{R}_{j,k}^{*-1}\xi)| \prod_{j=j_0(k)+1}^{\infty} |\widehat{\delta_{D_j}}(\mathbf{R}_{j,k}^{*-1}\xi)| \\ &\ge \eta^{j_0(k)-k}\prod_{j=j_0(k)+1}^{\infty}\left|\frac{1}{\#D_j}\sum_{d\in D_j}\cos 2\pi\langle\mathbf{R}_{j,k}^{-1}d, \xi\rangle\right| \qquad \text{(by (1.9))} \\ &\ge \eta^{j_0(k)-k}\prod_{j=1}^{\infty}\cos\frac{\pi}{4\kappa^j} \qquad \text{(by (4.22))} \\ &= \eta^{j_0(k)-k}\prod_{j=1}^{\infty}\left(1-\sin^2\frac{\pi}{4\kappa^j}\right)^{\frac{1}{2}} \\ &\ge \eta^{j_0(k)-k}\prod_{j=1}^{\infty}\left(1-\frac{\pi^2}{16\kappa^{2j}}\right)^{\frac{1}{2}} =: c > 0.\end{aligned}$$

This completes the proof of (4.20), and therefore the first statement is proved.

Next, suppose that (4.20) holds, but there is a $k_0 \geq 1$ such that $\mathbf{Z}(\nu_{k_0}) \neq \emptyset$. Then there is an $\xi_0 \in \mathbb{R}^n$ such that

$$\widehat{\nu_{k_0}}(\xi_0 + \mathbf{m}) = 0 \qquad \text{for all } \mathbf{m} \in \mathbb{Z}^n.$$

This clearly yields that there is an $\xi_1 \in \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ such that $\widehat{\nu_{k_0}}(\xi_1) = 0$, contradicting to (4.20). Thus, we get that $\mathbf{Z}(\nu_k) = \emptyset$ for all $k \in \mathbb{N}$.

Moreover, if there is a subsequence $\{\nu_{k_j}\}_{j=1}^{\infty}$ of $\{\nu_k\}$ such that ν_{k_j} converges to a measure ν in the weak-star topology, we get, from (4.20), that

$$|\widehat{\nu}(\xi)| = \lim_{j\to\infty} |\widehat{\nu_{k_j}}(\xi)| \geq c$$

for all $\xi \in \left[-\frac{1}{2}, \frac{1}{2}\right]^n$, which yields that $\mathbf{Z}(\nu) = \emptyset$.

The proof of Lemma 4.4 is finished. □

The idea of the proof of Theorem 1.7 is essentially identical to that of Theorem 1.6.

Proof of Theorem 1.7 Let ν_k be as in (1.5). By Theorem 1.4, it suffices to show that the sequence of measures $\{\nu_k\}$ is equi-positive on the 2-norm closed ball $B(\kappa)$ as in (1.7).

As in the proof of Theorem 1.6, it also requires the following two claims.

Claim 1. for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{k\geq 0} \sup_{\|x-y\|_2<\delta} |\widehat{\nu_k}(x) - \widehat{\nu_k}(y)| < \epsilon.$$

Claim 2. For each $x \in B(\kappa)$, there exists a $\epsilon_x > 0$ and an integral vector $h(x, \nu_k) \in \mathbb{Z}^n$ such that

$$\inf_{k\geq 1} |\widehat{\nu_k}(x + h(x, \nu_k))| > \epsilon_x.$$

For the claim 1, it follows from (4.21), (1.5) and (1.6) that

$$\bigcup_{k\geq 1} \text{supp } (\nu_k) \subset \left\{\xi \in \mathbb{R}^n : \|\xi\|_2 \leq \frac{\kappa M_1}{\kappa - 1}\right\}.$$

Then, a similar argument as in (4.17) will give the desired result of Claim 1.

For the claim 2, we assume on the contrary that there is a $x_0 \in B(\kappa)$ and a strictly increasing sequence k_j such that

$$|\widehat{\nu_{k_j}}(x_0 + \mathbf{m})| \leq \frac{1}{j} \qquad \text{for all } \mathbf{m} \in \mathbb{Z}^n.$$

By applying Banach-Alaoglu's theorem (cf. [15, Theorem 5.18]), we can extract from the sequence $\{\nu_{k_j}\}$ a subsequence tending to a measure ν in the weak-star topology.

Here, in order to avoid complicating the notations, we assume that this subsequence is $\{\nu_{k_j}\}$. Whence,

$$\widehat{\nu}(x_0 + \mathbf{m}) = \lim_{j\to\infty} \widehat{\nu_{k_j}}(x_0 + \mathbf{m}) = 0$$

holds for all $\mathbf{m} \in \mathbb{Z}^n$, that is, $x_0 \in \mathbf{Z}(\nu)$, or $\mathbf{Z}(\nu) \neq \emptyset$. However, $\mathbf{Z}(\nu) = \emptyset$ by Lemma 4.4. This contradiction yields claim 2.

The proof left is the same to that of Theorem 1.6, so we omit the details of the proof. This completes the proof of Theorem 1.7. □

Proof of Corollary 1.8 Since $\kappa > 1$ and the assumptions of Corollary 1.8 mean that (1.9) holds. Thus, we get Corollary 1.8 by Theorem 1.7. □

4.3 Proof of Example 1.10

We end this section by giving the proof of Example 1.10.

Proof of Example 1.10 For the first statement, by Theorem 1.7 or Corollary 1.8, it suffices to show that the zero set $\mathcal{Z}(\widehat{\delta_D})$ has a positive distance to the 2-normed closed ball

$$B(\mathbf{0}, (\sqrt{2}\kappa)^{-1}) := \{\xi \in \mathbb{R}^n : \|\xi\|_2 \le (\sqrt{2}\kappa)^{-1}\}.$$

In fact, this is clearly true since $\kappa > \sqrt{2}$ implies that $(\sqrt{2}\kappa)^{-1} < 1/2$, and the zero set of the function

$$\widehat{\delta_D}(\xi_1, \xi_2) = \frac{1}{4}\left(1 + e^{2\pi i \xi_1}\right)\left(1 + e^{2\pi i \xi_2}\right)$$

is the union of two family of parallel lines

$$\mathcal{Z}(\widehat{\delta_D}) := \left\{\begin{pmatrix} \frac{1}{2} + n \\ y \end{pmatrix} : n \in \mathbb{Z}, y \in \mathbb{R}\right\} \bigcup \left\{\begin{pmatrix} x \\ \frac{1}{2} + n \end{pmatrix} : x \in \mathbb{R}, n \in \mathbb{Z}\right\},$$

which means that $\inf\{\|\xi\|_2 : \xi \in \mathcal{Z}(\widehat{\delta_D})\} \ge \frac{1}{2}$.

For the second statement, with easy computations, one gets that

$$\min \sigma(R(1)) = \min \sigma(R(2)) = 2\sqrt{3 - \sqrt{5}} \approx 1.748 > \sqrt{2},$$
$$\min \sigma(R(3)) = \min \sigma(R(4)) = \sqrt{2(9 - \sqrt{17})} \approx 3.123 > \sqrt{2}.$$

By setting

$$C(1) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right\}, \qquad C(2) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\},$$

$$C(3)=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}\right\},\qquad C(4)=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}2\\0\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix}\right\},$$

one check that, for each $k=1,2,3,4$, the matrix

$$H_{R(k)^{-1}D,C}:=\left[e^{2\pi i\langle R(k)^{-1}d,c\rangle}\right]_{d\in D,c\in C}=\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}$$

is unitary, i.e., all $(R(k),D)$ are admissible pairs for all $k=1,2,3,4$. Therefore, the second statement follows from the first. This completes the proof of Example 1.10. □

5 Proof of Theorem 1.9

The proof of the sufficient part of Theorem 1.9(i), (ii) depends on Lemma 5.1, whose one-dimensional version has been obtained by Deng and Li in [9, Lemma 3.1].

Lemma 5.1 *Let $A\in GL(n,\mathbb{Z})$. For for each $k\geq 0$, we let $R_k\in GL(n,\mathbb{Z})$ be $n\times n$ matrices and $D_k\subseteq\mathbb{Z}^n$ be finite digit set, and set*

$$\nu_k:=\delta_{R_{k+1}^{-1}D_{k+1}}*\delta_{R_{k+1}^{-1}R_{k+2}^{-1}D_{k+2}}*\cdots,$$
$$\nu_k^{(A)}:=\delta_{A^{-1}D_{k+1}}*\delta_{A^{-1}R_{k+2}^{-1}D_{k+2}}*\delta_{A^{-1}R_{k+2}^{-1}R_{k+3}^{-1}D_{k+3}}*\cdots.$$

Then (ν_k,Λ) is a spectral pair if and only if $(\nu_k^{(A)},A^*(R_{k+1}^*)^{-1}\Lambda)$ is a spectral pair.

Proof Since $\langle x,y\rangle=\langle R_{k+1}x,(R_{k+1}^*)^{-1}y\rangle$ and $\langle B^{-1}x,y\rangle=\langle x,(B^*)^{-1}y\rangle$ hold for all $B\in GL(n,\mathbb{R})$ and for all $x,y\in\mathbb{R}^n$, it follows that, for any $\xi\in\mathbb{R}^n$,

$$\begin{aligned}\widehat{\nu}_k(\xi)&=\prod_{j=1}^{\infty}\widehat{\delta}_{R_{k+1}^{-1}\cdots R_{k+j}^{-1}D_{k+j}}(\xi)\\&=\widehat{\delta_{D_{k+1}}}((A^*)^{-1}A^*(R_{k+1}^*)^{-1}\xi)\cdot\prod_{j=2}^{\infty}\widehat{\delta}_{R_{k+2}^{-1}\cdots R_{k+j}^{-1}D_{k+j}}((A^*)^{-1}A^*(R_{k+1}^*)^{-1}\xi)\\&=\widehat{\delta_{A^{-1}D_{k+1}}}(A^*(R_{k+1}^*)^{-1}\xi)\prod_{j=2}^{\infty}\widehat{\delta}_{A^{-1}R_{k+2}^{-1}\cdots R_{k+j}^{-1}D_{k+j}}(A^*(R_{k+1}^*)^{-1}\xi)\\&=\widehat{\nu_k^{(A)}}(A^*(R_{k+1}^*)^{-1}\xi).\end{aligned}$$

Thus, if we define $\Gamma=A^*(R_{k+1}^*)^{-1}\Lambda$, we get that

$$\sum_{\lambda\in\Lambda}|\widehat{\nu_k}(\xi+\lambda)|^2=\sum_{\lambda\in\Lambda}|\widehat{\nu_k^{(A)}}(A^*(R_{k+1}^*)^{-1}(\xi+\lambda))|^2$$

$$= \sum_{\gamma\in\Gamma} |\widehat{\nu_k^{(A)}}(A^*(R_{k+1}^*)^{-1}\xi + \gamma)|^2.$$

This yields the desired result by Lemma 2.3. □

We first give the proof of the sufficient part of the statements (i), (ii) of Theorem 1.9.

Proof of Theorem 1.9 (admissible pairs ⇒ spectral measure) Assume that (R_k, D_k) is an admissible pair for each $k \geq 2$. By Lemma 5.1, it suffices to show that the following infinite convolution

$$\widetilde{\mu} := \delta_{A^{-1}D_1} * \delta_{A^{-1}R_2^{-1}D_2} * \delta_{A^{-1}R_2^{-1}R_2^{-1}D_3} * \cdots, \tag{5.1}$$

is a spectral measure, where $A = \text{diag}[3, 3]$ if $D_1 = D^{(3)}$; and $A = \text{diag}[2, 2]$ if $D_1 = D^{(4)}$.

We will prove the sufficient part of Theorem 1.9(i) (*resp.* Theorem 1.9(ii)) by checking that all assumptions of Theorem 1.6 (*resp.* Corollary 1.8) are satisfied by the measure $\widetilde{\mu}$. Obviously, $\mathbb{Z}[D^{(3)}] = \mathbb{Z}[D^{(4)}] = \mathbb{Z}^2$, i.e., assumption (i) of Theorem 1.6 is satisfied. And (ii) of Theorem 1.6 also holds, because it follows from the fact that

$$\mathcal{Z}(\widehat{\delta_{D^{(3)}}}) = \pm\frac{1}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \mathbb{Z}^2 \tag{5.2}$$

and

$$\mathcal{Z}(\widehat{\delta_{D^{(4)}}}) = \left\{\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\right\} + \mathbb{Z}^2 \tag{5.3}$$

are both uniformly discrete set in $\mathbb{R}^2$. In particular, if $\kappa > \frac{3}{2}$, then $(\sqrt{2}\kappa)^{-1} < \frac{\sqrt{2}}{3}$, and (5.2) and (5.3) yield that

$$\inf\left\{\|\xi\|_2 : \xi \in \mathcal{Z}(\widehat{\delta_{D^{(3)}}})\right\} \geq \frac{\sqrt{2}}{3}, \qquad \inf\left\{\|\xi\|_2 : \xi \in \mathcal{Z}(\widehat{\delta_{D^{(4)}}})\right\} \geq \frac{1}{2}.$$

This means the condition of Corollary 1.8 is satisfied.

It remains to show that (A, D_1) is an admissible pair. For this, by setting

$$C^{(3)} = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}, \qquad C^{(4)} = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\},$$

we obtain, from Definition 1.2, that $(\text{diag}[3, 3], D^{(3)}, C^{(3)})$ and $(\text{diag}[2, 2], D^{(4)}, C^{(4)})$ are Hadamard triples, i.e., (A, D_1) is an admissible pair.

Overall, the measure $\widetilde{\mu}$ in (5.1) satisfies all assumptions of Theorem 1.6 (*resp.* Corollary 1.8). This finishes the proof of Theorem 1.9 for which admissible pairs imply spectral measures. □

In additional to Lemma 5.1, the following Theorem 5.2 plays a crucial role in establishing the proof of "*spectral measure ⇒ admissible pairs*" for Theorem 1.9, which of course includes the proof of the necessary part of the statements (i), (ii) for Theorem 1.9.

Theorem 5.2 *Let $\{R_k\}_{k=1}^{\infty} \subseteq GL(n, \mathbb{Z})$ be a sequence of matrices satisfies* (1.6) *(i.e., $\kappa > 1$), and let $\{D_k\}$ be a sequence of digit sets chosen from $\{D^{(3)}, D^{(4)}\}$, where*

$$D^{(3)} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ and } D^{(4)} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}.$$

If the associated measure $\mu(R_k, D_k)$ in (1.4) *is a spectral measure, then all measures ν_k written as in* (1.5) *are spectral measures.*

It is remarked here that Theorem 5.2 will be proved by induction. More precisely, we set $\nu_0 := \mu(R_k, D_k)$ and prove Theorem 5.2 by the following two cases: for each $k \geq 0$,

- if ν_k with $D_{k+1} = D^{(3)}$ is spectral, then ν_{k+1} is also spectral (Theorem 5.5(ii)).
- if ν_k with $D_{k+1} = D^{(4)}$ is spectral, then ν_{k+1} is also spectral (Theorem 5.7(ii)).

Here and after, for each nonnegative integer $k \geq 0$, the measure ν_k is written as in (1.5), i.e,

$$\nu_k = \delta_{R_{k+1}^{-1} D_{k+1}} * \delta_{R_{k+1}^{-1} R_{k+2}^{-1} D_{k+2}} * \delta_{R_{k+1}^{-1} R_{k+2}^{-1} R_{k+3}^{-1} D_{k+3}} * \cdots,$$

where $D_{k+1} = D^{(3)}$ or $D_{k+1} = D^{(4)}$.

Standard Hypothesis in the Rest of this Section We always assume that

$$A = \mathrm{diag}[6, 6] = 6I_2 \quad \text{and} \quad \mathcal{S} = \{0, 1, \ldots, 5\} \times \{0, 1, \ldots, 5\}, \tag{5.4}$$

where I_2 is the 2×2 identity matrix; and for each ν_k we define a new probability measure

$$\nu_k^{(6)} := \nu_k^{(A)} := \delta_{6^{-1} D_{k+1}} * \delta_{6^{-1} R_{k+2}^{-1} D_{k+2}} * \delta_{6^{-1} R_{k+2}^{-1} R_{k+3}^{-1} D_{k+3}} * \cdots, \tag{5.5}$$

where $D_{k+1} = D^{(3)}$ or $D_{k+1} = D^{(4)}$.

Lemma 5.3 below gives a simple characterization on the spectra of ν_k or $\nu_k^{(6)}$, which makes sense to make assumptions as in Theorems 5.5 and 5.7.

Lemma 5.3 *With notations above, the measure ν_k admits a spectrum Λ_k with $\mathbf{0} \in \Lambda_k$ if and only if the discrete set*

$$\Gamma_k := A^*(R_{k+1}^*)^{-1} \Lambda_k = 6(R_{k+1}^*)^{-1} \Lambda_k \qquad \text{with } \mathbf{0} \in \Gamma_k$$

is a spectrum for the measure $\nu_k^{(6)}$. Moreover, $\Gamma_k \subseteq \mathbb{Z}^2$, and hence Γ_k can be uniquely decomposed into the following disjoint union

$$\Gamma_k = \bigcup_{s \in \mathcal{S}} (s + 6\Gamma_s^{(k)}), \tag{5.6}$$

where $\Gamma_s^{(k)}$ *satisfies that*

$$\Gamma_s^{(k)} = \left\{ \omega \in \mathbb{Z}^2 : s + 6\omega \in \Gamma_k \right\},$$

with the interpretation that $s + 6\Gamma_s^{(k)} = \emptyset$ *if* $\Gamma_s^{(k)} = \emptyset$*. In particular, since* $\boldsymbol{0} \in \Gamma_k$*, it follows from* (5.6) *that* $\boldsymbol{0} \in \Gamma_{\boldsymbol{0}}^{(k)}$*, therefore,*

$$\Gamma_{\boldsymbol{0}}^{(k)} \neq \emptyset. \tag{5.7}$$

Proof The first statement clearly follows from Lemma 5.1. By the orthogonality of $E(\Gamma_k)$ in $L^2(\nu_k^{(6)})$, we obtain, from (5.2) and (5.3), that

$$\Gamma_k - \Gamma_k \subseteq 6\mathcal{Z}(\widehat{\delta_{D_{k+1}}}) \cup \bigcup_{j=2}^{\infty} 6R_{k+2}^* \cdots R_{k+j}^* \mathcal{Z}(\widehat{\delta_{D_{k+j}}}) \subseteq \mathbb{Z}^2.$$

Since $\mathbf{0} \in \Gamma_k$, we get that $\Gamma_k \subseteq \mathbb{Z}^2$ and hence (5.7) holds. The representation (5.6) is also clear because $|\det A| = 36$ and $\mathcal{S}$ is a standard digit set associated with A, that is, $\mathcal{S}$ is a complete set of coset representatives for $\mathbb{Z}^2/A\mathbb{Z}^2$, it follows that each $\gamma \in \Gamma_k$ corresponds to a unique $s \in \mathcal{S}$ and $\omega \in \mathbb{Z}^2$ such that $\gamma = s + A\omega = s + 6\omega$. This completes the proof of Lemma 5.3. □

The arguments below will be divided into two cases: $D_{k+1} = D^{(3)}$ and $D_{k+1} = D^{(4)}$.

5.1 The Case that $D_{k+1} = D^{(3)}$.

For the measure ν_k with $D_{k+1} = D^{(3)}$, we divide the set $\mathcal{S}$ in (5.4) into the following 12 sets.

$$\begin{aligned}
&\mathcal{S}_1^{(3)} = \{(0,0)^*, (2,4)^*, (4,2)^*\}, \ \mathcal{S}_2^{(3)} = \{(1,0)^*, (3,4)^*, (5,2)^*\}, \\
&\mathcal{S}_3^{(3)} = \{(2,0)^*, (4,4)^*, (0,2)^*\}, \\
&\mathcal{S}_4^{(3)} = \{(3,0)^*, (5,4)^*, (1,2)^*\}, \ \mathcal{S}_5^{(3)} = \{(4,0)^*, (0,4)^*, (2,2)^*\}, \\
&\mathcal{S}_6^{(3)} = \{(5,0)^*, (1,4)^*, (3,2)^*\}, \\
&\mathcal{S}_7^{(3)} = \{(0,1)^*, (2,5)^*, (4,3)^*\}, \ \mathcal{S}_8^{(3)} = \{(1,1)^*, (3,5)^*, (5,3)^*\}, \\
&\mathcal{S}_9^{(3)} = \{(2,1)^*, (4,5)^*, (0,3)^*\}, \\
&\mathcal{S}_{10}^{(3)} = \{(3,1)^*, (5,5)^*, (1,3)^*\}, \ \mathcal{S}_{11}^{(3)} = \{(4,1)^*, (0,5)^*, (2,3)^*\}, \\
&\mathcal{S}_{12}^{(3)} = \{(5,1)^*, (1,5)^*, (3,3)^*\}.
\end{aligned}$$

The basic properties of the sets $\{\mathcal{S}_j^{(3)}\}_{j=1}^{12}$ are stated as follows, which will be used to prove Theorem 5.5.

Lemma 5.4 *Let A and $\mathcal{S}$ be as in (5.4). With notations above, the following statements hold.*

(i) $\mathcal{S} = \bigcup_{j=1}^{12} \mathcal{S}_j^{(3)}$, *where the unions are pairwise disjoint.*
(ii) *For each $j = 1, \ldots, 12$, there is a unique $(j_1, j_2) \in \{0, 1, \ldots, 5\} \times \{0, 1\}$ such that*

$$\mathcal{S}_j^{(3)} \equiv \mathcal{S}_1^{(3)} + (j_1, j_2)^* \pmod{A}.$$

(iii) *If $a \in \mathcal{S}_j^{(3)}$ and $a' \in \mathcal{S}_{j'}^{(3)}$ for distinct j, j', then $a - a' \notin \mathcal{Z}(\widehat{\delta_{6^{-1}D^{(3)}}})$.*
(iv) *For each $j = 1, 2, \ldots, 12$, one has $(A^{-1}D^{(3)}, \mathcal{S}_j^{(3)})$ is a compatible pair.*

Proof The statements (i) and (ii) are obvious, and (iv) is easily checked by the definition of compatible pairs in Definition 1.2.

To prove (iii), if $a \in \mathcal{S}_j^{(3)}$ and $a' \in \mathcal{S}_{j'}^{(3)}$ with $j \neq j'$, it follows from (ii) that there exist $a_1, a_2 \in \mathcal{S}_1^{(3)}$ and distinct $(j_1, j_2), (j_1', j_2') \in \{0, 1, \ldots, 5\} \times \{0, 1\}$ such that

$$a = a_1 + (j_1, j_2)^* + 6t_1 \qquad \text{and} \qquad a' = a_2 + (j_1', j_2')^* + 6t_2$$

for some $t_1, t_2 \in \mathbb{Z}^2$. Thus,

$$(a - a')^* = (a_1 - a_2)^* + (j_1 - j_1', j_2 - j_2') + 6(t_1 - t_2)^*, \tag{5.8}$$

where

$$(j_1 - j_1', j_2 - j_2') \subseteq \pm\{0, 1, \ldots, 5\} \times \{0, \pm 1\} \setminus \{(0, 0)\}. \tag{5.9}$$

There are two possible cases:

- If $a_1 = a_2$, it follows from (5.8) and (5.9) that

$$(a - a')^* \in \pm\{0, 1, \ldots, 5\} \times \{0, \pm 1\} \setminus \{(0, 0)\} + 6\mathbb{Z}^2. \tag{5.10}$$

- If $a_1 \neq a_2$, then it is easy to see from $\mathcal{S}_1^{(3)}$ that $(a_1 - a_2)^* \in \{(4, 2)^*, (2, 4)^*\} + 6\mathbb{Z}^2$. This, together with (5.8) and (5.9), yields that

$$(a - a')^* \notin \{(2, 4)^*, (4, 2)^*\} + 6\mathbb{Z}^2. \tag{5.11}$$

Notice that (5.2) implies that $\mathcal{Z}(\widehat{\delta_{6^{-1}D^{(3)}}}) = \{(2, 4)^*, (4, 2)^*\} + 6\mathbb{Z}^2$. Therefore, by (5.10) and (5.11), we get that $(a - a')^* \notin \mathcal{Z}(\widehat{\delta_{6^{-1}D^{(3)}}})$, this proves (iii). □

In terms of Lemma 5.3, we assume that $\mathbf{0} \in \Gamma_k$ is a spectrum of $\nu_k^{(6)}$ with $D_{k+1} = D^{(3)}$. By Lemma 5.4(i) and (5.6) in Lemma 5.3, the set Γ_k can be furthermore decomposed into the following pairwise disjoint union

$$\Gamma_k = \bigcup_{j=1}^{12} \bigcup_{s_j \in \mathcal{S}_j^{(3)}} (s_j + 6\Gamma_{s_j}^{(k)}), \tag{5.12}$$

where $\Gamma_{s_j}^{(k)} = \left\{\omega \in \mathbb{Z}^2 : s_j + 6\omega \in \Gamma_k\right\}$. Also, $\Gamma_{\mathbf{0}}^{(k)} \neq \emptyset$ since $\mathbf{0} = (0,0)^* \in \Gamma_k$. Associated to each sequence $(s_j) := (s_1, s_2, \ldots, s_{12}) \in \prod_{j=1}^{12} \mathcal{S}_j^{(3)}$, we define a new discrete set

$$\Gamma_k((s_j)) := \bigcup_{j=1}^{12} \left(\frac{1}{6}s_j + \Gamma_{s_j}^{(k)}\right). \tag{5.13}$$

Theorem 5.5 *Let $\mathbf{0} \in \Gamma_k$ be a spectrum for $\nu_k^{(6)}$ with $D_{k+1} = D^{(3)}$, and write Γ_k as in (5.12). Then the following statements hold.*

(i) *For any $s_1 \in \mathcal{S}_1^{(3)}$, $\Gamma_{s_1}^{(k)} \neq \emptyset$;*
(ii) *For any sequence $(s_j) \in \prod_{j=1}^{12} \mathcal{S}_j^{(3)}$, the set $\Gamma_k((s_j))$ forms a spectrum for ν_{k+1}.*

Proof Fix $\xi \in (\mathbb{R} \setminus \mathbb{Q})^2$. Applying Lemma 2.3(ii) to the spectral pair $(\nu_k^{(6)}, \Gamma_k)$, we get that

$$1 = Q_{\nu_k^{(6)},\Gamma_k}(\xi) = \sum_{j=1}^{12} \sum_{s_j \in \mathcal{S}_j^{(3)}} \sum_{\omega \in \Gamma_{s_j}^{(k)}} \left|\widehat{\nu_k^{(6)}}(\xi + s_j + 6\omega)\right|^2.$$

By (5.5), we compute that $\widehat{\nu_k^{(6)}}(\xi) = \widehat{\delta_{D^{(3)}}}(\frac{\xi}{6})\widehat{\nu_{k+1}}(\frac{\xi}{6})$ holds for all $\xi \in \mathbb{R}^2$. This, together with the integral-periodicity of $\widehat{\delta_{D^{(3)}}}$, yields that

$$\begin{aligned}
1 &= \sum_{j=1}^{12} \sum_{s_j \in \mathcal{S}_j^{(3)}} \left|\widehat{\delta_{D^{(3)}}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j\right)\right|^2 \sum_{\omega \in \Gamma_{s_j}^{(k)}} \left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j + \omega\right)\right|^2 \\
&\leq \sum_{j=1}^{12} \sum_{s_j \in \mathcal{S}_j^{(3)}} \left|\widehat{\delta_{D^{(3)}}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j\right)\right|^2 \cdot \max\left\{\sum_{\omega \in \Gamma_{s_j}^{(k)}} \left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j + \omega\right)\right|^2 : s_j \in \mathcal{S}_j^{(3)}\right\} \\
&= \sum_{j=1}^{12} \max\left\{\sum_{\omega \in \Gamma_{s_j}^{(k)}} \left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j + \omega\right)\right|^2 : s_j \in \mathcal{S}_j^{(3)}\right\} \leq 1,
\end{aligned} \tag{5.14}$$

where the last equality uses the fact that $(A^{-1}D^{(3)}, \mathcal{S}_j^{(3)})$ is a compatible pair for each $j = 1, \ldots, 12$ by Lemma 5.4(iv), which yields, from Lemma 2.1(iv), that

$$\sum_{s_j \in \mathcal{S}_j^{(3)}} \left|\widehat{\delta_{D^{(3)}}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j\right)\right|^2 = 1 \qquad (\xi \in \mathbb{R}^2),$$

and the last inequality follows from the following claim:

Claim *Under the assumption of Theorem 5.5, for each sequence* $(s_j) \in \prod_{j=1}^{12} \mathcal{S}_j^{(3)}$*, the non-empty set* $\Gamma_k((s_j))$ *as in* (5.13) *forms an orthogonal set of* ν_{k+1}*. Consequently, by Lemma* 2.3(i)*,*

$$Q_{\nu_{k+1},\Gamma_k((s_j))}(\xi) := \sum_{\gamma\in\Gamma_k((s_j))} \left|\widehat{\nu_{k+1}}\left(\tfrac{1}{6}\xi+\gamma\right)\right|^2 = \sum_{j=1}^{12}\sum_{\omega\in\Gamma_{s_j}^{(k)}} \left|\widehat{\nu_{k+1}}\left(\xi+\tfrac{1}{6}s_j+\omega\right)\right|^2 \le 1 \qquad (\xi\in\mathbb{R}^2).$$

Proof of Claim. Fix a non-empty set $\Gamma_k((s_j))$, it suffices to show that

$$\widehat{\nu_{k+1}}(\gamma-\gamma') = 0 \qquad \text{for two distinct } \gamma,\gamma'\in\Gamma_k((s_j)). \tag{5.15}$$

First, by (5.12) and (5.13), we get that $6\gamma, 6\gamma' \in \Gamma_k$, therefore, the orthogonality of $E(\Gamma_k)$ in $L^2(\nu_k^{(6)})$ yields that

$$\widehat{\nu_k^{(6)}}(6\gamma-6\gamma') = 0.$$

Next, we show (5.15) in the following two cases.

- If $\gamma,\gamma' \in \frac{1}{6}s_j + \Gamma_{s_j}^{(k)}$ for some $1\le j\le 12$, it is clear that $\gamma-\gamma'\in\mathbb{Z}^2$, and hence the integral-periodicity of $\widehat{\delta_{D^{(3)}}}$ and $\widehat{\delta_{D^{(3)}}}(\mathbf{0})=1$ imply that

$$0 = \widehat{\nu_k^{(6)}}(6\gamma-6\gamma') = \widehat{\delta_{D^{(3)}}}(\gamma-\gamma')\widehat{\nu_{k+1}}(\gamma-\gamma') = \widehat{\nu_{k+1}}(\gamma-\gamma'). \tag{5.16}$$

- If $\gamma\in\frac{1}{6}s_j+\Gamma_{s_j}^{(k)}$ and $\gamma'\in\frac{1}{6}s_{j'}+\Gamma_{s_{j'}}^{(k)}$ for two distinct $j\neq j'$, by the integral-periodicity of $\widehat{\delta_{D^{(3)}}}$, we get that

$$0 = \widehat{\nu_k^{(6)}}(6\gamma-6\gamma') = \widehat{\delta_{6^{-1}D^{(3)}}}(6\gamma-6\gamma')\widehat{\nu_{k+1}}(\gamma-\gamma') = \widehat{\delta_{6^{-1}D^{(3)}}}(s_j-s_{j'})\widehat{\nu_{k+1}}(\gamma-\gamma').$$

  Since $\widehat{\delta_{6^{-1}D^{(3)}}}(s_j-s_{j'})\neq 0$ by Lemma 5.4(iii), we get that $\widehat{\nu_{k+1}}(\gamma-\gamma')=0$.

This finished the proof of the claim.

Moreover, by (5.2), we obtain, for all $j=1,2,\ldots,12$ and for all $s_j\in\mathcal{S}_j^{(3)}$ and for all $\xi\in(\mathbb{R}\setminus\mathbb{Q})^2$, that

$$\left|\widehat{\delta_{D^{(3)}}}\left(\frac{1}{6}\xi+\frac{1}{6}s_j\right)\right|^2 \neq 0,$$

which yields from (5.14) that

$$\sum_{\omega\in\Gamma_{s_j}^{(k)}}\left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi+\frac{1}{6}s_j+\omega\right)\right|^2=\max\left\{\sum_{\omega\in\Gamma_{s_j}^{(k)}}\left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi+\frac{1}{6}s_j+\omega\right)\right|^2: s_j\in\mathcal{S}_j^{(3)}\right\}. \tag{5.17}$$

Notice that $\mathbf{0}=(0,0)^*\in\Gamma_k$ implies that $\Gamma_{\mathbf{0}}^{(k)}\neq\emptyset$ by Lemma 5.3. Thus, the right hand of (5.17) must be positive, which clearly yields that $\Gamma_{s_1}^{(k)}\neq\emptyset$ for any $s_1\in\mathcal{S}_1^{(3)}$, i.e., (i) is proved.

Combining (5.14) with (5.17), we obtain, for each sequence $(s_j)\in\prod_{j=1}^{12}\mathcal{S}_j^{(3)}$ and for all $\xi\in(\mathbb{R}\setminus\mathbb{Q})^2$, that

$$\begin{aligned}&Q_{\nu_{k+1},\Gamma_k((s_j))}(\xi)\\&=\sum_{\gamma\in\Gamma_k((s_j))}\left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi+\gamma\right)\right|^2=\sum_{j=1}^{12}\sum_{\omega\in\Gamma_{s_j}^{(k)}}\left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi+\frac{1}{6}s_j+\omega\right)\right|^2=1.\end{aligned}$$

Furthermore, because $(\mathbb{R}\setminus\mathbb{Q})^2$ is dense in $\mathbb{R}^2$ and $Q_{\nu_{k+1},\Gamma_k((s_j))}$ is a continuous function on $\mathbb{R}^2$ by Lemma 2.3(iii), it follows that $Q_{\nu_{k+1},\Gamma_k((s_j))}(\xi)=1$ for all $\xi\in\mathbb{R}^2$. By Lemma 2.3(ii), each set $\Gamma_k((s_j))$ forms a spectrum for ν_{k+1}. The proof of Theorem 5.5 is complete. □

5.2 The Case that $D_{k+1}=D^{(4)}$.

In this case, we divide the set $\mathcal{S}$ into the following 9 subsets:

$$\begin{aligned}&\mathcal{S}_1^{(4)}=\{(0,0)^*,(3,0)^*,(0,3)^*,(3,3)^*\},\ \mathcal{S}_2^{(4)}=\{(1,0)^*,(4,0)^*,(1,3)^*,(4,3)^*\},\\&\mathcal{S}_3^{(4)}=\{(2,0)^*,(5,0)^*,(2,3)^*,(5,3)^*\},\ \mathcal{S}_4^{(4)}=\{(0,1)^*,(3,1)^*,(0,4)^*,(3,4)^*\},\\&\mathcal{S}_5^{(4)}=\{(0,2)^*,(3,2)^*,(0,5)^*,(3,5)^*\},\ \mathcal{S}_6^{(4)}=\{(1,1)^*,(4,1)^*,(1,4)^*,(4,4)^*\},\\&\mathcal{S}_7^{(4)}=\{(2,1)^*,(5,1)^*,(2,4)^*,(5,4)^*\},\ \mathcal{S}_8^{(4)}=\{(1,2)^*,(4,2)^*,(1,5)^*,(4,5)^*\},\\&\mathcal{S}_9^{(4)}=\{(2,2)^*,(5,2)^*,(2,5)^*,(5,5)^*\}.\end{aligned}$$

The basic properties of the sets $\{\mathcal{S}_j^{(4)}\}_{j=1}^{9}$ are stated as follows, which will be used to prove Theorem 5.7.

Lemma 5.6 *Let A and $\mathcal{S}$ be as in (5.4). With notations above, one has*

(i) $\mathcal{S}=\bigcup_{j=1}^{9}\mathcal{S}_j^{(4)}$ *and the union is disjoint.*

(ii) *For each $\mathcal{S}_j^{(4)}$, there exists a unique $(j_1,j_2)\in\{0,1,2\}\times\{0,1,2\}$ such that $\mathcal{S}_j^{(4)}=\mathcal{S}_1^{(4)}+(j_1,j_2)^*$.*

(iii) *For any $a\in\mathcal{S}_j^{(4)}$, $a'\in\mathcal{S}_{j'}^{(4)}$ with $\mathcal{S}_j^{(4)}\neq\mathcal{S}_{j'}^{(4)}$, we have $a-a'\notin\mathcal{Z}(\widehat{\delta_{6^{-1}D^{(4)}}})$.*

(iv) *For each* $j = 1, 2, \ldots, 9$, *one has* $(A^{-1}D^{(4)}, \mathcal{S}_j^{(4)})$ *is a compatible pair.*

Proof The statements (i), (ii) and (iv) can be easily checked.

To prove (iii), it is first noted that (5.3) implies that

$$\mathcal{Z}(\widehat{\delta_{6^{-1}D^{(4)}}}) = \{(0,3)^*, (3,0)^*, (3,3)^*\} + 6\mathbb{Z}^2. \tag{5.18}$$

Thus, if $a \in \mathcal{S}_j^{(4)}$, $a' \in \mathcal{S}_{j'}^{(4)}$ with $j \neq j'$, by (ii), there exist $a_1, a_2 \in \mathcal{S}_1^{(4)}$ and two distinct elements $(j_1, j_2), (j_1', j_2') \in \{0,1,2\} \times \{0,1,2\}$ such that

$$a = a_1 + (j_1, j_2)^* \qquad a' = a_2 + (j_1', j_2')^*. \tag{5.19}$$

It is clear that

$$(j_1 - j_1', j_2 - j_2') \subseteq \{0, \pm 1, \pm 2\} \times \{0, \pm 1, \pm 2\} \setminus \{(0,0)\}. \tag{5.20}$$

Thus,

- If $a_1 = a_2$, it follows from (5.18) that

$$(a - a')^* = (j_1 - j_1', j_2 - j_2') \notin \mathcal{Z}(\widehat{\delta_{6^{-1}D^{(4)}}}).$$

- If $a_1 \neq a_2$, then $a_1 - a_2 \in \pm\{(0,3)^*, (3,0)^*, (3,3)^*\}$, and hence one gets, from (5.18), (5.19) and (5.20), that

$$(a - a')^* \notin \mathcal{Z}(\widehat{\delta_{6^{-1}D^{(4)}}}).$$

This completes the proof of Lemma 5.6. □

In terms of Lemma 5.3, in what follows, we assume $\mathbf{0} \in \Gamma_k$ is a spectrum of $\nu_k^{(6)}$ with $D_{k+1} = D^{(4)}$. By Lemma 5.6(i) and (5.6) in Lemma 5.3, we now write the set Γ_k as a pairwise disjoint union

$$\Gamma_k = \bigcup_{j=1}^{9} \bigcup_{s_j \in \mathcal{S}_j^{(4)}} (s_j + 6\Gamma_{s_j}^{(k)}), \tag{5.21}$$

where $\Gamma_{s_j}^{(k)} = \{\omega \in \mathbb{Z}^2 : s_j + 6\omega \in \Gamma_k\}$. Associated to each sequence $(s_j) := (s_1, s_2, \ldots, s_9) \in \prod_{j=1}^{9} \mathcal{S}_j^{(4)}$, we define a new discrete set

$$\Gamma_k((s_j)) := \bigcup_{j=1}^{9} \left(\frac{1}{6}s_j + \Gamma_{s_j}^{(k)}\right). \tag{5.22}$$

It is remarked here that $\Gamma_{\mathbf{0}}^{(k)} \neq \emptyset$ since $\mathbf{0} = (0,0)^* \in \Gamma_k$.

The following Theorem 5.7 below is essentially parallel to that of Theorem 5.5. We will give its details of proof for the sake of completeness.

Theorem 5.7 *Suppose $\mathbf{0} \in \Gamma_k$ is a spectrum of $\nu_k^{(6)}$ with $D_{k+1} = D^{(4)}$, and we write Γ_k as in (5.21). Then the following statements hold.*

(i) *For any $s_1 \in \mathcal{S}_1^{(4)}$, $\Gamma_{s_1} \neq \emptyset$;*
(ii) *For any sequence $(s_j) \in \prod_{j=1}^{9} \mathcal{S}_j^{(4)}$, the set $\Gamma_k((s_j))$ forms a spectrum for ν_{k+1}.*

Proof Fix $\xi \in (\mathbb{R} \setminus \mathbb{Q})^2$. As we do in the proof of Theorem 5.5, one can similarly yield that

$$\begin{aligned} 1 = Q_{\nu_k^{(6)},\Gamma_k}(\xi) &= \sum_{j=1}^{9} \sum_{s_j \in \mathcal{S}_j^{(4)}} \sum_{\omega \in \Gamma_{s_j}^{(k)}} |\widehat{\nu_k^{(6)}}(\xi + s_j + 6\omega)|^2 \\ &= \sum_{j=1}^{9} \sum_{s_j \in \mathcal{S}_j^{(4)}} \left|\widehat{\delta_{D^{(4)}}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j\right)\right|^2 \sum_{\omega \in \Gamma_{s_j}^{(k)}} \left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j + \omega\right)\right|^2 \\ &\leq \sum_{j=1}^{9} \sum_{s_j \in \mathcal{S}_j^{(4)}} \left|\widehat{\delta_{D^{(4)}}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j\right)\right|^2 \max\left\{ \sum_{\omega \in \Gamma_{s_j}^{(k)}} \left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j + \omega\right)\right|^2 : s_j \in \mathcal{S}_j^{(4)} \right\} \\ &= \sum_{j=1}^{9} \max\left\{ \sum_{\omega \in \Gamma_{s_j}^{(k)}} \left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j + \omega\right)\right|^2 : s_j \in \mathcal{S}_j^{(4)} \right\} \leq 1. \end{aligned} \tag{5.23}$$

Here, the last equality follows from Lemma 5.6(iv) that $(A^{-1}D^{(4)}, \mathcal{S}_j^{(4)})$ is a compatible pair for each $j = 1, \ldots, 9$, which is equivalent to that (by Lemma 2.1(iv))

$$\sum_{s_j \in \mathcal{S}_j^{(4)}} \left|\widehat{\delta_{D^{(4)}}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j\right)\right|^2 = 1 \qquad (\forall \xi \in \mathbb{R}^2),$$

and the last inequality follows from the following claim:

Claim *Under the assumptions of Theorem 5.7, for each sequence $(s_j) \in \prod_{j=1}^{9} \mathcal{S}_j^{(3)}$, the non-empty set $\Gamma_k((s_j))$ as in (5.22) forms an orthogonal set of ν_{k+1}. Consequently, by Lemma 2.3(i),*

$$\begin{aligned} Q_{\nu_{k+1},\Gamma_k((s_j))}(\xi) := \sum_{\gamma \in \Gamma_k((s_j))} \left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \gamma\right)\right|^2 = \sum_{j=1}^{9} \sum_{\omega \in \Gamma_{s_j}^{(k)}} \left|\widehat{\nu_{k+1}}\left(\xi + \frac{1}{6}s_j + \omega\right)\right|^2 \\ \leq 1 \qquad (\forall \xi \in \mathbb{R}^2). \end{aligned}$$

Proof of Claim. Fixing a non-empty set $\Gamma_k((s_j))$. It is enough for us to show, for any two distinct elements $\gamma, \gamma' \in \Gamma_k((s_j))$, that

$$\widehat{\nu_{k+1}}(\gamma - \gamma') = 0.$$

Because $6\gamma, 6\gamma' \in \Gamma_k$ by (5.21) and (5.22), we prove this claim by the following two cases:

- If $\gamma, \gamma' \in \frac{1}{6}s_j + \Gamma_{s_j}^{(k)}$ for some $1 \le j \le 9$, one has that $\gamma - \gamma' \in \mathbb{Z}^2$. Now it follows from $\widehat{\delta_{D^{(4)}}}(\mathbf{0}) = 1$ and the integral-periodicity of $\widehat{\delta_{D^{(4)}}}$ that

$$0 = \widehat{\nu_k^{(6)}}(6\gamma - 6\gamma') = \widehat{\delta_{D^{(4)}}}(\gamma - \gamma')\widehat{\nu_{k+1}}(\gamma - \gamma') = \widehat{\nu_{k+1}}(\gamma - \gamma').$$

- If $\gamma \in \frac{1}{6}s_j + \Gamma_{s_j}^{(k)}$ and $\gamma' \in \frac{1}{6}s_{j'} + \Gamma_{s_{j'}}$ for some distinct j and j', it follows from the integral-periodicity of $\widehat{\delta_{D^{(4)}}}$ that

$$0 = \widehat{\nu_k^{(6)}}(6\gamma - 6\gamma') = \widehat{\delta_{6^{-1}D^{(4)}}}(6\gamma - 6\gamma')\widehat{\nu_{k+1}}(\gamma - \gamma') = \widehat{\delta_{6^{-1}D^{(4)}}}(s_j - s_{j'})\widehat{\nu_{k+1}}(\gamma - \gamma').$$

  Because $\widehat{\delta_{6^{-1}D^{(4)}}}(s_j - s_{j'}) \neq 0$ by Lemma 5.6 (iii), we conclude that $\widehat{\nu_{k+1}}(\gamma - \gamma') = 0$.

We complete the proof of the claim.

In this case, it follows from (5.3) that, for all $j = 1, 2, \ldots, 9$ and for all $s_j \in \mathcal{S}_j^{(4)}$ and for all $\xi \in (\mathbb{R} \setminus \mathbb{Q})^2$,

$$\left|\widehat{\delta_{D^{(3)}}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j\right)\right|^2 \neq 0,$$

and hence this yields, together with (5.23), that

$$\sum_{\omega \in \Gamma_{s_j}^{(k)}} |\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j + \omega\right)|^2 = \max\left\{\sum_{\omega \in \Gamma_{s_j}^{(k)}} |\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j + \omega\right)|^2 : s_j \in \mathcal{S}_j^{(4)}\right\}. \tag{5.24}$$

Since $\mathbf{0} = (0, 0)^* \in \Gamma_k$ implies that $\Gamma_{\mathbf{0}}^{(k)} \neq \emptyset$, it follows that the right hand of (5.24) is positive. Thus (5.24) implies that $\Gamma_{s_1}^{(k)} \neq \emptyset$ for any $s_1 \in \mathcal{S}_1^{(3)}$, i.e., (i) is proved.

Whence, by (5.23) and (5.24), we get, for each sequence $(s_j) \in \prod_{j=1}^{12} \mathcal{S}_j^{(3)}$, that

$$Q_{\nu_{k+1},\Gamma_k((s_j))}(\xi) = \sum_{\gamma \in \Gamma_k((s_j))} \left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \gamma\right)\right|^2 = \sum_{j=1}^{9} \sum_{\omega \in \Gamma_{s_j}^{(k)}} \left|\widehat{\nu_{k+1}}\left(\frac{1}{6}\xi + \frac{1}{6}s_j + \omega\right)\right|^2 = 1$$

holds for all $\xi \in (\mathbb{R}\setminus\mathbb{Q})^2$. Since $Q_{\nu_{k+1},\Gamma_k((s_j))}$ is continuous on $\mathbb{R}^2$ by Lemma 2.3(iii), it follows that $Q_{\nu_{k+1},\Gamma_k((s_j))}(\xi) = 1$ for all $\xi \in \mathbb{R}^2$. Therefore, each set $\Gamma_k((s_j))$ forms a spectrum for ν_{k+1} by Lemma 2.3(ii). This completes the proof of Theorem 5.7. □

Proof of Theorem 5.2 Since $\nu_0 := \mu(R_k, D_k)$ is a spectral measure, the proof of Theorem 5.2 follows from Theorem 5.5 and 5.7 by induction. □

The proof of Theorem 1.9 also needs the following Corollary 5.8, which is a direct result of Theorems 5.5(i) and 5.7(i).

Corollary 5.8 *For each nonnegative integer $k \geq 0$, assume that Λ_k is a spectrum of ν_k such that $\boldsymbol{0} \in \Lambda_k$. Then the following two statements hold.*

(i) *If $D_{k+1} = D^{(3)}$, there are integer vectors $t_1^{(k)}, t_2^{(k)} \in \mathbb{Z}^2$ such that*

$$\frac{1}{6}R_{k+1}^*(2,4)^* + R_{k+1}^T t_1^{(k)} \in \Lambda_k, \qquad \frac{1}{6}R_{k+1}^*(4,2)^* + R_{k+1}^T t_2^{(k)} \in \Lambda_k. \tag{5.25}$$

(ii) *If $D_{k+1} = D^{(4)}$, there are integer vectors $t_1^{(k)}, t_2^{(k)}, t_3^{(k)} \in \mathbb{Z}^2$ such that*

$$\frac{1}{6}R_{k+1}^*(3,0)^* + R_{k+1}^T t_1^{(k)}, \quad \frac{1}{6}R_{k+1}^*(0,3)^* + R_{k+1}^T t_2^{(k)}, \quad \frac{1}{6}R_{k+1}^*(3,3)^* + R_{k+1}^T t_3^{(k)} \in \Lambda_k. \tag{5.26}$$

Proof By Lemma 5.3, the set

$$\Gamma_k := A^*(R_{k+1}^*)^{-1}\Lambda_k = 6(R_{k+1}^*)^{-1}\Lambda_k$$

is a spectrum for $\nu_k^{(6)}$.

(i) If $D_{k+1} = D^{(3)}$, it follows from Theorem 5.5(i) that $\Gamma_{s_1}^{(k)} \neq \emptyset$ for all $s_1 \in \mathcal{S}_1^{(3)} = \{(0,0)^*, (2,4)^*, (4,2)^*\}$, and hence (5.12) implies that

$$\frac{1}{6}R_{k+1}^* \bigcup_{s_1 \in \mathcal{S}_1^{(3)}} (s_1 + 6\Gamma_{s_1}^{(k)}) \subseteq \Lambda_k.$$

This yields (5.25) holds.

(ii) If $D_{k+1} = D^{(4)}$, by Theorem 5.7(i), $\Gamma_{s_1}^{(k)} \neq \emptyset$ for all $s_1 \in \mathcal{S}_1^{(4)} = \{(0,0)^*, (3,0)^*, (0,3)^*, (3,3)^*\}$, and hence (5.21) implies that

$$\frac{1}{6}R_{k+1}^* \bigcup_{s_1 \in \mathcal{S}_1^{(4)}} (s_1 + 6\Gamma_{s_1}^{(k)}) \subseteq \Lambda_k.$$

This yields (5.26) holds.

□

Now we have all ingredients for the proof of the rest of Theorem 1.9.

Proof of Theorem 1.9 (spectral measure ⇒ admissible pairs)

Suppose $\nu_0 := \mu(R_k, D_k)$ is a spectral measure. By Theorem 5.2, for each nonnegative integer $k \geq 0$, the ν_k written as in (1.5) is a spectral measure.

Fix $k \geq 0$. We next give the proof of Theorem 1.9 by proving the following two cases.

Case I. In the case that $D_{k+1} = D^{(3)}$, we define

$$\mathcal{A}^{(3)} := \left\{(\mathbf{0}, s_2, \ldots, s_{12}) : s_j \in \mathcal{S}_j^{(3)}, 2 \leq j \leq 12\right\}.$$

By Theorem 5.5(ii), for any sequence $(\mathbf{0}, s_2, \ldots, s_{12}) \in \mathcal{A}^{(3)}$, the pairwise disjoint union set

$$\Gamma(\mathbf{0}s_2 \ldots s_{12}) := \Gamma_{\mathbf{0}} \bigcup \left(\bigcup_{j=2}^{12} \left(\frac{1}{6}s_j + \Gamma_{s_j}^{(k)}\right)\right)$$

as in (5.13) forms a spectrum of ν_{k+1}. By Lemma 5.3, it is easy to see that

$$\Gamma(\mathbf{0}s_2 \ldots s_{12}) \subseteq \mathbb{Z}^2 \bigcup \left(\bigcup_{j=2}^{12} \left(\frac{1}{6}s_j + \mathbb{Z}^2\right)\right). \tag{5.27}$$

We next show that (R_{k+2}, D_{k+2}) is an admissible pair for all $k \geq 0$.

(i) If $D_{k+2} = D^{(3)}$ in ν_{k+1}, it follows from Corollary 5.8(i) that, for any $(\mathbf{0}, s_2, \ldots, s_{12}) \in \mathcal{A}^{(3)}$, there are two integers $t_1^{(k+1)}, t_2^{(k+1)} \in \mathbb{Z}^2$ depending on $(\mathbf{0}, s_2, \ldots, s_{12})$ such that

$$\left\{\frac{1}{6}R_{k+2}^*(2,4)^* + R_{k+2}^* t_1^{(k+1)}, \frac{1}{6}R_{k+2}^*(4,2)^* + R_{k+2}^* t_2^{(k+1)}\right\} \subseteq \Gamma(\mathbf{0}s_2 \ldots s_{12}).$$

Therefore, we conclude from (5.27) and Lemma 5.4(ii) that

$$\left\{R_{k+2}^*(2,4)^*,\ R_{k+2}^*(4,2)^*\right\} \subseteq \bigcap_{(\mathbf{0},s_2,\ldots,s_{12})\in\mathcal{A}^{(3)}} \left(6\mathbb{Z}^2 \bigcup \bigcup_{j=2}^{12} \left(s_j + 6\mathbb{Z}^2\right)\right) \subseteq 6\mathbb{Z}^2,$$

that is, $\left\{R_{k+2}^*(1,2)^*,\ R_{k+2}^*(2,1)^*\right\} \subseteq 3\mathbb{Z}^2$. It follows that

$$R_{k+2}^*(1,-1)^* \in 3\mathbb{Z}^2 \quad \text{and} \quad R_{k+2}^*(-1,1)^* \in 3\mathbb{Z}^2. \tag{5.28}$$

Write the matrix R_{k+2} as

$$R_{k+2} = \begin{bmatrix} a & b \\ d & c \end{bmatrix} = 3\begin{bmatrix} a_1 & b_1 \\ d_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ d_2 & c_2 \end{bmatrix} =: 3R_1 + R_2, \tag{5.29}$$

where $a_1, b_1, c_1, d_1 \in \mathbb{Z}$ and $a_2, b_2, c_2, d_2 \in \{0, 1, -1\}$. Thus, by the residue system of modulo matrix diag[3, 3], it follows from (5.28) that the matrix R_2 in (5.29) must belong to one of the following matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \pm\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \pm\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \pm\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

By [3, Proposition 2.5] or [19, Theorem 1.8], $(R_{k+2}, D^{(3)})$ is an admissible pair for each $k \geq 0$.

(ii) If $D_{k+2} = D^{(4)}$, by Corollary 5.8(ii), we get that, for any $(\mathbf{0}, s_2, \ldots, s_{12}) \in \mathcal{A}^{(3)}$, there are two integer vectors $t_1^{(k+1)}, t_2^{(k+1)} \in \mathbb{Z}^2$ depending on $(\mathbf{0}, s_2, \ldots, s_{12})$, such that

$$\left\{\frac{1}{6}R_{k+2}^*(3,0)^* + R_{k+2}^* t_1^{(k+1)},\ \frac{1}{6}R_{k+2}^*(0,3)^* + R_{k+2}^* t_2^{(k+1)}\right\} \subseteq \Gamma(\mathbf{0}s_2\ldots s_{12}).$$

Similar to (i) above, we can get, from (5.27) and Lemma 5.4(ii), that

$$\left\{R_{k+2}^*(3,0)^*,\ R_{k+2}^*(0,3)^*\right\} \subseteq 6\mathbb{Z}^2 \quad \Leftrightarrow \quad \left\{R_{k+2}^*(1,0)^*,\ R_{k+2}^*(0,1)^*\right\} \subseteq 2\mathbb{Z}^2. \tag{5.30}$$

This clearly yields that each element of R_{k+2}^* belongs to $2\mathbb{Z}$, we denote it by $R_{k+2}^* \in M_2(2\mathbb{Z})$; in other words, if we write $R_{k+2} = \begin{bmatrix} a & b \\ d & c \end{bmatrix}$, then $a, b, c, d \in 2\mathbb{Z}$. By setting

$$C = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a/2 \\ b/2 \end{pmatrix}, \begin{pmatrix} d/2 \\ c/2 \end{pmatrix}, \begin{pmatrix} (a+d)/2 \\ (b+c)/2 \end{pmatrix}\right\} \subseteq \mathbb{Z}^2,$$

one can check that the the matrix

$$H_{R^{-1}D^{(4)},C} := [e^{2\pi i\langle R^{-1}d,c\rangle}]_{d\in D^{(4)},c\in C} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

is unitary, that is, $(R_{k+2}, D^{(4)})$ is an admissible pair for each $k \geq 0$.

Case II. In the case that $D_{k+1} = D^{(4)}$, we define

$$\mathcal{A}^{(4)} := \{(\mathbf{0}, s_2, \ldots, s_9) : s_j \in \mathcal{S}_j^{(4)}, 2 \leq j \leq 9\}.$$

It follows from Theorem 5.7(ii) that, for any sequence $(s_2, \ldots, s_9) \in \mathcal{A}^{(4)}$, the set

$$\Gamma(\mathbf{0}s_2\ldots s_9) := \Gamma_{\mathbf{0}} \bigcup \left(\bigcup_{j=2}^{9}\left(\frac{1}{6}s_j + \Gamma_{s_j}^{(k)}\right)\right) \tag{5.31}$$

is a spectrum of ν_{k+1}.

(i) If $D_{k+2} = D^{(3)}$, by Corollary 5.8(i), we obtain that, for any $(\mathbf{0}, s_2, \ldots, s_9) \in \mathcal{A}^{(4)}$, there are integer vectors $t_1^{(k+1)}, t_2^{(k+1)} \in \mathbb{Z}^2$ depending on $(\mathbf{0}, s_2, \ldots, s_9)$ such that

$$\left\{\frac{1}{6}R_{k+2}^*(2,4)^* + R_{k+2}^T t_1,\ \frac{1}{6}R_{k+2}^*(2,4)^* + R_{k+2}^T t_2\right\} \in \Gamma(\mathbf{0}s_2 \ldots s_9).$$

By (5.31) and Lemma 5.6(ii), we get that

$$\{R_{k+2}^*(2,4)^*, R_{k+2}^*(4,2)^*\} \subseteq \bigcap_{(\mathbf{0},s_2,\ldots,s_9)\in\mathcal{A}^{(4)}} \left(6\mathbb{Z}^2 \bigcup \left(\bigcup_{j=2}^{9}(s_j + 6\mathbb{Z}^2)\right)\right) \subseteq 6\mathbb{Z}^2.$$

By repeating the arguments after (5.28) and (5.29) in Case I (i), we will get that $(R_{k+2}, D^{(3)})$ is an admissible pair for each $k \geq 0$.

(ii) If $D_{k+2} = D^{(4)}$, by Corollary 5.8(ii), we get that, for any $(\mathbf{0}, s_2, \ldots, s_9) \in \mathcal{A}^{(4)}$, there are two integer vectors $t_1^{(k+1)}, t_2^{(k+1)} \in \mathbb{Z}^2$ depending on $(\mathbf{0}, s_2, \ldots, s_9)$, such that

$$\left\{\frac{1}{6}R_{k+2}^*(3,0)^* + R_{k+2}^* t_1^{(k+1)},\ \frac{1}{6}R_{k+2}^*(0,3)^* + R_{k+2}^* t_2^{(k+1)}\right\} \subseteq \Gamma(\mathbf{0}s_2 \ldots s_9).$$

By (5.31) and Lemma 5.6(ii),

$$\{R_{k+2}^*(3,0)^*, R_{k+2}^*(0,3)^*\} \subseteq \bigcap_{(\mathbf{0},s_2,\ldots,s_9)\in\mathcal{A}^{(4)}} \left(6\mathbb{Z}^2 \bigcup \left(\bigcup_{j=2}^{9}(s_j + 6\mathbb{Z}^2)\right)\right) \subseteq 6\mathbb{Z}^2.$$

By repeating the arguments after (5.30) in Case I (ii), we will get that $(R_{k+2}, D^{(4)})$ is an admissible pair for each $k \geq 0$. This completes the proof of Theorem 1.9 for the case that spectral measure implies that admissible pairs.

□

Acknowledgements The authors would like to thank the referees for his/her many valuable comments and suggestions.

Funding Yan-Song Fu is supported by the National Natural Science Foundation of China (Grant Nos. 12371090, 11801035) and the Fundamental Research Funds for the Central Universities (No. 2023ZKPYLX01). Min-Wei Tang is supported by National Natural Science Foundation of China (Grant No. 12201208) and the Hunan Provincial NSF (Grant Nos. 2023JJ40422, 2024JJ3023).

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