Journal of Fourier Analysis and Applications (2024) 30:29
https://doi.org/10.1007/s00041-024-10085-z

On the Sharp Estimates for Convolution Operators with Oscillatory Kernel

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Received: 10 November 2022 / Revised: 24 July 2023 / Accepted: 5 March 2024 /
Published online: 2 May 2024

Abstract

In this article, we studied the convolution operators M_k with oscillatory kernel, which are related to the solutions of the Cauchy problem for the strictly hyperbolic equations. The operator M_k is associated to the characteristic hypersurfaces $\Sigma \subset \mathbb{R}^3$ of a hyperbolic equation and smooth amplitude function, which is homogeneous of the order $-k$ for large values of the argument. We investigated the convolution operators assuming that the corresponding amplitude function is contained in a sufficiently small conic neighborhood of a given point $v \in \Sigma$ at which, exactly one of the principal curvatures of the surface Σ does not vanish. Such surfaces exhibit singularities of the type A in the sense of Arnold's classification. Denoting by k_p the minimal number such that M_k is $L^p \mapsto L^{p'}$-bounded for $k > k_p$, we showed that the number k_p depends on some discrete characteristics of the surface Σ.

Keywords Convolution operator · Hypersurface · Oscillatory integral · Singularity

Mathematics Subject Classification 42B10 · 42B20 · 42B37

Communicated by Elijah Liflyand.

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1 Introduction

It is well known that the solution operator of the Cauchy problem for homogeneous constant coefficient strictly hyperbolic equation, up to a regularizing operator, can be written as a sum of convolution operators of the type:

$$\mathcal{M}_k = F^{-1}[e^{it\varphi(\xi)}a_k]F, \tag{1.1}$$

where F is the Fourier transform operator, $\varphi \in C^\infty(\mathbb{R}^\nu \setminus \{0\})$ is homogeneous of order one function, $a_k \in C^\infty(\mathbb{R}^\nu_\xi)$ is a homogeneous function of order $-k$ for large ξ.

We demonstrate the motivation of the main problem in a simple example. Consider the classical example related to the Cauchy problem for the wave equation in $\mathbb{R} \times \mathbb{R}^\nu$:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \quad u(x,0) = g_0(x), \ \frac{\partial u(x,0)}{\partial t} = g_1(x),$$

where g_0, g_1 are distributions.

The solution to the Cauchy problem formally can be written as:

$$u(x,t) = F^{-1}\left(\frac{e^{it|\xi|} + e^{-it|\xi|}}{2} F(g_0)\right) + F^{-1}\left(\frac{e^{it|\xi|} - e^{-it|\xi|}}{2i|\xi|} F(g_1)\right).$$

Thus, the solution operator of the Cauchy problem, modulo a regularizing operator, can be written as a sum of Fourier multiplier operators having the form (1.1). Actually, the classical Strichartz estimates are related to the wave equation [18].

Surely, analogical issues require much more effort for the higher order strictly hyperbolic equation (see [12] and references therein).

After the scaling arguments for time $t > 0$ the operator $\mathcal{M}_k$ is reduced to the following convolution operator (see [16]):

$$M_k = F^{-1}[e^{i\varphi(\xi)}a_k]F. \tag{1.2}$$

Let $1 \leq p \leq 2$ be a fixed number: We consider the problem: *find the minimal number $k(p)$ such that the operator $M_k : L^p(\mathbb{R}^\nu) \to L^{p'}(\mathbb{R}^\nu)$ is bounded for any $k > k(p)$*. Where and further the p' denotes the conjugate exponent e.g. $1/p + 1/p' = 1$.

Similar problems have been considered by many authors including Strichartz [18, 19], in the case when the characteristic hypersurface is the unit sphere, Brenner [5], when the characteristic hypersurface has non-vanishing Gaussian curvature. These results were extended by Sugimoto [14–16], in which the characteristic hypersurface was convex (but, not necessarily strictly convex) and also for some non-convex hypersurfaces (see also [13]).

Nevertheless, the problem remains in general wide open. Also, the issue is related to many other open problems of harmonic analysis related to oscillatory integrals.

Note that if $a_k(\xi) = |\xi|^{-k}$ for large ξ with $0 < k < \nu$ and $\varphi \equiv 0$ then the problem can be solved by using the classical Hardy–Littlewood–Sobolev's inequality. Due to the classical Hardy–Littlewood–Sobolev's inequality if $k \geq 2n(1/p - 1/2)$ then the operator (1.2) is bounded from $L^p(\mathbb{R}^\nu)$ to $L^{p'}(\mathbb{R}^\nu)$. Moreover, if a_k is a classical symbol of PsDO and $\varphi \equiv 0$ then we deal with $L^p(\mathbb{R}^\nu) \mapsto L^{p'}(\mathbb{R}^\nu)$ boundedness problem for pseudo-differential operators (see [14]). It is well-known that if a_k is a classical symbol of the PsDO of order zero then the corresponding PsDO is bounded on $L^p(\mathbb{R}^\nu)$ for $1 < p < \infty$.

Further, we assume that the function φ preserves sign, e.g. $\varphi(\xi) \neq 0$ for any $\xi \in \mathbb{R}^\nu \backslash \{0\} (\nu \geq 2)$. Note that, due to the oscillation factor for a wider range of the order $-k$ of the amplitude a_k we get the $L^p(\mathbb{R}^\nu) \mapsto L^{p'}(\mathbb{R}^\nu)$ boundedness of the operator (1.2).

Next, without loss of generality we may assume that $\varphi(\xi) > 0$ for any $\xi \neq 0$. Since φ is a smooth homogeneous function of order one, then, due to the Euler's homogeneity relation we have:

$$\sum_{j=1}^{\nu} \xi_j \frac{\partial \varphi(\xi)}{\partial \xi_j} = \varphi(\xi),$$

and hence the set Σ defined by the following

$$\Sigma = \{\xi \in \mathbb{R}^\nu : \varphi(\xi) = 1\}$$

is a smooth or a real analytic hypersurface provided φ is a smooth or a real analytic function on $\mathbb{R}^\nu \setminus \{0\}$ respectively.

Further, we use notation:

$$k_p := k_p(\Sigma) := \inf_{k>0} \{k > 0 : M_k \text{ is } L^p(\mathbb{R}^\nu) \to L^{p'}(\mathbb{R}^\nu) \text{ bounded for any } a_k\}. \tag{1.3}$$

It turns out that the number $k_p(\Sigma)$ depends on geometric properties of the hypersurface Σ. More precisely, the number depends on behavior of the Fourier transform of measures supported on Σ. The monograph [10] contains many modern results related to the Fourier transform of surface-carried measures.

Since $\Sigma \subset \mathbb{R}^\nu \setminus \{0\}$ is a compact hypersurface, then following Sugimoto [16], it is enough to consider the local version of the problem. More precisely, we may assume that the amplitude function $a_k(\xi)$ is concentrated in a sufficiently small conic neighborhood Γ of a fixed point $v \in \Sigma$ and $\varphi(\xi) \in C^\infty(\Gamma)$. Fixing such a point $v \in \Sigma$, let us define the following local exponent $k_p(v)$ associated to this point:

$$\begin{gathered} k_p(v) := \inf_{k>0} \{k : \exists \Gamma,\ M_k : L^p(\mathbb{R}^\nu) \mapsto L^{p'}(\mathbb{R}^\nu) \\ \text{is bounded, whenever } supp(a_k) \subset \Gamma\}. \end{gathered}$$

The definition of $k_p(v)$ yields that it is an upper semicontinuous function of v for a fixed p.

Further, we use the following standard notation, assuming F being a sufficiently smooth function:

$$\partial^\gamma F(x) := \partial_1^{\gamma_1} \dots \partial_\nu^{\gamma_\nu} F(x) := \frac{\partial^{|\gamma|} F(x)}{\partial x_1^{\gamma_1} \dots \partial x_\nu^{\gamma_\nu}},$$

where $\gamma := (\gamma_1, \dots, \gamma_\nu) \in \mathbb{Z}_+^\nu$ is a multiindex, with $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$, and $|\gamma| := \gamma_1 + \dots + \gamma_\nu$.

Also, for the sake of being definite, we assume that $v = (0, \dots, 0, 1)$ and $\varphi(v) = 1$. After possible a linear transform in the space $\mathbb{R}_\xi^\nu$, which preserves the point v, we may assume that $\partial_j \varphi(v) = 0 (j = 1, \dots, \nu - 1)$. Thus, in a neighborhood of the point v the hypersurface Σ is given as the graph of a smooth function:

$$\Sigma \cap \Gamma = \{(\xi_1, \dots, \xi_{\nu-1}, 1 + \phi(\xi_1, \dots, \xi_{\nu-1})) \in \mathbb{R}^\nu : (\xi_1, \dots, \xi_{\nu-1}) \in U\},$$

where $U \subset \mathbb{R}^{\nu-1}$ is a sufficiently small neighborhood of the origin and, $\phi \in C^\infty(U)$ is a smooth function satisfying the conditions: $\phi(0) = 0$, $\nabla\phi(0) = 0$ (compare with [16]).

Further, we mainly consider the problem for the case $\nu = 3$. In order to state the main results, we need the following Proposition [9]:

Proposition 1.1 *Assume that ϕ is a smooth function defined in a neighborhood of the origin of $\mathbb{R}^2$ satisfying the conditions: $\partial_2^2 \phi(0, 0) \neq 0$ and also $\partial^\gamma \phi(0, 0) = 0$ for any $|\gamma| \leq 2$ with $\gamma \neq (0, 2)$.*

Then, ϕ can be written in the following form on a sufficiently small neighborhood of the origin:

$$\phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + b_0(x_1), \tag{1.4}$$

where b, b_0 and ψ are smooth functions with $b(0, 0) \neq 0$. The function ψ (resp. b_0) can be written as $\psi(x_1) = x_1^m \omega(x_1)$ with a smooth function ω satisfying $\omega(0) \neq 0$, $m \geq 2$ (resp. $b_0(x_1) = x_1^n \beta(x_1)$, with a smooth function β satisfying $\beta(0) \neq 0$, $n \geq 2$) unless ψ (resp. b_0) is a flat function.

Further, we assume that if Σ is a C^∞ hypersurface and b_0 is a flat function then $b_0 \equiv 0$. This condition agrees with so-called "$R-$condition" introduced in the monograph [9]. Surely, if ϕ is a real analytic function then the $R-$ condition is automatically fulfilled.

Also, we assume that the function ϕ has a singularity of type $A_n (1 \leq n \leq \infty)$ at the origin (see [3] for a definition of A type singularities). The last condition means that the hypersurface Σ has exactly one non-vanishing principal curvature at the point v, whenever $n \geq 2$ in the case $\nu = 3$.

Remark 1.2 It is easy to show that the numbers m and n are well-defined for arbitrary smooth function ϕ having A type singularity (see [16] and also [9]). Moreover, to each point $v \in \Sigma$ of the surface with at least one non-vanishing principal curvature we can attach a pair $(m(v), n(v))$ due to the Proposition 1.1.

1.1 The Main Results

In this paper we prove the following statement, which is the main our result.

Theorem 1.3 *Let $\Sigma \subset \mathbb{R}^3 \setminus \{0\}$ be a smooth surface having at least one non-vanishing principal curvature at the point $v = (0, 0, 1) \in \Sigma$ and $1 \leq p \leq 2$ be a fixed number and also (m, n) be the pair defined by the Proposition 1.1. Then the following statements hold:*
(i) *If $2m \geq n$ (with $2 \leq n \leq \infty$) then $k_p(v) = (5 - \frac{2}{n})(\frac{1}{p} - \frac{1}{2})$;*
(ii) *If Σ is a smooth hypersurface satisfying the $R-$condition and $m \geq 3$ and also $2m < n \leq \infty$ then*

$$k_p(v) = \max\left\{\left(5 - \frac{1}{m}\right)\left(\frac{1}{p} - \frac{1}{2}\right), \left(6 - \frac{2(m+1)}{n}\right)\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} + \frac{m}{n}\right\}. \tag{1.5}$$

Let $p \in [1, 2]$ be a fixed number. As noted before, the $k_p(v)$ is an upper semicontinuous function of v defined on a compact hypersurface $\Sigma \subset \mathbb{R}^3 \setminus \{0\}$. Then the main Theorem 1.3 yields.

Corollary 1.4 *If $\Sigma \subset \mathbb{R}^3 \setminus \{0\}$ is a smooth compact hypersurface satisfying the conditions of the main Theorem 1.3 at any point $v \in \Sigma$ then the following relation*

$$k_p(\Sigma) = \max_{v \in \Sigma} k_p(v)$$

holds true.

1.2 Classes of Hypersurfaces

Sugimoto [16] considered the problem for the case when $\Sigma \subset \mathbb{R}^3$ is an analytic surface having at least one non-vanishing principal curvature at every point and obtained an upper bound for the number $k_p(\Sigma)$. More precisely, Sugimoto introduced [16] three classes of hypersurfaces in $\mathbb{R}^3$ with at least one non-vanishing principal curvature.

Following [16] we can introduce the following classes of hypersurfaces by the function ϕ, defined in a small neighborhood of the origin, having the form given in the Eq. (1.4): We say that Σ is of type I with order n if $b_0(x_1) = x_1^n \beta(x_1)$, where β is a smooth function with $\beta(0) \neq 0$; Σ is of type II with order m if b_0 is a flat function at the origin and also $\psi(x_1) = x_1^m \omega(x_1)$, where ω is a smooth function with $\omega(0) \neq 0$, and finally, Σ is of type III if both functions ψ, b_0 are flat at the origin.

It has been obtained an upper bound for the number $k_p(v)$ for each class of hypersurfaces [16]. Moreover, Sugimoto suggested examples for each classes showing sharpness of the bounds for those examples.

The natural question is: *Whether the upper estimate for the number $k_p(v)$ given in* [16] *is the sharp bound for each hypersurface of the appropriate class ?*

Actually, due to the Theorem 1.3, we obtain the exact value of $k_p(v)$, improving the results proved in [16], for arbitrary analytic hypersurfaces having at least one non-vanishing principal curvature and smooth hypersurfaces under the so-called $R-$ condition introduced in [9].

Remark 1.5 Note that in the case (i) of the Theorem 1.3 formally it is possible $m = \infty$ e.g. the ψ can be a flat function. Sugimoto [16] suggested the example:

$$\phi_I(x) = 1 - (x_2^2 - x_1^n), \tag{1.6}$$

which corresponds to the case (i), with $\psi(x_1) \equiv 0$. From our results it follows that the Sugimoto result [16] is sharp in that case. Moreover, the Sugimoto result, for a surface of the class I with order n, is sharp if and only if $2m \geq n$.

Note that the first case (i) agrees with the so-called linearly adapted condition introduced in the monograph [9] (see also [8]). Also note that under the linearly adapted case the sharp uniform (with respect to directions of the frequencies) estimates for the Fourier transform of measures give the sharp bound for the exponent p in the $L^p \mapsto L^2$ Fourier restriction estimate. It had been shown in [9] that it is only the case.

If $n = \infty$ e.g. if b_0 is a flat function at the origin then so is ψ, under the condition $2m \geq n$. Hence, the Sugimoto result is sharp in that case also, in other words, the results of the paper [16] are sharp for arbitrary smooth surface of the class III.

On the other hand if $2m < n < \infty$ then the result of Sugimoto [16] is not sharp for the hypersurfaces Σ of the class I. Our results show that one can not be ignored influence of the number m for the surfaces of the class I.

For the case $n = \infty$ and $m < \infty$ e.g. for hypersurfaces of the class II Sugimoto obtained the sharp bound for $k_p(v)$, when Σ belongs to a subclass of analytic surfaces of the class II. It turns out that the analogical result holds true for arbitrary analytic hypersurfaces of the class II and also for arbitrary smooth surfaces of the class II under the $R-$ condition. More precisely, from our result it follows that actually the statement of the Theorem 2 proved by Sugimoto [16] (page no. 396) holds true for arbitrary analytic hypersurface having type II and also for analogical smooth hypersurfaces under the R-condition.

Consider the example of strictly hyperbolic equation of order 4 in the space $\mathbb{R} \times \mathbb{R}^3$, for which the corresponding surface Σ can be written as the graph of a function having A_∞ type singularities:

$$\frac{\partial^4 u}{\partial t^4} - 2\frac{\partial^2(\Delta + \partial_3^2)u}{\partial t^2} + \frac{1}{2}(\Delta^2 u + \partial_3^4 u) = 0, \tag{1.7}$$

where Δ is the standard Laplace operator in $\mathbb{R}^2$. It is the axisymmetric partial differential equation.

Consider one of the roots of the characteristic equation

$$P(\tau, \xi) := \tau^4 - 2\tau^2|\xi|^2 + \frac{1}{2}((\xi_1^2 + \xi_2^2)^2 + \xi_3^4) = 0$$

given by

$$\varphi(\xi) := \sqrt{|\xi|^2 - \sqrt{|\xi|^4 - \frac{1}{2}((\xi_1^2 + \xi_2^2)^2 + \xi_3^4)}}.$$

Then the corresponding hypersurface Σ can be obtained from the plane curve

$$\{(\xi_1, \xi_3) : \xi_1^2 + \xi_3^2 > 1, (\xi_1^2 - 2)^2 + (\xi_3^2 - 2)^2 = 6\}$$

by rotation around the $O\xi_3$ axis. It is easy to see that at least one of the principal curvatures of the surface Σ does not vanish at any point. Moreover, exactly one of the principal curvatures does not vanish at the points

$$\left(\sqrt{2}\cos\varsigma, \sqrt{2}\sin\varsigma, \sqrt{2+\sqrt{6}}\right) \in \Sigma,$$

with $0 \leq \varsigma \leq 2\pi$, and directions of the normals to the surface Σ at the points coincide with the direction of the $O\xi_3$ axis. Thus, the corresponding phase function ϕ has A_∞ type singularities at those points.

Actually, one can consider the analogical axisymmetric equation in the space $\mathbb{R}^n (n \geq 4)$. Then more complicated non-isolated singular points appear. In this case we can not get analogical estimates for the corresponding convolution operators. Moreover, one can construct examples of hypersurfaces in three dimensional space, for which both principal curvatures vanish at some point and our methods can not be applied.

We plan to consider the case of hypersurfaces in $\mathbb{R}^3$ for which both principal curvatures vanish at some point (work in progress).

This paper organized as follows: in Sect. 2 we give preliminary results on relations between decay rate of oscillatory integrals and upper estimates for the number $k_p(v)$. Then we obtain an upper bound for the number $k_p(v)$, for each class of surfaces in Sect. 3. Finally, in Sect. 4 we give lower bounds for the number $k_p(v)$, which agree with the upper bounds. The results of the last Sect. 4 finish a proof of the main Theorem 1.3.

Conventions: Throughout this article, we use the variable constant notation, i.e., many constants appearing in the course of our arguments, often denoted by $c, C, \varepsilon, \delta$; will typically have different values at different lines. Moreover, we use symbols such as $\sim$, $\lesssim$ or $<<$ in order to avoid writing down constants, as explained in [9] (Chapter 1). By χ_0 we denote a non-negative smooth cut-off function on $\mathbb{R}$ with typically small compact support which is identically 1 on a small neighborhood of the origin.

2 Preliminaries

The boundedness problem for the convolution operators is related to behaviour of the following convolution kernel:

$$K_k := F^{-1}(e^{i\varphi(\xi)}a_k(\xi)).$$

We define the Fourier transform operator and its inverse as [17]:

$$F(u)(\xi) := \frac{1}{\sqrt{(2\pi)^{\nu}}} \int_{\mathbb{R}^{\nu}} e^{i\xi\cdot x} u(x)dx,$$

and the inverse

$$u(x) := \frac{1}{\sqrt{(2\pi)^{\nu}}} \int_{\mathbb{R}^{\nu}} e^{-i\xi\cdot x} F(u)(\xi)d\xi$$

for a Schwartz function u, where $\xi \cdot x$ is the usual inner product of the vectors ξ and x. Then it has been defined for distributions by the standard arguments.

It is well known that (see [16]) the main contribution to K_k gives points x which belongs to a sufficiently small neighborhood of the set $-\nabla\varphi(supp(a_k) \setminus \{0\})$.

In the paper [16] it had been shown the relation between the boundedness of the convolution operator M_k and behaviour of the following oscillatory integral:

$$I(\lambda, z) = \int_{\mathbb{R}^{\nu-1}} e^{i\lambda(\phi(x)+z\cdot x)} g(x)dx, \, (\lambda > 0, \; z \in \mathbb{R}^{\nu-1}),$$

where $g \in C_0^{\infty}(U)$ and U is a sufficiently small neighborhood of the origin.

More precisely the following statements were proved in [16]:

Proposition 2.1 *Let $q \geq 2$ and $\alpha \geq 0$. Suppose for all $g \in C_0^{\infty}(U)$ and $\lambda > 1$,*

$$\|I(\lambda, \cdot)\|_{L^q(\mathbb{R}_z^{\nu-1})} \leq C_g \lambda^{-\alpha}, \tag{2.1}$$

where C_g is independent of λ. Then $K_k(\cdot) := F^{-1}[e^{i\varphi(\xi)} a_k(\xi)](\cdot) \in L^q(\mathbb{R}^{\nu})$ and $M_k : L^p(\mathbb{R}^{\nu}) \to L^{p'}(\mathbb{R}^{\nu})$ bounded for $p = \frac{2q}{2q-1}$, if $k > \nu - \alpha - \frac{1}{q}$.

Also, Sugimoto [16] proved another version of the Proposition 2.1 in the case $q = \infty$. One can define

$$K_{k,j}(x) = F^{-1}[e^{i\varphi(\xi)} a_k(\xi)\Phi_j(\xi)](x).$$

Here $\{\Phi_j(\xi)\}_{j=1}^{\infty}$ is a Littlewood–Paley partition of unity which is used to define the norm

$$\|\eta\|_{B^s_{p,q}} := \left(\sum_{j=0}^{\infty} (2^{js} \|F^{-1}(\Phi_j(\xi)F(\eta)\|_{L^p})^q \right)^{\frac{1}{q}}$$

of Besov space $B^s_{p,q}$ (see [4]).

Proposition 2.2 *Let $\alpha \geq 0$. Suppose, for all $g \in C_0^{\infty}(U)$ and $\lambda > 1$,*

$$\|I(\lambda; \cdot)\|_{L^{\infty}(\mathbb{R}_z^{\nu-1})} \leq C_g \lambda^{-\alpha}, \tag{2.2}$$

where C_g is independent of λ. Then $\{K_{k,j}\}_{j=1}^{\infty}$ is bounded in $L^{\infty}(\mathbb{R}^{\nu})$, if $k = \nu - \alpha$. Hence M_k is $L^p \mapsto L^{p'}$ bounded, if $k > (2\nu - 2\alpha)\left(\frac{1}{p} - \frac{1}{2}\right)$. This inequality can be replaced by an equation, if $p \neq 1$.

3 An Upper Bound for the Number $k_p(v)$

Note that we deal with the two-dimensional oscillatory integral $I(\lambda, z)$ e. g. $\nu = 3$. If ϕ has singularity of type A_{n-1} with $2 \leq n \leq \infty$ at the origin and $|z| > \delta$ (where δ is a fixed positive number) then the phase function $\phi(x_1, x_2) + x_1 z_1 + x_2 z_2$ has no critical points provided U is a sufficiently small neighborhood of the origin and $g \in C_0^{\infty}(U)$. Therefore we can use integration by parts arguments and obtain:

$$|I(\lambda, z)| \lesssim \frac{1}{|z\lambda|^2},$$

which is better than what we expected.

Further, we assume that $|z| << 1$ and U is a sufficiently small neighborhood of the origin. Then we use the stationary phase method in x_2 variable and obtain as:

$$I(\lambda, z) = \frac{C}{\lambda^{\frac{1}{2}}} \int_{\mathbb{R}} e^{i\lambda(\phi_1(x_1,z_2)+x_1^n\beta(x_1)+z_2 x_1^m \omega(x_1)+z_1 x_1))} g(x_2^c(x_1, z_2), x_1) dx_1 + R(\lambda, z),$$

where R is a remainder term satisfying the estimate $|R(\lambda, z)| \lesssim \lambda^{-\frac{3}{2}}$ and $x_2^c(x_1, z_2)$ is the unique critical point of the phase function with respect to x_2. Moreover, the phase function $\phi_1(x_1, z_2)$ can be written as:

$$\phi_1(x_1, z_2) = z_2^2 B(z_2) + z_2^2 x_1 q(x_1, z_2),$$

where B, q are smooth functions with $B(0) \neq 0$ (see [6]).

Then by using the Van der Corput type lemma [2] (the paper [7] contains analogical estimates for oscillatory integrals with more general phase function) we see that the estimate (2.2) holds true with $\alpha = \frac{1}{2} + \frac{1}{n}$ (see [11] and [8] for analogical estimates in the case $n = \infty$). It is the sharp uniform (with respect to the parameters z) bound for the oscillatory integrals with phase having A type singularities. In this case we can use Proposition 2.2 and have the following upper bound for the $k_p(v)$:

$$k_p(v) \leq \left(5 - \frac{2}{n}\right)\left(\frac{1}{p} - \frac{1}{2}\right). \tag{3.1}$$

This case includes the class of surfaces of type III e.g. the case $m = n = \infty$. Note that the upper bound (3.1) does not depend on the number m. It turns out that, it is the sharp bound for the $k_p(v)$ under the condition $2m \geq n$. However, if $2m < n$ then the bound (3.1) is not sharp. Thus, the sharp uniform estimates for the oscillatory integrals give the sharp bound for $k_p(v)$ if and only if $2m \geq n$.

Now, we consider the more subtle case $2m < n$. In this case we use the following Lemma (compare with the Theorem 2 of [16]):

Lemma 3.1 *Let ϕ be a smooth function satisfying the conditions of the Proposition 1.1, in addition the $R-$condition, in which $2\,m \le n \le \infty$ and $3 \le m < \infty$ and also $\varepsilon > 0$ be a fixed positive number. Then the following estimate*

$$\|I(\lambda, \cdot)\|_{L^{m+1}(\mathbb{R}^2)} \le C_\varepsilon \lambda^{-\left(\frac{1}{2}+\frac{2}{m+1}\right)+\varepsilon}.$$

holds true.

Remark 3.2 For the case $n = 2m$ the result of the Lemma 3.1 does not give a better estimate for $k_p(v)$ than the sharp uniform, with respect to the parameters z, estimate for the corresponding oscillatory integrals. On the other hand if $2m > n$ then the phase function may have more degenerate than A_{m-1} critical points and we can not get the sharp analogical estimates for $L^{m+1}(\mathbb{R}^2)-$norm of the corresponding oscillatory integrals.

Proof As noted before, we assume that $|z| << 1$ and ε is a fixed positive number. So, in order to prove the Lemma 3.1 we show the validity of the following estimate:

$$\|I(\lambda, \cdot)\|_{L^{m+1}(V)} \le C|\lambda|^{-\left(\frac{1}{2}+\frac{2}{m+1}\right)+\varepsilon},$$

where V is a sufficiently small neighborhood of the origin.

Due to the stationary phase arguments it is enough to estimate the integral

$$I_1(\lambda, z) := \int_{\mathbb{R}} e^{i\lambda\Phi_1(x_1,z)} g(x_1, x_2^c(z_2, x_1))dx_1 =: \int_{\mathbb{R}} e^{i\lambda\Phi_1(x_1,z)} a(x_1, z_2)dx_1,$$

where we use the notation:

$$a(x_1, z_2) := g(x_1, x_2^c(z_2, x_1)),$$
$$\Phi_1(x_1, z) := x_1^n \beta(x_1) + z_2 x_1^m \omega(x_1) + z_2^2 x_1 q(x_1, z_2) + z_1 x_1.$$

First, we assume that $\left\{|z_2| < \delta|z_1|^{\frac{n-m}{n-1}}\right\}$, where δ is a sufficiently small fixed number, which will be defined later.

If $\left\{\lambda|z_1|^{\frac{n}{n-1}} \le 1\right\}$, then the classical van der Corpute Lemma [2] yields:

$$|I_1| \lesssim \frac{1}{|\lambda|^{\frac{1}{n}}} \le \frac{1}{|\lambda|^{\frac{1}{n}}(\lambda|z_1|^{\frac{n}{n-1}})^{\frac{2}{m+1}-\frac{1}{n}}} = \frac{1}{|\lambda|^{\frac{2}{m+1}}|z_1|^{\frac{2n-m-1}{(n-1)(m+1)}}}. \tag{3.2}$$

We show that, the estimate (3.2) holds true for $\lambda|z_1|^{\frac{n}{n-1}} > 1$, whenever δ is a sufficiently small positive number.

Indeed, we use change of variables as $x_1 = |z_1|^{\frac{1}{n-1}} y_1$ in the integral I_1 and denoting y_1 again by x_1 obtain:

$$I_1 = |z_1|^{\frac{1}{n-1}} \int e^{i\lambda|z_1|^{\frac{n}{n-1}}\Phi_2(x_1,z)} a(|z_1|^{\frac{1}{n-1}} x_1, z_2) dx_1,$$

where

$$\Phi_2(x_1, z) = \beta(|z_1|^{\frac{1}{n-1}} x_1) x_1^n + \frac{z_2}{|z_1|^{\frac{n-m}{n-1}}} x_1^m \omega(|z_1|^{\frac{1}{n-1}} x_1) + \frac{z_2^2}{|z_1|} x_1 q(|z_1|^{\frac{1}{n-1}} x_1, z_2) + sgn(z_1) x_1.$$

Note that

$$\frac{|z_2|}{|z_1|^{\frac{n-m}{n-1}}} \le \delta << 1 \quad \text{and also} \quad \frac{z_2^2}{|z_1|} \le \delta^2 |z_1|^{\frac{n-2m+1}{n-1}} << 1.$$

There exists a number N such that the phase function Φ_2 has no critical point on the set $\{|x_1| \ge N\}$. Take a smooth non-negative function χ_0 such that

$$\chi_0(x) = \begin{cases} 1, \text{ for } & |x| \le 1.1 \\ 0, \text{ for } & |x| > 2. \end{cases}$$

We write the integral I_1 as the sum of two integrals using the function χ_0:

$$I_1 = |z_1|^{\frac{1}{n-1}} \int e^{i\lambda|z_1|^{\frac{n}{n-1}}\Phi_2(x_1,z)} a(|z_1|^{\frac{1}{n-1}} x_1, z_2) \chi_0\left(\frac{x_1}{N}\right) dx_1 +$$
$$|z_1|^{\frac{1}{n-1}} \int e^{i\lambda|z_1|^{\frac{n}{n-1}}\Phi_2(x_1,z)} a(|z_1|^{\frac{1}{n-1}} x_1, z_2) \left(1 - \chi_0\left(\frac{x_1}{N}\right)\right) dx_1 =: I_{11} + I_{12}.$$

Using the integration by parts formula in the integral I_{12} we get:

$$|I_{12}| \le \frac{c|z_1|^{\frac{1}{n-1}}}{|\lambda|z_1|^{\frac{n-1}{n}}|} \le \frac{c|z_1|^{\frac{1}{n-1}}}{|\lambda|z_1|^{\frac{n}{n-1}}|^{\frac{2}{m+1}}} = \frac{c}{|\lambda|^{\frac{2}{m+1}} |z_1|^{\frac{2n-m-1}{(n-1)(m+1)}}}.$$

Surely, it coincides with the estimate (3.2).

Now, we consider the estimate for the integral I_{11}. The phase function of the integral can be considered as a small perturbation of the function $\beta(0)x_1^n + sgn(z_1)x_1$. Hence, there exists a positive number $\delta > 0$ such that the function $\Phi_2(x_1, z)$ has only non-degenerate critical points, whenever the parameter z satisfies the condition: $|z_2| < \delta|z_1|^{\frac{n-m}{n-1}}$. Therefore we use Van der Corpute type estimate and obtain:

$$|I_{11}| \le \frac{c|z_1|^{\frac{1}{n-1}}}{|\lambda|^{\frac{1}{2}} |z_1|^{\frac{n}{2(n-1)}}} \le \frac{c}{|\lambda|^{\frac{2}{m+1}} |z_1|^{\frac{2n-m-1}{(n-1)(m+1)}}}.$$

This completes a proof of the estimate (3.2) in the considered case.

Now, suppose $\{|z_1|^{\frac{n-m}{n-1}} \le \frac{1}{\delta}|z_2|\}$.

If $|z_2|^{\frac{n}{n-m}}|\lambda| \le 1$ then by using Van der Corpute type estimate we obtain the following bound:

$$|I_1 \lesssim \frac{1}{|\lambda|^{\frac{1}{n}}} \le \frac{1}{|\lambda|^{\frac{1}{n}}(|z_2|^{\frac{n}{n-m}}|\lambda|)^{\frac{2}{m+1}-\frac{1}{n}}} = \frac{1}{|\lambda|^{\frac{2}{m+1}}|z_2|^{\frac{2n-m-1}{(n-m)(m+1)}}}.$$

Finally, we consider the case $|z_2|^{\frac{n}{n-m}}|\lambda| > 1$, where our arguments based on induction method over m (see Proposition 3.3 stated below and we refer readers to [1] for more general result with the detailed proof). In this case, it is natural to use the change of variables $x_1 \mapsto |z_2|^{\frac{1}{n-1}}x_1$ in the integral I_1 which can be written as:

$$I_1 = |z_2|^{\frac{1}{n-1}} \int e^{i\lambda|z_2|^{\frac{n}{n-m}}\Phi_2(x_1,z)}a(|z_2|^{\frac{1}{n-m}}x_1, z_2)dx_1,$$

where

$$\begin{aligned}\Phi_2(x_1, z) &:= x_1^n\beta(|z_2|^{\frac{1}{n-m}}x_1) + sgn(z_2)x_1^m\omega(|z_2|^{\frac{1}{n-m}}x_1)\\ &+|z_2|^{2-\frac{n-1}{n-m}}sgn(z_2)x_1q(|z_2|^{\frac{1}{n-m}}x_1, z_2) + \frac{z_1}{|z_2|^{\frac{n-1}{n-m}}}x_1.\end{aligned}$$

There exists a positive number N such that the phase function Φ_2 has no critical points on the set $\{|x_1| \ge N\}$. Again, as before we write the integral I_1 as the sum of two integrals I_{11}, I_{12} given by the formulas:

$$\begin{aligned}I_{11} &= |z_2|^{\frac{1}{n-m}} \int e^{i\lambda|z_2|^{\frac{n}{n-m}}\Phi_2(x_1,z)}a(|z_2|^{\frac{1}{n-m}}x_1)\chi_0\left(\frac{x_1}{N}\right)dx_1,\\ |I_{12}| &= |z_2|^{\frac{1}{n-m}} \int e^{i\lambda|z_2|^{\frac{n}{n-m}}\Phi_2(x_1,z)}a(|z_2|^{\frac{1}{n-m}}x_1)\left(1-\chi_0\left(\frac{x_1}{N}\right)\right)dx_1.\end{aligned}$$

For the integral I_{12} we get:

$$|I_{12}| \le \frac{c|z_2|^{\frac{1}{n-m}}}{|\lambda|z_2|^{\frac{n}{n-m}}|} \le \frac{c|z_2|^{\frac{1}{n-m}}}{|\lambda|z_2|^{\frac{n}{n-m}}|^{\frac{2}{m+1}}} = \frac{c}{|\lambda|^{\frac{2}{m+1}}|z_2|^{\frac{2n-m-1}{(n-m)(m+1)}}},$$

because on the support of the amplitude function of the integral I_{12} the function $\Phi_2(x_1, z)$ has no critical points.

Finally, we consider estimate for the integral I_{11}. Note that

$$\xi_1 := \frac{z_1}{|z_2|^{\frac{n-1}{n-m}}} \in \left[-\frac{1}{\delta^{\frac{n-1}{n-m}}}, \frac{1}{\delta^{\frac{n-1}{n-m}}}\right] = [-\delta^{-\frac{n-1}{n-m}}, \delta^{-\frac{n-1}{n-m}}].$$

Since the interval $\Big[-\delta^{-\frac{n-1}{n-m}}, \delta^{-\frac{n-1}{n-m}}\Big]$ is the compact set then the required estimate follows from the corresponding local estimates. Let $\xi_1 = \xi_1^0$ be a fixed point of the interval $\Big[-\delta^{-\frac{n-1}{n-m}}, \delta^{-\frac{n-1}{n-m}}\Big]$. Further, suppose that the parameter ξ_1 changes in a sufficiently small neighborhood of the fixed point ξ_1^0. Then the phase function Φ_2 can be considered as a small perturbation of the function

$$x_1^n\beta(0) + sgn(z_2)x_1^m\omega(0) + \xi_1^0 x_1.$$

If $\xi_1^0 \neq 0$, then the phase function has only singularities of type A_k with $(k \le 2)$. If $\xi_1^0 = 0$, then the phase function has singularities of type A_{m-1} at the origin and all other critical points are non-degenerate. In particular, if $2 \le m \le 3$ then the phase function has only singularities of type A_k with $k \le 2$.

Assume $m \ge 3$ and $\xi_1^0 = 0$. Consider a smooth function $\phi(x_1, s_2)$ satisfying the condition $\phi(x_1, 0) = x_1^m b_m(x_1)$, where b_m is a smooth function with $b_m(0) \neq 0$. We define the phase function

$$\Phi(x_1, s_1, s_2) := \phi(x_1, s_2) + s_1x_1 \tag{3.3}$$

and consider the oscillatory integral:

$$I(\lambda, s_1, s_2) := \int_{\mathbb{R}} a(x_1, s)e^{i\lambda\Phi(x_1,s_1,s_2)}dx_1,$$

where a is a smooth function concentrated in a sufficiently small neighborhood of the origin.

The following Proposition is analogy of the Lemma 4 of the paper [1]:

Proposition 3.3 *Assume $I(\lambda, s_1, s_2)$ is the oscillatory integral with phase (3.3). Then there exists a neighborhood $U \times V \subset \mathbb{R} \times \mathbb{R}^2$ of the origin of $\mathbb{R} \times \mathbb{R}^2$, (where $V := [-\Delta, \Delta]^2$ with a sufficiently small positive number Δ) and a function Ψ such that the following estimate:*

$$|I(\lambda, s_1, s_2)| \le \frac{\Psi(s_1, s_2)}{|\lambda|^{\frac{1}{2}}} \tag{3.4}$$

holds true. Moreover, the following relation $\int_{[-\Delta,\Delta]} \Psi(s_1, s_2)^p ds_1 \lesssim 1$ is fulfilled for any $1 < p < \frac{2(m-1)}{m-2}$.

Proof The proof of Proposition 3.3 directly follows from the more general Lemma 4 of the paper [1]. Also, Proposition 3.3 can be proved by induction method over m. Note that if $m = 2$, then due to Van der Corpute Lemma the analogical estimate (3.4) holds true with $\Psi \equiv C$. □

Therefore from the Proposition 3.3 there exists a function

$$\Psi(\cdot, z_2) \in L^{\frac{2(m-1)}{m-2}-0}\Big[-\delta^{-\frac{n-1}{n-m}}, \delta^{-\frac{n-1}{n-m}}\Big] := \bigcap_{p<\frac{2(m-1)}{m-2}} L^p[-\delta^{-\frac{n-1}{n-m}}, \delta^{-\frac{n-1}{n-m}}]$$

such that the following estimate:

$$|I_{11}| \leq \frac{|z_2|^{\frac{1}{n-m}} \Psi\left(\frac{z_1}{|z_2|^{\frac{n-1}{n-m}}}, z_2\right)}{\lambda^{\frac{1}{2}} |z_2|^{\frac{n}{2(n-m)}}} = \frac{\Psi\left(\frac{z_1}{|z_2|^{\frac{n-1}{n-m}}}, z_2\right)}{|\lambda|^{\frac{1}{2}} |z_2|^{\frac{n-2}{2(n-m)}}} \tag{3.5}$$

holds true for the integral I_{11}, whenever $m \geq 3$. If $m = 2$ then there exists a function $\Psi(\xi_1, z_2) \in L^{4-0}\Big[-\delta^{-\frac{n-1}{n-m}}, \delta^{-\frac{n-1}{n-m}}\Big]$ such that the estimate (3.5) holds true with the function Ψ.

On the other hand the Van der Corpute Lemma yields:

$$|I_{11}| \leq \frac{c|z_2|^{\frac{1}{n-m}}}{\left|\lambda z_2^{\frac{n}{n-m}}\right|^{\frac{1}{m}}}.$$

By interpolating the two bounds we get:

$$|I_{11}| \leq \frac{\Psi\left(\frac{z_1}{|z_2|^{\frac{n-1}{n-m}}}, z_2\right)^{\frac{2(m-1)}{(m-2)(m+1)}}}{\lambda^{\frac{2}{m+1}} |z_2|^{\frac{2n-m-1}{n-m}}}.$$

Thus, for the integral I_1 we have the estimate:

$$\begin{aligned}|I_1| &\leq \frac{c\chi_{|z_2|<\delta|z_1|^{\frac{n-m}{n-1}}}(z_2)}{|\lambda|^{\frac{2}{m+1}} |z_1|^{\frac{2n-m-1}{(n-1)(m+1)}}} + \frac{\chi_{\delta|z_1|^{\frac{n-m}{n-1}}<|z_2|}(z_1) \Psi\left(\frac{z_1}{|z_2|^{\frac{n-1}{n-m}}}, z_2\right)^{\frac{2(m-1)}{(m-2)(m+1)}}}{|\lambda|^{\frac{2}{m+1}} |z_2|^{\frac{2n-m-1}{(n-m)(m+1)}}} \\ &= \frac{\widetilde{\Psi}(z_1, z_2)}{|\lambda|^{\frac{2}{m+1}}},\end{aligned}$$

where

$$\widetilde{\Psi}(z_1, z_2) := \frac{c\chi_{|z_2|<\delta|z_1|^{\frac{n-m}{n-1}}}(z_2)}{|z_1|^{\frac{2n-m-1}{(n-1)(m+1)}}} + \frac{\chi_{\delta|z_1|^{\frac{n-m}{n-1}}<|z_2|}(z_1) \Psi\left(\frac{z_1}{|z_2|^{\frac{n-1}{n-m}}}, z_2\right)^{\frac{2(m-1)}{(m-2)(m+1)}}}{|z_2|^{\frac{2n-m-1}{(n-m)(m+1)}}}.$$

Now, we show that $\widetilde{\Psi} \in L^{m+1-0}(V)$. Indeed, let $1 < p < m+1$ be a fixed number. Then

$$\int_{|z_1|<1} \frac{dz_1}{|z_1|^{\frac{2n-m-1}{(n-1)(m+1)}p}} \int_0^{\delta|z_1|^{\frac{n-m}{n-1}}} dz_2 = 2\delta \int_0^1 \frac{dz_1}{z_1^{\frac{2n-m-1}{(n-1)(m+1)}p-\frac{n-m}{n-1}}}.$$

Obviously, the last integral converges, whenever $p < m+1$. Moreover,

$$\int_V \frac{\Psi\left(\frac{z_1}{|z_2|^{\frac{n-1}{n-m}}}, z_2\right)^{\frac{2(m-1)p}{(m-2)(m+1)}} \chi_{c|z_1|^{\frac{n-m}{n-1}}<|z_2|}(z_1)}{|z_2|^{\frac{2n-m-1}{(n-m)(m+1)}p}} dz_1 dz_2 =$$

$$= \int_0^1 dz_2 \frac{1}{|z_2|^{\frac{2n-m-1}{(n-m)(m+1)}p-\frac{n-1}{n-m}}} \int_0^{\delta^{\frac{n-1}{m-n}}} \Psi^{\frac{2(m-1)}{(m-2)(m+1)}p}(\xi_1, z_2) d\xi_1 \leq$$

$$c\int_0^1 \frac{dz_2}{z_2^{\frac{2n-m-1}{(n-m)(m+1)}p-\frac{n-1}{n-m}}} < +\infty$$

whenever $p < m+1$.

Actually, in summation of the obtained estimates we came to a proof of the Lemma 3.1.

Indeed, for the integral I_1 we have the following uniform, with respect to the parameters z, estimate:

$$|I_1| \lesssim \frac{1}{|\lambda|^{\frac{1}{n}}}.$$

If $\varepsilon \geq \frac{2}{m+1} - \frac{1}{n}$ then the last estimate enough to have a proof of the Lemma 3.1.

Suppose $0 < \varepsilon < \frac{2}{m+1} - \frac{1}{n}$. Then we use the estimate

$$|I_1| \lesssim \frac{\tilde{\Psi}_1(z)}{|\lambda|^{\frac{2}{m+1}}},$$

with $\tilde{\Psi}_1 \in L^{m+1-0}(V)$.

Finally, interpolating the last two inequalities we get:

$$|I_1| \lesssim \frac{\tilde{\Psi}_1^{1-\theta}}{|\lambda|^{\frac{\theta}{n}+\frac{2(1-\theta)}{m+1}}},$$

where $0 < \theta < 1$. We can choose the number θ such that the following relation

$$\frac{\theta}{n} + \frac{2(1-\theta)}{m+1} = \frac{2}{m+1} - \varepsilon \quad \text{or} \quad \varepsilon = \theta\left(\frac{2}{m+1} - \frac{1}{n}\right) > 0.$$

holds. Then the inclusion $\tilde{\Psi}_1^{1-\theta} \in L^{m+1}(V)$ is obviously valid.

Analogical result holds true for the case $n = \infty$.

Indeed, assume $n = \infty$ then b_0 is a flat function at the origin. Note that it is enough to use uniform, with respect to the parameter z, estimates obtained by Karpushkin for the case $n = m = \infty$ for an analytic phase function [11] and alternatively we use the estimates proved in the paper [8] for smooth functions. This case corresponds to the surfaces of the class III.

Further, we assume that $n = \infty$ and $3 \leq m < \infty$. In this case, we essentially use the $R-$condition. So, $b_0 \equiv 0$ and we have

$$\phi(x_1, x_2) = b(x_1, x_2)(x_2 - x_1^m \omega(x_1))^2.$$

In this case the phase function Φ_2 has the form:

$$\Phi_1(x_1, z) = z_2 x_1^m \omega(x_1) + z_2^2 x_1 q(x_1, z_2) + z_1 x_1.$$

Then if $|z_1| \geq |z_2|$ then the phase function has no critical point in x_1, provided that the amplitude function is concentrated in a sufficiently small neighborhood of the origin. Then we use the integration by parts formula and have:

$$|I_1| \lesssim \frac{1}{1 + |\lambda z_1|}.$$

The last estimate yields

$$|I_1| \lesssim \frac{1}{\sqrt[4]{|z_1 z_2|}\sqrt{|\lambda|}}.$$

Now, suppose $|z_1| \leq |z_2|$. Then we can pull out z_2 and due to Proposition 3.3 we have the following estimate

$$|I_1| \leq \frac{\Psi\left(\frac{z_1}{z_2}, z_2\right)}{|z_2 \lambda|^{\frac{1}{2}}} =: \frac{\tilde{\Psi}(z)}{|\lambda|^{\frac{1}{2}}},$$

where $\tilde{\Psi} \in L^{\frac{2(m-1)}{m-2}-0}(V)$. Thus, we have a conclusion of the Lemma 3.1 as before, which finishes a proof of the Lemma 3.1. □

From the Lemma 3.1 it follows the required upper bound for the number $k_p(v)$ in the case $2m < n$. Indeed, first, we use the Proposition 2.1 and obtain $L^{p_0} \mapsto L^{p_0'}$ boundedness of the convolution operator M_k with $k > \frac{5}{2} - \frac{3}{m+1}$ for $p_0 = \frac{2m+2}{2m+1}$. Also, we get $L^{p_1} \mapsto L^{p_1'}$ boundedness of the convolution operator with $k > \frac{5}{2} - \frac{1}{n}$ for $p_1 = 1$ and also $L^{p_2} \mapsto L^{p_2'}$ boundedness of the convolution operator with $k = 0$ for $p_2 = 2$. Then by analytic interpolation of the obtained estimates, we get the required upper bound for the number $k_p(v)$:

$$k_p(v) \le \max\left\{\left(5-\frac{1}{m}\right)\left(\frac{1}{p}-\frac{1}{2}\right), \left(6-\frac{2(m+1)}{n}\right)\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}+\frac{m}{n}\right\}. \tag{3.6}$$

Further, we consider a lower bound for the number $k_p(v)$.

4 On the Sharpness of Results (a Lower Bound for the Number $k_p(v)$)

Theorem 4.1 *If $2m \ge n$, then there exists an amplitude function a_k such that the associated operator M_k is not $L^p(\mathbb{R}^3) \mapsto L^{p'}(\mathbb{R}^3)$ bounded, whenever $k < (5-\frac{2}{n})(\frac{1}{p}-\frac{1}{2})$.*

Proof We show that the sequence of functions suggested in [16] can be used to prove sharpness of the upper bound for the $k_p(v)$ in the case (i) of the Theorem 1.3. Let us take a smooth function in $\mathbb{R}^3$ such that $a_k(\xi) = |\xi|^{-k}$ for large ξ. For instance, we can take $a_k(\xi) = (1-\chi_0(|\xi|))|\xi|^{-k}$. Following, Sugimoto [16] we introduce the function: $G(y) = 1 + \phi(y_1, y_2) - y\nabla\phi(y)$. Define smooth non-negative functions f, g with $f(0) = g(0) = 1$ concentrated in a sufficiently small neighborhood of the origin, and a smooth non-negative function with $\chi_1(1) = 1$ and with support in a sufficiently small neighborhood of the point 1.

We set

$$u_j(x) = 2^{j\left(\frac{5}{2}-\frac{1}{n}\right)\left(-\frac{1}{p'}\right)} F^{-1}(v_j(2^{-j}\cdot))(x),$$

where

$$v_j(\xi) = \frac{f\left(2^{\frac{j}{n}}\frac{\xi_1}{\varphi(\xi)}\right) g\left(2^{\frac{j}{2}}\frac{\xi_2}{\varphi(\xi)}\right)\chi_1(\varphi(\xi))|\xi|^k}{\varphi(\xi)^2 G\left(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)}\right)} \in C_0^\infty(\mathbb{R}^3).$$

The sequence $\left\{F^{-1}(v_j(2^{-\frac{j}{n}}\cdot, 2^{-\frac{j}{2}}\cdot, \cdot))\right\}_{j=1}^{\infty}$ is bounded in $L^p(\mathbb{R}^3)$. Indeed, the following inequality holds:

$$\|F^{-1}(v_j(2^{-j/n}\cdot, 2^{-j/2}\cdot, \cdot))\|_{L^p} \lesssim \|v_j(2^{-j/n}\cdot, 2^{-j/2}\cdot, \cdot)\|_{L^{p'}}.$$

On the other hand

$$\begin{aligned}
&\|v_j(2^{-j/n}\cdot, 2^{-j/2}\cdot, \cdot)\|_{L^{p'}}^{p'} \\
&= \int f^{p'}\left(\frac{\xi_1}{\varphi(2^{-j/n}\xi_1, 2^{-j/2}\xi_2, \xi_3)}\right) g^{p'}\left(\frac{\xi_2}{\varphi(2^{-j/n}\xi_1, 2^{-j/2}\xi_2, \xi_3)}\right) \\
&\frac{\chi_1^{p'}(\varphi(2^{-j/n}\xi_1, 2^{-j/2}\xi_2, \xi_3))((2^{-j/n}\xi_1)^2 + (2^{-j/2}\xi_2)^2 + \xi_3^2)^{k/2}}{\varphi(2^{-j/n}\xi_1, 2^{-j/2}\xi_2, \xi_3)^{2p'} G^{p'}(\xi_1/\varphi(2^{-j/n}\xi_1, 2^{-j/2}\xi_2, \xi_3), \xi_2/\varphi(2^{-j/n}\xi_1, 2^{-j/2}\xi_2, \xi_3))} d\xi.
\end{aligned}$$

Since χ_1 is concentrated in a sufficiently small neighborhood of one, then we have: $\frac{1}{2} \le \varphi(2^{-j/n}\xi_1, 2^{-j/2}\xi_2, \xi_3) \le 2$. On the other hand supports of the functions f and g are concentrated in a sufficiently small neighborhood of the origin. Hence, $|\xi_1| < 1$ and $|\xi_2| < 1$ and also $|\xi_3| \sim 1$, because $\varphi(0, 0, 1) = 1$. This yields:

$$\|v_j(2^{-j/n}\cdot, 2^{-j/2}\cdot, \cdot)\|_{L^{p'}} \lesssim 1.$$

Consequently,

$$\|F^{-1}(v_j(2^{-j}\cdot))\|_{L^p} \lesssim 2^{j\left(\frac{1}{2}+\frac{n-1}{n}+1\right)\frac{1}{p'}} = 2^{j\left(\frac{5}{2}-\frac{1}{n}\right)\frac{1}{p'}}.$$

Hence the sequence $\{u_j\}_{j=1}^{\infty}$ is bounded in the space $L^p(\mathbb{R}^3)$.

On the other hand there is a relation:

$$M_k u_j(x) = 2^{j\left(\frac{5}{2}-\frac{1}{n}\right)\left(-\frac{1}{p'}\right)-kj+2j} F^{-1} \left(e^{i\varphi(\xi)} \frac{f\left(2^{\frac{j}{n}}\frac{\xi_1}{\varphi(\xi)}\right) g\left(2^{\frac{j}{2}}\frac{\xi_2}{\varphi(\xi)}\right)\chi_1(2^{-j}\varphi(\xi))}{\varphi(\xi)^2 G\left(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)}\right)}\right).$$

We perform the change of variables given by the scaling $2^{-j}\xi \mapsto \xi$ and obtain:

$$M_k u_j(x) = \frac{2^{j\left(\left(\frac{5}{2}-\frac{1}{n}\right)\left(-\frac{1}{p'}\right)-k+3\right)}}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{i2^j(\varphi(\xi)-x\xi)} \frac{f\left(2^{\frac{j}{n}}\frac{\xi_1}{\varphi(\xi)}\right) g\left(2^{\frac{j}{2}}\frac{\xi_2}{\varphi(\xi)}\right)\chi_1(\varphi(\xi))}{\varphi^2(\xi) G\left(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)}\right)} d\xi.$$

Then following Sugimoto we use change of variables $\xi = (\lambda y, \lambda(1 + \phi(y)))$ and get:

$$M_k u_j(x) = \frac{2^{j\left(\left(\frac{5}{2}-\frac{1}{n}\right)\left(-\frac{1}{p'}\right)-k+3\right)}}{\sqrt{(2\pi)^3}} \int e^{i2^j\lambda(1-(x_1y_1+x_2y_2+x_3(1+\phi(y))))} f(2^{\frac{j}{n}}y_1)g(2^{\frac{j}{2}}y_2)\chi_1(\lambda)d\lambda dy.$$

Finally, we use change of variables $2^{j/n}y_1 \mapsto y_1$, $2^{j/2}y_2 \mapsto y_2$ and obtain:

$$M_k u_j(x) = 2^{j\left(\left(\frac{5}{2}-\frac{1}{n}\right)\left(-\frac{1}{p'}\right)-k-\frac{1}{2}-\frac{1}{n}+3\right)} \int_{\mathbb{R}^3} e^{2^j i\lambda((x_3-1)-2^{-\frac{j}{n}}y_1x_1-2^{-\frac{j}{2}}y_2x_2-x_3\phi(2^{-\frac{j}{n}}y_1, 2^{-\frac{j}{2}}y_2))} f(y_1)g(y_2)\chi_1(\lambda)d\lambda dy.$$

If $|x_3 - 1| \ll 2^{-j}$, $|x_1| \ll 2^{-j(n-1)/n}$, $|x_2| \ll 2^{-j/2}$, then the phase is the non-oscillating function, because $\phi(2^{-j/n}y_1, 2^{-j/2}y_2) = o(2^{-j})$ provided the supports of f, g are small enough.

Consequently, we have the following lower bound:

$$\|M_k u_j\|_{L^{p'}} \gtrsim 2^{j\left(\left(\frac{5}{2}-\frac{1}{n}\right)\left(-\frac{1}{p'}\right)-k+\frac{5}{2}-\frac{1}{n}-\left(\frac{5}{2}-\frac{1}{n}\right)\left(\frac{1}{p'}\right)\right)}$$
$$= 2^{j\left(\left(5-\frac{2}{n}\right)\left(-\frac{1}{p'}\right)+\frac{5}{2}-\frac{1}{n}-k\right)} = 2^{j\left(\left(5-\frac{2}{n}\right)\left(\frac{1}{p}-\frac{1}{2}\right)-k\right)}.$$

Therefore, if $k < k_p(v) := \left(5-\frac{2}{n}\right)\left(\frac{1}{p}-\frac{1}{2}\right)$, then $\|M_k u_j\|_{L^{p'}} \to \infty$(as $j \to +\infty$). Thus, the operator $M_k : L^p(\mathbb{R}^3) \to L^{p'}(\mathbb{R}^3)$ is unbounded. □

The Theorem 4.1 finishes a proof of the part (i) of the main Theorem 1.3 for the case $n < \infty$.

Remark 4.2 The proof of the Theorem 4.1 shows that if $2m \le n$ and $k < \left(5-\frac{1}{m}\right)\left(\frac{1}{p}-\frac{1}{2}\right)$, then $\|M_k u_j\|_{L^{p'}} \to \infty$(as $j \to +\infty$), for some bounded sequence $\{u_j\}$ in the space $L^p(\mathbb{R}^3)$. Thus, the operator $M_k : L^p(\mathbb{R}^3) \to L^{p'}(\mathbb{R}^3)$ is an unbounded operator, whenever $k < \left(5-\frac{1}{m}\right)\left(\frac{1}{p}-\frac{1}{2}\right)$. Indeed, we can repeat all arguments of the proof of the Theorem 4.1 taking the sequence of functions:

$$u_j(x) = 2^{j\left(\frac{5}{2}-\frac{1}{2m}\right)\left(-\frac{1}{p'}\right)} F^{-1}(v_j(2^{-j}\xi))(x),$$

with

$$v_j(\xi) = \frac{f\left(2^{\frac{j}{2m}}\frac{\xi_1}{\varphi(\xi)}\right) g\left(2^{\frac{j}{2}}\frac{\xi_2}{\varphi(\xi)}\right)\chi_1(\varphi(\xi))|\xi|^k}{\varphi(\xi)^2 G\left(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)}\right)} \in C_0^\infty(\mathbb{R}^3)$$

for the case $2m \le n$ and obtain the following lower bound:

$$k_p(v) \ge \left(5-\frac{1}{m}\right)\left(\frac{1}{p}-\frac{1}{2}\right) \tag{4.1}$$

for the number $k_p(v)$ whenever $2m \le n$.

The same arguments can be used for the case $n = m = \infty$. Then we have the following lower bound:

$$k_p(v) \ge 5\left(\frac{1}{p}-\frac{1}{2}\right) \tag{4.2}$$

for the number $k_p(v)$, which corresponds to the class III of surfaces.

The lower bound (4.2) finishes a proof of the part (i) of Theorem 1.3.

Further, we consider the case $2m < n$. We prove the following statement.

Theorem 4.3 *If* $2m < n$, *and* $m \geq 3$ *then*

$$k_p(v) = \max\left\{\left(5 - \frac{1}{m}\right)\left(\frac{1}{p} - \frac{1}{2}\right), \left(6 - \frac{2m+2}{n}\right)\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} + \frac{m}{n}\right\}. \tag{4.3}$$

Proof Since we already got the upper bound for $k_p(v)$ (see (3.6)), then it is enough to prove a lower bound for that number.

If $k < \left(5 - \frac{1}{m}\right)\left(\frac{1}{p} - \frac{1}{2}\right)$, then the operator M_k is not $L^p(\mathbb{R}^3) \mapsto L^{p'}(\mathbb{R}^3)$ bounded (see Remark 4.2).

Assume $k < \left(6 - \frac{2(m+1)}{n}\right)\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} + \frac{m}{n}$. We show that M_k is not $L^p(\mathbb{R}^3) \mapsto L^{p'}(\mathbb{R}^3)$ bounded.

We slightly modified the Sugimoto [16] arguments and consider the sequence

$$u_j = 2^{-\frac{3j}{p'} + \frac{j(m+1)}{n}} F^{-1}(v_j(2^{-j}\cdot))(x),$$

where

$$v_j(\xi) = f\left(2^{\frac{j}{n}} \frac{\xi_1}{\varphi(\xi)}\right) g\left(2^{\frac{jm}{n}}\left(\frac{\xi_2}{\varphi(\xi)} - \left(\frac{\xi_1}{\varphi(\xi)}\right)^m \omega\left(\frac{\xi_1}{\varphi(\xi)}\right)\right)\right) \frac{\chi_1(\varphi(\xi))|\xi|^k}{\varphi^2(\xi) G\left(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)}\right)},$$

where $f, g, \chi_1 \in C_0^\infty(\mathbb{R})$ are non-negative smooth functions satisfying the conditions: $f(0) = g(0) = 1$ and supports of functions f, g lie in a sufficiently small neighborhood of the origin of $\mathbb{R}$ and χ_1 is a non-negative smooth function concentrated in a sufficiently small neighborhood of 1 and identically vanishes in a neighborhood of the origin and also $\chi_1(1) = 1$ (cf. [16]). Obviously $v_j \in C_0^\infty(\mathbb{R}^3)$ and $\|v_j\|_{L^{p'}(\mathbb{R}^3)} \sim 2^{-j\frac{m+1}{p'n}}$, where the symbol " $\sim''$ means that there exist non-zero constants $c_1, c_2 > 0$ such that

$$c_1 2^{-j\frac{m+1}{n}} \leq \int_{\mathbb{R}^3} |v_j(\xi)|^{p'} d\xi \leq c_2 2^{-j\frac{m+1}{n}}.$$

Indeed, we use change of variables $\xi = \lambda(y_1, y_2, 1 + \phi(y_1, y_2))$ in the integral $\int_{\mathbb{R}^3} |v_j(\xi)|^{p'} d\xi$. Note that on the support of v_j make sense the change of variables, provided j is big enough. Then we get:

$$\int_{\mathbb{R}^3} |v_j(\xi)|^{p'} d\xi = \int_{\mathbb{R}^3} f^{p'}(2^{\frac{j}{n}} y_1) g^{p'}(2^{j\frac{m}{n}}(y_2 - y_1^m \omega(y_1))) \chi_1^{p'}(\lambda)$$
$$\lambda^{(k-2)p'+2}(y_1^2 + y_2^2 + (1 + \phi(y_1, y_2))^2)^{\frac{kp'}{2}} G^{2-p'}(y_1, y_2) dy_1 dy_2 d\lambda \sim 2^{-j\frac{m+1}{n}}.$$

Thus, for large j we have

$$\|u_j\|_{L^p(\mathbb{R}^3)} \sim 1.$$

Now, we consider the lower estimate for $\|M_k u_j\|_{L^{p'}(\mathbb{R}^3)}$.

We have:

$$M_k u_j = F^{-1} e^{i\varphi(\xi)} a_k(\xi) F u_j = 2^{-\frac{3j}{p'} + j\frac{m+1}{np'}} F^{-1}(e^{i\varphi(\xi)} a_k(\xi) v_j(2^{-j}\xi))(x).$$

We perform change of variables given by the scaling $2^j\xi \to \xi$ and obtain:

$$\begin{aligned} M_k u_j(x) = \frac{2^{\frac{3j}{p} + \frac{j(m+1)}{np'} - kj}}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} & e^{i2^j(\varphi(\xi) - \xi x)} \\ & f\left(2^{\frac{j}{n}} \frac{\xi_1}{\varphi(\xi)}\right) g\left(2^{\frac{jm}{n}} \left(\frac{\xi_2}{\varphi(\xi)} - \left(\frac{\xi_1}{\varphi(\xi)}\right)^m \right.\right. \\ & \left.\left.\left. \omega\left(\frac{\xi_1}{\varphi(\xi)}\right)\right)\right) \frac{\chi_1(\varphi(\xi))}{\varphi^2(\xi) G\left(\frac{\xi_1}{\varphi(\xi)}, \frac{\xi_2}{\varphi(\xi)}\right)} d\xi. \end{aligned}$$

Finally, we use the change of variables $\xi \to \lambda(y_1, y_2, 1 + \phi(y_1, y_2))$ and we have:

$$\begin{aligned} M_k u_j(x) &= \frac{2^{\frac{3j}{p} + \frac{j(m+1)}{p'} - kj}}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} e^{i2^j\lambda(1 - x_3 - (y_1x_1 + y_2x_2 + x_3\phi(y_1, y_2)))} \times \\ & \times f(2^{\frac{j}{n}} y_1) g(2^{\frac{jm}{n}}(y_2 - y_1^m \omega(y_1))) \chi_1(\lambda) d\lambda dy_1 dy_2. \end{aligned}$$

Now, we perform the change of variables

$$y_1 = 2^{-\frac{j}{n}} z_1,\ y_2 = y_1^m \omega(y_1) + 2^{-j\frac{m}{n}} z_2.$$

Then we get

$$M_k u_j(x) = 2^{\frac{3j}{p} + \frac{m+1}{np'} j - \frac{m+1}{n} j - kj} \int e^{i2^j\lambda\Phi_3(z,x,j)} f(z_1) g(z_2) \chi_1(\lambda) d\lambda dz_1 dz_2,$$

where

$$\begin{aligned} &\Phi_3(z, x, j) := 1 - x_3 - (2^{-\frac{j}{n}} x_1 z_1 + x_2 2^{-\frac{jm}{n}} z_1^m \omega(2^{-\frac{j}{n}} z_1) + z_2 2^{-\frac{jm}{n}} x_2 + \\ &x_3 2^{-\frac{2jm}{n}} z_2^2 b(2^{-\frac{j}{n}} z_1, 2^{-\frac{jm}{n}}(z_1^m \omega(2^{-\frac{j}{n}} z_1) + z_2)) + 2^{-j} z_1^n \beta(2^{-\frac{j}{n}} z_1)). \end{aligned}$$

We use the stationary phase method in z_2 assuming,

$$|1 - x_3| << 2^{-j},\ |x_1| << 2^{-\frac{n-1}{n} j},\ |x_2| << 2^{-\frac{j(n-m)}{n}} \tag{4.4}$$

and, reminding that $2m < n$, to obtain:

$$M_k u_j(x) = 2^{j\left(\frac{3}{p}+\frac{m+1}{np'}-\frac{1}{n}-\frac{1}{2}-k\right)}$$
$$\left(\int_{\mathbb{R}^2} e^{i2^j\lambda\Phi_4} f(z_1)g(z_2^c(z_1,x_2))\chi_1(\lambda)d\lambda dz_1 + O(2^{j\left(\frac{2m}{n}-1\right)})\right),$$

where

$$\Phi_4 := \Phi_4(z_1,x,j) := 1 - x_3 - x_1z_12^{-\frac{j}{n}} - x_22^{-\frac{jm}{n}}z_1^m\omega(2^{-\frac{j}{n}}z_1)$$
$$-2^{-j}z_1^n\beta(2^{-\frac{j}{n}}z_1) + x_2^2B(z_1,x_2,x_3,2^{-j}),$$

and B is a smooth function satisfying the condition $|B| \backsim 1$. Consequently, accounting the conditions (4.4) and the inequality $2m < n$, we establish the following lower bound:

$$\|M_k u_j\|_{L^{p'}(\mathbb{R}^3)} \geq 2^{j\left(\frac{3}{p}+\frac{m+1}{np'}-\frac{1}{n}-\frac{1}{2}-\frac{1}{p'}\left(3-\frac{m+1}{n}\right)-k\right)}c,$$

where $c > 0$ is a constant which does not depend on j. Thus if

$$k < \left(6 - \frac{2(m+1)}{n}\right)\left(\frac{1}{p}-\frac{1}{2}\right) - \frac{1}{2} + \frac{m}{n}$$

then the operator M_k is not $L^p(\mathbb{R}^3) \mapsto L^{p'}(\mathbb{R}^3)$ bounded.

Analogical result holds true for the case $n = \infty$.

Thus, if $k < k_p(v)$ then the M_k is not $L^p - L^{p'}$ bounded operator. This completes a proof of the Theorem 4.3. □

Theorem 4.3 finishes a proof of the part (ii) of the main Theorem 1.3.

Thus, the main Theorem 1.3 is proved.

Acknowledgements The authors wish to thank to Sh. A. Alimov for useful discussions of the obtained results. Also, we thank to anonymous referees for very valuable remarks and comments.

Funding The research is not funded from any foundation.

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