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On the Boundedness of Non-standard Rough Singular Integral Operators

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Abstract

Let Ω be a homogeneous function of degree zero, have vanishing moment of order one on the unit sphere $\mathbb{S}^{d-1}(d \geq 2)$. In this paper, our object of investigation is the following rough non-standard singular integral operator

$$T_{\Omega, A} f(x) = \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} \big(A(x) - A(y) - \nabla A(y)(x-y)\big) f(y) \mathrm{d}y,$$

where A is a function defined on $\mathbb{R}^d$ with derivatives of order one in $\mathrm{BMO}(\mathbb{R}^d)$. We show that $T_{\Omega, A}$ enjoys the endpoint $L \log L$ type estimate and is L^p bounded if $\Omega \in L(\log L)^2(\mathbb{S}^{d-1})$. These results essentially improve the previous known results given by Hofmann (Stud Math 109:105–131, 1994) for the L^p boundedness of $T_{\Omega, A}$ under the condition $\Omega \in L^q(\mathbb{S}^{d-1})$ $(q > 1)$, Hu and Yang (Bull Lond Math Soc 35:759–769, 2003) for the endpoint weak $L \log L$ type estimates when $\Omega \in \mathrm{Lip}_\alpha(\mathbb{S}^{d-1})$ for some $\alpha \in (0, 1]$.

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1 Introduction

This paper will be devoted to study the boundedness of certain non-standard Calderón-Zygmund operators with rough kernels. To begin with, let $d \geq 2$, $\mathbb{R}^d$ be the d-dimensional Euclidean space and $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$. Let Ω be a function of homogeneous of degree zero, $\Omega \in L^1(\mathbb{S}^{d-1})$ and satisfy the vanishing condition

$$\int_{\mathbb{S}^{d-1}} \Omega(x)x_j dx = 0, \quad j = 1, ..., d. \tag{1.1}$$

Define the non-standard rough Calderón-Zygmund operator by

$$T_{\Omega,A} f(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} \big(A(x) - A(y) - \nabla A(y)(x-y)\big) f(y)dy, \tag{1.2}$$

where A is a function on $\mathbb{R}^d$ such that $\nabla A \in \text{BMO}(\mathbb{R}^d)$, that is, $\partial_n A \in \text{BMO}(\mathbb{R}^d)$ for all n with $1 \leq n \leq d$. This class of singular integrals is of interest in Harmonic analysis. It was well-known that $T_{\Omega,A}$ is closely related to the study of Calderón commutators [1, 2]. Even for smooth kernel Ω, since $L^\infty(\mathbb{R}^d) \subsetneq \text{BMO}(\mathbb{R}^d)$, the kernel of the operator $T_{\Omega,A}$ may fail to satisfy the classical standard kernel conditions. This is the main reason why one calls them nonstandard singular integral operators.

Recall that if $\nabla A \in L^\infty(\mathbb{R}^d)$, then the $L^p(\mathbb{R}^d)$ boundedness of $T_{\Omega,A}$ follows by using the methods of rotation in the nice work of Caldéron [2], Bainshansky and Coifman [1]. Since the method of rotations doesn't work in the case of $\nabla A \in \text{BMO}(\mathbb{R}^d)$, Cohen [7] and Hu [24] obtained the $L^p(\mathbb{R}^d)$ boundedness of $T_{\Omega,A}$ with smooth kernels by means of a good-λ inequality. More precisely, if $\Omega \in \text{Lip}_\alpha(\mathbb{S}^{d-1})$ $(0 < \alpha \leq 1)$, then Cohen [7] proved that $T_{\Omega,A}$ is a bounded operator on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. Later on, the result of Cohen [7] was improved by Hofmann [19]. It was shown that $\Omega \in \cup_{q>1} L^q(\mathbb{S}^{d-1})$ is a sufficient condition for the $L^p(\mathbb{R}^d)$ boundedness of $T_{\Omega,A}$. If $\Omega \in L^\infty(\mathbb{S}^{d-1})$, Hofmann [19] demonstrated that $T_{\Omega,A}$ is bounded on $L^p(\mathbb{R}^d, w)$ for all $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^d)$, where and in what follows, $A_p(\mathbb{R}^d)$ denotes the weight function class of Muckenhoupt, see [12, Chap. 9] for properties of $A_p(\mathbb{R}^d)$.

It is quite natural to ask if one can establish weak type inequalities for $T_{\Omega,A}$ or not. Hu and Yang [23] considered the operator

$$T_a f(x) = \text{p. v.} \int_{\mathbb{R}} \frac{a(x) - a(y) - a'(y)(x-y)}{(x-y)^2} f(y)dy,$$

where a is a function on $\mathbb{R}$ such that $a' \in \mathrm{BMO}(\mathbb{R})$. Hu and Yang showed that, T_a may fail to be of weak type $(1, 1)$, which differs in this aspect from the property of the classical singular integral operators, see Remark 3 in [23, p. 762]. As a replacement of weak $(1, 1)$ boundedness, it was shown in [23] that, when $\Omega \in \mathrm{Lip}_\alpha(S^{d-1})$ with $\alpha \in (0, 1]$, $T_{\Omega,A}$ still enjoys the endpoint $L \log L$ type estimates. This, tells us that, when Ω satisfies suitable regularity condition, the endpoint estimates of $T_{\Omega,A}$ parallels to that of the commutator of Calderón-Zygmund operators with symbol in $\mathrm{BMO}(\mathbb{R}^d)$. For the endpoint estimates of the commutator of Calderón-Zygmund operators, see [22, 29] and the references therein.

Now, we recall some known results of classical singular integrals and make a comparative analysis. It was first shown by Calderón and Zygmund [3] that the singular integrals T_Ω defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(y/|y|)}{|y|^d} f(x-y)dy$$

is bounded on $L^p(\mathbb{R}^d)$ $(1 < p < \infty)$ either when Ω is an odd function and $\Omega \in L^1(\mathbb{S}^{d-1})$, or Ω is an even function with $\int_{\mathbb{S}^{d-1}} \Omega \, d\sigma = 0$ and $\Omega \in L \log L(\mathbb{S}^{d-1})$. Later on, the condition $\Omega \in L \log L(\mathbb{S}^{d-1})$ was improved to $\Omega \in H^1(\mathbb{S}^{d-1})$ by Connett [8], Ricci and Weiss [30], independently. Since then, great achievements have been made in this field. Among them are the celebrated works of the weak type $(1, 1)$ bounds given by Christ [5], Christ and Rubio de Francia [6], Hofmann [17], Seeger [31], and Tao [33]. It was shown that $\Omega \in L \log L(\mathbb{S}^{d-1})$ is sufficient condition for the weak type $(1, 1)$ estimate of T_Ω. Recently, this result was generalized by Ding and Lai [9] for the operator T^*_Ω defined by

$$T^*_\Omega f(x) = \text{p. v.} \int_{\mathbb{R}^d} \Omega(x-y) K(x, y) f(y) dy,$$

where the kernel $\Omega \in L \log L(\mathbb{S}^{d-1})$ and K needs to satisfy some size and regularity conditions. For other related contributions, we refer the readers to references [10, 11, 15, 22, 25–28, 32, 34, 35] and the references therein.

Consider now the $L^p(\mathbb{R}^d)$ boundedness and endpoint estimates for the operator $T_{\Omega, A}$ when Ω satisfies only size condition, things become more subtle. Hu [21] considered the $L^2(\mathbb{R}^d)$ boundedness of $T_{\Omega,A}$ when $\Omega \in GS_\beta(\mathbb{S}^{d-1})$, which means,

$$\sup_{\zeta \in S^{d-1}} \int_{\mathbb{S}^{d-1}} |\Omega(\theta)| \log^\beta \left(\frac{1}{|\zeta \cdot \theta|} \right) d\theta < \infty. \tag{1.3}$$

The main result in [21] can be summarized as follows:

Theorem A *Let Ω be homogeneous of degree zero which satisfies the vanishing condition* (1.1)*, A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Suppose that $\Omega \in GS_\beta(S^{d-1})$ for some $\beta > 3$, then $T_{\Omega, A}$ is bounded on $L^2(\mathbb{R}^d)$.*

This size condition was introduced by Grafakos and Stefanov [14], to study the $L^p(\mathbb{R}^d)$ boundedness of the homogeneous singular integral operator. As it was pointed out in [14], there exist integrable functions on $\mathbb{S}^{d-1}$ which are not in $H^1(\mathbb{S}^{d-1})$ but satisfy (1.3) for all $\beta \in (1, \infty)$. Thus, $GS_\beta(\mathbb{S}^{d-1})$ is also a minimum size condition for functions on $\mathbb{S}^{d-1}$. It is easy to verify that

$$\cup_{q>1} L^q(\mathbb{S}^{d-1}) \subset \cap_{\beta>1} GS_\beta(\mathbb{S}^{d-1}), \;\; L(\log L)^\beta(\mathbb{S}^{d-1}) \subset GS_\beta(\mathbb{S}^{d-1}).$$

For the $L^p(\mathbb{R}^d)$ $(1 < p < \infty)$ boundedness of $T_{\Omega,A}$, the best known condition $\Omega \in \cup_{q>1} L^q(\mathbb{S}^{d-1})$ is given in [19]. There is no any endpoint estimate for $T_{\Omega,A}$ when Ω only satisfies some size condition, even if $\Omega \in L^\infty(\mathbb{S}^{d-1})$. Note that the following inclusion relationship holds

$$\begin{aligned} &\mathrm{Lip}_\alpha(\mathbb{S}^{d-1})(0 < \alpha \le 1) \subsetneq L^q(\mathbb{S}^{d-1})(q > 1) \subsetneq L(\log L)^2(\mathbb{S}^{d-1}) \\ &\subsetneq L\log L(\mathbb{S}^{d-1}) \subsetneq H^1(\mathbb{S}^{d-1}). \end{aligned} \tag{1.4}$$

Therefore, it is quite natural to ask the following question:

Question: What is the minimal condition such that $T_{\Omega,\,A}$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$? Does the endpoint estimate of $L\log L$ type still holds true when Ω only satisfies size condition?

The main purpose of this paper is to show that $\Omega \in L(\log L)^2(\mathbb{S}^{d-1})$ is a sufficient condition for the $L^p(\mathbb{R}^d)$ boundedness and weak type $L\log L$ estimate for $T_{\Omega,A}$. Our first result can be stated as follows.

Theorem 1.1 *Let Ω be homogeneous of degree zero, satisfy the vanishing moment (1.1), and A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Suppose that $\Omega \in L(\log L)^2(\mathbb{S}^{d-1})$. Then $T_{\Omega,\,A}$ is bounded on $L^2(\mathbb{R}^d)$.*

Let $\widetilde{T}_{\Omega,\,A}$ be the dual operator of $T_{\Omega,\,A}$, defined as

$$\widetilde{T}_{\Omega,A} f(x) = \mathrm{p.\,v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} \big(A(x) - A(y) - \nabla A(x)(x-y)\big) f(y)dy. \tag{1.5}$$

Theorem 1.2 *Let Ω be homogeneous of degree zero, satisfy the vanishing condition (1.1), and A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Suppose that $\Omega \in L(\log L)^2(\mathbb{S}^{d-1})$. Then for any $\lambda > 0$ and $\Phi(\mathrm{t}) = \mathrm{t}\log(\mathrm{e}+\mathrm{t})$, the following inequalities hold*

$$\big|\{x \in \mathbb{R}^d : |T_{\Omega,A} f(x)| > \lambda\}\big| \lesssim \int_{\mathbb{R}^d} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx; \tag{1.6}$$

$$\big|\{x \in \mathbb{R}^d : |\widetilde{T}_{\Omega,A} f(x)| > \lambda\}\big| \lesssim \lambda^{-1} \|f\|_{L^1(\mathbb{R}^d)}. \tag{1.7}$$

As far as we know, there is no previous study concerning the weak type endpoint estimates for $\widetilde{T}_{\Omega,A}$, even if $\Omega \in \mathrm{Lip}_\alpha(\mathbb{S}^{d-1})$ for $\alpha \in (0, 1]$. We consider this operator mainly to deduce the following precise $L^p(\mathbb{R}^d)$ bounds of $T_{\Omega,\,A}$.

Theorem 1.3 *Let Ω be homogeneous of degree zero, satisfy the vanishing condition (1.1), and A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Suppose that $\Omega \in L(\log L)^2(\mathbb{S}^{d-1})$. Then*

$$\|T_{\Omega,\,A} f\|_{L^p(\mathbb{R}^d)} \lesssim \begin{cases} p'^2\|f\|_{L^p(\mathbb{R}^d)}, & p \in (1,\ 2]; \\ p\|f\|_{L^p(\mathbb{R}^d)}, & p \in (2,\ \infty). \end{cases}$$

Remark 1.4 Theorem 1.1, along with Theorem 1.3, shows that $\Omega \in L(\log L)^2(\mathbb{S}^{d-1})$ is a sufficient condition such that $T_{\Omega,\,A}$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1,\ \infty)$. This improves essentially the result obtained in [19, Theorem 1.1], in which, it was shown that if $\Omega \in \cup_{q>1} L^q(\mathbb{S}^{d-1})$, then $T_{\Omega,\,A}$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1,\ \infty)$.

Remark 1.5 As it was pointed out, for $\beta \in [1,\ \infty)$, $L(\log L)^\beta(\mathbb{S}^{d-1}) \subset GS_\beta(S^{d-1})$. However, it is unknown whether $L(\log L)^\beta(\mathbb{S}^{d-1}) \subset GS_{\beta'}(S^{d-1})$ when $\beta' > \beta$. We conjecture that there is no inclusion relationship between $L(\log L)^\beta(\mathbb{S}^{d-1})$ and $GS_{\beta'}(S^{d-1})$ when $\beta' > \beta$, and believe Theorem A and Theorem 1.3 do not imply each other in the case $p = 2$.

We believe that the condition $\Omega \in L(\log L)^2(\mathbb{S}^{d-1})$ is the weakest condition for these weak type results to hold, in the following sense.

Conjecture 1.6 $\Omega \in L(\log L)^2(\mathbb{S}^{d-1})$ is the minimal condition for the weak $L \log L$ type estimate of $T_{\Omega,\,A}$, and weak (1, 1) estimate of $\widetilde{T}_{\Omega,A}$, in the sense that the power 2 can't be replaced by any real number smaller than 2.

The article is organized as follows. Section 2 will be devoted to demonstrate the L^2 boundedness of $T_{\Omega,A}$. In Sect. 3, we will prove Theorem 1.2 and Theorem 1.3. The proof of Theorem 1.2 is not short and will be divided into several cases and steps. Smoothness trunction method will play an important role and will be used several times.

Let's explain a little bit about the proofs of the main results. In Sect. 2, we will introduce a convolution operator Q_s with the property that

$$\int_0^\infty Q_s^4 \frac{ds}{s} = I.$$

This makes it possible to commutate with the paraproducts appeared in the proof and thus obtains more freedom in dealing with the estimates of the L^2 boundedness. Moreover, the method of dyadic analysis has been applied in the delicate decomposition of L^2 norm of $T_{\Omega,\,A}$. At some key points, we will use some properties of Carleson measure.

The key ingredient in our proof of Theorem 1.2 is to estimate the bad part in the Calderón-Zygmund decomposition of f. In the work of [31], Seeger showed that if $\Omega \in L \log L(\mathbb{S}^{d-1})$, then T_Ω is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\,\infty}(\mathbb{R}^d)$. Ding and Lai [9] proved that if $\Omega \in L \log L(\mathbb{S}^{d-1})$ and for some $\delta \in (0,\ 1]$, the function K satisfies

$$|K(x,\ y)| \lesssim \frac{1}{|x-y|^d}; \tag{1.8}$$

$$|K(x_1,\ y)-K(x_2,\ y)|\lesssim\frac{|x_1-x_2|^\delta}{|x_1-y|^{d+\delta}},\ \ |x_1-y|\geq 2|x_1-x_2|, \tag{1.9}$$

$$|K(x,\ y_1)-K(x,\ y_2)|\lesssim\frac{|y_1-y_2|^\delta}{|x-y_1|^{d+\delta}},\ \ |x-y_1|\geq 2|y_1-y_2|, \tag{1.10}$$

and T_Ω^* is bounded on $L^2(\mathbb{R}^d)$, then T_Ω^* is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. However, when A has derivatives of order one in BMO($\mathbb{R}^d$), the function $[A(x)-A(y)-\nabla A(y)(x-y)]|x-y|^{-d-1}$ does not satisfy the conditions (1.8)–(1.10). Let f be a bounded function with compact support, $b=\sum_L b_L$ be the bad part in the Calderón-Zygmund decomposition of f. In order to overcome this essential difficulty, we write

$$T_{\Omega,\,A}b(x)=\sum_L\sum_s\int_{\mathbb{R}^d}\frac{\Omega(x-y)}{|x-y|^{d+1}}\phi_s(x-y)\big(A_L(x)-A_L(y)\big)b_L(y)dy$$
$$+\text{error terms},$$

where $A_L(y)=A(y)-\sum_{n=1}^d\langle\partial_nA\rangle_Ly_n$. $\phi_s(x)=\phi(2^{-s}x)$. Here, $\langle\partial_nA\rangle_L$ denotes the mean value of ∂_nA on the cube L, ϕ is a smooth radial nonnegative function on $\mathbb{R}^d$ such that supp $\phi\subset\{x:\frac14\leq|x|\leq 1\}$ and $\sum_s\phi_s(x)=1$ for all $x\in\mathbb{R}^d\backslash\{0\}$. Then, our key observation is that, for each $s\in\mathbb{Z}$ and L with side length $\ell(L)=2^{s-j}$, the kernel $|x-y|^{-d-1}\phi_s(x-y)\big(A_L(x)-A_L(y)\big)\chi_L(y)$ instead satisfies (1.9) and (1.10).

In what follows, C always denotes a positive constant which is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A\lesssim B$ to denote that there exists a positive constant C such that $A\leq CB$. Specially, we use $A\lesssim_{n,p}B$ to denote that there exists a positive constant C depending only on $n,\ p$ such that $A\leq CB$. Constant with subscript such as c_1, does not change in different occurrences. For any set $E\subset\mathbb{R}^d$, χ_E denotes its characteristic function. For a cube $Q\subset\mathbb{R}^d$, $\ell(Q)$ denotes the side length of Q, and for $\lambda\in(0,\ \infty)$, we use λQ to denote the cube with the same center as Q and whose side length is λ times that of Q. For a suitable function f, $\widehat{f}$ denotes the Fourier transform of f. For $p\in[1,\ \infty]$, p' denotes the dual exponent of p, namely, $1/p'=1-1/p$.

2 Proof of Theorem 1.1

This section will be devoted to prove Theorem 1.1, the $L^2(\mathbb{R}^d)$ boundedness of $T_{\Omega,\,A}$ when $\Omega\in L(\log L)^2(\mathbb{S}^{d-1})$. We will employ some ideas from [19], together with many more refined estimates. We begin with some notions and lemmas. Let $\psi\in C_0^\infty(\mathbb{R}^d)$ be a radial function with integral zero, supp $\psi\subset B(0,\ 1)$, $\psi_s(x)=s^{-d}\psi(s^{-1}x)$ and assume that

$$\int_0^\infty[\widehat{\psi}(s)]^4\frac{ds}{s}=1.$$

Consider the convolution operator $Q_s f(x) = \psi_s * f(x)$. It enjoys the property that

$$\int_0^\infty Q_s^4 \frac{ds}{s} = I. \tag{2.1}$$

Moreover, by the classical Littlewood-Paley theory, it follows that

$$\Big\| \Big(\int_0^\infty |Q_s f|^2 \frac{ds}{s} \Big)^{1/2} \Big\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}. \tag{2.2}$$

Let ϕ be a smooth radial nonnegative function on $\mathbb{R}^d$ with supp $\phi \subset \{x : \frac{1}{4} \le |x| \le 1\}$, $\sum_s \phi_s(x) = 1$ with $\phi_j(x) = 2^{-jd}\phi(2^{-j}x)$ for all $x \in \mathbb{R}^d \backslash \{0\}$. For each fixed $j \in \mathbb{Z}$, define

$$T_{\Omega,\, A;\, j} f(x) = \int_{\mathbb{R}^d} K_{A,\, j}(x,\ y) f(y) dy, \tag{2.3}$$

where

$$K_{A,\, j}(x,\ y) = \frac{\Omega(x-y)}{|x-y|^{d+1}} \big(A(x) - A(y) - \nabla A(y)(x-y)\big)\phi_j(x-y).$$

The following lemmas are needed in our analysis.

Lemma 2.1 ([19]) *Let Ω be homogeneous of degree zero, satisfies the vanishing condition (1.1) and $\Omega \in L^1(\mathbb{S}^{d-1})$. Let A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Then for any $k_1,\ k_2 \in \mathbb{Z}$ with $k_1 < k_2$, the following inequality holds*

$$\Big| \sum_{k_1 \le j \le k_2} \int_{\mathbb{R}^d} K_{A,\, j}(x,\ y) dy \Big| \lesssim \|\Omega\|_{L^1(\mathbb{S}^{d-1})}.$$

Lemma 2.2 ([19]) *Let Ω be homogeneous of degree zero, integrable on $\mathbb{S}^{d-1}$ and satisfy the vanishing moment (1.1). Let A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Then there exists a constant $\epsilon \in (0,\ 1)$, such that for $s \in (0,\ \infty)$ and $j \in \mathbb{Z}$ with $s2^{-j} \le 1$,*

$$\|Q_s T_{\Omega,\, A;\, j} 1\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\Omega\|_{L^1(\mathbb{S}^{d-1})} (2^{-j}s)^\epsilon.$$

Lemma 2.3 ([19]) *Let Ω be homogeneous of degree zero and $\Omega \in L^\infty(\mathbb{S}^{d-1})$. Let A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Then there exists a constant $\varepsilon \in (0,\ 1)$, such that for $s \in (0,\ \infty)$ and $j \in \mathbb{Z}$ with $2^{-j}s \le 1$,*

$$\|Q_s T_{\Omega,\, A;\, j} f\|_{L^2(\mathbb{R}^d)} \lesssim \|\Omega\|_{L^\infty(\mathbb{S}^{d-1})} (2^{-j}s)^\varepsilon \|f\|_{L^2(\mathbb{R}^d)}.$$

Lemma 2.4 ([20]) *Let Ω be homogeneous of degree zero, have mean value zero on $\mathbb{S}^{d-1}$ and $\Omega \in L(\log L)^2(S^{d-1})$. Then for $b \in \mathrm{BMO}(\mathbb{R}^d)$, $[b, T_\Omega]$, the commutator of T_Ω with symbol b, defined by*

$$[b, T_\Omega]f(x) = b(x)T_\Omega f(x) - T_\Omega(bf)(x), \quad f \in C_0^\infty(\mathbb{R}^d),$$

is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$.

Lemma 2.5 ([19]) *Let Ω be homogeneous of degree zero, and integrable on $\mathbb{S}^{d-1}$ and satisfy the vanishing moment (1.1), A be a function in $\mathbb{R}^d$ with derivatives of order one in $\mathrm{BMO}(\mathbb{R}^d)$. Then for any $r \in (0, \infty)$, functions $\widetilde{\eta}_1, \widetilde{\eta}_2 \in C_0^\infty(\mathbb{R}^d)$ whose supported on balls of radius r,*

$$\Big|\int_{\mathbb{R}^d} \widetilde{\eta}_2(x)T_{\Omega,A}\widetilde{\eta}_1(x)dx\Big| \lesssim \|\Omega\|_{L^1(\mathbb{S}^{d-1})}r^{-d}\prod_{j=1}^{2}\big(\|\widetilde{\eta}_j\|_{L^\infty(\mathbb{R}^d)} + r\|\nabla\widetilde{\eta}_j\|_{L^\infty(\mathbb{R}^d)}\big).$$

The following lemma plays an important role in our analysis.

Lemma 2.6 ([4]) *Let A be a function on $\mathbb{R}^d$ with derivatives of order one in $L^q(\mathbb{R}^d)$ for some $q \in (d, \infty]$. Then*

$$|A(x) - A(y)| \lesssim |x-y|\Big(\frac{1}{|I_{(x,|x-y|)}|}\int_{I_{(x,|x-y|)}} |\nabla A(z)|^q \mathrm{d}z\Big)^{\frac{1}{q}},$$

where $I_{(x,|x-y|)}$ is a cube which is centered at x with length $2|x-y|$.

We need a lemma from the book of Grafakos.

Lemma 2.7 ([12, p. 140]) *Let Φ be a function on $\mathbb{R}^d$ satisfying for some $0 < C, \delta < \infty$, $|\Phi(x)| \le C(1+|x)|)^{-d-\delta}$. For $t > 0$, set $\Phi_t(x) = t^{-d}\Phi(t^{-1}x)$. Then a measure μ on $\mathbb{R}_+^{d+1}$ is a Carleson if and only if for every p with $1 < p < \infty$ there is a constant $C_{p,d,\mu}$ such that for all $f \in L^p(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}_+^{d+1}} |\Phi_t * f(x)|^p d\mu(x,t) \le C_{p,d,\mu}\int_{\mathbb{R}^d} |f(x)|^p dx.$$

Proof of Theorem 1.1 Invoking (2.1), to prove that $T_{\Omega,A}$ is bounded on $L^2(\mathbb{R}^d)$, it suffices to show the following inequalities hold for $f, g \in C_0^\infty(\mathbb{R}^d)$,

$$\Big|\int_0^\infty\int_0^t\int_{\mathbb{R}^d} Q_s^4 T_{\Omega,A}Q_t^4 f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}; \quad (2.4)$$

$$\Big|\int_0^\infty\int_t^\infty\int_{\mathbb{R}^d} Q_s^4 T_{\Omega,A}Q_t^4 f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \quad (2.5)$$

First, we will prove (2.4). To this aim, the kernel Ω will be decomposed into disjoint forms. Let

$$E_0 = \{\theta \in \mathbb{S}^{d-1} : |\Omega(\theta)| \le 1\} \text{ and } E_i = \{\theta \in \mathbb{S}^{d-1} : 2^{i-1} < |\Omega(\theta)| \le 2^i\}, \quad i \in \mathbb{N}.$$

Set

$$\Omega_0(\theta) = \Omega(\theta)\chi_{E_0}(\theta), \quad \Omega_i(\theta) = \Omega(\theta)\chi_{E_i}(\theta) \ (i \in \mathbb{N}).$$

For $i \in \mathbb{N} \cup \{0\}$, let $T^i_{\Omega,\,A;\,j}$ be the same as in (2.3) for $T_{\Omega,\,A;\,j}$ with Ω replaced by Ω_i. Then

$$\begin{aligned}&\int_0^\infty \int_0^t \int_{\mathbb{R}^d} Q_s^4 T_{\Omega,\,A} Q_t^4 f(x) g(x) dx \frac{ds}{s}\frac{dt}{t}\\ &\quad = \sum_i \sum_j \int_0^\infty \int_0^t \int_{\mathbb{R}^d} Q_s^4 T^i_{\Omega,\,A;\,j} Q_t^4 f(x) g(x) dx \frac{ds}{s}\frac{dt}{t}.\end{aligned} \tag{2.6}$$

Let $\alpha \in \left(\frac{d+1}{d+2},\ 1\right)$ be a constant. Fix $j \in \mathbb{Z}$, we decompose the set $\{(s,t) : 0 < t < \infty,\ 0 < s \le t\}$ into three regions:

$$\begin{aligned}E_1(j,s,t) &= \{(s,\ t) : 0 \le t \le 2^j,\ 0 < s \le t\};\\ E_2(j,s,t) &= \left\{(s,\ t) : 2^j \le t < (2^j s^{-\alpha})^{\frac{1}{1-\alpha}},\ 0 < s \le t\right\};\\ E_3(j,s,t) &= \left\{(s,\ t) : \max\{2^j,\ (2^j s^{-\alpha})^{\frac{1}{1-\alpha}}\} \le t < \infty,\ 0 < s \le t\right\}.\end{aligned}$$

In the following three subsections, we will discuss the contribution of each $E_{j,s,t}$ on the right ride of (2.6) to inequality (2.4). □

2.1 Contribution of $E_1(j, s, t)$

Let ε be the same constant appeared in Lemma 2.3 and denote $N = 2(\lfloor \varepsilon^{-1} \rfloor + 1)$. For each fixed $i \in \mathbb{N}$, we introduce the notion $E^i_{1,1}$ and $E^i_{1,2}$ as follows

$$\begin{aligned}E^i_{1,1}(j,s,\ t) &= \{(j,s,\ t) : 0 \le t \le 2^j,\ 0 \le s \le t,\ 2^j \le s2^{iN}\};\\ E^i_{1,2}(j,s,\ t) &= \{(j,s,\ t) : 0 \le t \le 2^j,\ 0 \le s \le t,\ 2^j > s2^{iN}\}.\end{aligned}$$

Then, one gets obviously that $E_1(j,s,t) = E^i_{1,1}(j,s,\ t) \cup E^i_{1,2}(j,s,\ t) := E^i_{1,1} \cup E^i_{1,2}$. Therefore

$$\Big|\sum_{i=0}^\infty \sum_j \int_0^\infty \int_0^t \int_{\mathbb{R}^d} \chi_{E_1(j,s,t)} Q_s^4 T^i_{\Omega,\,A;\,j} Q_t^4 f(x) g(x) dx \frac{ds}{s}\frac{dt}{t}\Big|$$

$$\leq \sum_{i=1}^{\infty}\sum_{j}\int_0^\infty\int_0^\infty \chi_{E^i_{1,1}}\Big|\int_{\mathbb{R}^d} Q_s^4 T^i_{\Omega,\,A;\,j}Q_t^4 f(x)g(x)dx\Big|\frac{ds}{s}\frac{dt}{t}$$
$$+\sum_{i=1}^{\infty}\sum_{j}\int_0^\infty\int_0^\infty \chi_{E^i_{1,2}}\Big|\int_{\mathbb{R}^d} Q_s^4 T^i_{\Omega,\,A;\,j}Q_t^4 f(x)g(x)dx\Big|\frac{ds}{s}\frac{dt}{t}$$
$$+\sum_{j}\int_0^\infty\int_0^\infty \chi_{E_1(j,s,t)}\Big|\int_{\mathbb{R}^d} Q_s^4 T^0_{\Omega,\,A;\,j}Q_t^4 f(x)g(x)dx\Big|\frac{ds}{s}\frac{dt}{t} =: \mathrm{I}+\mathrm{II}+\mathrm{III}.$$

We first consider term I. Let $\{I_l\}_l$ be a sequence of cubes having disjoint interiors and side lengths 2^j, such that

$$\mathbb{R}^d = \bigcup_l I_l. \tag{2.7}$$

For each fixed l, let $\zeta_l \in C_0^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\zeta_l \subset 48dI_l$, $0 \le \zeta_l \le 1$ and $\zeta_l(x) \equiv 1$ when $x \in 32dI_l$. Let x_l be a point on the boundary of $50dI_l$ and

$$\widetilde{A}_{I_l}(y) = A(y) - \sum_{m=1}^{d}\langle \partial_m A\rangle_{I_l} y_m,\ \ A_{I_l}(y) = A^*_{I_l}(y)\zeta_l(y),\ \ y\in\mathbb{R}^d,$$

with $A^*_{I_l}(y) = \widetilde{A}_{I_l}(y) - \widetilde{A}_{I_l}(x_l)$. Note that for $x \in 30dI_l$ and $y \in \mathbb{R}^d$ with $|x-y| \le 2^j$, we have

$$A(x)-A(y)-\nabla A(y)(x-y) = A_{I_l}(x)-A_{I_l}(y)-\nabla A_{I_l}(y)(x-y).$$

An application of Lemma 2.6 then implies that $\|A_{I_l}\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^j$.

For each fixed $j\in\mathbb{Z}$, consider the operators $W^i_{\Omega,\,j}$ and $U^i_{\Omega,\,m;j}$ defined by

$$W^i_{\Omega,j}h(x) = \int_{\mathbb{R}^d}\frac{\Omega_i(x-y)}{|x-y|^{d+1}}\phi_j(x-y)h(y)dy$$

and

$$U^i_{\Omega,m;j}h(x) = \int_{\mathbb{R}^d}\frac{\Omega_i(x-y)(x_m-y_m)}{|x-y|^{d+1}}\phi_j(x-y)h(y)dy.$$

The method of rotation of Caldeón-Zygmund states that for $p\in(1,\,\infty)$, they enjoy the following properties:

$$\|W^i_{\Omega,j}h\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-j}\|\Omega_i\|_{L^1(S^{d-1})}\|h\|_{L^p(\mathbb{R}^d)};$$
$$\|U^i_{\Omega,\,m,\,j}h\|_{L^p(\mathbb{R}^d)} \lesssim \|\Omega_i\|_{L^1(S^{d-1})}\|h\|_{L^p(\mathbb{R}^d)},$$

see [12, pp. 272–274]. For each fixed l, let $h_{s,l}(x) = Q_s g(x)\chi_{I_l}(x)$ and $I_l^* = 60dI_l$. For $x \in \operatorname{supp} h_{s,l}$, we have

$$T^i_{\Omega,A,j} Q_t^4 f(x) = A_{I_l}(x) W^i_{\Omega,j} Q_t^4 f(x) - W^i_{\Omega,j}(A_{I_l} Q_t^4 f)(x) - \sum_{m=1}^{d} U^i_{\Omega,m,j}(\partial_m A_{I_l} Q_t^4 f)(x).$$

Hence, to show the estimate for I, we need to consider the following three terms.

$$\mathrm{R}_i^1 = \sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^{j-Ni}}^{2^j} \Big| \sum_l \int_{\mathbb{R}^d} A_{I_l}(x) Q_s^3 h_{s,l}(x) W^i_{\Omega,j} Q_t^4 f(x) dx \Big| \frac{dt}{t} \frac{ds}{s};$$

$$\mathrm{R}_i^2 = \sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^{j-Ni}}^{2^j} \Big| \sum_l \int_{\mathbb{R}^d} Q_s^3 h_{s,l}(x) W^i_{\Omega,j}(A_{I_l} Q_t^4 f)(x) dx \Big| \frac{dt}{t} \frac{ds}{s};$$

and

$$\mathrm{R}_i^3 = \sum_{m=1}^{d} \sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^{j-Ni}}^{2^j} \Big| \sum_l \int_{\mathbb{R}^d} Q_s^3 h_{s,l}(x) U^i_{\Omega,m,j}(\partial_m A_{I_l} Q_t^4 f)(x) dx \Big| \frac{ds}{s} \frac{dt}{t} =: \sum_{m=1}^{d} \mathrm{R}^3_{i,m}.$$

For R_i^1, note that

$$\sum_j \sum_l \int_{2^{j-iN}}^{2^j} \|Q_s^3 h_{s,l}\|^2_{L^2(\mathbb{R}^d)} \frac{ds}{s} \lesssim iN \sum_j \int_{2^{j-1}}^{2^j} \sum_l \|h_{s,l}\|^2_{L^2(\mathbb{R}^d)} \frac{ds}{s} \lesssim i \int_0^\infty \|Q_s g\|^2_{L^2(\mathbb{R}^d)} \frac{ds}{s}.$$

Then, the well-known Littlewood-Paley theory for g-function leads to that

$$\sum_j \sum_l \int_{2^{j-iN}}^{2^j} \|Q_s^3 h_{s,l}\|^2_{L^2(\mathbb{R}^d)} \frac{ds}{s} \lesssim i \Big\| \Big(\int_0^\infty |Q_s g(\cdot)|^2 \frac{ds}{s} \Big)^{1/2} \Big\|^2_{L^2(\mathbb{R}^d)} \lesssim i \|g\|^2_{L^2(\mathbb{R}^d)}.$$

For $x \in 48dI_l$, since $\sup\{\phi_j\} \subset [2^{j-2}, 2^j]$ and note that $\phi_j(x-y)Q_t^4 f(y) = \chi_{I_l^*}(y)\phi_j(x-y)Q_t^4 f(y)$, then, $W^i_{\Omega,j}(Q_t^4 f) = W^i_{\Omega,j}(\chi_{I_l^*} Q_t^4 f)$. It then follows from Hölder's inequality, Cauchy-Schwarz inequality and the boundedness of $W^i_{\Omega,j}$ that

$$|\mathrm{R}_i^1| \le \Big(\sum_j \sum_l \int_{2^{j-iN}}^{2^j} \int_{2^{j-iN}}^{2^j} \|Q_s^3 h_{s,l}\|^2_{L^2(\mathbb{R}^d)} \frac{ds}{s} \frac{dt}{t} \Big)^{1/2}$$

$$\times\Big(\sum_j\sum_l\int_{2^{j-iN}}^{2^j}\int_{2^{j-iN}}^{2^j}\|A_{I_l}W^i_{\Omega,j}(\chi_{I_l^*}Q_t^4f)\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\Big)^{1/2}$$
$$\lesssim\|\Omega_i\|_{L^1(S^{d-1})}\Big(\sum_j\sum_l\int_{2^{j-iN}}^{2^j}\int_{2^{j-iN}}^{2^j}\|Q_s^3h_{s,l}\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\Big)^{1/2}$$
$$\times\Big(\sum_j\sum_l\int_{2^{j-iN}}^{2^j}\int_{2^{j-iN}}^{2^j}\|\chi_{I_l^*}Q_t^4f\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\Big)^{1/2}$$
$$\lesssim i^2\|\Omega_i\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)},$$

where in the last inequality we have used the fact that the cubes $\{60dI_l\}_l$ have bounded overlaps.

The same reasoning applies to R_i^2 with small and straightforward modifications yields that

$$|\mathrm{R}_i^2|\lesssim i\|\Omega_i\|_{L^1(S^{d-1})}\Big(\sum_j\int_{2^{j-iN}}^{2^j}\|Q_sg\|^2_{L^2(\mathbb{R}^d)}\frac{ds}{s}\Big)^{1/2}$$
$$\times\Big(\sum_j\sum_l\int_{2^{j-iN}}^{2^j}\|\zeta_lQ_t^4f\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\Big)^{1/2}$$
$$\lesssim i^2\|\Omega_i\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

Now we are in a position to consider each term $\mathrm{R}^3_{i,m}$. For $x\in 32dI_l$, it is easy to check

$$\partial_mA_{I_l}(x)Q_t^4f(x)=\zeta_l(x)[\partial_mA,\ Q_t]Q_t^3f(x)+\zeta_l(x)Q_t([\partial_mA,\ Q_t]Q_t^2f)(x)$$
$$+\zeta_l(x)Q_t^2(\partial_m\widetilde{A}_{I_l}Q_t^2f)(x).$$

Therefore $\mathrm{R}^3_{i,m}$ can be controlled by the sum of the following terms:

$$\mathrm{R}^{3,1}_{i,m}=\sum_j\int_{2^{j-Ni}}^{2^j}\int_{2^{j-iN}}^{2^j}\Big|\sum_l\int_{\mathbb{R}^d}Q_s^3h_{s,l}(x)U^i_{\Omega,m,j}\big([\partial_mA,\ Q_t]Q_t^3f\big)(x)dx\Big|\frac{dt}{t}\frac{ds}{s};$$
$$\mathrm{R}^{3,2}_{i,m}=\sum_j\int_{2^{j-Ni}}^{2^j}\int_{2^{j-iN}}^{2^j}\Big|\sum_l\int_{\mathbb{R}^d}Q_s^3h_{s,l}(x)U^i_{\Omega,m,j}Q_t\big([\partial_mA,\ Q_t]Q_t^2f\big)(x)dx\Big|\frac{dt}{t}\frac{ds}{s};$$
$$\mathrm{R}^{3,3}_{i,m}=\sum_j\int_{2^{j-Ni}}^{2^j}\int_{2^{j-iN}}^{2^j}\Big|\sum_l\int_{\mathbb{R}^d}Q_s^3h_{s,l}(x)U^i_{\Omega,m,j}Q_t^2(\partial_m\widetilde{A}_{I_l}Q_t^2)f(x)dx\Big|\frac{dt}{t}\frac{ds}{s}.$$

Observe that $|[\partial_m A,\ Q_t]h(x)| \lesssim M_{\partial_m A}h(x)$, where $M_{\partial_m A}$ is the commutator of the Hardy-Littlewood maximal operator defined by

$$M_{\partial_m A}h(x) = \sup_{r>0} r^{-d} \int_{|x-y|<r} |\partial_m A(x) - \partial_m A(y)||h(y)|dy.$$

Hölder's inequality, along with the $L^2(\mathbb{R}^d)$ boundedness of $M_{\partial_m A}$ and $U^i_{\Omega,m,j}$, it yields that

$$\begin{aligned}|\mathrm{R}^{3,1}_{i,m}| &\le \Big(\sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^{j-iN}}^{2^j} \Big\| Q_s^3\big(\sum_l h_{s,l}\big)\Big\|^2_{L^2(\mathbb{R}^d)} \frac{dt}{t}\frac{ds}{s}\Big)^{1/2} \\ &\quad \times\Big(\sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^{j-iN}}^{2^j} \|U^i_{\Omega,m,j}([\partial_m A,\ Q_t]Q_t^3 f\|^2_{L^2(\mathbb{R}^d)} \frac{dt}{t}\frac{ds}{s}\Big)^{1/2} \\ &\lesssim i^2\|\Omega_i\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Exactly the same reasoning applies to $\mathrm{R}^{3,2}_{i,m}$, we obtain

$$|\mathrm{R}^{3,2}_{i,m}| \lesssim i^2\|\Omega_i\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

As for $\mathrm{R}^{3,3}_{i,m}$, observing that for fixed $l \in \mathbb{Z}$, $s,\ t \le 2^j$, one gets

$$\begin{aligned}Q_t(\partial_m \widetilde{A}_{I_l} Q_t^2 f)(x) &= Q_t(\partial_m \widetilde{A}_{I_l} \chi_{I_l^*} Q_t^2 f)(x), \\ U^i_{\Omega,\,m,\,j} Q_s = Q_s U^i_{\Omega,m,j} &\text{ and } \ Q_s Q_t = Q_t Q_s.\end{aligned}$$

Henceforth we have

$$\begin{aligned}\mathrm{R}^{3,3}_{i,m} &= \sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^{j-iN}}^{2^j} \Big|\sum_l \int_{\mathbb{R}^d} Q_t Q_s^2 h_{s,l}(x) Q_s U^i_{\Omega,m,j} Q_t(\partial_m \widetilde{A}_{I_l}\chi_{I_l^*} Q_t^2 f)(x)dx\frac{dt}{t}\frac{ds}{s}\Big| \\ &\le \Big(\sum_j\sum_l \int_{2^{j-Ni}}^{2^j} \int_{2^{j-iN}}^{2^j} \|Q_t Q_s^2 h_{s,l}\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\Big)^{1/2} \\ &\quad \times\Big(\sum_j\sum_l \int_{2^{j-Ni}}^{2^j} \int_{2^{j-iN}}^{2^j} \Big\|Q_s\big(U^i_{\Omega,m,j} Q_t(\partial_m \widetilde{A}_{I_l}\chi_{I_l^*} Q_t^2 f)\big)\Big\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\Big)^{1/2}.\end{aligned}$$

Let $x \in 48dI_l$, $q \in (1,\ 2)$ and $s \in (2^{j-1}, 2^j)$. A straightforward computation involving Hölder's inequality and the John-Nirenberg inequality gives us that

$$\begin{aligned}|Q_s(\partial_m \tilde{A}_{I_l} h)(x)| \le &\int_{\mathbb{R}^d} |\psi_s(x-y)||\partial_m A(y) - \langle \partial_m A\rangle_{I(x,\,s)}||h(y)|dy \\ &+ |\langle \partial_m A\rangle_{I_l} - \langle \partial_m A\rangle_{I(x,\,s)}| \int_{\mathbb{R}^d} |\psi_s(x-y)||h(y)|dy\end{aligned}$$

$$\begin{aligned}&\lesssim M_q h(x)+\log(1+2^j/s)Mh(x)\\&\lesssim M_q h(x),\end{aligned}\tag{2.8}$$

where $I(x,\ s)$ is the cube center at x and having side length s.

This inequality, together with the boundedness of $U^i_{\Omega,m,j}$ and maximal function $M_q h$, implies that

$$\begin{aligned}&\Big(\sum_j\sum_l\int_{2^{j-Ni}}^{2^j}\int_{2^{j-iN}}^{2^j}\|Q_s\big(U^i_{\Omega,m,j}Q_t(\partial_m\tilde{A}_{I_l}\chi_{I_l^*}Q_t^2f)\big)\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\Big)^{1/2}\\&\lesssim\Big(i\sum_j\sum_l\int_{2^{j-1}}^{2^j}\|U^i_{\Omega,m,j}Q_t(\partial_m\tilde{A}_{I_l}\chi_{I_l^*}Q_t^2f)\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\Big)^{1/2}\\&\lesssim i\|\Omega_i\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

On the other hand, by the L^2 boundedness of convolution operators and the Littlewood-Paley theory for g-function again, we have that

$$\begin{aligned}&\sum_j\sum_l\int_{2^{j-Ni}}^{2^j}\int_{2^{j-iN}}^{2^j}\|Q_tQ_s^2h_{s,l}\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\\&\qquad\lesssim\ i^2\int_0^\infty\|Q_sg\|^2_{L^2(\mathbb{R}^d)}\frac{ds}{s}\lesssim i^2\|g\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Therefore

$$\mathrm{R}^{3,3}_{i,m}\lesssim i^2\|\Omega_i\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

Combining the estimates for R^1_i, R^2_i and $\mathrm{R}^{3,n}_{i,m}$ (with $1\le m\le d,\ n=1,\ 2,\ 3$) in all yields that

$$\mathrm{I}\lesssim\sum_{i=1}^\infty i^2\|\Omega_i\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}\lesssim\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)},\tag{2.9}$$

since

$$\sum_{i=1}^\infty i^2\|\Omega_i\|_{L^1(S^{d-1})}\lesssim\|\Omega\|_{L(\log L)^2(\mathbb{S}^{d-1})}.$$

It remains to discuss the contribution of terms II and III. For $i\in\mathbb{N}\cup\{0\}$, by Lemma 2.3, one gets

$$\sum_j\int_0^\infty\int_0^\infty\chi_{E^i_{1,2}}\|Q_sT^i_{\Omega,\,A;\,j}Q_t^4f\|_{L^2(\mathbb{R}^d)}\|Q_s^3g\|_{L^2(\mathbb{R}^d)}\frac{ds}{s}\frac{dt}{t}$$

$$\lesssim \|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})}\Big(\int_0^\infty\int_0^\infty \sum_j \chi_{E_{1,2}^i}(2^{-j}s)^\varepsilon \|Q_s^3 g\|_{L^2(\mathbb{R}^d)}^2 \frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}$$
$$\times\Big(\int_0^\infty\int_0^\infty \sum_j \chi_{E_{1,2}^i}(2^{-j}s)^\varepsilon \|Q_t^4 f\|_{L^2(\mathbb{R}^d)}^2 \frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}. \tag{2.10}$$

Note that

$$E_{1,2}^i(j,s,t)\subset\big\{(j,s,t):\, 0\le t\le 2^j,\, 0\le s\le t,\, 2^j\ge \max\{t,\, s2^{iN}\}\big\},$$

Thus

$$\sum_j \chi_{E_{1,2}^i}(2^{-j}s)^\varepsilon \le 2^{-iN\varepsilon/2}\big(\frac{s}{t}\big)^{\varepsilon/2}\chi_{\{(s,t):\, s\le t\}}(s,\, t),$$

which further implies that

$$\Big(\int_0^\infty\int_0^\infty \sum_j \chi_{E_{1,2}^i}(2^{-j}s)^\varepsilon \|Q_s^3 g\|_{L^2(\mathbb{R}^d)}^2 \frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}$$
$$\lesssim 2^{-Ni\varepsilon/4}\Big(\int_0^\infty\int_s^\infty \big(\frac{s}{t}\big)^{\varepsilon/2}\frac{dt}{t}\|Q_s g\|_{L^2(\mathbb{R}^d)}^2\frac{ds}{s}\Big)^{1/2}\lesssim 2^{-Ni\varepsilon/4}\|g\|_{L^2(\mathbb{R}^d)}.$$

Similarly, we have that

$$\Big(\int_0^\infty\int_0^\infty \sum_j \chi_{E_{1,2}^i}(2^{-j}s)^\varepsilon \|Q_t^4 f\|_{L^2(\mathbb{R}^d)}^2 \frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\lesssim 2^{-Ni\varepsilon/4}\|f\|_{L^2(\mathbb{R}^d)}.$$

Therefore, these inequalities, together with the fact that $E_{1,1}^0=\emptyset$ may lead to

$$\mathrm{II}+\mathrm{III}\lesssim \sum_{i=0}^\infty 2^i 2^{-Ni\varepsilon/2}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.11}$$

Inequality (2.11), together with the estimate (2.9) for I, gives that

$$\Big|\sum_i\sum_j\int_0^\infty\int_0^t\int_{\mathbb{R}^d}\chi_{E_1(j,s,t)}Q_s^4 T_{\Omega,\,A;\,j}^i Q_t^4 f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big|$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.12}$$

2.2 Contribution of $E_2(j, s, t)$

Let $\alpha \in (\frac{d+1}{d+2},\ 1)$, $i \in \mathbb{N} \cup \{0\}$, and write

$$\begin{aligned}
&\sum_i \sum_{j\in\mathbb{Z}} \int_{2^j}^{\infty} \int_0^{(2^j t^{\alpha-1})^{\frac{1}{\alpha}}} \left| \int_{\mathbb{R}^d} Q_s^4 T^i_{\Omega,A,\,j} Q_t^4 f(x) g(x) dx \right| \frac{ds}{s}\frac{dt}{t}\\
&\quad \le \sum_i \sum_{j\in\mathbb{Z}} \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{(2^j s^{-\alpha})^{\frac{1}{1-\alpha}}} \left| \int_{\mathbb{R}^d} Q_s^4 T^i_{\Omega,A,\,j} Q_t^4 f(x) g(x) dx \right| \frac{dt}{t}\frac{ds}{s}\\
&\quad\quad + \sum_i \sum_{j\in\mathbb{Z}} \int_0^{2^{j-Ni}} \int_{2^j}^{(2^j s^{-\alpha})^{\frac{1}{1-\alpha}}} \left| \int_{\mathbb{R}^d} Q_s^4 T^i_{\Omega,A,\,j} Q_t^4 f(x) g(x) dx \right| \frac{dt}{t}\frac{ds}{s}\\
&\quad =: \mathrm{IV} + \mathrm{V}.
\end{aligned} \tag{2.13}$$

Firstly, we consider the term IV. When $i = 0$, the integral $\int_{2^{j-Ni}}^{2^j} \int_{2^j}^{(2^j s^{-\alpha})^{1/(1-\alpha)}} \frac{dt}{t}\frac{ds}{s}$ vanishes, we only need to consider the case $i \in \mathbb{N}$. Since $s > 2^{j-Ni}$, then $(2^j s^{-\alpha})^{\frac{1}{1-\alpha}} \le 2^j 2^{iN\frac{\alpha}{1-\alpha}}$. Therefore

$$\begin{aligned}
\mathrm{IV} &= \sum_i \sum_{j\in\mathbb{Z}} \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{(2^j s^{-\alpha})^{\frac{1}{1-\alpha}}} \left| \int_{\mathbb{R}^d} Q_s^4 T^i_{\Omega,A,\,j} Q_t^4 f(x) g(x) dx \right| \frac{dt}{t}\frac{ds}{s}\\
&\le \sum_i \sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \left| \int_{\mathbb{R}^d} T^i_{\Omega,A,j} Q_t^4 f(x) Q_s^4 g(x) dx \right| \frac{dt}{t}\frac{ds}{s}\\
&\le \sum_i \sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \left| \sum_l \int_{\mathbb{R}^d} T^i_{\Omega,A,j} Q_t^4 f(x) Q_s^3 h_{s,l}(x) dx \right| \frac{dt}{t}\frac{ds}{s},
\end{aligned}$$

where $h_{s,l}(x) = Q_s g(x)\chi_{I_l}(x)$, and $\{I_l\}_l$ be the cubes in (2.7).

Observe that when $x \in 4dI_l$, $T^i_{\Omega,A,j}(Q_t^4 f)(x) Q_s^3 h_{s,l}(x) = T^i_{\Omega,A,j}(\zeta_l Q_t^4 f)(x) Q_s^3 h_{s,l}(x)$, we rewrite

$$\begin{aligned}
&T^i_{\Omega,A,j}(\zeta_l Q_t^4 f)(x)\\
&= \Big(A_{I_l}(x) W^i_{\Omega,j} Q_t^4 f(x) - W^i_{\Omega,j}(A_{I_l} Q_t^4 f)(x) - \sum_{m=1}^d U^i_{\Omega,m,j}(\zeta_l[\partial_m A,\ Q_t] Q_t^3 f)(x)\\
&\quad - \sum_{m=1}^d U^i_{\Omega,m,j}(\zeta_l Q_t [\partial_m A,\ Q_t] Q_t^2 f)(x)\\
&\quad - \sum_{m=1}^d U^i_{\Omega,\,m,\,j}(\zeta_l Q_t Q_t (\partial_m \widetilde{A}_{I_l} Q_t^2 f)(x) \Big) \chi_{4dI_l}(x).
\end{aligned}$$

Similar to the estimate for R_i^1 and R_i^2, we know that

$$\sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \Big| \sum_l \int_{\mathbb{R}^d} W^i_{\Omega,j}(A_{I_l} Q_t^4 f)(x) Q_s^3 h_{s,l}(x) dx \Big| \frac{dt}{t}\frac{ds}{s} \lesssim i^2 \|\Omega_i\|_{L^1(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}, \tag{2.14}$$

and

$$\sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \sum_l \Big| \int_{\mathbb{R}^d} A_{I_l}(x) W^i_{\Omega,j} Q_t^4 f(x) Q_s^3 h_{s,l}(x) dx \Big| \frac{dt}{t}\frac{ds}{s} \lesssim i^2 \|\Omega_i\|_{L^1(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}. \tag{2.15}$$

On the other hand, for each fixed $1 \le m \le d$, the same reasoning as what we have done for $\mathrm{R}^{3,1}_{i,m}$ and $\mathrm{R}^{3,2}_{i,m}$ yields that

$$\sum_j \sum_l \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \Big| \int_{\mathbb{R}^d} U^i_{\Omega,m,j}(\zeta_l [\partial_m A, Q_t] Q_t^3 f)(x) Q_s^3 h_{s,l}(x) dx \Big| \frac{dt}{t}\frac{ds}{s} \lesssim i^2 \|\Omega_i\|_{L^1(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}, \tag{2.16}$$

and

$$\sum_j \sum_l \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \Big| \int_{\mathbb{R}^d} U^i_{\Omega,m,j}(\zeta_l Q_t [\partial_m A, Q_t] Q_t^2 f)(x) Q_s^3 h_{s,l}(x) dx \Big| \frac{dt}{t}\frac{ds}{s} \lesssim i^2 \|\Omega_i\|_{L^1(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}, \tag{2.17}$$

Note that if $x \in 4dI_l(x)$, then $U^i_{\Omega,m,j}(Q_t Q_t(\partial_m \widetilde{A}_{I_l} Q_t^2 f))(x) = U^i_{\Omega,m,j}(\zeta_l Q_t Q_t(\partial_m \widetilde{A}_{I_l} Q_t^2 f))(x)$. Since the kernel of Q_t is radial and it enjoys the property that

$$< U^i_{\Omega,m,j}(\zeta_l Q_t f), g >=< U^i_{\Omega,m,j}(\zeta_l f), Q_t g > .$$

Hence, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} U^i_{\Omega,m,j}(\zeta_l Q_t Q_t(\partial_m \widetilde{A}_{I_l} Q_t^2 f))(x) Q_s^3 h_{s,l}(x) dx \\ &\quad = \int_{\mathbb{R}^d} U^i_{\Omega,m,j} Q_s(\partial_m \widetilde{A}_{I_l} Q_s Q_t^2 f)(x) Q_t^2 Q_s h_{s,l}(x) dx \\ &\qquad - \int_{\mathbb{R}^d} U^i_{\Omega,m,j} Q_s[\partial_m A, Q_s] Q_t^2 f(x) Q_t^2 Q_s h_{s,l}(x) dx. \end{aligned}$$

A trivial argument then yields that

$$\Big|\sum_j\sum_l\int_{2^{j-iN}}^{2^j}\int_{2^j}^{2^j2^{iN\frac{\alpha}{1-\alpha}}}\int_{\mathbb{R}^d}U^i_{\Omega,m,j}Q_s[\partial_mA,\ Q_s]Q_t^2f(x)Q_t^2Q_sh_{s,l}(x)dx\frac{dt}{t}\frac{ds}{s}\Big| \lesssim i^2\|\Omega_i\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.18}$$

Now we write

$$\begin{aligned}&\int_{\mathbb{R}^d}U^i_{\Omega,m,j}Q_s(\partial_m\widetilde{A_{I_l}}Q_sQ_t^2f)(x)Q_t^2Q_sh_{s,l}(x)dx\\&=\int_{\mathbb{R}^d}Q_sQ_t^2f(x)[\partial_mA,\ Q_s]U^i_{\Omega,m,j}Q_t^2Q_sh_{s,l}(x)dx\\&\quad+\int_{\mathbb{R}^d}Q_sQ_t^2f(x)Q_s[\partial_mA,\ U^i_{\Omega,m,j}]Q_t^2Q_sh_{s,l}(x)dx\\&\quad+\int_{\mathbb{R}^d}Q_sQ_t^2f(x)Q_sU^i_{\Omega,m,j}[\partial_mA,\ Q_t^2]Q_sh_{s,l}(x)dx\\&\quad+\int_{\mathbb{R}^d}Q_sQ_t^2f(x)Q_sU^i_{\Omega,m,j}Q_t^2[\partial_mA,\ Q_s]h_{s,l}(x)dx\\&\quad+\int_{\mathbb{R}^d}Q_sQ_t^2f(x)Q_sU^i_{\Omega,m,j}Q_t^2Q_s(\partial_m\widetilde{A}_{I_l}h_{s,l})(x)dx:=\sum_{k=1}^5\mathrm{S}^k_{i,m,l}.\end{aligned}$$

A standard argument involving Hölder's inequality leads to that

$$\begin{aligned}&\sum_j\int_{2^{j-Ni}}^{2^j}\int_{2^j}^{2^j2^{iN\frac{\alpha}{1-\alpha}}}\Big|\sum_l\mathrm{S}^1_{i,m,l}\Big|\frac{dt}{t}\frac{ds}{s}\\&\lesssim\sum_j\int_{2^{j-Ni}}^{2^j}\int_{2^j}^{2^j2^{iN\frac{\alpha}{1-\alpha}}}\|Q_sQ_t^2f\|_{L^2(\mathbb{R}^d)}\Big\|[\partial_mA,\ Q_s]U^i_{\Omega,m,j}Q_t^2Q_s^2g\Big\|_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\\&\lesssim\|\Omega_i\|_{L^1(S^{d-1})}\Big(\sum_j\int_{2^{j-Ni}}^{2^j}\int_{2^j}^{2^j2^{iN\frac{\alpha}{1-\alpha}}}\|Q_sQ_t^2f\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\Big)^{1/2}\\&\quad\times\Big(\sum_j\int_{2^{j-Ni}}^{2^j}\int_{2^j}^{2^j2^{iN\frac{\alpha}{1-\alpha}}}\Big\|Q_t^2Q_s^2g\Big\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\Big)^{1/2}\\&\lesssim i^2|\Omega_i\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.\end{aligned}\tag{2.19}$$

Similarly, one can verify that

$$\sum_j\int_{2^{j-Ni}}^{2^j}\int_{2^j}^{2^j2^{iN\frac{\alpha}{1-\alpha}}}\Big|\sum_l\mathrm{S}^3_{i,m,l}\Big|\frac{dt}{t}\frac{ds}{s}\lesssim i^2\|\Omega_i\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.20}$$

and

$$\sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \Big|\sum_l \mathrm{S}^4_{i,m,l}\Big| \frac{dt}{t}\frac{ds}{s} \lesssim i^2 \|\Omega_i\|_{L^1(S^{d-1})} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}. \tag{2.21}$$

On the other hand, the fact (see [20, Lemma 4 and Lemma 3])

$$\|[\partial_m A,\ U^i_{\Omega,m,j}]h\|_{L^2(\mathbb{R}^d)} \lesssim \big(2^{-i} + i\|\Omega_i\|_{L^1(S^{d-1})}\big)\|h\|_{L^2(\mathbb{R}^d)},$$

implies that

$$\sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \Big|\sum_l \mathrm{S}^2_{i,m,l}\Big| \frac{dt}{t}\frac{ds}{s} \lesssim \big(i2^{-i} + i^2\|\Omega_i\|_{L^1(S^{d-1})}\big)\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.22}$$

Applying Hölder's inequality and inequality (2.8) in the case $s \in (2^{j-1}, 2^j)$, we obtain

$$\begin{aligned}
&\sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \Big|\sum_l \mathrm{S}^5_{i,m,l}\Big| \frac{dt}{t}\frac{ds}{s} \\
&\lesssim \|\Omega_i\|_{L^1(S^{d-1})} \Big(\sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \|Q_s Q_t^2 f\|^2_{L^2(\mathbb{R}^d)} \frac{dt}{t}\frac{ds}{s}\Big)^{1/2} \\
&\quad \times \Big(\sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{2^j 2^{iN\frac{\alpha}{1-\alpha}}} \Big\|Q_t^2 Q_s\Big(\sum_l \partial_m \widetilde{A}_{I_l} h_{s,l}\Big)\Big\|^2_{L^2(\mathbb{R}^d)} \frac{dt}{t}\frac{ds}{s}\Big)^{1/2} \\
&\lesssim i^{\frac{3}{2}} \|\Omega_i\|_{L^1(S^{d-1})} \|f\|_{L^2(\mathbb{R}^d)} \Big(\sum_j \int_{2^{j-Ni}}^{2^j} \Big\|\sum_l Q_s\big(\partial_m \widetilde{A}_{I_l} h_{s,l}\big)\Big\|^2_{L^2(\mathbb{R}^d)} \frac{ds}{s}\Big)^{1/2} \\
&\lesssim i^2 \|\Omega_i\|_{L^1(S^{d-1})} \|f\|_{L^2(\mathbb{R}^d)} \Big(\sum_j \int_{2^{j-1}}^{2^j} \|M_q h\|^2_{L^2(\mathbb{R}^d)} \frac{ds}{s}\Big)^{1/2} \\
&\lesssim i^2 \|\Omega_i\|_{L^1(S^{d-1})} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}.
\end{aligned} \tag{2.23}$$

Collecting the estimates from (2.14) to (2.23) in all, we deduce that

$$\begin{aligned}
\mathrm{IV} &= \sum_i \sum_j \int_{2^{j-Ni}}^{2^j} \int_{2^j}^{(2^j s^{-\alpha})^{\frac{1}{1-\alpha}}} \Big|\int_{\mathbb{R}^d} Q_t^4 f(x) T^i_{\Omega,A,j} Q_s^4 g(x)dx\Big| \frac{dt}{t}\frac{ds}{s} \\
&\lesssim \Big(\sum_i i2^{-i} + \sum_i i^2 \|\Omega_i\|_{L^1(S^{d-1})}\Big) \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

$$\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.24}$$

To show the estimate for V, note that for each fixed j, it holds that

$$\begin{aligned}&\left\{(s,\ t):\ 0\le s\le 2^{j-Ni},\ 2^j\le t<(2^js^{-\alpha})^{\frac{1}{1-\alpha}}\right\}\\&\quad\subset\left\{(s,\ t):\ 2^j\le t<\infty, 0<s\le \min\{2^{j-Ni},\ (2^jt^{\alpha-1})^{\frac{1}{\alpha}}\}\right\}.\end{aligned}$$

It then follows that

$$\begin{aligned}&\sum_j\int_0^{2^{j-Ni}}\int_{2^j}^{(2^js^{-\alpha})^{\frac{1}{1-\alpha}}}\|Q_t^4f\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}(2^{-j}s)^{\varepsilon}\frac{ds}{s}\\&\quad\le 2^{-Ni\varepsilon/2}\int_0^\infty\sum_{j:\,2^j\le t}\int_0^{(2^jt^{\alpha-1})^{\frac{1}{\alpha}}}(2^{-j}s)^{\frac{\varepsilon}{2}}\frac{ds}{s}\|Q_t^4f\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}\\&\quad\lesssim 2^{-Ni\varepsilon/2}\|f\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

and

$$\begin{aligned}&\sum_j\int_0^{2^{j-Ni}}\int_{2^j}^{(2^js^{-\alpha})^{\frac{1}{1-\alpha}}}\frac{dt}{t}\|Q_s^3g\|^2_{L^2(\mathbb{R}^d)}(2^{-j}s)^{\varepsilon}\frac{ds}{s}\\&\quad\le 2^{-Ni\varepsilon/2}\int_0^\infty\Big(\sum_{j:\,2^j\ge s2^{Ni}}\int_{2^j}^{(2^js^{-\alpha})^{\frac{1}{1-\alpha}}}\frac{dt}{t}(2^{-j}s)^{\frac{\varepsilon}{2}}\Big)\|Q_s^3g\|^2_{L^2(\mathbb{R}^d)}\frac{ds}{s}\\&\quad\lesssim 2^{-Ni\varepsilon/2}\|g\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

Thus, by Lemma 2.3, we obtain

$$\begin{aligned}\mathrm{V}&\le\sum_i\sum_j\int_0^{2^{j-Ni}}\int_{2^j}^{(2^js^{-\alpha})^{\frac{1}{1-\alpha}}}\|Q_sT^i_{\Omega,\,A;\,j}Q_t^4f\|_{L^2(\mathbb{R}^d)}\|Q_s^3g\|_{L^2(\mathbb{R}^d)}\frac{dt}{t}\frac{ds}{s}\\&\le\sum_i 2^i\Big(\sum_j\int_0^{2^{j-Ni}}\int_{2^j}^{(2^js^{-\alpha})^{\frac{1}{1-\alpha}}}\|Q_t^4f\|^2_{L^2(\mathbb{R}^d)}\frac{dt}{t}(2^{-j}s)^{\varepsilon}\frac{ds}{s}\Big)^{\frac12}\\&\quad\times\Big(\sum_j\int_0^{2^{j-Ni}}\int_{2^j}^{(2^js^{-\alpha})^{\frac{1}{1-\alpha}}}\|Q_s^3g\|^2_{L^2(\mathbb{R}^d)}(2^{-j}s)^{\varepsilon}\frac{dt}{t}\frac{ds}{s}\Big)^{1/2}\\&\lesssim\sum_i 2^i2^{-Ni\varepsilon/2}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.\end{aligned}\tag{2.25}$$

Combining estimates (2.24)–(2.25) yields

$$\Big|\sum_i\sum_j\int_{2^j}^{\infty}\int_0^{(2^j t^{\alpha-1})^{1/\alpha}}\int_{\mathbb{R}^d} Q_s^4 T^i_{\Omega,A,j} Q_t^4 f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big| \\ \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.26}$$

Therefore, by (2.13), (2.24) and (2.26), it holds that

$$\Big|\sum_i\sum_j\int_0^{\infty}\int_0^{t}\int_{\mathbb{R}^d} \chi_{E_2(j,,s,t)} Q_s^4 T^i_{\Omega,\,A;\,j} Q_t^4 f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big| \\ \lesssim IV+V\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)},$$

which gives the contribution of $E_2(j,s,t)$.

To finish the proof of (2.4), it remains to show the contribution of the term $E_3^i(j,s,t)$.

2.3 Contribution of $E_3(j,s,t)$

Our aim is to prove

$$\Big|\sum_{i=0}^{\infty}\sum_j\int_0^{\infty}\int_0^{t}\int_{\mathbb{R}^d} \chi_{E_3}(j,s,t) Q_s^4 T^i_{\Omega,\,A;\,j} Q_t^4 f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big| \\ \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}, \tag{2.27}$$

where

$$T_{\Omega_i,\,A;\,j}f(x)=\int_{\mathbb{R}^d}\frac{\Omega_i(x-y)}{|x-y|^{d+1}} \\ \times\big(A(x)-A(y)-\nabla A(y)(x-y)\big)\phi_j(|x-y|)f(y)dy, \tag{2.28}$$

Since the sum of i and the sum of j are independent and the sum of j depends only on the functions ϕ_j and $\chi_{E_3}(j,s,t)$, one may put $\phi_j\cdot\chi_{E_3}(j,s,t)$ together in the place of ϕ_j in (2.28), and temporary moves the summation over j before $\phi_j\cdot\chi_{E_3}(j,s,t)$, which indicates that it is possible to move the summation over i inside the integral again before Ω_i to obtain Ω. After that, one may move the sum of j outside the integral. Therefore, to prove (2.27), it suffices to show that

$$\sum_j\int_{2^j}^{\infty}\int_{(2^j t^{\alpha-1})^{1/\alpha}}^{t}\Big|\int_{\mathbb{R}^d} Q_s^4 T_{\Omega,A,j} Q_t^4 f(x)g(x)dx\Big|\frac{ds}{s}\frac{dt}{t} \\ \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.29}$$

To this purpose, we set

$$h^{(1)}(x,\ y)=\int\int\psi_s(x-z)\sum_{j:2^j\le s^\alpha t^{1-\alpha}}K_{A,\,j}(z,\ u)\big[\psi_t(u-y)-\psi_t(x-y)]dudz.$$

Let $H^{(1)}$ be the integral operator corresponding to kernel $h^{(1)}$. It then follows that

$$\begin{aligned}&\Big|\sum_j\int_{2^j}^\infty\int_{(2^jt^{\alpha-1})^{1/\alpha}}^t\int_{\mathbb{R}^d}Q_s^4T_{\Omega,A,j}Q_t^4f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big|\\&\le\int_0^\infty\int_0^t\|H^{(1)}Q_t^3f\|_{L^2(\mathbb{R}^d)}\|Q_s^3g\|_{L^2(\mathbb{R}^d)}\frac{ds}{s}\frac{dt}{t}\\&\quad+\sum_j\int_{2^j}^\infty\int_{(2^jt^{\alpha-1})^{1/\alpha}}^t\int_{\mathbb{R}^d}(Q_sT_{\Omega,A;\,j}1)(x)Q_t^4f(x)Q_s^3g(x)dx\frac{ds}{s}\frac{dt}{t}\end{aligned}\tag{2.30}$$

Applying Lemma 2.5 and reasoning as the same argument as in [18, p. 1282] give us that

$$|h^{(1)}(x,\ y)|\lesssim\Big(\frac{s}{t}\Big)^\gamma t^{-d}\chi_{\{(x,\,y):\,|x-y|\le Ct\}}(x,\ y),$$

where $\gamma=(d+2)\alpha-d-1$. This in turn indicates that $|H^{(1)}Q_tf(x)|\lesssim\big(\frac{s}{t}\big)^\gamma M(Q_tf)(x)$. Therefore

$$\begin{aligned}&\int_0^\infty\int_0^t\|H^{(1)}Q_t^3f\|_{L^2(\mathbb{R}^d)}\|Q_s^3g\|_{L^2(\mathbb{R}^d)}\frac{ds}{s}\frac{dt}{t}\\&\lesssim\Big(\int_0^\infty\int_0^t\Big(\frac{s}{t}\Big)^\gamma\|M(Q_t^3f)\|^2_{L^2(\mathbb{R}^d)}\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\\&\quad\times\Big(\int_0^\infty\|Q_s^3g\|^2_{L^2(\mathbb{R}^d)}\int_s^\infty\Big(\frac{s}{t}\Big)^\gamma\frac{dt}{t}\frac{ds}{s}\Big)^{\frac12}\\&\lesssim\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.\end{aligned}\tag{2.31}$$

It remains to show the corresponding estimate for the second term on the rightside of (2.30).

Let $F_x^j(s,t)=(Q_sT_{\Omega,A;\,j}1)(x)Q_t^4f(x)Q_s^3g(x)$. Then

$$\begin{aligned}&\int_{2^j}^\infty\int_{(2^jt^{\alpha-1})^{1/\alpha}}^t F_x^j(s,t)\frac{dsdt}{st}\\&=\int_0^\infty\int_0^t F_x^j(s,t)\frac{dsdt}{st}-\int_0^{2^j}\int_0^t F_x^j(s,t)\frac{dsdt}{st}\\&\quad-\int_{2^j}^\infty\int_0^{(2^jt^{\alpha-1})^{1/\alpha}}F_x^j(s,t)\frac{dsdt}{st}\end{aligned}\tag{2.32}$$

Therefore, it is sufficient to consider the contributions of each terms in Eq. (2.32) to the second term in (2.30).

Consider the first term in (2.32). Let $P_s = \int_s^\infty Q_t^4 \frac{dt}{t}$. Han and Sawyer [16] observed that the kernel Φ of the convolution operator P_s is a radial bounded function with bound cs^{-d}, supported on a ball of radius Cs and has integral zero. Therefore, it is easy to see that Φ is a Schwartz function. Since $P_s g = \Phi_s * g$, it then follows from the Littlewood-Paley theory that

$$\int_0^\infty \|P_s g\|_{L^2(\mathbb{R}^d)}^2 \frac{ds}{s} \lesssim \|g\|_{L^2(\mathbb{R}^d)}^2.$$

On the other hand, whenever $\Omega \in L^1(\mathbb{R}^d)$, it was shown in [19, p.121, Lemma 4.1] that $T_{\Omega,A}1 \equiv b \in \mathrm{BMO}(\mathbb{R}^d)$. Therefore, by [19, p.114, (3.1)], $\int_0^\infty Q_s^3(Q_s b P_s)\frac{ds}{s}$ defines an operator which is bounded on $L^2(\mathbb{R}^d)$. However, we can't use this boundedness directly in our case, since once using Hölder's inequality, we have to put the absolute value inside the integral and the $L^2(\mathbb{R}^d)$ boundedness may fail in this case. To overcome this obstacle, we apply the property of Carleson measure.

Note that $|Q_s T_{\Omega,\,A}1(x)|^2 \frac{dxds}{s}$ is a Carleson measure since $T_{\Omega,\,A}1 \in \mathrm{BMO}(\mathbb{R}^d)$. By Hölder's inequality, Lemma 2.7, it yields that

$$\begin{aligned} &\Big|\int_0^\infty \int_0^t \int_{\mathbb{R}^d} \sum_j Q_s T_{\Omega,\,A;\,j}1(x) Q_t^4 f(x) Q_s^3 g(x) dx \frac{ds}{s}\frac{dt}{t}\Big| \\ &\lesssim \Big(\int_0^\infty \int_{\mathbb{R}^d} |Q_s^3 g(x)|^2 dx \frac{ds}{s}\Big)^{\frac12} \Big(\int_{\mathbb{R}^{n+1}_+} |P_s f(x)|^2 |Q_s T_{\Omega,\,A}1(x)|^2 \frac{dxds}{s}\Big)^{\frac12} \\ &\lesssim \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)} \end{aligned} \tag{2.33}$$

On the other hand, by Lemma 2.2, one gets

$$\|Q_s T_{\Omega,\,A;\,j}1\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\Omega\|_{L^1(\mathbb{S}^{d-1})} (2^{-j}s)^\epsilon.$$

Denote by $D^1_{j,s,t} = \{(j,s,t) : s \le t \le 2^j\}$, $D^2_{j,s,t} = \{(j,s,t) : s \le t, s^\alpha t^{1-\alpha} \le 2^j \le t\}$. It then follows that

$$\begin{aligned} &\sum_j \int_0^{2^j} \int_0^t \int_{\mathbb{R}^d} \Big| Q_s T_{\Omega,\,A;\,j}1(x) Q_t^4 f(x) Q_s^3 g(x)\Big| dx \frac{ds}{s}\frac{dt}{t} \\ &\quad + \sum_j \int_{2^j}^\infty \int_0^{(2^j t^{\alpha-1})^{1/\alpha}} \int_{\mathbb{R}^d} \Big| Q_s T_{\Omega,\,A;\,j}1(x) Q_t^4 f(x) Q_s^3 g(x)\Big| dx \frac{ds}{s}\frac{dt}{t} \\ &\lesssim \sum_{i=1}^2 \Big\{\Big(\int_0^\infty \int_0^\infty \sum_j (2^{-j}s)^\epsilon \chi_{D^i_{j,s,t}}(j,s,t) \|Q_t f\|_{L^2(\mathbb{R}^d)}^2 \frac{ds}{s}\frac{dt}{t}\Big)^{1/2} \\ &\quad \times \Big(\int_0^\infty \int_0^\infty \sum_j (2^{-j}s)^\epsilon \chi_{D^i_{j,s,t}}(j,s,t) \|Q_s g\|_{L^2(\mathbb{R}^d)}^2 \frac{ds}{s}\frac{dt}{t}\Big)^{1/2}\Big\} \end{aligned}$$

$$\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}, \tag{2.34}$$

where in the last inequality, we used the property (2.2).

Combining (2.32)–(2.34), we have

$$\Big|\sum_j \int_{2^j}^{\infty}\int_{(2^j t^{\alpha-1})^{1/\alpha}}^{t}\int_{\mathbb{R}^d}(Q_s T_{\Omega,A;\,j}1)(x)Q_t^4 f(x)Q_s^3 g(x)dx\frac{ds}{s}\frac{dt}{t}\Big|$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)},$$

which, together with (2.31), leads to (2.27). This finishes the proof of $E_3(j,s,t)$, and also completes the proof of inequality (2.4).

2.4 Proof of (2.5)

To finish the proof of Theorem 1.1, it remains to show the estimate (2.5). Observe that

$$\int_0^{\infty}\int_t^{\infty}\int_{\mathbb{R}^d} Q_s^4 T_{\Omega,\,A} Q_t^4 f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}$$
$$= -\int_0^{\infty}\int_0^{s}\int_{\mathbb{R}^d} Q_t^4 \widetilde{T}_{\widetilde{\Omega},\,A} Q_s^4 g(x)f(x)dx\frac{dt}{t}\frac{ds}{s},$$

where $\widetilde{\Omega}(x)=\Omega(-x)$ and $\widetilde{T}_{\widetilde{\Omega},\,A}$ is the operator defined by (1.5), with Ω replaced by $\widetilde{\Omega}$. Let $T_{\widetilde{\Omega},\,m}$ be the operator defined by

$$T_{\widetilde{\Omega},\,m}h(x)=\text{p. v.}\int_{\mathbb{R}^d}\frac{\widetilde{\Omega}(x-y)(x_m-y_m)}{|x-y|^{d+1}}h(y)dy.$$

It then follows that

$$\widetilde{T}_{\widetilde{\Omega},\,A}h(x)=T_{\widetilde{\Omega},\,A}h(x)-\sum_{m=1}^{d}[\partial_m A,\ T_{\widetilde{\Omega},\,m}]h(x).$$

Inequality (2.4) tells us that

$$\Big|\int_0^{\infty}\int_0^{t}\int_{\mathbb{R}^d} Q_s^4 T_{\widetilde{\Omega},\,A} Q_t^4 g(x)f(x)dx\frac{ds}{s}\frac{dt}{t}\Big|\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{2.35}$$

For each fixed m with $1\le m\le d$, by duality, involving Lemma 2.4 and Hölder's inequality may lead to

$$\Big|\int_0^{\infty}\int_0^{t}\int_{\mathbb{R}^d} Q_s^4[\partial_m A,\ T_{\widetilde{\Omega},\,m}]Q_t^4 g(x)f(x)dx\frac{ds}{s}\frac{dt}{t}\Big|$$
$$\lesssim \int_0^{\infty}\|[\partial_m A,\ T_{\widetilde{\Omega},\,m}]Q_s^4 f\|_{L^2(\mathbb{R}^d)}\Big\|\int_s^{\infty}Q_t^4 g\frac{dt}{t}\Big\|_{L^2(\mathbb{R}^d)}\frac{ds}{s}$$

$$\lesssim \Big(\int_0^\infty \|Q_s^4 f\|_{L^2(\mathbb{R}^d)}^2 \frac{ds}{s}\Big)^{1/2}\Big(\int_0^\infty \|P_s g\|_{L^2(\mathbb{R}^d)}^2 \frac{ds}{s}\Big)^{1/2}$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

This estimate, together with (2.35), leads to (2.5) and then completes the proof of Theorem 1.1.

3 Proof of Theorems 1.2 and 1.3

This section is devoted to prove Theorem 1.2, the weak type endpoint estimates for $T_{\Omega,\,A}$ and $\widetilde{T}_{\Omega,A}$. To this end, we first introduce the definition of standard dyadic grid. Recall that the standard dyadic grid in $\mathbb{R}^d$, denoted by $\mathcal{D}$, consists of all cubes of the form

$$2^{-k}([0,\ 1)^d + j),\ k \in \mathbb{Z},\ \ j \in \mathbb{Z}^d.$$

For each fixed $j \in \mathbb{Z}$, set $\mathcal{D}_j = \{Q \in \mathcal{D} :\ \ell(Q) = 2^j\}$.

3.1 Proof of (1.6) in Theorem 1.2

The key ingredient of our proof lies in the step of dealing with the bad part of the Calderón-Zygmund decomposition of f. By homogeneity, it suffices to prove (1.6) for the case $\lambda = 1$. Applying the Calderón-Zygmund decomposition to $|f|\log(\mathrm{e}+|f|)$ at level 1, we can obtain a collection of non-overlapping closed dyadic cubes $\mathcal{S} = \{\mathbb{L}\}$, such that

(i) $\|f\|_{L^\infty(\mathbb{R}^d\backslash\cup_{\mathbb{L}\in\mathcal{S}}\mathbb{L})} \lesssim 1$;
(ii) $\int_{\mathbb{L}} |f(x)|\log(\mathrm{e}+|f(x)|)dx \lesssim |\mathbb{L}|$;
(iii) $\sum_{\mathbb{L}\in\mathcal{S}} |\mathbb{L}| \lesssim \int_{\mathbb{R}^d} |f(x)|\log(\mathrm{e}+|f(x)|)dx$.

Let g be the good part and b be the bad part of the decomposition of f, which are defined by

$$g(x) = f(x)\chi_{\mathbb{R}^d\backslash\cup_{\mathbb{L}\in\mathcal{S}}\mathbb{L}}(x) + \sum_{\mathbb{L}\in\mathcal{S}}\langle f\rangle_{\mathbb{L}}\chi_{\mathbb{L}}(x) \quad \text{and}$$
$$b(x) = \sum_{\mathbb{L}\in\mathcal{S}}(f - \langle f\rangle_{\mathbb{L}})\chi_{\mathbb{L}}(x) = \sum_{\mathbb{L}\in\mathcal{S}} b_{\mathbb{L}}(x).$$

It is easy to see that $\|g\|_{L^\infty(\mathbb{R}^d)} \lesssim 1$, and for $E = \cup_{\mathbb{L}\in\mathcal{S}} 100d\mathbb{L}$, it holds that

$$|E| \lesssim \int_{\mathbb{R}^d} |f(x)|\log(\mathrm{e}+|f(x)|)dx.$$

The $L^2(\mathbb{R}^d)$ boundedness of $T_{\Omega,\,A}$ then yields that

$$\big|\{x \in \mathbb{R}^d :\ |T_{\Omega,\,A}g(x)| \geq 1/2\}\big| \lesssim \|T_{\Omega,\,A}g\|_{L^2(\mathbb{R}^d)}^2$$

$$\lesssim \|g\|_{L^2(\mathbb{R}^d)}^2 \lesssim \|f\|_{L^1(\mathbb{R}^d)}. \tag{3.1}$$

Therefore, it is sufficient to show that

$$\left|\{x \in \mathbb{R}^d : |T_{\Omega,\,A} b(x)| \geq 1/2\}\right| \lesssim \int_{\mathbb{R}^d} |f(x)| \log(\mathrm{e} + |f(x)|) dx. \tag{3.2}$$

To prove (3.2), let ϕ be a smooth radial nonnegative function on $\mathbb{R}^d$ with supp $\phi \subset \{x : \frac{1}{4} \leq |x| \leq 1\}$ and $\sum_s \phi_s(x) = 1$ with $\phi_s(x) = \phi(2^{-s}x)$ for all $x \in \mathbb{R}^d \backslash \{0\}$. Set $\mathcal{S}_j = \{\mathbb{L} \in \mathcal{S} : \ell(\mathbb{L}) = 2^j\}$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} (A(x) - A(y)) b(y) dy \\
&= \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} (A(x) - A(y)) \sum_s \phi_s(x-y) \sum_j \sum_{\mathbb{L} \in \mathcal{S}_{s-j}} b_{\mathbb{L}}(y) dy \\
&= \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} (A(x) - A(y)) \sum_j \sum_s \phi_s(x-y) \sum_{\mathbb{L} \in \mathcal{S}_{s-j}} b_{\mathbb{L}}(y) dy \\
&= \sum_j \sum_s \sum_{\mathbb{L} \in \mathcal{S}_{s-j}} T_{\Omega,\,A;\,s,j} b_{\mathbb{L}}(x),
\end{aligned}$$

where

$$T_{\Omega,\,A;\,s,j} b_{\mathbb{L}}(x) = \int_{\mathbb{R}^d} \phi_s(x-y) \frac{\Omega(x-y)}{|x-y|^{d+1}} (A(x) - A(y)) b_{\mathbb{L}}(y) dy. \tag{3.3}$$

Let $A_{\mathbb{L}}(y) = A(y) - \sum_{n=1}^d \langle \partial_n A \rangle_{\mathbb{L}} y_n$. A trivial computation leads to the fact that

$$A_{\mathbb{L}}(x) - A_{\mathbb{L}}(y) - \nabla A_{\mathbb{L}}(y) \cdot (x-y) = A(x) - A(y) - \nabla A(y) \cdot (x-y).$$

Now write $T_{\Omega,A} b$ as

$$T_{\Omega,A} b(x) = \sum_j \sum_s \sum_{\mathbb{L} \in \mathcal{S}_{s-j}} T_{\Omega,\,A_{\mathbb{L}};s,j} b_{\mathbb{L}}(x) - \sum_{n=1}^d T_\Omega^n \Big(\sum_{\mathbb{L} \in \mathcal{S}} b_{\mathbb{L}} \partial_n A_{\mathbb{L}} \Big)(x),$$

where

$$T_\Omega^n h(x) = \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} (x_n - y_n) h(y) dy, \quad \text{for } 1 \leq n \leq d.$$

Fixed $1 \leq n \leq d$, since the kernel $\Omega(x) x_n |x|^{-1}$ is still in $L \log L(\mathbb{S}^{d-1})$, homogenous of degree zero and satisfies the vanishing condition on the unit sphere, by the weak endpoint estimate of the operators T^n_Ω (see [31] or [9]), it follows that

$$\begin{aligned}&\Big|\Big\{x\in\mathbb{R}^d\backslash E:\Big|T^n_\Omega\Big(\sum_{\mathbb{L}\in\mathcal{S}}b_{\mathbb{L}}\partial_nA_{\mathbb{L}}\Big)(x)\Big|>\frac{1}{4d}\Big\}\Big|\lesssim\Big\|\sum_{\mathbb{L}\in\mathcal{S}}b_{\mathbb{L}}\partial_nA_{\mathbb{L}}\Big\|_{L^1(\mathbb{R}^d)}\\&\lesssim\sum_{\mathbb{L}\in\mathcal{S}}|\mathbb{L}|\|b_{\mathbb{L}}||_{L\log L,\,\mathbb{L}}\\&\lesssim\int_{\mathbb{R}^d}|f(x)|\log(\mathrm{e}+|f(x)|)dx,\end{aligned}\tag{3.4}$$

where in the last inequality, we have used the fact that $\|b_{\mathbb{L}}\|_{L\log L,\,\mathbb{L}}\lesssim 1$ for each cube $\mathbb{L}\in\mathcal{S}$.

Therefore, to prove inequality (1.6), by (3.1), (3.2) and (3.4), it is sufficient to show that

$$\Big|\Big\{x\in\mathbb{R}^d\backslash E:\Big|\sum_j\sum_s\sum_{\mathbb{L}\in\mathcal{S}_{s-j}}T_{\Omega,A_{\mathbb{L}};s,j}b_{\mathbb{L}}(x)\Big|>1/4\Big\}\Big|\lesssim\|f\|_{L^1(\mathbb{R}^d)}.\tag{3.5}$$

In order to prove inequality (3.5), we first give some estimate for $\sum_s\sum_{\mathbb{L}\in\mathcal{S}_{s-j}}T_{\Omega,\,A_{\mathbb{L}};s,j}b_{\mathbb{L}}$. For this purpose, we need to introduce some notations.

For $\mathbb{L}\in\mathcal{S}_{s-j}$, $s,\ j\in\mathbb{Z}$ with $j\ge\log_2(100d/2)=:j_0$. Let $L_{j,1}=2^{j+2}d\mathbb{L}$, $L_{j,2}=2^{j+4}d\mathbb{L}$, $L_{j,3}=2^{j+6}d\mathbb{L}$, and $y^j_{\mathbb{L}}$ be a point on the boundary of $L_{j,3}$. Set

$$A_{\varphi_{\mathbb{L}}}(y)=\varphi_{\mathbb{L}}(y)\big(A_{\mathbb{L}}(y)-A_{\mathbb{L}}(y^j_{\mathbb{L}})),$$

where $\varphi_{\mathbb{L}}\in C^\infty_c(\mathbb{R}^d)$, $\operatorname{supp}\varphi_{\mathbb{L}}\subset L_{j,1}$, $\varphi_{\mathbb{L}}\equiv 1$ on $3\cdot 2^jd\mathbb{L}$, and $\|\nabla\varphi_{\mathbb{L}}\|_{L^\infty(\mathbb{R}^d)}\lesssim 2^{-s}$. Let y_0 be the center point of $\mathbb{L}$. Observe that for $x\in\mathbb{R}^d\backslash E$, $j\le j_0$, $y\in\mathbb{L}$, we have $|x-y|\ge|x-y_0|-|y-y_0|>2^s$. The support condition of ϕ then implies that $T_{\Omega,A_{\mathbb{L}};s,j}b_{\mathbb{L}}(x)=0$ if $j\le j_0$. For $y\in\mathbb{L}\in\mathcal{S}_{s-j}$, $s,\ j\in\mathbb{Z}$ with $j>j_0$, we have $\varphi_{\mathbb{L}}(y)=1$. By the support condition of ϕ, it follows that $|x-y_0|\le|x-y|+|y-y_0|\le 1.5d2^s$. Hence $x\in 3\cdot 2^jd\mathbb{L}$ and $\varphi_{\mathbb{L}}(x)=1$. Collecting these facts in all, it follows that

$$\phi_s(x-y)(A_{\mathbb{L}}(x)-A_{\mathbb{L}}(y))=\phi_s(x-y)(A_{\varphi_{\mathbb{L}}}(x)-A_{\varphi_{\mathbb{L}}}(y)).$$

The kernel Ω will be decomposed into disjoint forms as in Section 2 as follows:

$$\Omega_0(\theta)=\Omega(\theta)\chi_{E_0}(\theta),\quad \Omega_k(\theta)=\Omega(\theta)\chi_{E_k}(\theta)\ (k\in\mathbb{N}),$$

where $E_0=\{\theta\in\mathbb{S}^{d-1}:\ |\Omega(\theta)|\le 1\}$ and $E_k=\{\theta\in\mathbb{S}^{d-1}:\ 2^{k-1}<|\Omega(\theta)|\le 2^k\}$ for $k\in\mathbb{N}$.

Let the operator $T^i_{\Omega,A_{\mathbb{L}};\,s,j}b_{\mathbb{L}}$ be defined in the same form as $T_{\Omega,A_{\mathbb{L}};\,s,j}b_{\mathbb{L}}$, with Ω replaced by Ω_i. Then we can divide the summation of $T_{\Omega,A_{\mathbb{L}};\,s}b_{\mathbb{L}}$ into two terms as follows

$$\begin{aligned}\sum_{j>j_0}\sum_{s}\sum_{\mathbb{L}\in\mathcal{S}_{s-j}} T_{\Omega,A_{\mathbb{L}};\,s,j}b_{\mathbb{L}}(x) &= \sum_{i=0}^{\infty}\sum_{j>j_0}\sum_{s}\sum_{\mathbb{L}\in\mathcal{S}_{s-j}} T^i_{\Omega,A_{\mathbb{L}};\,s,j}b_{\mathbb{L}}(x)\\ &=\sum_{i=0}^{\infty}\sum_{j_0<j\le Ni}\sum_{s}\sum_{\mathbb{L}\in\mathcal{S}_{s-j}} T^i_{\Omega,A_{\mathbb{L}};\,s,j}b_{\mathbb{L}}(x)\\ &\quad+\sum_{i=0}^{\infty}\sum_{j>Ni}\sum_{s}\sum_{\mathbb{L}\in\mathcal{S}_{s-j}} T^i_{\Omega,A_{\mathbb{L}};\,s,j}b_{\mathbb{L}}(x)\\ &:= \mathrm{D}_1(x)+\mathrm{D}_2(x),\end{aligned}$$

where N is some constant which will be chosen later. If we can verify that

$$\|\mathrm{D}_1\|_{L^1(\mathbb{R}^d)} \lesssim \|\Omega\|_{L(\log L)^2(\mathbb{S}^{d-1})}\|f\|_{L^1(\mathbb{R}^d)}. \tag{3.6}$$

and

$$|[x\in\mathbb{R}^d:\ |D_2(x)|>1/8\}| \lesssim \|f\|_{L^1(\mathbb{R}^d)}, \tag{3.7}$$

the inequality (1.6) then follows directly. The proofs of these two estimate will be given in the next two subsections respectively.

3.2 Proof of Inequality (3.6)

We first claim that if $\mathbb{L}\in\mathcal{S}_{s-j}$, then

$$\left|T^i_{\Omega,A_{\mathbb{L}};\,s,j}b_{\mathbb{L}}(x)\right| \lesssim j\int_{\{2^{s-2}\le|y|\le 2^{s+2}\}}\frac{|\Omega_i(y')|}{|y|^d}|b_{\mathbb{L}}(x-y)|dy.$$

This claim is a consequence of the following lemma, which will also be used several times later.

Lemma 3.1 *Let A be a function in $\mathbb{R}^d$ with derivatives of order one in* $\mathrm{BMO}(\mathbb{R}^d)$. *Let $s,\ j\in\mathbb{Z}$ and $\mathbb{L}\in\mathcal{S}_{s-j}$ with $j>j_0$ and let $R_{s,\mathbb{L};j}(x,y)$ be the function on $\mathbb{R}^d\times\mathbb{R}^d$ defined by*

$$R_{s,\mathbb{L};j}(x,y)=\phi_s(x-y)\frac{A_{\varphi_{\mathbb{L}}}(x)-A_{\varphi_{\mathbb{L}}}(y)}{|x-y|^{d+1}}.$$

Then, $R_{s,\mathbb{L};j}$ enjoys the properties that

(i) *For any $x,\ y\in\mathbb{R}^d$,*

$$\left|R_{s,\mathbb{L};j}(x,y)\right| \lesssim \frac{j}{|x-y|^d}\chi_{\{2^{s-2}\le|x-y|\le 2^{s+2}\}}(x,\ y);$$

(ii) *For any* $x,\ x' \in \mathbb{R}^d$ *and* $y \in \mathbb{L}$ *with* $|x-y| > 2|x-x'|$,

$$\left|R_{s,\mathbb{L};j}(x,y) - R_{s,\mathbb{L};j}(x',y)\right| \lesssim \frac{|x-x'|}{|x-y|^{d+1}}\Big(j + \Big|\log\left(2^{s-j}|x-x'|^{-1}\right)\Big|\Big);$$

(iii) *For any* $x,\ y' \in \mathbb{R}^d$ *and* $y \in \mathbb{L}$ *with* $|x-y| > 2|y-y'|$,

$$\left|R_{s,\mathbb{L};j}(x,y) - R_{s,\mathbb{L};j}(x,\ y')\right| \lesssim \frac{|y-y'|}{|x-y|^{d+1}}\Big(j + \Big|\log\left(2^{s-j}|y-y'|^{-1}\right)\Big|\Big).$$

Proof We first prove (i). It is obvious that supp $R_{s,\mathbb{L};j} \subset L_{j,2} \times L_{j,2}$. Fixed $x \in L_{j,1}$, we know that $2^{s-j} < |x - y^j_{\mathbb{L}}|$ and

$$\left|\langle\nabla A\rangle_{\mathbb{L}} - \langle\nabla A\rangle_{I_{(x,|x-y^j_{\mathbb{L}}|)}}\right| \le \left|\langle\nabla A\rangle_{\mathbb{L}} - \langle\nabla A\rangle_{I_{(x,2^{s-j})}}\right| + \left|\langle\nabla A\rangle_{I_{(x,2^{s-j})}} - \langle\nabla A\rangle_{I_{(x,|x-y^j_{\mathbb{L}}|)}}\right|.$$

Note that if $x \in 4\mathbb{L}$, then $I_{(x,2^{s-j})} \subset 8\mathbb{L}$ and it holds that

$$\left|\langle\nabla A\rangle_{\mathbb{L}} - \langle\nabla A\rangle_{I_{(x,2^{s-j})}}\right| \le \left|\langle\nabla A\rangle_{\mathbb{L}} - \langle\nabla A\rangle_{8\mathbb{L}}\right| + \left|\langle\nabla A\rangle_{8\mathbb{L}} - \langle\nabla A\rangle_{I_{(x,2^{s-j})}}\right| \lesssim 1.$$

If $x \in L_{j,1}\backslash 4\mathbb{L}$, then the center of $\mathbb{L}$ and the center of $I_{(x,2^{s-j})}$ are at a distance of $a2^{s-j}$ with $a > 1$. Hence, the results in [13, Proposition 3.1.5, p. 158 and 3.1.5–3.1.6, p. 166.] gives that

$$\left|\langle\nabla A\rangle_{\mathbb{L}} - \langle\nabla A\rangle_{I_{(x,2^{s-j})}}\right| \lesssim j \quad \text{and} \quad \left|\langle\nabla A\rangle_{I_{(x,2^{s-j})}} - \langle\nabla A\rangle_{I_{(x,|x-y^j_{\mathbb{L}}|)}}\right| \lesssim j,$$

since $2^s < |x - y^j_{\mathbb{L}}| < 2^{s+5+d^2}$.

Therefore, for $x \in \mathbb{L}_{j,1}$, it holds that

$$\left|\langle\nabla A\rangle_{\mathbb{L}} - \langle\nabla A\rangle_{I_{(x,|x-y^j_{\mathbb{L}}|)}}\right| \lesssim j. \tag{3.8}$$

Lemma 2.6, together with John-Nirenberg inequality then gives that

$$|A_{\varphi_{\mathbb{L}}}(x)| \lesssim |x - y^j_{\mathbb{L}}|\Big(\frac{1}{|I_{(x,|x-y^j_{\mathbb{L}}|)}|}\int_{I_{(x,|x-y^j_{\mathbb{L}}|)}} |\nabla A(z) - \langle\nabla A\rangle_{\mathbb{L}}|^q dz\Big)^{1/q} \lesssim j2^s, \tag{3.9}$$

which finishes the proof of (i).

Now we give the proof of (ii). For any $x,\ x' \in \mathbb{R}^d$ and $y \in \mathbb{L}$ with $|x-y| > 2|x-x'|$, it is easy to see that

(1) if $x \notin L_{j,1}$ and $x' \notin L_{j,1}$, then $R_{s,\mathbb{L};j}(x,y) = R_{s,\mathbb{L};j}(x',y) = 0$;
(2) if $x \notin L_{j,1}$, then $x' \notin 3 \cdot 2^j d\mathbb{L}$, hence $R_{s,\mathbb{L};j}(x,y) = R_{s,\mathbb{L};j}(x',y) = 0$;
(3) if $x' \notin L_{j,1}$, then $x \notin 3 \cdot 2^j d\mathbb{L}$, hence $R_{s,\mathbb{L};j}(x,y) = R_{s,\mathbb{L};j}(x',y) = 0$.

If $z \in I_{(x,|x-x'|)}$, another application of Lemma 2.6 and John-Nirenberg inequality indicates

$$\begin{aligned}|\nabla A_{\varphi_{\mathbb{L}}}(z)| &\lesssim 2^{-s}|A_{\mathbb{L}}(z) - A_{\mathbb{L}}(y_L^j)| + |\nabla A(z) - \langle \nabla A\rangle_{\mathbb{L}}| \\ &\lesssim j + |\nabla A(z) - \langle \nabla A\rangle_{\mathbb{L}}|,\end{aligned} \tag{3.10}$$

and the similar method as what was used in the proof of (3.8) further implies that

$$\begin{aligned}|\langle \nabla A\rangle_{\mathbb{L}} - \langle \nabla A\rangle_{I_{(x,|x-x'|)}}| &\le |\langle \nabla A\rangle_{\mathbb{L}} - \langle \nabla A\rangle_{I_{(x,2^{s-j})}}| + |\langle \nabla A\rangle_{I_{(x,|x-x'|)}} \langle \nabla A\rangle_{I_{(x,2^{s-j})}}| \\ &\lesssim \log 2^j + \Big|\log\big(2^{s-j}|x-x'|^{-1}\big)\Big|.\end{aligned} \tag{3.11}$$

By Lemma 2.6, (3.10) and (3.11), we have

$$\begin{aligned}|A_{\varphi_{\mathbb{L}}}(x) - A_{\varphi_{\mathbb{L}}}(x')| &\lesssim |x-x'|\Big(\frac{1}{|I_{(x,|x-x'|)}|}\int_{I_{(x,|x-x'|)}} |\nabla A_{\varphi_{\mathbb{L}}}(z)|^q dz\Big)^{\frac{1}{q}} \\ &\lesssim |x-x'|\Big(j + \frac{1}{|I_{(x,|x-x'|)}|}\int_{I_{(x,|x-x'|)}} |\nabla A(z) - \langle \nabla A_L\rangle|^q dz\Big)^{\frac{1}{q}}\end{aligned} \tag{3.12}$$

Similarly, we obtain

$$\begin{aligned}|A_{\varphi_{\mathbb{L}}}(x) - A_{\varphi_{\mathbb{L}}}(x')| &\lesssim |x-x'|\big(j + |\langle \nabla A\rangle_{\mathbb{L}} - \langle \nabla A\rangle_{I_{(x,|x-x'|)}}|\big) \\ &\lesssim |x-x'|\Big[j + \Big|\log\big(2^{s-j}|x-x'|^{-1}\big)\Big|\Big].\end{aligned} \tag{3.13}$$

In a similar way, we have

$$\begin{aligned}|A_{\varphi_{\mathbb{L}}}(x) - A_{\varphi_{\mathbb{L}}}(y)| &\lesssim |x-y|\Big[j + \Big|\log\big(2^{s-j}|x-y|^{-1}\big)\Big|\Big]; \\ |A_{\varphi_{\mathbb{L}}}(x') - A_{\varphi_{\mathbb{L}}}(y)| &\lesssim |x'-y|\Big[j + \Big|\log\big(2^{s-j}|x'-y|^{-1}\big)\Big|\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}&|R_{s,\mathbb{L};j}(x,\ y) - R_{s,\mathbb{L};j}(x',\ y)| \\ &\quad\le |\phi_s(x-y)|\Big|\frac{A_{\varphi_{\mathbb{L}}}(x) - A_{\varphi_{\mathbb{L}}}(y)}{|x-y|^{d+1}} - \frac{A_{\varphi_{\mathbb{L}}}(x') - A_{\varphi_{\mathbb{L}}}(y)}{|x'-y|^{d+1}}\Big| \\ &\quad+ \frac{|A_{\varphi_{\mathbb{L}}}(x') - A_{\varphi_{\mathbb{L}}}(y)|}{|x'-y|^{d+1}}\big|\phi_s(x-y) - \phi_s(x'-y)\big| \\ &\lesssim \frac{|x-x'|}{|x-y|^{d+1}}\Big(j + \Big|\log\big(2^{s-j}|x-x'|^{-1}\big)\Big|\Big).\end{aligned}$$

This completes the proof of (ii) in Lemma 3.1. (iii) can be proved in the same way as (ii). □

Let us turn back to the contribution of D_1. It follows from the method of rotation of Calderón-Zygmund that

$$\begin{aligned}\|\mathrm{D}_1\|_{L^1(\mathbb{R}^d)} &= \sum_{i=0}^{\infty} \sum_{j_0<j\le Ni} \sum_{s} \sum_{\mathbb{L}\in\mathcal{S}_{s-j}} \|T^i_{\Omega,A_{\mathbb{L}};\,s,j} b_{\mathbb{L}}(x)\|_{L^1(\mathbb{R}^d)} \\ &\lesssim \sum_{i=0}^{\infty} \sum_{j_0<j\le Ni} \sum_{s} \sum_{\mathbb{L}\in\mathcal{S}_{s-j}} j \\ &\quad \int_{\mathbb{R}^d} \int_{2^{s-2}}^{2^{s+2}} \int_{\mathbb{S}^{d-1}} \frac{|\Omega_i(y')|}{|r|} |b_{\mathbb{L}}(x-ry')|\, dy'\, dr\, dx \\ &\lesssim \sum_{i=0}^{\infty} \|\Omega_i\|_{L^1(\mathbb{S}^{d-1})} \sum_{j_0<j\le Ni} j \sum_{s} \sum_{\mathbb{L}\in\mathcal{S}_{s-j}} \|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)} \\ &\lesssim \|\Omega\|_{L(\log L)^2(\mathbb{S}^{d-1})} \|f\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

This verifies (3.6).

3.3 Proof of the Inequality (3.7)

The estimate of D_2 is long and complicated. We split the proof into three steps.

Step 1. A reduction for the estimate of D_2.

Let $l_\tau(j) = \tau j + 3$, where $0 < \tau < 1$ will be chosen later. Let ω be a nonnegative, radial $C_c^\infty(\mathbb{R}^d)$ function which is supported in $\{x \in \mathbb{R}^d : |x| \le 1\}$ and has integral 1. Set $\omega_t(x) = 2^{-td}\omega(2^{-t}x)$. For $s \in \mathbb{N}$ and a cube $\mathbb{L}$, we define $R^j_{s,\mathbb{L}}$ as

$$R^j_{s,\mathbb{L}}(x, y) = \int_{\mathbb{R}^d} \omega_{s-l_\tau(j)}(x-z) \frac{1}{|z-y|^{d+1}} \phi_s(z-y)\big(A_{\varphi_{\mathbb{L}}}(z) - A_{\varphi_{\mathbb{L}}}(y)\big)dz. \tag{3.14}$$

It is obvious that $\operatorname{supp} R^j_{s,\mathbb{L}}(x, y) \subset \{(x,\ y) : 2^{s-3} \le |x-y| \le 2^{s+3}\}$. Moreover, if $y \in \mathbb{L}$ with $\mathbb{L} \in \mathcal{S}_{s-j}$, then (i) of Lemma 3.1 implies that

$$|R^j_{s,\mathbb{L}}(x, y)| \lesssim j2^{-sd} \chi_{\{2^{s-3}\le|x-y|\le 2^{s+3}\}}(x,\ y). \tag{3.15}$$

We define the operator $T^{i,j}_{\Omega,\mathbb{L};\,s}$ by

$$T^{i,j}_{\Omega,\mathbb{L};\,s} h(x) = \int_{\mathbb{R}^d} \Omega_i(x-y) R^j_{s,\mathbb{L}}(x, y) h(y) dy,$$

and let D_2^* be the operator as follows

$$\mathrm{D}_2^*(x) = \sum_{i=0}^{\infty} \sum_{j>Ni} \sum_{s} \sum_{\mathbb{L}\in\mathcal{S}_{s-j}} T^{i,j}_{\Omega,\mathbb{L};\,s} b_{\mathbb{L}}(x).$$

The following lemma indicates the intrinsically close relationship in each subtract terms between D_2 and D_2^*. Thus, the corresponding proof is transferred to verify it for each term of D_2^*.

Lemma 3.2 *Let Ω be homogeneous of degree zero, A be a function on $\mathbb{R}^d$ with derivatives of order one in* $\mathrm{BMO}(\mathbb{R}^d)$. *For $j > j_0$ and $i \geq 0$, it holds that*

$$\|T^i_{\Omega,A_{\mathbb{L}};\,s,j} b_{\mathbb{L}} - T^{i,j}_{\Omega,\mathbb{L};\,s} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)} \lesssim j2^{-\tau j}\|\Omega_i\|_{L^1(\mathbb{S}^{d-1})}\|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.$$

Proof For each $y \in \mathbb{L}$ and $z \in \operatorname{supp}\omega_{s-l_\tau(j)}$, notice that $R_{s,\mathbb{L};j}(x,\ y) - R_{s,\mathbb{L};j}(x-z,\ y) = 0$ if $x \in L_{j,1}\backslash 3\cdot 2^j d\mathbb{L}$. In fact, since $|z| \leq 2^{s-\tau j-3}$, then we have $2^{s+1} < |x-y| < 3\cdot 2^s$ and $2^s < |x-y-z| < 2^{s+2}$.

By Lemma 3.1, we have

$$\left|R_{s,\mathbb{L};j}(x,\ y) - R_{s,\mathbb{L};j}(x-z,\ y)\right| \lesssim \frac{|z|}{2^{s(d+1)}}\left[j + \log\left(\frac{2^{s-j}}{|z|}\right)\right]\chi_{\{2^{s-2}\leq|x-y|\leq 2^{s+2}\}}(x,\ y).$$

Observing that the function $\Theta(t) = t\log(\mathrm{e} + \frac{1}{t})$ is bounded at $t \in (0,\ 1]$, and then for $0 < t \leq r$,

$$t\log\left(\mathrm{e} + \frac{r}{t}\right) \lesssim r,$$

we deduce that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \omega_{s-l_\tau(j)}(z)\Big(R_{s,\mathbb{L};j}(x,\ y) - R_{s,\mathbb{L};j}(x-z,\ y)\Big)dz\right| \\
&\quad\lesssim 2^{(-s+\tau j)d}\int_{\{|z|\leq 2^{s-\tau j}\}} \frac{|z|}{2^{s(d+1)}}\left[j + \log\left(\frac{2^{s-\tau j}}{|z|}\right)\right]dz \lesssim j2^{-sd-\tau j}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|T^i_{\Omega,A_{\mathbb{L}};\,s,j} b_{\mathbb{L}} - T^{i,j}_{\Omega,\mathbb{L};\,s} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)} \\
&\quad\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\Omega_i(x-y)|\left|\int_{\mathbb{R}^d} \omega_{s-l_\tau(j)}(z)\Big(R_{s,\mathbb{L};j}(x,\ y)\right. \\
&\qquad \left.- R_{s,\mathbb{L};j}(x-z,\ y)\Big)dz\right||b_{\mathbb{L}}(y)|dydx \\
&\quad\lesssim j2^{-sd-\tau j}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\Omega_i(y)|\chi_{\{2^{s-2}\leq|y|\leq 2^{s+2}\}}(y)|b_{\mathbb{L}}(x-y)|dydx \\
&\quad\lesssim j2^{-\tau j}\|\Omega_i\|_{L^1(\mathbb{S}^{d-1})}\|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

This leads to the desired conclusion of Lemma 3.2. □

With Lemma 3.2 in hand, we only need to estimate D_2^*. This is the content of the second step.

Step 2. Estimate for each term of D_2^*.

Define $P_t f(x) = \omega_t * f(x)$. Now we split

$$T^{i,j}_{\Omega,\mathbb{L};\,s} = P_{s-j\kappa} T^{i,j}_{\Omega,\mathbb{L};\,s} + (I - P_{s-j\kappa}) T^{i,j}_{\Omega,\mathbb{L};\,s},$$

where $\kappa \in (0, 1)$ will be chosen later. In the following, we will estimate this two terms one by one. We have the following norm inequality for $P_{s-j\kappa} T^{i,j}_{\Omega,\mathbb{L};\,s}$.

Lemma 3.3 *Let Ω be homogeneous of degree zero, A be a function in $\mathbb{R}^d$ with derivatives of order one in $\mathrm{BMO}(\mathbb{R}^d)$, $b_{\mathbb{L}}$ satisfies the vanishing moment with $\ell(\mathbb{L}) = 2^{s-j}$. For each $j \in \mathbb{N}$ with $j > j_0$, we have*

$$\begin{aligned}&\left\| P_{s-j\kappa} T^{i,j}_{\Omega,\mathbb{L};\,s} b_{\mathbb{L}} \right\|_{L^1(\mathbb{R}^d)}\\ &\lesssim j\big(2^{-(1-\kappa)j} + 2^{-(1-\tau)j}\big) \|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})} \|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Before proving Lemma 3.3, we need the following lemma for $R^j_{s,\mathbb{L}}$.

Lemma 3.4 *Let $R^j_{s,\mathbb{L}}$ be defined as (3.14), $\theta \in \mathbb{S}^{d-1}$, $y,\ y' \in \mathbb{L}$ with $\ell(\mathbb{L}) = 2^{s-j}$. Then*

$$\begin{aligned}&\int_{\mathbb{L}} \int_{\mathbb{L}} |R^j_{s,\mathbb{L}}(y + r\theta,\ y) - R^j_{s,\mathbb{L}}(y' + r\theta,\ y')| |b_{\mathbb{L}}(y)| dy dy'\\ &\lesssim j 2^{-sd} 2^{\tau j} 2^{-j} |\mathbb{L}| \int_{\mathbb{L}} |b_{\mathbb{L}}(y)| dy.\end{aligned}$$

Proof By the triangle inequality, the mean value theorem and the support condition of ϕ, we get

$$\begin{aligned}&|R^j_{s,\mathbb{L}}(y' + r\theta,\ y) - R^j_{s,\mathbb{L}}(y' + r\theta,\ y')|\\ &\lesssim \int_{\mathbb{R}^d} |\omega_{s-l_\tau(j)}(y' + r\theta - z)| |\phi_s(z - y')| \frac{|A_{\varphi_{\mathbb{L}}}(y) - A_{\varphi_{\mathbb{L}}}(y')|}{|z - y'|^{d+1}} dz\\ &\quad + \int_{\mathbb{R}^d} |\omega_{s-l_\tau(j)}(y' + r\theta - z)| \frac{|A_{\varphi_{\mathbb{L}}}(z) - A_{\varphi_{\mathbb{L}}}(y)|}{|z - y|^{d+1}} |\phi_s(z - y) - \phi_s(z - y')| dz\\ &\quad + \int_{\mathbb{R}^d} |\omega_{s-l_\tau(j)}(y' + r\theta - z)| |\phi_s(z - y')| \frac{|A_{\varphi_{\mathbb{L}}}(z) - A_{\varphi_{\mathbb{L}}}(y)| |y - y'|}{|z - y|^{d+2}} dz\\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

If $r \notin [2^{s-4}, 2^{s+4}]$, by the support of $R^j_{s,\mathbb{L}}$, it gives that $|R^j_{s,\mathbb{L}}(y' + r\theta,\ y) - R^j_{s,\mathbb{L}}(y' + r\theta,\ y')| = 0$.

For $y,\ y' \in \mathbb{L}$, (3.13) gives us that

$$|A_{\varphi_{\mathbb{L}}}(y') - A_{\varphi_{\mathbb{L}}}(y)| \lesssim |y - y'|\left[j + \left|\log\left(\frac{2^{s-j}}{|y-y'|}\right)\right|\right].$$

For I, since $|z - y'| \geq 2^{s-2}$, $y,\ y' \in \mathbb{L}$, then (3.13) gives us that

$$\mathrm{I} \lesssim |y - y'|\left[j + \left|\log\left(\frac{2^{s-j}}{|y-y'|}\right)\right|\right]2^{-s(d+1)}.$$

Consider now the other two terms. If $y,\ y' \in \mathbb{L}$ and $|z - y'| \leq 2^s$, (i) of Lemma 3.1 gives us that

$$|A_{\varphi_{\mathbb{L}}}(y)| \lesssim j2^s, \quad |A_{\varphi_{\mathbb{L}}}(z)| \lesssim j2^s.$$

On the other hand, for $j > j_0$, when $y,\ y' \in \mathbb{L}$ and $|z - y'| \geq 2^{s-2}$, it holds that

$$|z - y| \geq |z - y'| - |y - y'| \geq 2^{s-2} - \sqrt{d}2^{s-j} > 2^{s-2} - \sqrt{d}2^{s-\log_2(100d/2)} > 2^{s-3}.$$

Therefore,

$$\begin{aligned}\mathrm{II} &\lesssim \frac{j2^s}{(2^s)^{d+1}}\int_{\mathbb{R}^d}|\omega_{s-l_\tau(j)}(y' + r\theta - z)||\phi_s(z - y) - \phi_s(z - y')|dz\\ &\lesssim j2^{-s(d+1)}|y - y'|\int_{\mathbb{R}^d}|\omega_{s-l_\tau(j)}(y' + r\theta - z)|dz \lesssim j2^{-s(d+1)}|y - y'|,\end{aligned}$$

where the second inequality follows from the fact that

$$|\phi_s(z - y) - \phi_s(z - y')| \lesssim \frac{|y - y'|}{2^s}\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{|y - y'|}{2^s}.$$

Similarly, we have

$$\mathrm{III} \lesssim j2^{-s(d+1)}|y - y'|.$$

Estimates for I, II and III above lead to that

$$|R^j_{s,\mathbb{L}}(y' + r\theta,\ y) - R^j_{s,\mathbb{L}}(y' + r\theta,\ y')| \lesssim \frac{j|y - y'|}{2^{s(d+1)}}\left[1 + \left|\log\left(\frac{2^{s-j}}{|y-y'|}\right)\right|\right]. \tag{3.16}$$

Similar to (3.16), we also have

$$\begin{aligned}&|R^j_{s,\mathbb{L}}(y + r\theta,\ y) - R^j_{s,\mathbb{L}}(y' + r\theta,\ y)|\\ &\quad\leq \int_{\mathbb{R}^d}|\omega_{s-l_\tau(j)}(y + r\theta - z) - \omega_{s-l_\tau(j)}(y' + r\theta - z)||\phi_s(z - y)|\end{aligned}$$

$$\frac{|A_{\varphi_{\mathbb{L}}}(z) - A_{\varphi_{\mathbb{L}}}(y)|}{|z-y|^{d+1}} dz$$
$$\le j 2^{-sd} 2^{-s+l_\tau(j)} |y-y'| \int_{\mathbb{R}^d} |\nabla \omega_{s-l_\tau(j)}(z)| dz$$
$$\lesssim j|y-y'| 2^{-s+l_\tau(j)} 2^{-sd}. \tag{3.17}$$

Notice that

$$\int_{\mathbb{L}} \int_{\mathbb{L}} |y-y'| \Big[1 + \Big|\log\Big(\frac{2^{s-j}}{|y-y'|}\Big)\Big|\Big] dy' |b_{\mathbb{L}}(y)| dy \le 2^{s-j} |\mathbb{L}| \int_{\mathbb{L}} |b_{\mathbb{L}}(y)| dy.$$

Combining (3.16) with (3.17), it gives that

$$\int_{\mathbb{L}} \int_{\mathbb{L}} |R^j_{s,\mathbb{L}}(y + r\theta,\ y) - R^j_{s,\mathbb{L}}(y' + r\theta,\ y')| |b_{\mathbb{L}}(y)| dy dy'$$
$$\lesssim j 2^{-s(d+1)} 2^{l_\tau(j)} \int_{\mathbb{L}} \int_{\mathbb{L}} |y-y'| dy' |b_{\mathbb{L}}(y)| dy$$
$$+ j 2^{-s(d+1)} \int_{\mathbb{L}} \int_{\mathbb{L}} |y-y'| \Big[1 + \Big|\log\Big(\frac{2^{s-j}}{|y-y'|}\Big)\Big|\Big] dy' |b_{\mathbb{L}}(y)| dy$$
$$\lesssim j 2^{-sd} 2^{l_\tau(j)} 2^{-j} |\mathbb{L}| \int_{\mathbb{L}} |b_{\mathbb{L}}(y)| dy.$$

This finishes the proof of Lemma 3.4. □

With Lemma 3.4, we are ready to prove Lemma 3.3 now.

Proof of Lemma 3.3 Write

$$P_{s-j\kappa} T^{i,j}_{\Omega,\mathbb{L};\,s} b_{\mathbb{L}}(x) = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \omega_{s-j\kappa}(x-z) \Omega_i(z-y) R^j_{s,\mathbb{L}}(z, y) dz \Big) b_{\mathbb{L}}(y) dy.$$

Let $z - y = r\theta$. By Fubini's theorem, $P_{s-j\kappa} T^{i,j}_{\Omega,\mathbb{L};\,s} b_{\mathbb{L}}(x)$ can be written as

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \int_0^\infty \Omega_i(\theta) \omega_{s-j\kappa}(x-y-r\theta) R^j_{s,\mathbb{L}}(y + r\theta, y) r^{d-1} b_{\mathbb{L}}(y) dr dy d\sigma_\theta.$$

Let $y' \in \mathbb{L}$. By the vanishing moment of $b_{\mathbb{L}}$, we have

$$|P_{s-j\kappa} T^{i,j}_{\Omega_i,\mathbb{L};\,s} b_L(x)|$$
$$\le \inf_{y'\in\mathbb{L}} \int_{\mathbb{S}^{d-1}} |\Omega_i(\theta)| \Big| \int_{\mathbb{R}^d} \int_0^\infty \Big(\omega_{s-j\kappa}(x-y-r\theta) R^j_{s,\mathbb{L}}(y + r\theta, y)$$
$$- \omega_{s-j\kappa}(x-y'-r\theta) R^j_{s,\mathbb{L}}(y' + r\theta, y')\Big) r^{d-1} dr b_{\mathbb{L}}(y) dy \Big| d\sigma_\theta$$
$$\le \int_{\mathbb{S}^{d-1}} |\Omega_i(\theta)| \frac{1}{|\mathbb{L}|} \int_{\mathbb{L}} \Big| \int_{\mathbb{R}^d} \int_0^\infty \Big(\omega_{s-j\kappa}(x-y-r\theta) R^j_{s,\mathbb{L}}(y + r\theta, y)$$

$$- \omega_{s-j\kappa}(x - y' - r\theta) R^j_{s,\mathbb{L}}(y' + r\theta, y') \Big) r^{d-1} dr b_{\mathbb{L}}(y) dy \Big| dy' d\sigma_\theta$$
$$\lesssim \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} =: \frac{1}{|\mathbb{L}|} \int_{\mathbb{S}^{d-1}} |\Omega_i(\theta)| \int_{\mathbb{L}} \Big| \int_{\mathbb{R}^d} \int_0^\infty \Big(\omega_{s-j\kappa}(x - y - r\theta) - \omega_{s-j\kappa}(x - y' - r\theta) \Big) \times R^j_{s,\mathbb{L}}(y + r\theta, y) r^{d-1} dr b_{\mathbb{L}}(y) dy \Big| dy' d\sigma_\theta,$$

and

$$\mathrm{II} =: \frac{1}{|\mathbb{L}|} \int_{\mathbb{S}^{d-1}} |\Omega_i(\theta)| \int_{\mathbb{L}} \Big| \int_{\mathbb{R}^d} \int_0^\infty \omega_{s-j\kappa}(x - y' - r\theta) \big(R^j_{s,\mathbb{L}}(y + r\theta, y) - R^j_{s,\mathbb{L}}(y' + r\theta, y') \big) r^{d-1} dr b_{\mathbb{L}}(y) dy \Big| dy' \, d\sigma_\theta.$$

Note that $|y - y'| \lesssim 2^{s-j}$, when $y, y' \in \mathbb{L}$. By (3.15) and the mean value formula, it follows that

$$\|I\|_{L^1(\mathbb{R}^d)} \lesssim j \int_{\mathbb{S}^{d-1}} |\Omega_i(\theta)| \int_{\mathbb{R}^d} \int_{2^{s-3}}^{2^{s+3}} 2^{-s+j\kappa} \|\nabla \omega\|_{L^1(\mathbb{R}^d)} 2^{s-j} 2^{-sd} r^{d-1} dr |b_{\mathbb{L}}(y)| dy d\sigma(\theta)$$
$$\lesssim j 2^{-(1-\kappa)j} \|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})} \|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.$$

By Lemma 3.4 and the Fubini's theorem one can get

$$\|II\|_{L^1(\mathbb{R}^d)} \lesssim \int_{\mathbb{S}^{d-1}} \int_{2^{s-3}}^{2^{s+3}} |\Omega_i(\theta)| \frac{1}{|\mathbb{L}|} \int_{\mathbb{L}} \int_{\mathbb{L}} \|\omega_{s-j\kappa}(\cdot - y' - r\theta)\|_{L^1(\mathbb{R}^d)} \times |\big(R^j_{s,\mathbb{L}}(y + r\theta, y) - R^j_{s,\mathbb{L}}(y' + r\theta, y') \big)| |b_{\mathbb{L}}(y)| dy dy' \, r^{d-1} dr d\sigma_\theta$$
$$\lesssim j 2^{-(1-\tau)j} \|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})} \|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.$$

This finishes the proof of Lemma 3.3. □

To estimate the term $(I - P_{s-j\kappa}) T^{i,j}_{\Omega,\mathbb{L};\,s}$, we introduce a partition of unity on the unit surface $\mathbb{S}^{d-1}$. For $j > j_0$, let $\mathfrak{E}^j = \{e^j_\nu\}$ be a collection of unit vectors on $\mathbb{S}^{d-1}$ such that

(a) for $\nu \neq \nu'$, $|e^j_\nu - e^j_{\nu'}| > 2^{-j\gamma-4}$;
(b) for each $\theta \in \mathbb{S}^{d-1}$, there exists an e^j_ν satisfying that $|e^j_\nu - \theta| \leq 2^{-j\gamma-4}$, where $\gamma \in (0,\ 1)$ is a constant.

The set $\mathfrak{E}^j$ can be constructed as in [31]. Observe that $\mathrm{card}(\mathfrak{E}^j) \lesssim 2^{j\gamma(d-1)}$.

Below, we will construct an associated partition of unity on the unit surface $\mathbb{S}^{d-1}$. Let ζ be a smooth, nonnegative, radial function with $\zeta(u) \equiv 1$ when $|u| \le 1/2$ and $\mathrm{supp}\,\zeta \subset \{|x| \le 1\}$. Set

$$\widetilde{\Gamma}_\nu^j(\xi) = \zeta\left(2^{j\gamma}\left(\frac{\xi}{|\xi|} - e_\nu^j\right)\right), \text{ and } \Gamma_\nu^j(\xi) = \widetilde{\Gamma}_\nu^j(\xi)\left(\sum_{e_\nu^j \in \mathfrak{E}^j} \widetilde{\Gamma}_\nu^j(\xi)\right)^{-1}.$$

It is easy to verify that Γ_ν^j is homogeneous of degree zero, and for all j and $\xi \in \mathbb{S}^{d-1}$, $\sum_\nu \Gamma_\nu^j(\xi) = 1$. Let $\widetilde{\psi} \in C_c^\infty(\mathbb{R})$ such that $0 \le \widetilde{\psi} \le 1$, $\mathrm{supp}\,\widetilde{\psi} \subset [-4, 4]$ and $\widetilde{\psi}(t) \equiv 1$ when $t \in [-2, 2]$. Define the multiplier operator G_ν^j by

$$\widehat{G_\nu^j f}(\xi) = \widetilde{\psi}\big(2^{j\gamma}\langle \xi/|\xi|, e_\nu^j\rangle\big)\widehat{f}(\xi).$$

Denote the operator $T_{\Omega,\mathbb{L};\,s,\nu}^{i,j}$ by

$$T_{\Omega,\mathbb{L};\,s,\nu}^{i,j}h(x) = \int_{\mathbb{R}^d} \Omega_i(x-y)\Gamma_\nu^j(x-y)R_{s,\mathbb{L}}^j(x,y)h(y)dy. \tag{3.18}$$

It is obvious that $T_{\Omega,\mathbb{L};\,s}^{i,j}h(x) = \sum_\nu T_{\Omega,\mathbb{L};\,s,\nu}^{i,j}h(x)$. For each fixed $i,\ s,\ j,\ \mathbb{L}$ and ν, $(I - P_{s-j\kappa})T_{\Omega,\mathbb{L};\,s,\nu}^{i,j}$ can be decomposed in the following way

$$(I - P_{s-j\kappa})T_{\Omega,\mathbb{L};\,s,\nu}^{i,j} = G_\nu^j(I - P_{s-j\kappa})T_{\Omega,\mathbb{L};\,s,\nu}^{i,j} + (1 - G_\nu^j)(I - P_{s-j\kappa})T_{\Omega,\mathbb{L};\,s,\nu}^{i,j}.$$

Estimate for the term $G_\nu^j(I - P_{s-j\kappa})T_{\Omega,\mathbb{L};\,s,\nu}^{i,j}$.

For the term $G_\nu^j(I - P_{s-j\kappa})T_{\Omega,\mathbb{L};\,s,\nu}^{i,j}$, we have the following lemma.

Lemma 3.5 *Let Ω be homogeneous of degree zero, A be a function in $\mathbb{R}^d$ with derivatives of order one in* $\mathrm{BMO}(\mathbb{R}^d)$*. For each $j \in \mathbb{N}$ with $j > j_0$, we have that,*

$$\begin{aligned}&\Big\|\sum_\nu\sum_s\sum_{\mathbb{L}\in\mathcal{S}_{s-j}} G_\nu^j(I - P_{s-j\kappa})T_{\Omega,\mathbb{L};\,s,\nu}^{i,j}b_{\mathbb{L}}\Big\|_{L^2(\mathbb{R}^d)}^2\\ &\quad\lesssim j^2 2^{-j\gamma}\|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})}^2\sum_s\sum_{\mathbb{L}\in\mathcal{S}_{s-j}}\|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Proof The proof is similar to the proof of Lemma 2.3 in [9]. For the sake of self-contained, we present the proof here. Observe that

$$\sup_{\xi\neq 0}\sum_\nu |\widetilde{\psi}(2^{j\gamma}\langle e_\nu^j, \xi/|\xi|\rangle)|^2 \lesssim 2^{j\gamma(d-2)}.$$

This, together with Plancherel's theorem and Cauchy-Schwartz inequality, leads to that

$$\begin{aligned}&\Big\|\sum_{\nu}\sum_{s}\sum_{\mathbb{L}\in\mathcal{S}_{s-j}} G_\nu^j(I-P_{s-j\kappa})T^{i,j}_{\Omega,\mathbb{L};\,s,\nu}b_{\mathbb{L}}\Big\|^2_{L^2(\mathbb{R}^d)}\\ &=\Big\|\sum_{\nu}\widetilde{\psi}\Big(2^{j\gamma}\langle e_\nu^j,\,\xi/|\xi|\rangle\Big)\mathcal{F}\Big(\sum_{s}\sum_{\mathbb{L}\in\mathcal{S}_{s-j}}(I-P_{s-j\kappa})T^{i,j}_{\Omega,\mathbb{L};\,s,\nu}b_{\mathbb{L}}\Big)(\xi)\Big\|^2_{L^2(\mathbb{R}^d)}\\ &\lesssim 2^{j\gamma(d-2)}\sum_{\nu}\Big\|\sum_{s}\sum_{\mathbb{L}\in\mathcal{S}_{s-j}}(I-P_{s-j\kappa})T^{i,j}_{\Omega,\mathbb{L};\,s,\nu}b_{\mathbb{L}}\Big\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Applying (3.15), we see that for each fixed s, $\mathbb{L}$, and $x\in\mathbb{R}^d$,

$$\begin{aligned}&\big|(I-P_{s-j\kappa})T^{i,j}_{\Omega,\mathbb{L};\,s,\nu}b_{\mathbb{L}}(x)\big|\\ &\lesssim\int_{\mathbb{R}^d}|\Omega_i(x-y)||\Gamma_\nu^j(x-y)||R^j_{s,\mathbb{L}}(x,y)||b_{\mathbb{L}}(y)|dy\\ &\quad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\Omega_i(z-y)||\omega_{s-j\kappa}(x-z)||\Gamma_\nu^j(z-y)||R^j_{s,\mathbb{L}}(z,y)|dz|b_{\mathbb{L}}(y)|dy\\ &\lesssim j\|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})}H^j_{s,\nu}*|b_{\mathbb{L}}|(x),\end{aligned}\tag{3.19}$$

where $H^j_{s,\nu}(x)=2^{-sd}\chi_{\mathcal{R}^j_{s\nu}}(x)$, and $\mathcal{R}^j_{s\nu}=\{x\in\mathbb{R}^d:\ |\langle x,e_\nu^j\rangle|\le 2^{s+3},\ |x-\langle x,e_\nu^j\rangle e_\nu^j|\le 2^{s+3-j\gamma}\}$. This means that $\mathcal{R}^j_{s\nu}$ is contained in a box having one long side of length $\lesssim 2^s$ and $(d-1)$ short sides of length $\lesssim 2^{s-j\gamma}$. Therefore, we have

$$\begin{aligned}&\Big\|\sum_{s}\sum_{\mathbb{L}\in\mathcal{S}_{s-j}}(I-P_{s-j\kappa})T^{i,j}_{\Omega,\mathbb{L};\,s,\nu}b_{\mathbb{L}}\Big\|^2_{L^2(\mathbb{R}^d)}\\ &\lesssim j^2\|\Omega_i\|^2_{L^\infty(\mathbb{S}^{d-1})}\sum_{s}\sum_{\mathbb{L}\in\mathcal{S}_{s-j}}\sum_{I\in\mathcal{S}_{s-j}}\int_{\mathbb{R}^d}\big(H^j_{s,\nu}*H^j_{s,\nu}*|b_I|\big)(x)|b_{\mathbb{L}}(x)|dx\\ &\quad+2j^2\|\Omega_i\|^2_{L^\infty(\mathbb{S}^{d-1})}\sum_{s}\sum_{\mathbb{L}\in\mathcal{S}_{s-j}}\sum_{i<s}\sum_{I\in\mathcal{S}_{i-j}}\int_{\mathbb{R}^d}\big(H^j_{s,\nu}*H^j_{i,\nu}*|b_I|\big)(x)|b_{\mathbb{L}}(x)|dx.\end{aligned}$$

Let $\widetilde{\mathcal{R}}^j_{s\nu}=\mathcal{R}^j_{s\nu}+\mathcal{R}^j_{s\nu}$. As in the proof of Lemma 2.3 in [9], for each fixed $\mathbb{L}\in\mathcal{S}_{s-j}$, $x\in\mathbb{L}$, ν and s, we obtain

$$\begin{aligned}\sum_{i\le s}\sum_{I\in\mathcal{S}_{i-j}}H^j_{s,\nu}*H^j_{i,\nu}*|b_I|(x)&\lesssim 2^{-j\gamma(d-1)}2^{-sd}\sum_{i\le s}\sum_{I\in\mathcal{S}_{i-j}}\int_{x+\widetilde{\mathcal{R}}^j_{s\nu}}|b_I(y)|dy\\ &\lesssim 2^{-2j\gamma(d-1)},\end{aligned}$$

where we have used the fact that $\int_{\mathbb{R}^d}|b_I(y)|dy\lesssim|I|$ and the cubes $I\in\mathcal{S}$ are pairwise disjoint.

This, in turn, implies further that

$$\Big\| \sum_s \sum_{\mathbb{L}\in\mathcal{S}_{s-j}} (I - P_{s-j\kappa}) T^{i,j}_{\Omega,\mathbb{L};\, s,\nu} b_{\mathbb{L}} \Big\|^2_{L^2(\mathbb{R}^d)} \lesssim j^2 \|\Omega_i\|^2_{L^\infty(\mathbb{S}^{d-1})} 2^{-2j\gamma(d-1)} \times \sum_s \sum_{\mathbb{L}\in\mathcal{S}_{s-j}} \|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.$$

which finishes the proof of Lemma 3.5. □

Estimate for the term $(I - G^j_\nu)(I - P_{s-j\kappa})T^{i,j}_{\Omega,\mathbb{L};\, s,\nu}$.

We need to present a lemma for $(I - G^j_\nu)(I - P_{s-j\kappa})T^{i,j}_{\Omega,\mathbb{L};\, s,\nu}$.

Lemma 3.6 *Let Ω be homogeneous of degree zero, A be a function in $\mathbb{R}^d$ with derivatives of order one in* $\mathrm{BMO}(\mathbb{R}^d)$. *For each $j \in \mathbb{N}$ with $j > j_0$, $\ell(\mathbb{L}) = 2^{s-j}$, some $s_0 > 0$, we have that*

$$\sum_\nu \|(I - G^j_\nu)(I - P_{s-j\kappa})T^{i,j}_{\Omega,\mathbb{L};\, s,\nu} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)} \lesssim j 2^{-s_0 j} \|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})} \|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.$$

Next we give the estimate of D^*_2 and postpone the proof of Lemma 3.6 later.

Let $\varepsilon = \min\{(1-\kappa), (1-\tau), s_0, \gamma\}$. With Lemma 3.3, Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned} \Big|\Big\{x \in \mathbb{R}^d : |\mathrm{D}^*_2| > 1/16\Big\}\Big| &\lesssim \sum_{i=0}^{\infty} \sum_{j>Ni} j^2 2^{-j\varepsilon} \|\Omega_i\|^2_{L^\infty(\mathbb{S}^{d-1})} \Big\| \sum_s \sum_{\mathbb{L}\in\mathcal{S}_{s-j}} b_{\mathbb{L}} \Big\|_{L^1(\mathbb{R}^d)} \\ &\lesssim \|f\|_{L^1(\mathbb{R}^d)}. \end{aligned} \tag{3.20}$$

The proof of Lemma 3.6 is similar to the proof of Lemma 2.4 in [9]. For the completeness of this paper, we give the proof for the remaining term $(1-G^j_\nu)(I-P_{s-j\kappa})T^{i,j}_{\Omega,\mathbb{L};\, s,\nu}$ here. Let's introduce the Littlewood-Paley decomposition first. Let α be a radial C^∞ function such that $\alpha(\xi) = 1$ for $|\xi| \le 1$, $\alpha(\xi) = 0$ for $|\xi| \ge 2$ and $0 \le \alpha(\xi) \le 1$ for all $\xi \in \mathbb{R}^d$. Define $\beta_k(\xi) = \alpha(2^k\xi) - \alpha(2^{k+1}\xi)$. Choose $\tilde{\beta}$ be a radial C^∞ function such that $\tilde{\beta}(\xi) = 1$ for $1/2 \le |\xi| \le 2$, $\operatorname{supp}\tilde{\beta} \in [1/4, 4]$ and $0 \le \tilde{\beta} \le 1$ for all $\xi \in \mathbb{R}^d$. Set $\tilde{\beta}_k(\xi) = \tilde{\beta}(2^k\xi)$, then it is easy to see $\beta_k = \tilde{\beta}_k\beta_k$. Define the convolution operators Λ_k and $\tilde{\Lambda}_k$ with Fourier multipliers β_k and $\tilde{\beta}_k$, respectively.

$$\widehat{\Lambda_k f}(\xi) = \beta_k(\xi)\widehat{f}(\xi), \qquad \widehat{\tilde{\Lambda}_k f}(\xi) = \tilde{\beta}_k(\xi)\widehat{f}(\xi).$$

It is easy to have $\Lambda_k = \tilde{\Lambda}_k\Lambda_k$.

Proof of Lemma 3.6 We first write $(I - G^j_\nu)T^{i,j}_{\Omega,\mathbb{L};\, s,\nu} = \sum_k (I - G^j_\nu)\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\, s,\nu}$. Then

$$\|(I - G^j_\nu)(I - P_{s-j\kappa})\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\, s,\nu} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}$$

$$\leq \|(I - P_{s-j\kappa})\tilde{\Lambda}_k (I - G_\nu^j)\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\,s,\nu} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}$$
$$\leq \|(I - P_{s-j\kappa})\tilde{\Lambda}_k\|_{L^1(\mathbb{R}^d)\to L^1(\mathbb{R}^d)} \|(I - G_\nu^j)\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\,s,\nu} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.$$

We can write

$$(I - G_\nu^j)\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\,s,\nu} b_{\mathbb{L}}(x) = \int_{\mathbb{L}} (I - G_\nu^j)\Lambda_k \Omega_i(x-y)\Gamma_\nu^j(x-y) R^j_{s,\mathbb{L}}(x,y) b_{\mathbb{L}}(y)dy$$
$$:= \int_{\mathbb{L}} M_k(x,y) b_{\mathbb{L}}(y)dy,$$

where M_k is the kernel of the operator $(I - G_\nu^j)\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\,s,\nu}$. Then

$$\|(I - G_\nu^j)\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\,s,\nu} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)} \leq \int_{\mathbb{L}} \|M_k(\cdot, y)\|_{L^1(\mathbb{R}^d)} |b_{\mathbb{L}}(y)|dy.$$

Applying the method of Lemma 4.2 in [9], there exists $M > 0$ such that

$$\|M_k(\cdot, y)\|_{L^1(\mathbb{R}^d)} \lesssim j 2^{\tau j - j\gamma(d-1) - s + k + j\gamma(1+2M)} \|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})}.$$

Hence, note that $\|(I - P_{s-j\kappa})\tilde{\Lambda}_k\|_{L^1(\mathbb{R}^d)\to L^1(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1}(\tilde{\beta}_k) - \omega_{s-j\kappa} * \mathcal{F}^{-1}(\tilde{\beta}_k)\|_{L^1(\mathbb{R}^d)} \lesssim 1$, we have

$$\|(I - G_\nu^j)(I - P_{s-j\kappa})\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\,s,\nu} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}$$
$$\lesssim j 2^{\tau j - j\gamma(d-1) - s + k + j\gamma(1+2M)} \|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})} \|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}. \tag{3.21}$$

On the other hand, we can write

$$\|(I - G_\nu^j)(I - P_{s-j\kappa})\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\,s,\nu} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}$$
$$\leq \|(I - P_{s-j\kappa})\tilde{\Lambda}_k\|_{L^1(\mathbb{R}^d)\to L^1(\mathbb{R}^d)} \|(I - G_\nu^j)\Lambda_k\|_{L^1(\mathbb{R}^d)\to L^1(\mathbb{R}^d)} \|T^{i,j}_{\Omega,\mathbb{L};\,s,\nu} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.$$

By (3.18), it is easy to show that

$$\|T^{i,j}_{\Omega,\mathbb{L};\,s,\nu} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)} \lesssim j 2^{-j\gamma(d-1)} \|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})} \|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}.$$

Let $W^j_{k,s,\kappa}$ be the kernel of $(I - P_{s-j\kappa})\tilde{\Lambda}_k$, then by the mean value formula, we obtain

$$\int_{\mathbb{R}^d} |W^j_{k,s,\kappa}(y)|dy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|\mathcal{F}^{-1}\tilde{\beta}_k(y) - \mathcal{F}^{-1}\tilde{\beta}_k(y-z)\right| \omega_{s-j\kappa}(z)dzdy$$
$$\lesssim 2^{s-j\kappa-k}. \tag{3.22}$$

By the proof of [26, Lemma 3.2], it holds that $\|(I - G_\nu^j)\Lambda_k\|_{L^1(\mathbb{R}^d)\to L^1(\mathbb{R}^d)} \lesssim 1$. Hence

$$\|(I - G_\nu^j)(I - P_{s-j\kappa})\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\,s,\nu} b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}$$

$$\lesssim j2^{-j\gamma(d-1)+s-k-j\kappa}\|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})}\|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}. \tag{3.23}$$

Let $m = s - [j\varepsilon_0]$, with $0 < \varepsilon_0 < 1$. Since $\operatorname{card}(\mathfrak{E}^j) \lesssim 2^{j\gamma(d-1)}$. Then (3.21) and (3.23) lead to

$$\begin{aligned}
&\sum_\nu \|(I-G_\nu^j)(I-P_{s-j\kappa})T^{i,j}_{\Omega,\mathbb{L};\,s,\nu}(b_{\mathbb{L}})\|_{L^1(\mathbb{R}^d)}\\
&\le \Big(\sum_\nu\sum_{k<m}+\sum_\nu\sum_{k\ge m}\Big)\|(I-P_{s-j\kappa})(I-G_\nu^j)\Lambda_k T^{i,j}_{\Omega,\mathbb{L};\,s,\nu}b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)}\\
&\lesssim (2^{s_1 j}+2^{s_2 j})j\|\Omega_i\|_{L^\infty(\mathbb{S}^{d-1})}\|b_{\mathbb{L}}\|_{L^1(\mathbb{R}^d)},
\end{aligned}$$

where $s_1 = \big(\tau - \varepsilon_0 + \gamma(1+2M)\big)$ and $s_2 = -\kappa + \varepsilon_0$.

We can now choose $0 \ll \tau \ll \gamma \ll \varepsilon_0 < \kappa < 1$ such that $\max\{s_1, s_2\} < 0$. Let $s_0 = -\max\{s_1, s_2\}$, then the proof of Lemma 3.6 is finished now. □

With Lemma 3.2 and (3.20) in hand, we can deduce (3.7) by

$$\begin{aligned}
\big|\{x\in\mathbb{R}^d\backslash E : |\mathrm{D}_2 > 1/8\}\big| &\le 16\|\mathrm{D}_2-\mathrm{D}_2^*\|_{L^1(\mathbb{R}^d)} + \big|\{x\in\mathbb{R}^d : |\mathrm{D}_2^*| > 1/16\}\big|\\
&\lesssim \|f\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

3.4 Proof of (1.7) in Theorem 1.2

It suffices to prove (1.7) for $\lambda = 1$. For a bounded function f with compact support, we employ the Calderón-Zygmund decomposition to $|f|$ at level 1 then obtain a collection of non-overlapping dyadic cubes $\mathcal{S} = \{Q\}$, such that

$$\|f\|_{L^\infty(\mathbb{R}^d\backslash\cup_{Q\in\mathcal{S}}Q)} \lesssim 1,\quad \int_Q |f(x)|dx \lesssim |Q|,\qquad \text{and } \sum_{Q\in\mathcal{S}}|Q| \lesssim \int_{\mathbb{R}^d}|f(x)|dx.$$

Let $E = \cup_{Q\in\mathcal{S}}100dQ$. With the same notations as in the proof of (1.6), for $x \in \mathbb{R}^d\backslash E$, we write

$$\widetilde{T}_{\Omega,A}b(x) = \sum_j\sum_{Q\in\mathcal{S}} T_{\Omega,A_Q,j}b_Q(x) - \sum_{Q\in\mathcal{S}}\sum_{n=1}^d \partial_n A_Q(x)T^n_\Omega b_Q(x).$$

By estimate (3.5), the proof of (1.7) can be reduced to show that for each n with $1 \le n \le d$,

$$\Big|\Big\{x\in\mathbb{R}^d\backslash E : \Big|\sum_{Q\in\mathcal{S}}\partial_n A_Q(x)T^n_\Omega b_Q(x)\Big| > 1/4d\Big\}\Big| \lesssim \int_{\mathbb{R}^d}|f(x)|dx.$$

But this inequality has already been proved in [22, inequality (3.3)]. Then the proof of (1.7) is finished. □

3.5 Proof of Theorem 1.3

The proof of Theorem 1.3 is now standard. We present the proof here mainly to make the constant of the norm inequality clearly. Consider the case $p \in (1, 2]$. Let

$$f_\lambda(x) = \begin{cases} f(x), & |f(x)| > \lambda \\ 0, & |f(x)| \le \lambda; \end{cases}$$

and

$$f^\lambda(x) = \begin{cases} 0, & |f(x)| > \lambda \\ f(x), & |f(x)| \le \lambda \end{cases}$$

By (1.6), we have

$$\begin{aligned} &p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R}^d : |T_{\Omega,A} f_\lambda(x)| > \lambda/2\}|d\lambda \\ &\lesssim p\int_0^\infty \lambda^{p-1}\int_{\mathbb{R}^d} \frac{|f_\lambda(x)|}{\lambda}\log\left(e + \frac{|f_\lambda(x)|}{\lambda}\right)dx\, d\lambda \\ &\le \left(\frac{p}{p-1}\right)^2 \|f\|^p_{L^p(\mathbb{R}^d)}. \end{aligned}$$

$L^2(\mathbb{R}^d)$ boundedness of $T_{\Omega,\,A}$ implies that

$$\begin{aligned} &p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R}^d : |T_{\Omega,A} f^\lambda(x)| > \lambda/2\}|d\lambda \\ &\lesssim\; p\int_{|f(x)|}^\infty \lambda^{p-1}\lambda^{-2}\|f^\lambda\|^2{}_{L^2(\mathbb{R}^d)}\, d\lambda \le \frac{p}{2-p}\|f\|^p_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Since $p \in (1, 2)$, we have

$$\begin{aligned} \|T_{\Omega,\,A} f\|_{L^p(\mathbb{R}^d)} &= \left(p\int_0^\infty \lambda^{p-1}\big|\{x \in \mathbb{R}^d : |T_{\Omega,A} f(x)| > \lambda\}\big|\, d\lambda\right)^{1/p} \\ &\le (p')^2\|f\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

When $p \in (2, \infty)$, by (1.7), we know $\widetilde{T}_{\Omega,A} f(x)$ is of weak type (1, 1). Combining the $L^2(\mathbb{R}^d)$ boundedness of $\widetilde{T}_{\Omega,A} f(x)$ and the Marcinkiewicz interpolation theorem, we have

$$\|\widetilde{T}_{\Omega,\,A} f\|_{L^{p'}(\mathbb{R}^d)} \le p'\|f\|_{L^{p'}(\mathbb{R}^d)}.$$

By duality, it holds that

$$\|T_{\Omega,\,A} f\|_{L^p(\mathbb{R}^d)} \le p\|f\|_{L^p(\mathbb{R}^d)}.$$

This completes the proof of Theorem 1.3. □

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