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Uniform Boundedness of Sequence of Operators Associated with the Walsh System and Their Pointwise Convergence

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Abstract

Revisiting the main point of the almost everywhere convergence, it becomes clear that a weak (1,1)-type inequality must be established for the maximal operator corresponding to the sequence of operators. The better route to take in obtaining almost everywhere convergence is by using the uniform boundedness of the sequence of operator, instead of using the mentioned maximal type of inequality. In this paper it is proved that a sequence of operators, defined by matrix transforms of the Walsh–Fourier series, is convergent almost everywhere to the function $f \in L_1$ if they are uniformly bounded from the dyadic Hardy space $H_1(\mathbb{I})$ to $L_1(\mathbb{I})$. As a further matter, the characterization of the points are put forth where the sequence of the operators of the matrix transform is convergent.

Keywords Walsh system · Boundedness of sequence of operators · Hardy spaces · Almost everywhere convergence

Mathematics Subject Classification 42C10

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1 Introduction

Walsh–Fourier series has some popular problems in relation to its convergence that has gained traction amongst many mathematicians. For instance, Stein [16] proved the existence of an integrable function whose Walsh–Fourier series is divergent at all points. In contrast, a sequence of linear operators on certain function spaces is defined as partial sums that are associated with Walsh–Fourier series obtained by a matrix transformation. Some of the most well known examples of sequences of this kind are Fejér means, Cesáro means, logarithmic means, Nörlund means, etc.

Hence, revisiting the main point of the almost everywhere convergence, for the cases mentioned, it becomes clear that a weak (1,1)-type inequality must be established for the maximal operator corresponding to the sequence of operators. Having said that, the better route to take in obtaining almost everywhere convergence is by using the uniform boundedness of the sequence of operator, instead of using the mentioned maximal type of inequality.

Therefore, this paper considers a general type of sequences of operators that are associated with the Walsh system. The operators are established to be convergent almost everywhere by establishing uniformly bounded inequalities in the dyadic Hardy space H_1.

Note, necessary information regarding the Walsh–Fourier series that is imperative to understanding the paper, is given below.

Note, $\mathbb{N}$ stands for the set of all non-negative integers. A dyadic interval in $\mathbb{I} := [0, 1)$ means an interval in the form $I(l,k) := \left[\frac{l}{2^k}, \frac{l+1}{2^k}\right)$ for some $k \in \mathbb{N}$, $0 \le l < 2^k$. Given $k \in \mathbb{N}$ and $x \in \mathbb{I}$, $I_k(x)$ denotes the dyadic interval of length 2^{-k} which contains the point x. For the sake of shortness, $I_n := I_n(0)$ $(n \in \mathbb{N})$ is denoted as $\overline{I}_k(x) := I \backslash I_k(x)$. Given $n \in \mathbb{N}$, $n \neq 0$ by $|n|$ which indicates $2^{|n|} \le n < 2^{|n|+1}$.

Let $L_0(\mathbb{I})$ be the set of all a. e. finite, Lebesgue measurable functions from $\mathbb{I}$ into $[-\infty, \infty]$. For $0 < p < \infty$ by $L_p(I)$ the set of all $f \in L_0(\mathbb{I})$ is denoted such that

$$\|f\|_p := \left(\int_{\mathbb{I}} |f(x)|^p \, dx\right)^{1/p} < \infty.$$

As usual, $L_\infty(I)$ denotes the set of all $f \in L_0(\mathbb{I})$ such that

$$\|f\|_\infty := \inf\left\{y \in \mathbb{R}^1 : |f(x)| \le y \text{ for a. e. } x \in \mathbb{I}\right\} < \infty.$$

The space $L_{1,\infty}(\mathbb{I})$ consists of all measurable functions $f \in L_0(\mathbb{I})$ such that

$$\|f\|_{1,\infty} := \sup_{\lambda>0} \lambda \, |(|f| > \lambda)| < +\infty. \tag{1}$$

Let

$$x=\sum_{n=0}^{\infty}x_n 2^{-(n+1)}$$

be the dyadic expansion of $x \in \mathbb{I}$, where $x_n \in \{0,1\}$. If x is a dyadic rational number the expansion which terminate in $0'$s is chosen.

By $\dotplus$, the logical addition on $\mathbb{I}$ is denoted, i.e. for any $x, y \in \mathbb{I}$

$$x \dotplus y := \sum_{n=0}^{\infty}\left|x_n - y_n\right| 2^{-(n+1)}.$$

For every $n \in \mathbb{N}$ the following binary expansion can be written

$$n=\sum_{k=0}^{\infty}\varepsilon_k\left(n\right)2^k,$$

where $\varepsilon_k\left(n\right)=0$ or 1 for $k \in \mathbb{N}$. The numbers $\varepsilon_k\left(n\right)$ will be called the binary coefficients of n.

The Rademacher system is defined by

$$\rho_n\left(x\right):=\left(-1\right)^{x_n} \quad \left(x \in \mathbb{I}, n \in \mathbb{N}\right).$$

The Walsh-Paley system is defined as the sequence of the Walsh-Paley functions:

$$w_0(x)=1, \quad w_n\left(x\right):=\prod_{k=0}^{\infty}\left(\rho_k\left(x\right)\right)^{\varepsilon_k(n)}=\left(-1\right)^{\sum_{k=0}^{|n|}\varepsilon_k(n)x_k} \quad \left(x \in \mathbb{I}, n \in \mathbb{N}\right).$$

The Walsh-Dirichlet kernel is defined by

$$D_n\left(x\right):=\sum_{k=0}^{n-1}w_k\left(x\right) \quad \left(n \in \mathbb{N}\right), \; D_0=0.$$

It is well-known [9, 15] that

$$D_{2^n}\left(x\right)=\begin{cases}2^n, & \text{if } x \in I_n\\ 0, & \text{if } x \in \overline{I}_n\end{cases}. \tag{2}$$

Given $f \in L_1\left(\mathbb{I}\right)$ its partial sums of the Walsh–Fourier series are defined by

$$S_m\left(f,x\right):=\sum_{i=0}^{m-1}\widehat{f}\left(i\right)w_i\left(x\right),$$

where

$$\widehat{f}(i)=\int_{\mathbb{I}} f(t)\, w_i(t)\, dt$$

which is referred as the $i-th$ Walsh–Fourier coefficient of the function f.

Denote

$$E_n(f,x):=S_{2^n}(f,x), \quad E^*(f,x):=\sup_{n\in\mathbb{N}}|E_n(f,x)|.$$

Recall, that for $0<p<\infty$, the Hardy space $H_p(\mathbb{I})$ is the set of all functions $f\in L_1(\mathbb{I})$ such that

$$\|f\|_{H_p}:=\left\|E^*(f)\right\|_p<\infty.$$

The Fejér means and kernel, respectively, are defined by

$$\sigma_n(f,x):=\frac{1}{n}\sum_{k=1}^{n}S_k(f,x), \quad K_n(t):=\frac{1}{n}\sum_{k=1}^{n}D_k(t).$$

It is well-known [9, 15] that the operator $\sigma_n(f):=\sigma_n(f,x)$ can be represented as a convolution of f and K_n, i.e.,

$$\sigma_n(f)=f*K_n.$$

Conclusively, the following maximal operators will be considered

$$\sigma_*(f):=\sup_{n\in\mathbb{N}}|\sigma_n(f)|$$

and

$$\sigma_*^{abc}(f):=\sup_{n\in\mathbb{N}}|f*|K_n||.$$

The validity of the following inequality has been proven in [6].

$$\left\|\sigma_*^{abc}(f)\right\|_p\leq c_p\|f\|_p \quad \left(f\in H_p(\mathbb{I}),\, p>0\right). \tag{3}$$

Now, recall certain properties of various sequences of operators associated with Walsh systems.

A very well known fact is that [9, 15] L_1 norms of Fejér kernels are uniformly bounded i.e.,

$$\|K_n\|_1\leq c \quad \text{for all } n\in\mathbb{N}. \tag{4}$$

We notice that in [17] the constant c is explicitly calculated as $c = 17/15$. The last condition implies

$$\sup_{n\in\mathbb{N}} \|\sigma_n(f)\|_\infty \le c\,\|f\|_\infty \quad (f \in L_\infty(\mathbb{I}))\,. \tag{5}$$

This means that the sequence of operators $\{\sigma_n\}$ are uniformly bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$. One can check that the conditions (4) and (5) are equivalent.

Schipp [14] showed that the maximal operator σ_* is of weak type (1, 1), i. e.

$$\left\|\sup_{n\in\mathbb{N}} |\sigma_n(f)|\right\|_{1,\infty}(\mathbb{I}) \le c\,\|f\|_1 \quad (f \in L_1(\mathbb{I}))\,, \tag{6}$$

This inequality by standard argument [11] implies the a. e. convergence of $\{\sigma_n\}$. The Schipp's result together with interpolation yields the boundedness of $\sigma_* : L_p(\mathbb{I}) \to L_p(\mathbb{I})$ $(1 < p \le \infty)$. However, at $p = 1$, this fails to hold. However Fujii [2] proved a weaker estimation

$$\|\sigma_*(f)\|_1 \le c\,\|f\|_{H_1} \quad (f \in H_1(\mathbb{I}))\,. \tag{7}$$

In addition, Fujii's theorem has been extended by Weisz [20]. In other words, the maximal operator of the Fejér means of the one-dimensional Walsh–Fourier series is bounded from the martingale Hardy space $H_p(\mathbb{I})$ to the space $L_p(\mathbb{I})$ for $p > 1/2$.

Recall that the (C, α_n) means of the Walsh–Fourier series of the function f is given by

$$\sigma_n^{\alpha_n}(f, x) = \frac{1}{A_n^{\alpha_n}} \sum_{j=0}^{n} A_{n-j}^{\alpha_n - 1} S_j(f, x),$$

where $\{\alpha_n\}$ is some sequence, and

$$A_n^\alpha := \frac{(1+\alpha)\ldots(n+\alpha)}{n!}$$

for any $n \in N$, $\alpha \ne -1, -2, \ldots$.

Denote

$$\sigma_*^{\alpha_n} f = \sup_{n\in\mathbb{N}} |\sigma_n^{\alpha_n}(f)|.$$

Weisz [20] investigated the maximal operator σ_*^α with $0 < \alpha < 1$, where the sequence $\{\alpha_n\}$ is stationary, i.e., $\alpha_n = \alpha$ for all $n \in \mathbb{N}$. His results prove the boundedness of $\sigma_*^\alpha : L_\infty(\mathbb{I}) \to L_\infty(\mathbb{I})$.

This result implies

$$\sup_{n\in\mathbb{N}} \left\|\sigma_n^\alpha(f)\right\|_\infty \le c\,\|f\|_\infty \quad (f \in L_\infty(\mathbb{I}))\,. \tag{8}$$

Moreover, Weisz also established the boundedness of $\sigma_*^\alpha : H_p(\mathbb{I}) \to L_p(\mathbb{I})$ when $p > 1/(1+\alpha)$.

However, if $\alpha_n \to 0$, then in [4], the sequence of operators $\{\sigma_n^{\alpha_n}\}$ is proved to not be bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$. Moreover, all subsequences $\{n_a : a \in \mathbb{N}\}$ are characterized which provide the uniform boundedness of $\{\sigma_{n_a}^{\alpha_{n_a}}\}$ from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$.

Recall that the Nörlund logarithmic means are defined by

$$M_n(f,x) := \frac{1}{l_n}\sum_{k=0}^{n-1}\frac{S_k(f,x)}{n-k}, \quad l_n := \sum_{k=1}^{n}(1/k).$$

In [5] the sequence of operators $M_n : L_\infty(\mathbb{I}) \to L_\infty(\mathbb{I})$ is proved to not be bounded. However, all subsequences $\{n_a : a \in \mathbb{N}\}$ have been characterized for which $\{M_{n_a} : L_\infty(\mathbb{I}) \to L_\infty(\mathbb{I})\}$ is uniformly bounded. Moreover, given $\{n_a : a \in \mathbb{N}\}$ the following

$$\left\|M_{n_a}\right\|_{L_\infty(\mathbb{I})\to L_\infty(\mathbb{I})} \sim \frac{1}{|n_a|}\sum_{k=1}^{|n_a|}\left|\varepsilon_k(n_a) - \varepsilon_{k+1}(n_a)\right| k \tag{9}$$

has been proved.

The results above have been provided as an aid as well as to gain an appreciation for the the subject matter. Therefore this paper aims to explore the sequences of general operators associated with the Walsh–Fourier series.

Let $\mathbb{T} := (t_{k,n})$ be an infinite triangular matrix satisfying the following conditions:

(a) $t_{k,n} \geq 0$, $k, n \in \mathbb{N}$;
(b) $t_{k,n} = 0$, $k > n$;
(c) $\sum_{k=1}^{n} t_{k,n} = 1$.

Define the sequence of operators associated with Walsh–Fourier series as follows

$$T_n(f;x) := \sum_{k=1}^{n} t_{k,n} S_k(f;x) \quad (n \in \mathbb{N}). \tag{10}$$

In what follows, always assume that

$$0 \leq t_{k,n} \leq t_{k+1,n}, k = 1, 2, \ldots, n-1, \ n > 1. \tag{11}$$

Remark 1 It's important to emphasize that, for each fixed n, the sequence $\{t_{k,n}\}$ is non-increasing, then the sequence $\{T_n(f;x)\}$ is convergent almost everywhere for all $f \in L_1$ [8]. Hence, it becomes pertinent to explore the scenario where $\{t_{k,n}\}$ is non-decreasing.

In [8] the following estimation has been proved

$$\|T_n\|_{L_\infty(\mathbb{I})\to L_\infty(\mathbb{I})} \sim V(n,\mathbb{T}), \tag{12}$$

where

$$V(n,\mathbb{T}) := \sum_{k=1}^{|n|} |\varepsilon_k(n) - \varepsilon_{k+1}(n)| \tau_n^{(n^{(k)})},$$

$$\tau_n^{(k)} := \sum_{l=k}^{n} t_{l,n}, \quad n^{(s)} := \sum_{j=s}^{\infty} \varepsilon_j(n) 2^j.$$

For each $n \in \mathbb{N}$, the triangular matrix transform kernel is defined by

$$F_n(t) := \sum_{k=1}^{n} t_{k,n} D_k(t).$$

Then

$$T_n(f,x) = (f * F_n)(x) = \int_{\mathbb{I}} f(x \dotplus t) F_n(t) d(t). \tag{13}$$

Denote

$$T^*(f) := \sup_{n\in\mathbb{N}} |T_n(f)|.$$

Remark 2 Emphasis and consideration is heavily put on the mentioned operators which are particular cases of the sequence of operators $\{T_n(f)\}$.

(I) Assume that

$$t_{k,n} = \begin{cases} \frac{1}{n}, & k \le n \\ 0, & k > n \end{cases} \tag{14}$$

then $T_n(f) = \sigma_n(f)$ (Fejér means);

(II) Now, let

$$t_{k,n} = \begin{cases} \frac{A_{n-k}^{\alpha_n - 1}}{A_n^{\alpha_n}}, & k \le n \\ 0, & k > n \end{cases}, \quad \alpha_n \to 0 \text{ as } n \to \infty, \tag{15}$$

then $T_n(f) = \sigma_n^{\alpha_n}(f)$ ((C,α_n)-means);

(III) Define

$$t_{k,n} = \begin{cases} \frac{1}{l_n}\frac{1}{(n-k)}, & k < n \\ 0, & k \geq n \end{cases}, \tag{16}$$

then $T_n(f) = M_n(f)$ (Nörlund logarithmic means).

Time and time again, the almost everywhere convergence of Fourier series have been explored where authors proved the boundedness of maximal operators on Hardy spaces and weak type inequalities for them. However they have left a gap which this paper aims to fill. The main goal of this paper is a novel outlook at the almost everywhere convergence of Fourier series. Moreover, to consider more general sequences of operators and for which to establish connections between their uniformly boundedness on Hardy spaces and the weak type inequality for its maximal operator.

Henceforth, the following is the main move result that has achieved this aim.

Theorem 1 *Let* $\{n_a : a \in \mathbb{N}\}$ *be a subsequence of natural numbers. Then the following statements are equivalent:*

(i) *The sequence of operators* $\{T_{n_a}\}$ *is uniformly bounded from* $H_1(\mathbb{I})$ *to* $L_1(\mathbb{I})$, *i. e. there exist a constant* $C > 0$ *such that*

$$\sup_{a\in\mathbb{N}} \|T_{n_a}(f)\|_1 \leq C\|f\|_{H_1} \quad (f \in H_1);$$

(ii) *The sequence of operators* $\{T_{n_a}\}$ *is uniformly bounded from* $L_\infty(\mathbb{I})$ *to* $L_\infty(\mathbb{I})$, *i. e. there exist a constant* $C > 0$ *such that*

$$\sup_{a\in\mathbb{N}} \|T_{n_a}(f)\|_\infty \leq C\|f\|_\infty \quad (f \in L_\infty);$$

(iii) *There are* C_1 *and* C_2 *positive constants, such that for any* $f \in L_1(\mathbb{I})$ *the following inequality is true*

$$\sup_{a\in\mathbb{N}} \left|T_{n_a}(f)\right| \leq C_1 E^*(|f|) + C_2\sigma_*^{abc}(|f|). \tag{17}$$

(iv) $\sup_{a\in\mathbb{N}} V(n_a, \mathbb{T}) < \infty$.

Due to [15, Ch. 3]

$$\left\|E^*(|f|)\right\|_p \leq C_p \||f|\|_{H_p} \quad \left(|f| \in H_p, p > 0\right),$$

by (see [6])

$$\left\| \sigma_{*}^{abc}(f) \right\|_{p} \leq C_p \, \| |f| \|_{H_p} \quad \left(|f| \in H_p, p > 0 \right),$$

and from Theorem 1, the following ensues

Theorem 2 *Let* $\{n_a : a \in \mathbb{N}\}$ *be a subsequence of natural numbers for which the sequence of operators* $\left\{T_{n_a}\right\}$ *is uniformly boundend from* $H_1(\mathbb{I})$ *to* $L_1(\mathbb{I})$*. Then there is a constant* $C_p > 0$ *such that*

$$\left\| \sup_a |T_{n_a}(f)| \right\|_{H_p} \leq C_p \| |f| \|_{H_p} \quad \left(|f| \in H_p(\mathbb{I}), p > 0 \right). \tag{18}$$

From inequality (18), using the interpolation theorem (for details, see [19, 21]), the following is obtained

$$\left\| \sup_a |T_{n_a}(f)| \right\|_{1,\infty} \leq C \left\| E^{*}(|f|) \right\|_{1,\infty} \leq C \| f \|_1 \quad (f \in L_1(\mathbb{I})). \tag{19}$$

Consequently, the next theorem follows from inequality (19) by the well-known density argument due to Marcinkiewicz and Zygmund [11].

Theorem 3 *Let* $\{n_a : a \in \mathbb{N}\}$ *be a subsequence of natural numbers for which the sequence of operators* $\left\{T_{n_a}\right\}$ *is uniformly bounded from* $H_1(\mathbb{I})$ *to* $L_1(\mathbb{I})$*. Then for each* $f \in L_1(\mathbb{I})$ *we have*

$$\lim_{a \to \infty} T_{n_a}(f, x) = f(x) \ \ \textit{for a. e. } x \in \mathbb{I}.$$

Remark 3 In the section 3, as an application of this theorem, a description of points at which the sequence of operators is pointwise convergent is also provided.

2 Proof of Theorem 1

In this section, a proof of Theorem 1 is provided.

Proof (i)⇒(ii). To prove it by indirect method assume that the sequence of operators $\{T_{n_a}\}$ is not uniformly bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$. Due to (12) this implies

$$\sup_{a \in \mathbb{N}} V(n_a, \mathbb{T}) = \infty. \tag{20}$$

From the last equality, without loss of generality, for the sake of simplicity the above given sequence $\{n_a\}$ satisfies

$$V(n_a, \mathbb{T}) \geq a^4 \tag{21}$$

and

$$|n_a| > |n_{a-1}| + 1. \tag{22}$$

Let us define

$$f := \sum_{a=1}^{\infty} \lambda_a f_a, ,$$

where

$$\lambda_a := \frac{1}{\sqrt{V(n_a, \mathbb{T})}}$$

and

$$f_a := D_{2^{|n_a|+1}} - D_{2^{|n_a|}} = w_{2^{|n_a|}} D_{2^{|n_a|}}.$$

Then, one can write

$$\begin{aligned} S_{2^n}(f) &= \sum_{\{a:|n_a|<n\}} \lambda_a S_{2^n}\left(D_{2^{|n_a|+1}} - D_{2^{|n_a|}}\right) \\ &= \sum_{\{a:|n_a|<n\}} \lambda_a f_a. \end{aligned}$$

Hence,

$$\sup_n |S_{2^n}(f)| \leq \sum_a \lambda_a |f_a| = \sum_a \lambda_a D_{2^{|n_a|}}.$$

Now, applying (2) and (21) one finds

$$\left\| \sup_n |S_{2^n}(f)| \right\|_1 \leq \sum_a \lambda_a \leq \sum_a \frac{1}{a^2} < \infty.$$

which yields that $f \in H_1(\mathbb{I})$.

On the other hand, one has the following

$$T_{n_a}(f) = \lambda_a T_{n_a}(f_a) + \sum_{j=0}^{a-1} \lambda_j T_{n_a}\left(f_j\right) + \sum_{j=a+1}^{\infty} \lambda_j T_{n_a}\left(f_j\right). \tag{23}$$

Now, consider each term one by one.

If $j > a$, then

$$T_{n_a}\left(f_j\right) = f_j * F_{n_a} = \left(D_{2^{|n_j|+1}} - D_{2^{|n_j|}}\right) * F_{n_a}$$
$$= S_{2^{|n_j|+1}}\left(F_{n_a}\right) - S_{2^{|n_j|}}\left(F_{n_a}\right) = 0. \tag{24}$$

Firstly, notice that by [8, Theorem 3] one has

$$\begin{aligned}\|T_n(f) - f\|_1 \leq c_1 V(n, \mathbb{T}) &\left\|E_{|n_a|}(f) - f\right\|_1 \\ &+c_2 \left\|E_{|n_a|-1}(f) - f\right\|_1 \\ &+c_3 \sum_{r=0}^{|n|-2} 2^r t_{2^{r+1}-1,n} \|E_r(f) - f\|_1 .\end{aligned}$$

Therefore, if $j < a$, from the last inequality, one finds

$$\begin{aligned}\left\|T_{n_a}\left(f_j\right)\right\|_1 &\leq \left\|T_{n_a}\left(f_j\right) - f_j\right\|_1 + 1 \\ &\leq c_1 V\left(n_a, \mathbb{T}\right) \left\|E_{|n_a|}\left(f_j\right) - f_j\right\|_1 \\ &\quad +c_2 \left\|E_{|n_a|-1}\left(f_j\right) - f_j\right\|_1 \\ &\quad +c_3 \sum_{r=0}^{|n_a|} 2^r t_{2^{r+1}-1,n_a} \left\|E_r\left(f_j\right) - f_j\right\|_1 + 1.\end{aligned} \tag{25}$$

According to

$$E_r\left(f_j\right) = \begin{cases} f_j, & \text{if } r > \left|n_j\right| \\ 0, & \text{if } r \leq \left|n_j\right| \end{cases},$$

from (25) it follows that

$$\begin{aligned}\left\|T_{n_a}\left(f_j\right)\right\|_1 &\leq c_3 \sum_{r=0}^{|n_j|} 2^r t_{2^{r+1}-1,n_a} + c_4 \\ &\leq c_3 \sum_{r=0}^{|n_j|-2} \sum_{l=2^{r+1}-1}^{2^{r+2}} t_{l,n_a} + c_4 \\ &\leq c_3 \sum_{l=1}^{2^{|n_j|}} t_{l,n_a} + c_4 \leq c < \infty, \quad j < a.\end{aligned} \tag{26}$$

Now, let us estimate $T_{n_a}(f_a)$. Assume that $n_a = 2^{|n_a|} + n'_a, n'_a < 2^{|n_a|}$. Then

$$\begin{aligned}
T_{n_a}(f_a) &= \sum_{k=1}^{n_a} t_{k,n_a} S_k(f_a) \\
&= \sum_{k=1}^{n_a} t_{k,n_a} S_k\left(D_{2^{|n_a|+1}} - D_{2^{|n_a|}}\right) \\
&= \sum_{k=2^{|n_a|}}^{n_a} t_{k,n_a} S_k\left(D_{2^{|n_a|+1}} - D_{2^{|n_a|}}\right) \\
&= \sum_{k=2^{|n_a|}}^{n_a} t_{k,n_a} \left(S_{2^{|n_a|+1}}(D_k) - S_k\left(D_{2^{|n_a|}}\right)\right) \\
&= \sum_{k=2^{|n_a|}}^{n_a} t_{k,n_a} \left(D_k - D_{2^{|n_a|}}\right) \\
&= \sum_{k=0}^{n'_a} t_{k+2^{|n_a|},n_a} \left(D_{k+2^{|n_a|}} - D_{2^{|n_a|}}\right) \\
&= w_{2^{|n_a|}} \sum_{k=0}^{n'_a} t_{k+2^{|n_a|},n_a} D_k.
\end{aligned}$$

Now, keeping in mind the following estimation (see [8, Theorem 2])

$$\left\| \sum_{k=1}^{n} t_{k,n} D_k \right\|_1 \sim \sum_{s=0}^{|n|} |\varepsilon_s(n) - \varepsilon_{s+1}(n)| \, \tau_n^{\left(n^{(s)}\right)}.$$

one finds

$$\begin{aligned}
\left\| T_{n_a}(f_a) \right\|_1 &= \left\| \sum_{k=1}^{n'_a} t_{k+2^{|n_a|},n_a} D_k \right\|_1 \\
&\sim \sum_{s=1}^{|n'_a|} \left|\varepsilon_s\left(n'_a\right) - \varepsilon_{s+1}\left(n'_a\right)\right| \, \widetilde{\tau}_{n'_a}^{\left(\left(n'_a\right)^{(s)}\right)},
\end{aligned}$$

where

$$\widetilde{\tau}_{n'_a}^{\left(\left(n'_a\right)^{(s)}\right)} = \sum_{k=\left(n'_a\right)^{(s)}}^{n'_a} t_{k+2^{|n_a|},n_a} = \sum_{k=(n_a)^{(s)}}^{n_a} t_{k,n_a} = \tau_{n_a}^{\left(n_a^{(s)}\right)}.$$

Consequently,

$$\left\|T_{n_a}(f_a)\right\|_1 \sim \sum_{s=1}^{|n'_a|}\left|\varepsilon_s\left(n'_a\right)-\varepsilon_{s+1}\left(n'_a\right)\right|\tau_{n_a}^{\left(n_a^{(s)}\right)} \sim \sum_{s=1}^{|n_a|}\left|\varepsilon_s(n_a)-\varepsilon_{s+1}(n_a)\right|\tau_{n_a}^{\left(n_a^{(s)}\right)} \sim V(n_a,\mathbb{T}). \tag{27}$$

From (24) and (26) we infer that the second and the third terms of (23) are bounded (here we have used $\lambda_a \le \frac{1}{a^2}$). On the other hand, the first term of (23) is not bounded due to (20) and (27). Therefore, one finds

$$\sup_{a\in\mathbb{N}}\left\|T_{n_a}(f)\right\|_1=\infty.$$

The obtained contradiction proves the assertion.

(ii) ⇒(i). Indeed, due to $V(n_a,\mathbb{T})\sim\left\|F_{n_a}\right\|_1$ (see [8]), the estimate from (12) can be obtained that the sequence of operators $\left\{T_{n_a}\right\}$ is bounded from $L_1(\mathbb{I})$ to $L_1(\mathbb{I})$. On the other hand, the following inequality $\|f\|_1\le\|f\|_{H_1}$ holds [15]. Therefore, the sequence of operators $\left\{T_{n_a}\right\}$ also bounded from $H_1(\mathbb{I})$ to $L_1(\mathbb{I})$.

(ii)⇒(iii). At present, (17) will be established. Indeed, the following is proved in [8]

$$T_{n_a}(f)=f*F_{n_a,1}+f*F_{n_a,2}, \tag{28}$$

where

$$F_{n_a,1}:=w_{n_a}\sum_{s=0}^{|n_a|}\varepsilon_s(n_a)\,\tau_{n_a}^{\left(n_a^{(s)}\right)}\left(D_{2^{s+1}}-D_{2^s}\right), \tag{29}$$

$$\begin{aligned}\left|F_{n_a,2}\right| \le& \sum_{s=0}^{|n_a|}\varepsilon_s(n_a)\,D_{2^s}\sum_{l=1}^{2^s-1}t_{n_a^{(s)}-l,n_a}\\ &+\sum_{s=0}^{|n|}\varepsilon_s(n_a)\sum_{l=1}^{2^s-2}(t_{n_a^{(s)}-l,n_a}-t_{n_a^{(s)}-l-1,n_a})l\,|K_l|\\ &+\sum_{s=0}^{|n_a|}\varepsilon_s(n_a)\,t_{n_a^{(s)}-2^s+1,n_a}(2^s-1)\,|K_{2^s-1}|\end{aligned} \tag{30}$$

and

$$\sup_{a\in\mathbb{N}}\left(|f|*\left|F_{n_a,2}\right|\right)\le c\left(\sup_{k\in\mathbb{N}}(|f|*|K_k|)+E^*(f;x)\right). \tag{31}$$

Then

$$\begin{aligned} f * F_{n_a,1} &= \sum_{s=0}^{|n_a|} \varepsilon_s\left(n_a\right) \tau_{n_a}^{\left(n_a^{(s)}\right)}\left(S_{2^{s+1}}\left(f w_{n_a}\right)-S_{2^s}\left(f w_{n_a}\right)\right) \\ &= \sum_{s=0}^{|n_a|-1}\left(\varepsilon_s\left(n_a\right)-\varepsilon_{s+1}\left(n_a\right)\right) \tau_{n_a}^{\left(n_a^{(s)}\right)} S_{2^{s+1}}\left(f w_{n_a}\right) \\ &\quad+\sum_{s=0}^{|n_a|-1} \varepsilon_{s+1}\left(n_a\right)\left(\tau_{n_a}^{\left(n_a^{(s)}\right)}-\tau_{n_a}^{\left(n_a^{(s+1)}\right)}\right) S_{2^{s+1}}\left(f w_{n_a}\right) \\ &\quad-\varepsilon_0\left(n_a\right) \tau_{n_a}^{\left(n_a^{(0)}\right)} S_{2^0}\left(f w_{n_a}\right)+\varepsilon_{|n_a|}\left(n_a\right) \tau_{n_a}^{\left(n_a^{(|n_a|)}\right)} S_{2^{|n_a|+1}}\left(f w_{n_a}\right). \end{aligned}$$

Consequently,

$$\begin{aligned} \sup_{a\in\mathbb{N}}\left|f * F_{n_a,1}\right| \leq c\Bigg(& V(n_a,\mathbb{T}) E^*(|f|) \\ & +E^*(|f|) \sum_{s=0}^{|n_a|-1} \varepsilon_{s+1}(n_a)\left(\tau_{n_a}^{\left(n_a^{(s)}\right)}-\tau_{n_a}^{\left(n_a^{(s+1)}\right)}\right) \\ & +\left(\tau_{n_a}^{\left(n_a^{(0)}\right)}+\tau_{n_a}^{\left(n_a^{(|n_a|)}\right)}\right) E^*(|f|)\Bigg). \end{aligned}$$

According to the assumption T_{n_a} is uniformly bounded from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$. Therefore by (12)

$$\sup_{a\in\mathbb{N}} V\left(n_a,\mathbb{T}\right)<\infty.$$

On the other hand,

$$\sup_{a\in\mathbb{N}} \sum_{s=0}^{|n_a|-1} \varepsilon_{s+1}\left(n_a\right)\left(\tau_{n_a}^{\left(n_a^{(s)}\right)}-\tau_{n_a}^{\left(n_a^{(s+1)}\right)}\right)<\infty.$$

Hence,

$$\sup_{a\in\mathbb{N}}\left|f * F_{n_a,1}\right| \leq c_1 E^*(|f|). \tag{32}$$

Consequently, combining (28), (31) and (32) one arrives at the required assertion (iii).

(iii)⇒(ii). According to $\|E^*(|f|)\|_\infty \leq C\|f\|_\infty$ and $\left\|\sigma_*^{abc}(|f|)\right\|_\infty \leq C\|f\|_\infty$ (see [6]) the boundedness of the operator $\sup_a\left|T_{n_a}(f)\right|$ from $L_\infty(\mathbb{I})$ to $L_\infty(\mathbb{I})$ follows immediately from (17).

Note that the equivalence of items (ii) and (iv) follows directly from (12).

This completes the proof of Theorem 1. □

3 Walsh–Lebesgue Points and Pointwise Convergence

In this section a characterization of points is given in which the sequence of operators associated with Walsh system is pointwise convergent.

Recall that [18] an element $x \in \mathbb{I}$ is a *Walsh–Lebesgue point* of an integrable function $f \in L_1(\mathbb{I})$ if

$$\lim_{n\to\infty}\sum_{k=0}^{n}2^k\int\limits_{I_n(x\dot{+}e_k)}|f(t)-f(x)|\,dt=0,$$

where $e_k := 2^{-k-1}$.

Weisz in [22] proved that for every integrable function almost every point is a Walsh–Lebesgue point. Moreover, the following estimation was established as well

$$\int\limits_{\mathbb{I}}|f(t)-f(x)|\,|K_l(x\dot{+}t)|\,dt\le\frac{c}{l}\sum_{k=0}^{|l|}2^kW_k(x), \tag{33}$$

here

$$W_nf(x):=\sum_{k=0}^{n}2^k\int\limits_{I_n(x\dot{+}e_k)}|f(t)-f(x)|\,dt.$$

Notice that the estimation (33) implies the convergence of the Fejér means at Walsh–Lebesgue points. This result gives where a.e. convergence occurs. Inspiring from this result, aim of this section is to establish an analogous result for more general sequence of operators.

Theorem 4 *Let $\{n_a : a \in \mathbb{N}\}$ be a subsequence of natural numbers with $t_{n_a,n_a} = o(1)$ as $a \to \infty$. Assume that one of the statments of Theorem 1 is fulfilled.*

Then for each $f \in L_1(\mathbb{I})$ the sequence $\{T_{n_a}(f)\}$ converges to f at every Walsh–Lebesgue point.

Proof From (29), one can write

$$\begin{aligned}&\int\limits_{\mathbb{I}}|f(t)-f(x)|\left|F_{n_a,1}(x\dot{+}t)\right|dt \\ &\le\sum_{s=1}^{|n_a|}\left(\tau_{n_a}^{\left(n_a^{(s)}\right)}-\tau_{n_a}^{\left(n_a^{(s-1)}\right)}\right)\int\limits_{\mathbb{I}}|f(t)-f(x)|\,D_{2^s}(x\dot{+}t)\,dt \\ &+\sum_{s=1}^{|n_a|}\left|\varepsilon_{s-1}(n_a)-\varepsilon_s(n_a)\right|\tau_{n_a}^{\left(n_a^{(s-1)}\right)}\int\limits_{\mathbb{I}}|f(t)-f(x)|\,D_{2^s}(x\dot{+}t)\,dt \\ &+\tau_{n_a}^{\left(n_a^{(|n_a|)}\right)}\int\limits_{\mathbb{I}}|f(t)-f(x)|\,D_{2^{|n_a|+1}}(x\dot{+}t)\,dt \\ &+\tau_{n_a}^{\left(n_a^{(0)}\right)}\int\limits_{\mathbb{I}}|f(t)-f(x)|\,D_{2^0}(x\dot{+}t)\,dt.\end{aligned} \tag{34}$$

One can check that

$$\int_{\mathbb{I}} |f(t) - f(x)| D_{2^s}(x \dotplus t)\, dt \le W_s f(x) \quad s \in \mathbb{N}.$$

It then follows from (34) that

$$\begin{aligned}
\int_{\mathbb{I}} |f(t) - f(x)| \left|F_{n,1}(x \dotplus t)\right| dt &\le \sum_{s=1}^{|n_a|} \left(\tau_{n_a}^{\left(n_a^{(s)}\right)} - \tau_{n_a}^{\left(n_a^{(s-1)}\right)} \right) W_s f(x) \\
&\quad + \sum_{s=1}^{|n_a|} |\varepsilon_{s-1}(n_a) - \varepsilon_s(n_a)|\, \tau_{n_a}^{\left(n_a^{(s-1)}\right)} W_s f(x) \\
&\quad + \tau_{n_a}^{\left(n_a^{(|n_a|)}\right)} W_{|n_a|+1} f(x) \\
&\quad + \tau_{n_a}^{\left(n_a^{(0)}\right)} W_0 f(x) \\
&=: J_1(n_a) + J_2(n_a) + J_3(n_a) + J_4(n_a).
\end{aligned}$$

Set

$$\eta(n_a) := \left[\frac{1}{2} \log_2 \left(\frac{1}{t_{n_a,n_a}} \right) \right].$$

Due to

$$\tau_{n_a}^{\left(n_a^{(0)}\right)} = \tau_{n_a}^{(n_a)} = t_{n_a,n_a},$$

one can write (x is fixed)

$$\begin{aligned}
J_1(n_a) &= \sum_{s=1}^{|n_a|} \left(\sum_{l=n_a^{(s)}}^{n_a^{(s-1)}-1} t_{l,n_a} \right) W_s f(x) \\
&= \sum_{s=1}^{\eta(n_a)} \left(\sum_{l=n_a^{(s)}}^{n_a^{(s-1)}-1} t_{l,n_a} \right) W_s f(x) + \sum_{s=\eta(n_a)+1}^{|n_a|} \left(\sum_{l=n_a^{(s)}}^{n_a^{(s-1)}-1} t_{l,n_a} \right) W_s f(x) \\
&\le c(f,x)\, t_{n_a,n_a} \sum_{s=1}^{\eta(n_a)} \left(n_a^{(s-1)} - n_a^{(s)} \right) + \sup_{s>\eta(n_a)} W_s f(x) \left(\sum_{l=0}^{n_a} t_{l,n_a} \right) \\
&\le c(f,x)\, t_{n_a,n_a} \left(n_a - n_a^{(\eta(n_a))} \right) + \sup_{s>\eta(n_a)} W_s f(x).
\end{aligned}$$

From the condition of the theorem, we find

$$t_{n_a,n_a}\left(n_a-n_a^{(\eta(n_a))}\right)\leq t_{n_a,n_a}2^{\eta(n_a)}\leq c\sqrt{t_{n_a,n_a}}\to 0$$

and

$$\sup_{s>\eta(n_a)}W_sf(x)\to 0\ \text{ as }\ a\to\infty.$$

Consequently,

$$J_1(n_a)\to 0\quad\text{as}\ a\to\infty. \tag{35}$$

Analogously, the condition of the theorem yields

$$\begin{aligned}J_2(n_a)&\leq\sum_{s=1}^{\eta(n_a)}\tau_{n_a}^{\left(n_a^{(s-1)}\right)}W_sf(x)\\&\quad+\sup_{s>\eta(n_a)}W_sf(x)\left(\sum_{s=1}^{|n_a|}\left|\varepsilon_{s-1}(n_a)-\varepsilon_s(n_a)\right|\tau_{n_a}^{\left(n_a^{(s-1)}\right)}\right)\\&\leq c(f,x)\,\eta(n_a)\,t_{n_a,n_a}+c\sup_{s>\eta(n_a)}W_sf(x)\to 0\quad(a\to\infty).\end{aligned}$$

By the same argument, we obtain

$$J_3(n_a),\,J_4(n_a)\to 0\,(a\to\infty). \tag{36}$$

Combining (34)-(36), one concludes that

$$f*F_{n_a,1}\to 0\quad(a\to\infty). \tag{37}$$

By

$$\begin{aligned}\tau_{n_a}^{\left(n_a^{(s-1)}\right)}&=\sum_{l=n_a^{(s-1)}}^{n_a}t_{l,n_a}\leq t_{n_a,n_a}\left(n_a-n_a^{(s-1)}\right)\\&\leq t_{n_a,n_a}2^s\leq t_{n_a,n_a}2^{\eta(n_a)}\leq c\sqrt{t_{n_a,n_a}},\end{aligned}$$

from (30) one gets

$$\begin{aligned}&\int_{\mathbb{I}} |f(t)-f(x)| \left|F_{n_a,2}(x\dotplus t)\right| dt \\ &\leq \sum_{s=0}^{|n_a|} \varepsilon_s(n_a) \sum_{l=1}^{2^s-1} t_{n_a^{(s)}-l,n_a} \int_{\mathbb{I}} |f(t)-f(x)| D_{2^s}(x\dotplus t)\,dt \\ &\quad+\sum_{s=0}^{|n_a|} \varepsilon_s(n_a) \sum_{l=1}^{2^s-2} (t_{n_a^{(s)}-l,n_a} - t_{n_a^{(s)}-l-1,n_a}) \int_{\mathbb{I}} |f(t)-f(x)|\, l\, |K_l(x\dotplus t)|\, dt \\ &\quad+\sum_{s=0}^{|n_a|} \varepsilon_s(n_a)\, t_{n_a^{(s)}-2^s+1,n_a} (2^s-1) \int_{\mathbb{I}} |f(t)-f(x)| K_{2^s-1}(x\dotplus t)\,dt \\ &=: M_1(n_a)+M_2(n_a)+M_3(n_a).\end{aligned} \tag{38}$$

The estimation of $M_1(n_a)$ is similar to the estimation of $J_1(n_a)$, therefore, one gets

$$M_1(n_a) \to 0 \quad \text{as} \quad a \to \infty \tag{39}$$

at every Walsh–Lebesgue points.

Since $W_k(x) \to 0$ as $k \to \infty$, it is easy to see that (see [22])

$$\delta_l(x) := \frac{c}{l} \sum_{k=0}^{|l|} 2^k W_k(x) \to 0 \quad (l \to \infty).$$

Hence, by (33) we obtain

$$\begin{aligned} M_2(n_a)+M_3(n_a) &\leq \sum_{s=0}^{|n_a|} \varepsilon_s(n_a) \sum_{l=1}^{2^s-2} (t_{n_a^{(s)}-l,n_a} - t_{n_a^{(s)}-l-1,n_a}) l\delta_l(x) \\ &\quad+\sum_{s=0}^{|n_a|} t_{n_a^{(s)}-2^s+1,n_a} (2^s-1)\delta_{2^s-1}(x) \\ &= \sum_{s=0}^{\eta(n_a)} \varepsilon_s(n_a) \sum_{l=1}^{2^s-2} (t_{n_a^{(s)}-l,n_a} - t_{n_a^{(s)}-l-1,n_a}) l\delta_l(x) \\ &\quad+\sum_{s=\eta(n_a)+1}^{|n_a|} \varepsilon_s(n_a) \sum_{l=1}^{2^{\eta(n_a)}} (t_{n_a^{(s)}-l,n_a} - t_{n_a^{(s)}-l-1,n_a}) l\delta_l(x) \\ &\quad+\sum_{s=\eta(n_a)+1}^{|n_a|} \varepsilon_s(n_a) \sum_{l=2^{\eta(n_a)}+1}^{2^s-2} (t_{n_a^{(s)}-l,n_a} - t_{n_a^{(s)}-l-1,n_a}) l\delta_l(x) \\ &\quad+\sum_{s=0}^{\eta(n_a)} \varepsilon_s(n_a)\, t_{n_a^{(s)}-2^s+1,n_a} (2^s-1)\delta_{2^s-1}(x)\end{aligned}$$

$$+\sum_{s=\eta(n_a)+1}^{|n_a|}\varepsilon_s(n_a)\,t_{n_a^{(s)}-2^s+1,n_a}(2^s-1)\delta_{2^s-1}(x)$$
$$=:N_1(n_a)+N_2(n_a)+N_3(n_a)+N_4(n_a)+N_5(n_a). \tag{40}$$

The following is readily visible

$$\begin{aligned}N_1(n_a)&\le c(f,x)\sum_{s=0}^{\eta(n_a)}\varepsilon_s(n_a)\sum_{l=1}^{2^s-2}(t_{n_a^{(s)}-l,n_a}-t_{n_a^{(s)}-l-1,n_a})l\\&\le c(f,x)\,2^{\eta(n_a)}\sum_{s=0}^{\eta(n_a)}\varepsilon_s(n_a)(t_{n_a^{(s)},n_a}-t_{n_a^{(s+1)},n_a})\\&\le c(f,x)\,2^{\eta(n_a)}t_{n_a,n_a}\le c(f,x)\sqrt{t_{n_a,n_a}}\to 0\quad (a\to\infty).\end{aligned}$$

Analogously, one can prove that

$$N_2(n_a),\,N_4(n_a)\to 0\qquad (a\to\infty). \tag{41}$$

On the other hand, the term $N_3(n_a)$ can be estimated as follows

$$N_3(n_a)\le\sup_{l>2^{\eta(n_a)}}\delta_l(x)\sum_{s=\eta(n_a)+1}^{|n_a|}\varepsilon_s(n_a)\sum_{l=2^{\eta(n_a)}+1}^{2^s-2}(t_{n_a^{(s)}-l,n_a}-t_{n_a^{(s)}-l-1,n_a})l.$$

Due to

$$\sup_{l>2^{\eta(n_a)}}\delta_l(x)\to 0\qquad (a\to\infty)$$

and

$$\sum_{s=\eta(n_a)+1}^{|n_a|}\varepsilon_s(n_a)\sum_{l=2^{\eta(n_a)}+1}^{2^s-2}(t_{n_a^{(s)}-l,n_a}-t_{n_a^{(s)}-l-1,n_a})l\le c<\infty,$$

we get

$$N_3(n_a)\to 0\qquad (a\to\infty). \tag{42}$$

The estimation of $N_5(n_a)$ is analogous to the estimation of $N_3(n_a)$, so

$$N_5(n_a)\to 0\qquad (a\to\infty). \tag{43}$$

The estimations (38)-(43) yield

$$f*F_{n_a,2}\to 0\qquad (a\to\infty). \tag{44}$$

Finally, by combining (28), (37) and (44) we complete the proof of Theorem 4. □

Remark 4 Let us notice the followings:

(a) If one considers the Fejér means (see (14)), It follows from Theorem 4 that the Walsh-Fejér means are convergent at every Walsh–Lebesgue point. This result was obtained by Weisz [18].

(b) If we look at the (C, α_n)-means $\alpha_n = \alpha > 0$ in case of constant, then their almost everywhere convergence was proved by Fine [1]. Our Theorem 4 reveals that the (C, α)-means of Walsh–Fourier series is convergent at every Walsh–Lebesgue point.

(c) Now, let us consider (C, α_n)-means in case when $\alpha_n \in (0, 1)$ and $\alpha_n \to 0$ as $n \to \infty$. According to the recently result [7] no matter how slowly the sequence $\{\alpha_n\}$ tends to zero, there exists an integrable function f, for which (C, α_n)-means of Walsh–Fourier series is almost everywhere divergent. On the other hand, by Theorem 1 if $\sup_{a\in\mathbb{N}} V(n_a, \mathbb{T}) < \infty$ is fulfilled, then the sequence $\left\{T_{n_a}(f)\right\}$ is convergent almost everywhere for any $f \in L_1(\mathbb{I})$. Moreover, Theorem 4 provides a characterization of points in which the sequence $\left\{T_{n_a}(f)\right\}$ is convergent. We point out that a.e. converges of (C, α_n)-means for certain subsequences was investigated in [10].

(d) Let us now consider the Nörlund logarithmic means (16). The condition (9) provides that the Nörlund logarithmic means $T_{n_a}(f)$ of Walsh–Fourier series is convergent to the f at every Walsh–Lebesgue point. The almost everywhere convergence of the Nörlund logarithmic means $T_{n_a}(f)$ of Walsh–Fourier series along subsequences was studied in [5]. In general, the issues of the almost everywhere divergence of Nörlund logarithmic means of Walsh–Fourier series have been studied in [3].

Example 1 Let $t_{k,n} := \frac{q_{n-k}}{Q_n}$, where $Q_n := \sum_{k=0}^{n-1} q_k$ $(n \geq 1)$. Then one has

$$T_n(f) := \frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k}S_k(f).$$

It is always assumed that $q_0 > 0$, $\lim_{n\to\infty} Q_n = \infty$ and the sequence $\{q_k\}$ is non-increasing. In the literature, these averages are known as Nörlund means of the Walsh–Fourier series. By Theorems 1 and 4, we infer that if $\sup_{a\in\mathbb{N}} V(n_a, \mathbb{T}) < \infty$, then the subsequence $\left\{T_{n_a}(f)\right\}$ of Walsh-Nörlund mean is convergent to the f at every Walsh–Lebesgue point.

We point out in [12, 13] it has been investigated the case when the sequence $\{q_k\}$ is non-decreasing.

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