Journal of Fourier Analysis and Applications (2024) 30:28
https://doi.org/10.1007/s00041-024-10080-4

Fast Decreasing Trigonometric Polynomials and Applications

D. Leviatan[1] · **O. V. Motorna**[2] · **I. A. Shevchuk**[3]

Received: 11 July 2023 / Revised: 1 December 2023 / Accepted: 3 December 2023 /
Published online: 23 April 2024

Abstract

We construct trigonometric polynomials that fast decrease towards $\pm\pi$. We apply them to construct a trigonometric polynomial the derivative of which interpolates the derivative of a given 2π-periodic function, at some prescribed distinct points in $[-\pi, \pi)$, and vanishes at some other prescribed points in that interval. The construction requires that the function possesses derivatives where the interpolation is supposed to take place. Still, we are able to apply the result to trigonometric approximation of a 2π-periodic piecewise algebraic polynomial which is merely continuous, while interpolating its derivative at some points (that, obviously, are not knots).

Keywords Fast decreasing trigonometric polynomials · Interpolation by trigonometric polynomials · Jackson-type estimates.

Mathematics Subject Classification 42A10 · 41A17 · 41A25

Communicated by Hans G. Feichtinger.

✉ D. Leviatan
leviatan@tauex.tau.ac.il

O. V. Motorna
omotorna@ukr.net

I. A. Shevchuk
shevchuk@univ.kiev.ua

1 Raymond and Beverly Sackler School of Mathematical Sciences, Tel Aviv University, Tel Aviv 6139001, Israel

2 Faculty of Radio Physics, Electronics and Computer Systems, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine

3 Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine

1 Introduction and the main results

Let $I := [-\pi, \pi]$ and let $C[a, b]$ and $C^l[a, b]$ denote, respectively, the space of continuous functions and of l times continuously differentiable functions, and denote by $\tilde{C}$ the space of 2π-periodic continuous functions. As usual all spaces are equipped with the sup-norm, i.e., $\|f\|_{[a,b]} := \max_{x\in[a,b]} |f(x)|$ and $\|f\| := \max_{x\in\mathbb{R}} |f(x)|$, respectively.

For $\beta \in \mathbb{N}$ and $n \in \mathbb{N}$, let

$$J(x) = J_{n,\beta}(x) = \frac{n}{\gamma_{n,\beta}} \left(\frac{\sin(nx/2)}{n\sin(x/2)}\right)^{2\beta}, \tag{1.1}$$

be a Jackson-type kernel, where $C_*(\beta) \le \gamma_{n,\beta} \le C^*(\beta)$ is a normalizing factor, so that J is a trigonometric polynomial of degree $\beta(n-1)$, and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} J(x)\,dx = 1, \tag{1.2}$$

(see, e.g., [1, p. 204]). For the asymptotic behavior of $\gamma_{n,\beta}$, see [3, Theorem 1].

Put $h := \pi/n$. Clearly,

$$J(x) \ge \frac{C_1(\beta)}{h}, \quad |x| \le h, \tag{1.3}$$

and Bernstein's inequality implies, for all $\nu \in \mathbb{N}_0$,

$$\|J^{(\nu)}\| \le \frac{C_2(\beta,\nu)}{h^{\nu+1}}. \tag{1.4}$$

In addition, we will show that (see the end of Sect. 2) for $0 \le \nu \le 2\beta$,

$$|J^{(\nu)}(x)| \le \frac{C_3(\beta)}{h^{\nu+1}} \frac{1}{(n|x|)^{2\beta-\nu}}, \quad 0 < |x| \le \pi, \tag{1.5}$$

where $C_1(\beta)$ and $C_3(\beta)$ depend only on β, and $C_2(\beta,\nu)$ may depend also on ν.

We wish to construct a trigonometric polynomial L_n, of degree $\beta(n-1)$, which satisfies analogues of (1.3) through (1.5), not only for $h \approx \frac{\pi}{n}$, but rather, for any $\frac{\pi}{n} \le h \le \text{const}$.

Theorem 1.1 *For each $m \in \mathbb{N}$, $\beta \in \mathbb{N}$ and $0 < \epsilon \le 1$, there are constants $K_1 > 0$ and K_2, depending only on m, β and ϵ, such that if $\frac{\pi}{n} \le h \le \frac{\pi}{m}$, then the trigonometric polynomial*

$$\begin{aligned} L_n(x) &:= \frac{1}{h^m} \underbrace{\int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2}}_{m \text{ times}} J(x + t_1 + \cdots + t_m)\,dt_1 \cdots dt_m \\ &=: \frac{1}{h^m} W_m(x, h, J), \end{aligned}$$

of degree $\leq \beta(n-1)$*, satisfies*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} L_n(t)\,dt = 1, \tag{1.6}$$

$$L_n(x) \geq \frac{K_1}{h}, \quad |x| \leq \frac{(m-\epsilon)h}{2}, \tag{1.7}$$

and for all $0 \leq \nu \leq m-1$*,*

$$\|L_n^{(\nu)}\| \leq 2^\nu \pi h^{-(\nu+1)}, \tag{1.8}$$

and

$$|L_n^{(\nu)}(x)| \leq K_2 \frac{h^{-(\nu+1)}}{(n|x|)^{2\beta-1}}, \quad \frac{(m+\epsilon)h}{2} \leq |x| \leq \pi. \tag{1.9}$$

Remark 1.2 The above constants may be replaced by $K_1 := C_1\epsilon^{m-1}$ and $K_2 := C_2\epsilon^{-2\beta}$, where C_1 and C_2 depend only on m and β.

We apply Theorem 1.1 to obtain an interpolation result. Namely,

Theorem 1.3 *Given* $n, s, \eta \in \mathbb{N}$ *and* $0 < \epsilon \leq 1/2$*. For* $\frac{\pi}{n} \leq h \leq \frac{\pi}{s+2}$*, denote*

$$\dot{O} := [-(s+1/2-\epsilon)h, (s+1/2-\epsilon)h] \quad \textit{and} \quad \ddot{O} := (-(s+1/2+\epsilon)h, (s+1/2+\epsilon)h).$$

Given a collection $\{z_i\}_{i=1}^{2s}$*, of distinct points in* $[-\pi, \pi)$*. Let* l*,* $0 \leq l \leq 2s$*, be such that*

$$z_q \in \dot{O}, \quad 1 \leq q \leq l, \quad (\textit{no } z_q \textit{ in } \dot{O}, \textit{ if } l = 0), \tag{1.10}$$

and

$$z_q \in [-\pi, \pi) \setminus \ddot{O}, \quad l+1 \leq q \leq 2s \quad (\textit{no } z_q \textit{ in } [-\pi, \pi) \setminus \ddot{O}, \textit{ if } l = 2s). \tag{1.11}$$

Assume that f *is defined in* $\dot{O}$*, and if* $l \geq 1$*, assume that* $f \in C^{l-1}(\dot{O})$ *and satisfies,*

$$|f^{(\nu)}(x)| \leq h^{-\nu}, \quad x \in \dot{O}, \quad \nu = 0 \ldots l-1. \tag{1.12}$$

Then, there exists a constant $c = c(s, \eta, \epsilon)$ *and a trigonometric polynomial* D_l *of degree* $\leq ([\eta/2] + 2s + 1)n$*, such that*

$$\|D_l\| \leq ch, \tag{1.13}$$

and its derivative $d_l := D_l'$ *satisfies,*

$$d_l(z_q) = f(z_q), \quad 1 \le q \le l, \tag{1.14}$$
$$d_l(z_q) = 0, \quad l+1 \le q \le 2s, \tag{1.15}$$

and, for all $\nu = 0, \ldots, 2s$,

$$|d_l^{(\nu)}(x)| \le ch^{-\nu} M^{\eta}(x), \quad x \in [-\pi, \pi], \tag{1.16}$$

where

$$M(x) := \begin{cases} 1 & x \in \ddot{O} \\ \frac{1}{n|x|} & x \in [-\pi, \pi] \setminus \ddot{O}. \end{cases} \tag{1.17}$$

Finally, in Sect. 6, we apply Theorem 1.3 to obtain a trigonometric polynomial which approximates a 2π-periodic continuous piecewise algebraic polynomial, and the derivative of which, interpolates the derivative of the latter at a given collection of points (obviously, not knots).

Throughout the paper we will have positive constants c and C that may differ from one another on different occurrences even if they appear in the same line.

2 Pointwise Bernstein Inequality

We extend the well known Bernstein inequality

$$\|(T_n^r)^{(\nu)}\| \le (rn)^{\nu} \|T_n^r\|,$$

which is valid for all trigonometric polynomials T_n, of degree $\le n$, and for all $r \in \mathbb{N}$ and $\nu \in \mathbb{N}$, into a pointwise version. Namely,

Lemma 2.1 *For arbitrary trigonometric polynomial T_n of degree $\le n$, any $r \in \mathbb{N}$ and each natural $\nu \le r$ the inequality*

$$|(T_n^r)^{(\nu)}(x)| \le (rn)^{\nu} \|T_n\|^{\nu} |T_n^{r-\nu}(x)|, \quad x \in \mathbb{R}, \tag{2.1}$$

holds.

Proof Without loss of generality assume that $\|T_n\| = 1$, so we have to prove the inequality

$$|(T_n^r)^{(\nu)}(x)| \le (rn)^{\nu} |T_n^{r-\nu}(x)|, \quad x \in \mathbb{R}. \tag{2.2}$$

Then Bernstein inequality implies $\|T_n^{(\nu)}\| \le n^{\nu}$ and

$$|(T_n^r)'(x)| = r|T_n^{r-1}(x) T_n'(x)| \le rn|T_n^{r-1}(x)|, \quad r \in \mathbb{N}.$$

Assuming by induction, that for some $\nu - 1 < r$ the inequality (2.1) holds for all $r \in \mathbb{N}$, we get

$$
\begin{aligned}
|(T_n^r)^{(\nu)}(x)| &= r \left| \left(T_n^{r-1}(x) T_n'(x) \right)^{(\nu-1)} \right| \\
&= r \left| \sum_{j=0}^{\nu-1} \binom{\nu-1}{j} (T_n^{r-1})^{(\nu-1-j)}(x) (T_n)^{(j+1)}(x) \right| \\
&\le r \sum_{j=0}^{\nu-1} \binom{\nu-1}{j} ((r-1)n)^{\nu-1-j} |T_n^{r-\nu+j}(x)| n^{j+1} \\
&\le r n^{\nu} |T_n^{r-\nu}(x)| \sum_{j=0}^{\nu-1} \binom{\nu-1}{j} (r-1)^{\nu-1-j} = (rn)^{\nu} |T_n^{r-\nu}(x)|,
\end{aligned}
$$

which is (2.2). □

Applying Lemma 2.1 to the polynomial $T_n(u) = \frac{\sin nu}{n \sin u}$, we readily obtain (1.5).

3 Fast Decreasing Trigonometric Polynomials

Proof of Theorem 1.1 Since J is a trigonometric polynomial of degree $< \beta n$, L_n is also a trigonometric polynomial of degree $< \beta n$.

First, we have

$$
\begin{aligned}
\int_{-\pi}^{\pi} L_n(x)\,dx &= \frac{1}{h^m} \int_{-\pi}^{\pi} \left(\underbrace{\int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2}}_{m \text{ times}} J(x + t_1 + \cdots + t_m)\,dt_1 \cdots dt_m \right) dx \\
&= \frac{1}{h^m} \underbrace{\int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2}}_{m \text{ times}} \left(\int_{-\pi}^{\pi} J(x + t_1 + \cdots + t_m)\,dx \right) dt_1 \cdots dt_m \\
&= \frac{1}{h^m} \underbrace{\int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2}}_{m \text{ times}} \pi\,dt_1 \cdots dt_m = \pi,
\end{aligned}
$$

and (1.6) is proved.

Evidently, for every $\nu = 1, \ldots, m-1$, there is a $\theta = \theta_\nu \in [x - \nu h/2, x + \nu h/2]$, such that

$$
|W_m^{(\nu)}(x, h, f)| \le 2^{\nu} |W_{m-\nu}(\theta, h, f)|. \tag{3.1}
$$

Now, for any $a \in \mathbb{R}$,

$$\int_{-h/2}^{h/2} J(a+t)\,dt \le \int_{-\pi}^{\pi} J(a+t)\,dt = \pi.$$

Thus, (3.1) implies,

$$|L_n^{(\nu)}(x)| \le \frac{2^\nu}{h^m} W_{m-\nu}(\theta, h, J) \le \frac{2^\nu \pi}{h^m} \underbrace{\int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2}}_{m-\nu-1 \text{ times}} dt_1 \cdots dt_{m-\nu-1} = 2^\nu \pi h^{-(\nu+1)},$$

which is (1.8).

In order to prove (1.9), take $\frac{(m+\epsilon)h}{2} \le |x| \le \pi$ and $|\theta - x| \le \nu h/2$. If

$$|t_j| \le h/2, \quad 1 \le j \le m-\nu, \quad \text{and} \quad \Theta := \theta + t_1 + \cdots + t_{m-\nu} \in I,$$

then

$$|\Theta| \ge |x| - (m-\nu+\nu)\frac{h}{2} = |x| - \frac{mh}{2},$$

which implies,

$$J(\Theta) \le \frac{1}{\gamma_{n,\beta}} \frac{n}{\big(n \sin \Theta/2\big)^{2\beta}} \le \frac{Cn}{\big(n(|x| - \frac{mh}{2})\big)^{2\beta}}. \tag{3.2}$$

Here and in the rest of the proof C and C^* depend only on β.

If, on the other hand, $|\Theta| > \pi$, then $\pi < |\Theta| \le \frac{3\pi}{2}$, which implies

$$|\sin \Theta/2| \ge \sin \pi/4 \ge \sin \frac{|x| - \frac{mh}{2}}{4} \ge \frac{|x| - \frac{mh}{2}}{2\pi},$$

so that (3.2) is valid in this case too.

Combined, (3.1) and (3.2) yield

$$\begin{aligned} |L_n^{(\nu)}(x)| &\le \frac{2^\nu}{h^m} W_{m-\nu}(\theta, h, J) \le \frac{2^\nu}{h^m} \frac{Cn}{\big(n(|x| - \frac{mh}{2})\big)^{2\beta}} \underbrace{\int_{-h/2}^{h/2} \cdots \int_{-h/2}^{h/2}}_{m-\nu \text{ times}} dt_1 \cdots dt_{m-\nu} \\ &= C \frac{2^\nu h^{-\nu}}{\big(n(|x| - \frac{mh}{2})\big)^{2\beta-1}} \frac{1}{|x| - \frac{mh}{2}} \le C \frac{2^\nu h^{-\nu}(m+1)^{2\beta-1}}{(\epsilon n|x|)^{2\beta-1}} \frac{2}{\epsilon h} \\ &\le K_2 \frac{h^{-(\nu+1)}}{(n|x|)^{2\beta-1}}, \end{aligned}$$

where, for $|x| \ge \frac{(m+\epsilon)h}{2}$, we applied the inequalities $|x| - \frac{mh}{2} \ge \frac{\epsilon|x|}{m+1}$ and $|x| - \frac{mh}{2} \ge \frac{\epsilon h}{2}$. Thus, we obtain (1.9).

In order to prove (1.7), we observe that if $|a| \le h/2$, then

$$\int_{-h/2}^{h/2} J(t+a)\,dt \ge \int_{0}^{h/2} J(t)\,dt \ge \int_{0}^{\pi/(2n)} J(t)\,dt > C^*. \tag{3.3}$$

For $m = 1$, this readily yields (1.7). Thus, let $m > 1$ and denote

$$H := \left[-\frac{h}{2}, \frac{h}{2}\right] \bigcap \left[\frac{-x}{m-1} - \frac{h}{2(m-1)}, \frac{-x}{m-1} + \frac{h}{2(m-1)}\right].$$

Note that if $t_j \in H$, $1 \le j \le m-1$, then $-h/2 \le a := x + t_1 + \cdots + t_{m-1} \le h/2$.

Hence, for $|x| \le \frac{(m-1)h}{2}$,

$$\begin{aligned} L_n(x) &\ge \frac{1}{h^m} \underbrace{\int_H \cdots \int_H}_{m-1 \text{ times}} \left(\int_{-h/2}^{h/2} J(x + t_1 + \cdots + t_m)\,dt_m\right) dt_1 \cdots dt_{m-1} \qquad (3.4)\\ &\ge \frac{1}{h^m} \underbrace{\int_H \cdots \int_H}_{m-1 \text{ times}} C^*\,dt_1 \cdots dt_{m-1} = \frac{1}{h^m} C^* |H|^{m-1} \ge \frac{C}{h}, \end{aligned}$$

where we used the fact that $|H| \ge \frac{h}{2(m-1)}$.

Finally, if $\frac{(m-1)h}{2} < |x| \le \frac{(m-\epsilon)h}{2}$, then

$$|H| = \frac{h}{2} - \left(\frac{|x|}{m-1} - \frac{h}{2(m-1)}\right) \ge \frac{h}{2} - \left(\frac{(m-\epsilon)h}{2(m-1)} - \frac{h}{2(m-1)}\right) = \frac{\epsilon h}{2(m-1)}.$$

Substituting in (3.4), completes the proof. □

4 Interpolating Trigonometric Polynomials

Proof of Theorem 1.3 If $l = 0$, then (1.14) is empty, so we may take $D_0(x) \equiv 0$.

We proceed by induction. By the induction assumption, there is a polynomial D_{l-1}, $1 \le l \le 2s$, satisfying (1.14) through (1.16) with $l-1$ instead of l, and with any $\tilde{z}_l \in \ddot{O}$, $\tilde{z}_l \notin \{z_i\}_{i=l+1}^{2s}$, instead of the given $z_l \in \dot{O}$.

We will construct the derivative d_l and then put $D_l(x) := \int_{-\pi}^{x} d_l(t)\,dt$.

To construct d_l we first note that for the polynomial L_n, defined by Theorem 1.1 with $\beta = [\eta/2] + 2s + 1$ and $m = 2s + 1$, we have for $x \in [-\pi, \pi]$,

$$|L_n^{(\nu)}(x)| \le c h^{-(\nu+1)} M^{2\beta-1}(x), \quad 0 \le \nu \le 2s. \tag{4.1}$$

Here and in the rest of the proof $c = c(s, \eta, \epsilon)$.

We will show that the desired polynomial d_l may be taken in the form

$$d_l = d_{l-1} + \hat{d}_l - \frac{\hat{B}_l}{\breve{B}_l}\breve{d}_l,$$

where

$$\hat{d}_l(x) := \underbrace{\frac{f(z_l) - d_{l-1}(z_l)}{\prod_{q=1}^{l-1}\sin\big((z_l - z_q)/2\big)}}_{=:F_l} \underbrace{\prod_{q=1}^{l-1}\sin\big((x - z_q)/2\big)\frac{\sin\big((x-\tilde{z}_l)/2\big)}{\sin\big((z_l-\tilde{z}_l)/2\big)}\prod_{q=l+1}^{2s}\frac{\sin\big((x-z_q)/2\big)}{\sin\big((z_l-z_q)/2\big)}}_{=:\hat{I}(x)}$$
$$\times\frac{L_n(x)}{L_n(z_l)} = F_l\hat{I}(x)\frac{L_n(x)}{L_n(z_l)},$$

$$\breve{d}_l(x) := h^{1-2l}L_n(x)\underbrace{\prod_{q=1}^{l}\sin^2\big((x-z_q)/2\big)\prod_{q=l+1}^{2s}\frac{\sin^2\big((x-z_q)/2\big)}{\sin^2\big((z_l-z_q)/2\big)}}_{=:\breve{I}(x)} = h^{1-2l}L_n(x)\breve{I}(x),$$

$$\hat{B}_l := \int_{-\pi}^{\pi}\hat{d}_l(x)\,dx \quad \text{and} \quad \breve{B}_l := \int_{-\pi}^{\pi}\breve{d}_l(x)\,dx.$$

If $l = 1$, we mean $\prod_{q=1}^{0} = 1$, and recall that $d_{l-1} = d_0 \equiv 0$. Similarly, $\prod_{q=2s+1}^{2s} = 1$.

Evidently, d_l is a trigonometric polynomial of degree $\le \beta(n-1) + 2s$, and (1.14) and (1.15) hold.

We first estimate the polynomials $\hat{d}_l$ and $\breve{d}_l$, and their derivatives.

By the induction assumption, (1.12) and (1.16) imply,

$$|f^{(l-1)}(x) - d_{l-1}^{(l-1)}(x)| \le ch^{1-l}, \quad x \in \dot{O},$$

while (1.14) yields,

$$f(z_q) - d_{l-1}(z_q) = 0, \quad q < l.$$

Thus, we have

$$\begin{aligned} |F_l| = \left|\frac{f(z_l) - d_{l-1}(z_l)}{\prod_{q=1}^{l-1}(z_l - z_q)}\right| &= \big|[z_1, \dots, z_l; f - d_{l-1}]\big| \\ &= \left|\frac{f^{(l-1)}(\theta) - d_{k-1}^{(l-1)}(\theta)}{(l-1)!}\right| \le ch^{1-l}, \end{aligned} \tag{4.2}$$

where the middle term is the divided difference of $f - d_{l-1}$, and in the last inequality we used the fact that $\theta \in \dot{O}$.

In order to estimate $\hat{I}(x)$ and $\breve{I}(x)$, and their derivatives, we observe that

$$|z| < (s+1)h, \quad z \in \dot{O},$$

and

$$|z - z_l| \geq 2h\epsilon, \quad z \in I \backslash \ddot{O}.$$

Also

$$|x| M(x) \leq (s+1)h, \quad x \in I.$$

Hence, for $x \in I$ and $z \in \dot{O}$, we have

$$\frac{|\sin\big((x-z)/2\big)|}{h} M(x) \leq \frac{|x|}{2\,h} M(x) + \frac{|z|}{2\,h} M(x) \leq \frac{|x|}{2\,h} M(x) + \frac{s+1}{2} \leq s+1 \leq c,$$

and for $x \in I$ and $z \in I \setminus \ddot{O}$, we have

$$\begin{aligned} \frac{2}{\pi} \left| \frac{\sin\big((x-z)/2\big)}{\sin\big((z_l-z)/2\big)} \right| M(x) &\leq \frac{|x-z|}{|z_l - z|} M(x) \leq 1 + \frac{|x| + |z_l|}{|z_l - z|} M(x) \\ &\leq 1 + \frac{|x| + |z_l|}{2h\epsilon} M(x) \leq c. \end{aligned}$$

Therefore, we write $\hat{I}(x) = h^{l-1} \prod_{q=1}^{2s} \alpha_q(x)$, $x \in I$, where for each $1 \leq q \leq 2s$,

$$|\alpha_q(x)| \leq \frac{c}{M(x)} \quad \text{and} \quad |\alpha_q^{(\nu)}(x)| \leq \frac{c}{h}, \quad \nu \in \mathbb{N}, \quad x \in I, .$$

This, in turn, yields for each $0 \leq \nu \leq 2s$,

$$|\hat{I}^{(\nu)}(x)| \leq c h^{l-1} \frac{h^{-\nu}}{M^{2\,s}(x)}, \quad x \in I.$$

Combining with (4.2), (1.7) and (4.1), we obtain for each $0 \leq \nu \leq 2s$,

$$\begin{aligned} |\hat{d}_l^{(\nu)}(x)| &\leq \frac{2^\nu |F_l|}{|L_n(z_l)|} \sum_{\mu=0}^{\nu} |L_n^{(\mu)}(x) \hat{I}^{(\nu-\mu)}(x)| \\ &\leq c \frac{h^{1-l}}{1/h} h^{l-1} \sum_{\mu=0}^{\nu} \frac{M^{2\beta-1}(x)}{h^{\mu+1}} \frac{h^{\mu-\nu}}{M^{2s}(x)} \\ &= c(\nu+1) h^{-\nu} M^{2\beta-2s-1}(x) \leq c h^{-\nu} M^{\eta}(x), \quad x \in I. \end{aligned} \tag{4.3}$$

Similarly,

$$|\breve{I}^{(\nu)}(x)| \leq c h^{2\,l-\nu} \frac{1}{M^{4\,s}(x)}, \quad x \in I,$$

whence,

$$|\breve{d}_l^{(\nu)}(x)| \leq c(\nu+1) h^{-\nu} M^{2\beta-4s-1}(x) \leq c h^{-\nu} M^{\eta}(x), \quad x \in I. \tag{4.4}$$

It follows by (1.13) with $l-1$, that

$$\int_{-\pi}^{\pi} d_l(x)\,dx = 0.$$

By virtue of (4.3) and (4.4), we obtain

$$\int_{-\pi}^{x} |\hat{d}_l(t)|\,dt \le ch \quad \text{and} \quad \int_{-\pi}^{x} \breve{d}_l(t)\,dt \le ch, \quad x \in I,$$

and, in particular,

$$|\hat{B}_l| \le ch \quad \text{and} \quad \breve{B}_l \le ch.$$

So, in order to complete the proof of (1.13) and (1.16), we will prove that

$$\breve{B}_l \ge ch. \tag{4.5}$$

To this end, we note that if $x \in [-sh, sh] \subset \dot{O}$, then

$$L_n(x) > \frac{c}{h},$$

and

$$\left|\frac{\sin\big((x-z)/2\big)}{\sin\big((z_l-z)/2\big)}\right| > c, \quad z \in I\setminus\ddot{O}.$$

Hence,

$$\breve{B}_l \ge \int_{-sh}^{sh} \breve{d}_l(x)\,dx \ge \frac{c}{h^{2l}} \int_{-sh}^{sh} \prod_{q=1}^{l} (x-z_q)^2\,dx.$$

Now, the algebraic polynomial,

$$Q(t) := \int_{-sh}^{t} \prod_{q=1}^{l} (x-z_q)^2\,dx, \quad -sh \le t \le sh,$$

of degree $2l+1$, satisfies, by Markov's inequality,

$$\|Q\|_{[-sh,sh]} \ge (sh)^{2l+1} c \|Q^{(2l+1)}\|_{[-sh,sh]} = c(2l)!(sh)^{2l+1} = ch^{2l+1},$$

and (4.5) is proved. Thus, the proofs of (1.13) and (1.16) are complete. □

5 An Auxiliary Lemma

For $j \in \mathbb{Z}$, let

$$x_j := \frac{j\pi}{n}, \quad I_j := [x_j, x_{j+1}], \quad \text{and} \quad |I_j| = \frac{\pi}{n}.$$

Denote by $\mathbb{P}_k$, the space of algebraic polynomials of degree $< k$, and by $\widetilde{\Sigma}_{k,n}$, the space of 2π-periodic continuous piecewise algebraic polynomials S, of degree $< k$, with knots x_j, that is,

$$S\Big|_{I_j} = p_j, \quad p_j \in \mathbb{P}_k, \quad j \in \mathbb{Z}.$$

Given $S \in \widetilde{\Sigma}_{k,n}$, $n_1, \beta \in \mathbb{N}$ and $J_{n_1} := J_{n_1,\beta}$, let $\nu \in \mathbb{N}_0$ and denote

$$B_{\nu,n_1}(x) := \frac{d^\nu}{dx^\nu}\int_{-\pi}^{\pi} S(x+\sigma t)J_{n_1}(t)\,dt - \int_{-\pi}^{\pi} S^{(\nu)}(x+\sigma t)J_{n_1}(t)\,dt. \quad (5.1)$$

Lemma 5.1 *Let $S \in \widetilde{\Sigma}_{k,n}$and let $\sigma \in \mathbb{N}$. For each $\nu \in \mathbb{N}$ we have*

$$B_{\nu,n_1}(x) = \frac{1}{\sigma}\sum_{l=1}^{\nu}\sum_{j=-n\sigma+1}^{n\sigma}\left(p_j^{(l-1)}(x_j) - p_{j-1}^{(l-1)}(x_j)\right)\frac{d^{\nu-l}}{dx^{\nu-l}}J_{n_1}\left(\frac{x-x_j}{\sigma}\right). \quad (5.2)$$

Proof We first prove that for each $\nu \in \mathbb{N}$,

$$B_{\nu,n_1}(x) = \frac{1}{\sigma}\sum_{j=-n\sigma+1}^{n\sigma}\left(p_j^{(\nu-1)}(x_j) - p_{j-1}^{(\nu-1)}(x_j)\right)J_{n_1}\left(\frac{x-x_j}{\sigma}\right) + \frac{d}{dx}B_{\nu-1,n_1}(x). \quad (5.3)$$

To this end, we observe that since S is differentiable of any degree $l \in \mathbb{N}$, except at a final number of points in any compact interval, the following integrals exist, for all $l \in \mathbb{N}_0$, and are equal.

$$\begin{aligned}\int_{-\pi}^{\pi} S^{(l)}(x+\sigma t)J_{n_1}(t)\,dt &= \frac{1}{\sigma}\int_{x-\sigma\pi}^{x+\sigma\pi} S^{(l)}(u)J_{n_1}\left(\frac{u-x}{\sigma}\right)du \quad (5.4)\\ &= \frac{1}{\sigma}\int_{-\sigma\pi}^{\sigma\pi} S^{(l)}(u)J_{n_1}\left(\frac{u-x}{\sigma}\right)du\\ &= \frac{1}{\sigma}\sum_{j=-n\sigma}^{n\sigma-1}\int_{x_j}^{x_{j+1}} p_j^{(l)}(u)J_{n_1}\left(\frac{u-x}{\sigma}\right)du,\end{aligned}$$

where for the second equation we used the fact that the integrand is $2\pi\sigma$-periodic.

Now,

$$\begin{aligned}&\int_{x_j}^{x_{j+1}} p_j^{(l)}(u)J_{n_1}\left(\frac{u-x}{\sigma}\right)du - \frac{d}{dx}\int_{x_j}^{x_{j+1}} p_j^{(l-1)}(u)J_{n_1}\left(\frac{u-x}{\sigma}\right)du\\ &= \int_{x_j}^{x_{j+1}} \frac{\partial}{\partial u}\left(p_j^{(l-1)}(u)J_{n_1}\left(\frac{u-x}{\sigma}\right)\right)du\\ &= p_j^{(l-1)}(x_{j+1})J_{n_1}\left(\frac{x-x_{j+1}}{\sigma}\right) - p_j^{(l-1)}(x_j)J_{n_1}\left(\frac{x-x_j}{\sigma}\right).\end{aligned}$$

Hence,

$$
\begin{aligned}
&B_{\nu,n_1}(x) - \frac{d}{dx} B_{\nu-1,n_1}(x) \\
&= -\int_{-\pi}^{\pi} S^{(\nu)}(x+\sigma t) J_{n_1}(t)\,dt + \frac{d}{dx}\int_{-\pi}^{\pi} S^{(\nu-1)}(x+\sigma t) J_{n_1}(t)\,dt \\
&= \frac{1}{\sigma}\sum_{j=-n\sigma}^{n\sigma-1} \left(p_j^{(\nu-1)}(x_j) J_{n_1}\left(\frac{x-x_j}{\sigma}\right) - p_j^{(\nu-1)}(x_{j+1}) J_{n_1}\left(\frac{x-x_{j+1}}{\sigma}\right)\right) \\
&= \frac{1}{\sigma}\sum_{j=-n\sigma+1}^{n\sigma} \left(p_j^{(\nu-1)}(x_j) - p_{j-1}^{(\nu-1)}(x_j)\right) J_{n_1}\left(\frac{x-x_j}{\sigma}\right),
\end{aligned}
$$

where for the last equation we used the fact that $p_{-n\sigma}^{(l)}(x_{-n\sigma}) = p_{n\sigma}^{(l)}(x_{n\sigma})$.

Thus (5.3) is proved.

Since $B_{0,n_1}(x) \equiv 0$, the lemma follows by induction. □

Remark 5.2 Since S is a continuous function, the $(l = 1)$-term of (5.2) vanishes, so that the sum begins with $l = 2$.

6 Approximating a Piecewise Polynomial

For $f \in C[a, b]$, let

$$
\Delta_h(f,x) = \Delta_h^1(f,x) := \begin{cases} f(x+h) - f(x), & x, x+h \in [a,b] \\ 0, & \text{otherwise,} \end{cases}
$$

and, for $k > 1$, let

$$
\Delta_h^k(f,x) := \Delta_h\left(\Delta_h^{k-1}(f,\cdot), x\right).
$$

Denote by

$$
\omega_k(f,t;[a,b]) := \sup_{\substack{0\le h\le t\\ x\in[a,b]}} \left|\Delta_h^k(f,x)\right|, \quad k \ge 1,
$$

the kth modulus of smoothness of f. (See [1, Chapter 2, Section 7] for properties of moduli of smoothness.)

Similarly, for $f \in \widetilde{C}$, the space of 2π-periodic functions on $\mathbb{R}$, let $\Delta_h(f,x) = \Delta_h^1(f,x) := f(x+h) - f(x)$, and for $k > 1$, let $\Delta_h^k(f,x) := \Delta_h\left(\Delta_h^{k-1}(f,\cdot), x\right)$.

Finally, denote by

$$
\omega_k(f,t) := \sup_{\substack{0\le h\le t\\ x\in\mathbb{R}}} \left|\Delta_h^k(f,x)\right|, \quad k \ge 1,
$$

the kth modulus of smoothness of f. Note that for such f, $\omega_k(f,t) = \omega_k(f,t;[-2\pi, 2\pi])$ for $0 < t < 2\pi/k$.

We call a closed interval E a *proper* interval, if $E = [x_{j_*}, x_{j^*}]$ for some indices j_* and j^*, and $x_{j^*} - x_{j_*} < 2\pi$.

Let $Y_s := \{y_i\}_{i\in\mathbb{Z}}$, $s \geq 1$, be a set of points, such that $y_i < y_{i+1}$ and $y_{i+2s} = y_i + 2\pi, \quad i \in \mathbb{Z}$.

For each $i \in \mathbb{Z}$, let j_i be the index such that $y_i \in [x_{j_i}, x_{j_i+1})$. We denote by O the interior of the union

$$\cup_{i\in\mathbb{Z}}[x_{j_i-1}, x_{j_i+2}].$$

We will write $S \in \widetilde{\Sigma}_{k,n}(Y_s)$, if $S \in \widetilde{\Sigma}_{k,n}$ and $p_{j\pm1} \equiv p_j$ for $x_j \in O$.

For $x \in E$ and $\eta \in \mathbb{N}$, denote

$$A(x, E) := \omega_k(S, 1/n; E) + \omega_k(S, 1/n)\left(\frac{1}{n_1 \operatorname{dist}(x, \mathbb{R}\setminus E)}\right)^{\eta}, \tag{6.1}$$

and, finally, let

$$\pi(t) := \prod_{i=1}^{2s} \frac{|\sin\frac{1}{2}(t-y_i)|}{|\sin\frac{1}{2}(t-y_i)| + 1/n}. \tag{6.2}$$

We devote this section to proving,

Theorem 6.1 *Let $n_1 \geq n$ and $S \in \widetilde{\Sigma}_{k,n}(Y_s)$. Then there is a trigonometric polynomial $\mathcal{T}$ of degree $< cn_1$, such that*

$$\|S - \mathcal{T}\| \leq c\omega_k(S, 1/n), \tag{6.3}$$

and if E is a proper interval, then

$$|S'(x) - \mathcal{T}'(x)| \leq cn\pi(x)A(x, E), \quad x \in E. \tag{6.4}$$

Here and in the sequel c and C denote constants which depend only on some or all the parameters k, s and η.

Let

$$O_\nu = (x_{\nu^-}, x_{\nu^+}), \quad \nu \in \mathbb{Z}, \tag{6.5}$$

be the connected components of the set O, enumerated from right to left, and set

$$\tilde{O}_\nu = (x_{\nu^-} + \pi/(2n), x_{\nu^+} - \pi/(2n)). \tag{6.6}$$

We need a few lemmas.

Lemma 6.2 *Let $n_1 \geq n$, $1 \leq \nu \leq 2s+1$, $1 \leq \sigma \leq k$, $S \in \tilde{\Sigma}_{k,n}(Y_s)$ $\eta \geq 2s$ and $J_{n_1} := J_{n_1,\eta}$. If E is a proper interval and $O_\mu \subset E$, then for all $x \in \tilde{O}_\mu$,*

$$|B_{\nu,n_1}(x)| \leq Cn^\nu A(x, E). \tag{6.7}$$

Proof Clearly, for any $j^0 \in \mathbb{Z}$, we may rewrite (5.2) as

$$B_{\nu,n_1}(x) = \frac{1}{\sigma}\sum_{l=2}^{\nu}\sum_{\substack{j=-n\sigma+j^0+1\\ x_j\notin O}}^{n\sigma+j^0}\left(p_j^{(l-1)}(x_j) - p_{j-1}^{(l-1)}(x_j)\right)\frac{d^{\nu-l}}{dx^{\nu-l}}J_{n_1}\left(\frac{x-x_j}{\sigma}\right),$$

where we use the fact that S is continuous and 2π-periodic, so that $p_j(x_j) = p_{j-1}(x_j)$, $j \in \mathbb{Z}$, and $p_j \equiv p_{j-1}$ for all j such that $x_j \in O$, and that $J_{n_1}(t/\sigma)$ is $2\pi\sigma$-periodic.

In particular we may take j^0 where $x \in I_{j^0}$. Thus, without loss of generality, we may assume that $x \in I_0$ and $j^0 = 0$. Then, for each j, $-n\sigma + 1 \le j \le n\sigma$, we have $\frac{|x-x_j|}{\sigma} \le \pi$.

By virtue of (1.5) we obtain for each $\hat{j}$, $-n\sigma + 1 \le \hat{j} \le \mu^-$,

$$\begin{aligned}
&\sum_{j=-n\sigma+1}^{\hat{j}}\left|J_{n_1}^{(\nu-l)}\left(\frac{x-x_j}{\sigma}\right)\right| \le C\sum_{j=-n\sigma+1}^{\hat{j}}\frac{n_1^{\nu-l+1}}{(n_1(x-x_j))^{2\eta-\nu+l}}\\
&\le C\sum_{j=-n\sigma+1}^{\hat{j}}\frac{n_1 n^{\nu-l}}{(n_1(x-x_j))^{\eta+1}} \le C\frac{n^{\nu-l}}{n_1(n_1(x-x_{\hat{j}})^{\eta-1}}\sum_{j=-n\sigma+1}^{\hat{j}}\frac{1}{(x-x_j)^2}\\
&\le C\frac{n^{\nu-l+1}}{(n_1(x-x_{\hat{j}}))^{\eta}}.
\end{aligned}$$

Similarly, (1.5) implies for each $\check{j}$, $\mu^+ \le \check{j} \le n\sigma$,

$$\sum_{j=\check{j}}^{n\sigma}\left|J_{n_1}^{(\nu-l)}\left(\frac{x-x_j}{\sigma}\right)\right| \le C\frac{n^{\nu-l+1}}{(n_1(x_{\check{j}}-x))^{\eta}}.$$

Let $E = [x_{j_*}, x_{j^*}]$. By Markov's inequality $|p_j^{(l)}(x_j) - p_{j-1}^{(l)}(x_j)| \le cn^l\omega_k(S, 1/n; E)$, if either $j_* < j \le \mu^-$, or $\mu^+ \le j < j^*$. Also $|p_j^{(l)}(x_j) - p_{j-1}^{(l)}(x_j)| \le cn^l\omega_k(S, 1/n)$ for all $j \in \mathbb{Z}$.

We have to separate the proof for $\sigma = 1$ and $\sigma \ge 2$. We begin with the latter, so that

$$-n\sigma + 1 \le j_* \le \mu^- < \mu^+ \le j^* \le n\sigma.$$

Hence,

$$\begin{aligned}
&\sum_{j=-n\sigma+1}^{j_*}\left|\left(p_{j-1}^{(l-1)}(x_j) - p_j^{(l-1)}(x_j)\right)J_{n_1}^{(\nu-l)}\left(\frac{x-x_j}{\sigma}\right)\right| \qquad (6.8)\\
&\le C\omega_k(S, 1/n)\frac{n^{l-1}n^{\nu-l+1}}{(n_1(x-x_{j_*}))^{\eta}} \le C\omega_k(S, 1/n)n^{\nu}\left(\frac{1}{n_1\operatorname{dist}(x, \mathbb{R}\setminus E)}\right)^{\eta},
\end{aligned}$$

and

$$\sum_{j=j_*+1}^{\mu^-} \left| \left(p_{j-1}^{(l-1)}(x_j) - p_j^{(l-1)}(x_j) \right) J_{n_1}^{(\nu-l)} \left(\frac{x - x_j}{\sigma} \right) \right| \\ \le C\omega_k(S, 1/n; E) \frac{n^{l-1} n^{\nu-l+1}}{(n_1(x - x_{j_{\mu^-}}))^\eta} \le C\omega_k(S, 1/n; E) n^\nu. \tag{6.9}$$

Similarly,

$$\sum_{j=\mu^+}^{n\sigma} \left| \left(p_j^{(l-1)}(x_j) - p_{j-1}^{(l-1)}(x_j) \right) J_{n_1}^{(\nu-l)} \left(\frac{x - x_j}{\sigma} \right) \right| \le C n^\nu A(x, E).$$

If $\sigma = 1$, then we may have $-n + 1 \le j_* < j^* \le n$, and the proof follows verbatim as above. Otherwise, we may have that $j_* \le -n$, so that (6.8) is irrelevant, while in the summation in (6.9) we replace $j = -j_* + 1$ by $j = -n + 1$, where we note that $\mu^- > -n$; or we may have $j^* > n$, so that we replace the upper end of the summation j^* with n, where we note that $\mu^+ < n$. This completes the proof. □

Lemma 6.3 *Let $n_1 \ge n$, $1 \le \nu \le 2s + 1$, $S \in \tilde{\Sigma}_{k,n}(Y_s)$ $\eta \ge 2s$ and $J_{n_1} := J_{n_1,\eta}$, and let*

$$T(x) := \frac{1}{\pi} \int_{-\pi}^{\pi} (S(x) - (-1)^k \Delta_t^k(S, x)) J_{n_1}(t)\, dt,$$

be the trigonometric polynomial of degree $< \eta n_1$. Then

$$\|S - T\| \le c\omega_k(S, 1/n). \tag{6.10}$$

If E is a proper interval, then

$$|S'(x) - T'(x)| \le cnA(x, E), \quad x \in E, \tag{6.11}$$

where $A(x, E)$ was defined in (6.1).

Moreover, if $O_\mu \subset E$, then for all $x \in \tilde{O}_\mu$ and $1 \le \nu \le 2s + 1$,

$$|S^{(\nu)}(x) - T^{(\nu)}(x)| \le cn^\nu A(x, E), \quad x \in \tilde{O}_\mu. \tag{6.12}$$

Proof The inequality $\|S - T\| \le c\omega_k(S, 1/n_1) \le c\omega_k(S, 1/n)$ is well-known.

Since S is a piecewise algebraic polynomial, it possesses all left- and right-hand derivatives at each x_j, $j \in \mathbb{Z}$. Thus, if it happens that $x + \nu t = x_j$ for some $j \in \mathbb{Z}$, then $\Delta_t^k(S^{(\nu)}, x)$ may not be well defined. But, for each fixed x, this may happen only for finitely many values of t, and would not influence the integration below. However, when we wish to estimate $\Delta_t^k(S^{(\nu)}, x)$ we must consider the collection of all (finitely many) possible values we may have by assigning the various appropriate left- or right-hand values that may occur.

Recall that in [2, Lemma 5.5], it was proved for a proper interval E, that if $x, x+kt \in E$ and $(x+\ell t) \notin \{x_j\}_{j=-\infty}^{\infty}$, $0 \le \ell \le k$, then

$$|\Delta_t^k(S^{(\nu)}, x)| \le cn^\nu (1+n|t|)^k \omega_k(S, 1/n; E). \tag{6.13}$$

Closely observing the proof of [2, Lemma 5.5], we see that (6.13) is valid also with our above relaxation. In addition, the restriction on the length of E, is irrelevant for the next inequality. Thus,we obtain,

$$|\Delta_t^k(S^{(\nu)}, x)| \le cn^\nu (1+n|t|)^k \omega_k(S, 1/n), \quad |t| \le \pi. \tag{6.14}$$

If $x \in E$, and $d := \operatorname{dist}(x, \mathbb{R} \setminus E) \ge \pi/(2n)$, then

$$\begin{aligned}
&\left| \int_\pi^\pi \Delta_t^k(S^{(\nu)}, x) J_{n_1}(t)\, dt \right| \qquad (6.15)\\
&\le \left(\int_{|t| \le \frac{\pi}{2kn}} + \int_{\frac{\pi}{2kn} \le |t| \le \frac{d}{k}} + \int_{\frac{d}{k} \le |t| \le \pi} \right) |\Delta_t^k(S^{(\nu)}, x)| J_{n_1}(t)\, dt \\
&\le cn^\nu \omega_k(S, 1/n; E) \int_I J_{n_1}(t)\, dt + c\frac{n^{\nu+k}}{n_1^{2\beta-1}} \omega_k(S, 1/n; E) \int_{\frac{\pi}{2kn}}^\infty \frac{t^k}{t^{2\beta}}\, dt \\
&+ c\frac{n^{\nu+k}}{n_1^{2\beta-1}} \omega_k(S, 1/n) \int_{\frac{d}{k}}^\infty \frac{t^k}{t^{2\beta}}\, dt \\
&\le cn^\nu A(x, E).
\end{aligned}$$

In particular, this is the case when $x \in \tilde{O}_\mu$.

Similarly, if $0 < d < \pi/(2n)$, then

$$|S'(x) - T_n'(x)| = \frac{1}{\pi} \left| \int_{-\pi}^\pi \Delta_t^k(S', x) J_{n_1}(t)\, dt \right| \le cnA(x, E).$$

Thus, (6.11) is proved.

Finally, if $x \in \tilde{O}_\mu \subset E$, then by (6.15) and (6.7),

$$\begin{aligned}
&\pi |S^{(\nu)}(x) - T^{(\nu)}(x)| = \left| \frac{d^\nu}{dx^\nu} \int_\pi^\pi \Delta_t(S, x) J_{n_1}(t)\, dt \right| \\
&\le \left| \int_\pi^\pi \Delta_t^k(S^{(\nu)}, x) J_{n_1}(t)\, dt \right| \\
&+ \left| \frac{d^\nu}{dx^\nu} \int_\pi^\pi \Delta_t(S, x) J_{n_1}(t)\, dt - \int_\pi^\pi \Delta_t^k(S^{(\nu)}, x) J_{n_1}(t)\, dt \right| \\
&\le cn^\nu A(x, E) + \sum_{\sigma=1}^k \binom{k}{\sigma} \left| \frac{d^\nu}{dx^\nu} \int_\pi^\pi S(x+\sigma t) J_{n_1}(t)\, dt - \int_\pi^\pi S^{(\nu)}(x+\sigma t) J_{n_1}(t)\, dt \right| \\
&\le cn^\nu A(x, E),
\end{aligned}$$

and (6.12) follows. □

If a proper interval E is such that its endpoints are not in O, we will call it a Y_s-proper interval.

For each $\mu \in \mathbb{Z}$, let $x_{\mu^\circ} := \frac{1}{2}(x_{\mu^-} + x_{\mu^-})$ be the midpoint of $O_\mu = (x_{\mu^-}, x_{\mu^-})$, and for each Y_s-proper interval, such that $O_\mu \subset E$, let

$$A_\mu(E) := A(x_{\mu^\circ}, E).$$

Since $\operatorname{dist}(x_{\mu^\circ}, \mathbb{R} \setminus E) \le C \operatorname{dist}(x, \mathbb{R} \setminus E)$, for all $x \in \tilde{O}_\mu$, and $\operatorname{dist}(x, \mathbb{R} \setminus E) \le C \operatorname{dist}(x_{\mu^\circ}, \mathbb{R} \setminus E)$, for all $x \in O_\mu$, it follows that

$$A(x, E) \le c A_\mu(E), \quad x \in \tilde{O}_\mu. \tag{6.16}$$

and

$$A_\mu(E) \le c A(x, E), \quad x \in O_\mu. \tag{6.17}$$

Define

$$A_\mu := \min_{E: O_\mu \subset E} A_\mu(E). \tag{6.18}$$

Finally, denote $J_\mu := [x_{\mu^\circ} - \pi, x_{\mu^\circ} + \pi]$, and let M_μ be the 2π-periodic function, defined on J_μ by

$$M_\mu(x) = \begin{cases} 1, & x \in \tilde{O}_\mu, \\ \frac{1}{n_1 |x - x_{\mu^\circ}|} & x \in J_\mu \backslash \tilde{O}_\mu. \end{cases}$$

Lemma 6.4 *Let $\mu \in \mathbb{Z}$ and $n_1 \ge n$. Then, for every Y_s-proper interval E and each $x \in E$, we have*

$$A_\mu M_\mu^\eta(x) \le C A(x, E). \tag{6.19}$$

Proof It is sufficient to prove (6.19) for $x \in J_\mu$, and we let a Y_s-proper interval E be such that $x \in E$.

First, assume that $O_\mu \not\subset E$. Thus, there is an endpoint of E, say γ, lying between x and x_{μ°. Then $\operatorname{dist}(x, \mathbb{R} \setminus E) \le |x - \gamma| \le |x - x_{\mu^\circ}|$.

Hence,

$$\begin{aligned} \frac{1}{2} A_\mu M_\mu^\eta(x) &\le \omega_k(S, 1/n) M_\mu^\eta(x) = \omega_k(S, 1/n) \left(\frac{1}{n_1 |x - x_{\mu^\circ}|} \right)^\eta \\ &\le \omega_k(S, 1/n) \left(\frac{1}{n_1 \operatorname{dist}(x, \mathbb{R} \setminus E)} \right)^\eta \le A(x, E). \end{aligned}$$

Otherwise, $O_\mu \subset E$.

If $x \in O_\mu$, then (6.19) is trivial, since $\|M_\mu\| = 1$, and by (6.17)

$$A_\mu M_\mu^\eta(x) \le A_\mu \le A_\mu(E) \le CA(x, E).$$

Similarly, if $x \in E \setminus O_\mu$ and $|x-\gamma| \le |x_{\mu^\circ} - \gamma|$, where now γ is the endpoint of E, closest to x_{μ°, then

$$A_\mu M_\mu^\eta(x) \le A_\mu \le A_\mu(E) \le A(x, E),$$

that yields (6.19).

Finally, if $x \in E \setminus O_\mu$ and $|x-\gamma| > |x_{\mu^\circ} - \gamma|$, then assume, without loss of generality, that $x_{\mu^\circ} < \gamma$. Then, it follows that $x + 3\pi/(2n) \le x_{\mu^\circ} \le \gamma - 3\pi/(2n)$. Since $g(u) := \frac{\gamma-u}{x_{\mu^\circ}-u}$ is an increasing function for $u < x_{\mu^\circ}$, we have,

$$\begin{aligned}\frac{\gamma-x}{x_{\mu^\circ}-x} &\le \frac{\gamma-(x_{\mu^\circ}-3\pi/(2n))}{x_{\mu^\circ}-(x_{\mu^\circ}-3\pi/(2n))} = \frac{2n}{3\pi}(\gamma-(x_{\mu^\circ}-3\pi/(2n)))\\ &\le \frac{2n}{3\pi}2(\gamma-x_{\mu^\circ}) < n(\gamma-x_{\mu^\circ}).\end{aligned}$$

Hence,

$$\frac{1}{n_1^2|x_{\mu^\circ}-\gamma||x-x_{\mu^\circ}|} < \frac{1}{n_1|x-\gamma|}.$$

Therefore, $A_\mu M_\mu^\eta(x) \le A(x, E)$. □

We are ready to prove Theorem 6.1. It is easy to show that if an endpoint of a proper interval E belongs to O, say, to its connected component O_μ, then $\omega_k(S, 1/n; \overline{E \cup O_\mu}) \le c\omega_k(S, 1/n; E)$, whence $A(x, \overline{E \cup O_\mu}) \le CA(x, E)$, $x \in E$. Therefore, we prove Theorem 6.1 for Y_s-proper intervals E and, without loss of generality, we assume that $\eta \ge 2s$.

Proof of Theorem 6.1 We apply Theorem 1.3 for each fixed $\mu \in \mathbb{Z}$, with $\epsilon = 1/12$, and n_1 instead of n and h such that,

$$\dot{O}_\mu := [x_{\mu^\circ} - (s + 5/(12))h, x_{\mu^\circ} + (s + 5/(12))h] := [x_{\mu^-} + \pi/n, x_{\mu^+} - \pi/n].$$

Thus,

$$(2s + 5/6)h = |\dot{O}_\mu| = |O_\mu| - 2\pi/n.$$

Since $\frac{3\pi}{n} \le |O_\mu| \le \frac{6s\pi}{n}$, we conclude that

$$\frac{\pi}{2(s+1)n} < h < \frac{6s\pi}{2sn} = 3\pi/n,$$

and

$$(2s + 7/6)h = |O_\mu| - 2\pi/n + h/3 < |O_\mu| - \pi/n.$$

Hence,

$$\ddot{O}_\mu := [x_{\mu^\circ} - (s + 7/(12))h, x_{\mu^\circ} + (s + 7/(12))h] \subset [x_{\mu^-} + \pi/(2n), x_{\mu^+} - \pi/(2n)] = \tilde{O}_\mu.$$

Note that all points $y_i \in J_\mu$ lie either in $\dot{O}_\mu$ or outside $\ddot{O}_\mu$. Let l be the number of points $y_i \in \dot{O}_\mu$.

Let T be the polynomial, guaranteed by Lemma 6.3, and denote

$$R_\mu := \max_{1 \le i \le l} h^{i-1} \|S^{(i)} - T^{(i)}\|_{\dot{O}_\mu} \quad \text{and} \quad f(x) := \frac{S'(x) - T'(x)}{R_\mu},$$

so that, for all $0 \le \nu \le l - 1$, we have

$$|f^{(\nu)}(x)| \le \frac{|S^{(\nu+1)}(x) - T^{(\nu+1)}(x)|}{R_\mu} \le \frac{\|S^{(\nu+1)} - T^{(\nu+1)}|_{\dot{O}_\mu}}{h^\nu \|S^{(\nu+1)} - T^{(\nu+1)}\|_{\dot{O}_\mu}} = h^{-\nu}, \quad x \in \dot{O}_\mu.$$

Thus, f satisfies (1.12). Hence, (1.14) through (1.17), imply the existence of a polynomial d_l, of degree $< cn_1$, such that

$$d_l(y_i) = f(y_i), \quad y_i \in \dot{O}_\mu, \quad d_l(y_i) = 0, \quad y_i \in J_\mu \setminus \ddot{O}_\mu, \quad \left| \int_{-\pi}^x d_l(t)\, dt \right| \le ch, \quad x \in \mathbb{R},$$

and for all $0 \le \nu \le 2s$,

$$|d_l^{(\nu)}(x)| \le ch^{-\nu} M_\mu^\eta(x), \quad x \in \mathbb{R}.$$

By (6.12), $R_\mu \le ch^{-1} A_\mu$. Therefore, the polynomial

$$\tau_\mu := R_\mu \int_{-\pi}^x d_l(t)\, dt$$

satisfies

$$\|\tau_\mu\| \le c\omega_k(S, 1/n), \tag{6.20}$$

$$\tau_\mu'(y_i) = S'(y_i) - T'(y_i), \quad y_i \in O_\mu,$$
$$\tau_\mu'(y_i) = 0, \quad y_i \in J_\mu \setminus O_\mu,$$

and for all $1 \le \nu \le 2s + 1$,

$$|\tau_\mu^{(\nu)}(x)| \le cn^\nu A_\mu M_\mu^\eta(x), \quad x \in \mathbb{R},$$

where in the last inequality we used the fact that $\dot{O}_\mu \subset \tilde{O}_\mu$.

Finally, Lemma 6.4 implies that for every Y_s-proper interval E, we have

$$|\tau_\mu^{(\nu)}(x)| \le Cn^\nu A(x, E) \quad x \in E. \tag{6.21}$$

We will prove that the desired polynomial $\mathcal{T}$ may be taken in the form

$$\mathcal{T} := T + \sum_{\substack{\mu \text{ s.t.} \\ x_{\mu^\circ} \in [-\pi,\pi)}} \tau_\mu.$$

Indeed, (6.3) readily follows by (6.10) and (6.20).

We observe that,

$$c \le \pi(t) \le 1, \quad t \notin \tilde{O}, \quad \text{where} \quad \tilde{O} := \cup_{\mu \in \mathbb{Z}} \tilde{O}_\mu,$$

where $\pi(t)$ was defined in (6.2), and combined with (6.11) and (6.21) with $\nu = 1$, we obtain (6.4) for $x \in E \setminus \tilde{O}$.

On the other hand, if $x \in \tilde{O}_{\mu^*} \subset E$, for some $\mu^* \in \mathbb{Z}$, then let $y_{i_\ell} \in O_{\mu^*}$, $1 \le \ell \le l$, and note that $y_{i_\ell} \in \tilde{O}_{\mu^*}$. Evidently, $S'(y_{i_\ell}) = \mathcal{T}'(y_{i_\ell})$, $1 \le \ell \le l$.

Applying (6.12) and (6.21) for all μ, all with $\nu = l + 1$, we obtain

$$|S^{(l+1)}(x) - \mathcal{T}^{(l+1)}(x)| \le cn^{l+1} A(x, E), \quad x \in \tilde{O}_{\mu_*}.$$

Hence, for $x \in \tilde{O}_{\mu^*}$,

$$\frac{|S'(x) - \mathcal{T}'(x)|}{\prod_{\ell=1}^{l} |x - y_{i_\ell}|} = [x, y_{i_1}, \ldots, y_{i_l}; S' - \mathcal{T}'] = \frac{|f^{(l+1)}(\theta)|}{l!} \le Cn^{l+1} A(\theta, E) \le Cn^{l+1} A_{\mu^*}(E).$$

Thus, by (6.17), we conclude that

$$|S'(x) - \mathcal{T}'(x)| \le cn^{l+1} A_{\mu_*}(E) \prod_{\ell=1}^{l} |x - y_{i_\ell}| \le cn\pi(x) A_{\mu_*}(E) \le cn\pi(x) A(x, E), \quad x \in \tilde{O}_{\mu_*}.$$

This completes the proof. □

Funding Open access funding provided by Tel Aviv University. The authors have not disclosed any funding.

References

1. DeVore, R.A., Lorentz, G.G.: Constructive Approximation, p. 449. Springer, Berlin (1993)
2. Leviatan, D., Shevchuk, I.A.: Coconvex approximation of periodic functions. Constr. Approx. **57**, 695–726 (2023)

3. Vyazovskaya, M.S., Pupashenko, N.S.: On the normalizing multiplier of the generalized Jackson kernel. Math. Notes **80**, 19–26. Translated from Matematicheskie Zametki, vol. 80, no. 1, 20–28 (2006)