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Fourier Transform for L^p-Functions with a Vector Measure on a Homogeneous Space of Compact Groups

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Abstract
Let G be a compact group and G/H a homogeneous space where H is a closed subgroup of G. Define an operator $T_H : C(G) \to C(G/H)$ by $T_H f(tH) = \int_H f(th)\, dh$ for each $tH \in G/H$. In this paper, we extend T_H to a norm-decreasing operator between L^p-spaces with a vector measure for each $1 \leq p < \infty$. This extension will be used to derive properties of invariant vector measures on G/H. Moreover, a definition of the Fourier transform for L^p-functions with a vector measure is introduced on G/H. We also prove the uniqueness theorem and the Riemann–Lebesgue lemma.

Keywords Vector measure · Homogeneous space · Compact group · Fourier transform

Mathematics Subject Classification 46G10 · 43A15 · 43A85

1 Introduction

Let G be a topological group which is compact and Hausdorff. Consider a homogeneous space G/H where H is a closed subgroup of G. If we denote the normalized Haar measures on G and H by m and dh respectively, then there is an induced left

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invariant Radon measure $\widetilde{m}$ on G/H satisfying Weil's formula:

$$\int_{G/H}\int_H f(th)\,dh\,d\widetilde{m}(tH) = \int_G f\,dm \qquad (f \in C(G)).$$

In this setting, Farashahi [5] introduced a method to obtain many of the well-known results on G/H from the ones on G. This method relies unavoidably on an extension of the operator $T_H : C(G) \to C(G/H)$ given by $T_H f(tH) = \int_H f(th)\,dh$. The extension is in fact a norm-decreasing operator from $L^p(G, m)$ onto $L^p(G/H, \widetilde{m})$ where $1 \leq p < \infty$. The crucial property for this method is the surjectivity of the extension as it provides a connection to all L^p-functions on G/H to those on G. The extension was used to study abstract Fourier analysis on homogeneous spaces in various aspects such as convolutions, Fourier transform operators, Fourier series and measure algebras, see [6–9].

A vector measure is a measure taking values in a Banach space. There are many studies about functions in L^p-spaces of a compact group associated to a vector measure and invariant properties under the group operations of the vector measure itself. For example, the Fourier transform and the convolution along with invariant properties were studied in [1–3] under the condition that G is an abelian compact group. Then they were generalized to a non-abelian case in [13, 14].

Let ν be a vector measure on G. In this paper, we initiate a study of an extension of the operator $T_H : C(G) \to C(G/H)$ to an operator with the domain $L^p(G, \nu)$. However, the codomain $C(G/H)$ must be extended as well. For this purpose, we will construct a corresponding vector measure $\tilde{\nu}$ on G/H and show that the codomain of the extended operator is $L^p(G/H, \tilde{\nu})$. We investigate whether the extended operator is surjective and whether Weil's formula is valid. It turns out that these are true for some vector measure ν. Fortunately, it is sufficient for the study of functions in $L^p(G/H, \mu)$ for any vector measure μ on G/H. We will employ the extension to obtain properties of invariant vector measures on G/H. Moreover, we introduce a new definition of a Fourier transform of functions in $L^1(G, \nu)$ which is a variant definition of [13]. In our definition, ν is taking values in a Banach space while in [13] ν is taking values in an operator space. The uniqueness theorem of the Fourier transform and the Riemann–Lebesgue lemma are considered. Finally, we provide an analogous definition for a Fourier transform of functions in $L^1(G/H, \mu)$ and once more employ the extension to obtain relations between the Fourier transforms of functions on G and G/H.

This paper is organized as follows. We give preliminary background in Sect. 2. In Sect. 3, an extension of the operator T_H to the space $L^p(G, \nu)$ is studied along with its properties. Then the obtained properties of the extension will be used to derive properties of invariant vector measures on G/H in Sect. 4. There are three types of invariant vector measures we consider in this paper: translation invariant, norm integral invariant and semivariation invariant measures. Section 5 concerns the Fourier transforms of functions on G and G/H.

2 Preliminaries

2.1 Fourier Analysis with Haar Measures

Let G be a compact group with the normalized Haar measure m. The **dual space** $\widehat{G}$ of G is the set of all unitary equivalence classes of irreducible unitary representations of G. For each $[\pi] \in \widehat{G}$, the representation space of π is denoted by $\mathcal{H}_\pi$ with the dimension $d_\pi = \dim \mathcal{H}_\pi$. For $[\pi] \in \widehat{G}$ and $u, v \in \mathcal{H}_\pi$, the function $\pi_{u,v} : G \to \mathbb{C}$ given by $\pi_{u,v}(t) = \langle \pi(t)v, u\rangle$ is called a **matrix element** of π. We write π_{ij} for π_{e_i,e_j}. Denote by $\mathrm{Trig}(G)$ the set of all finite linear combinations of matrix elements of irreducible representations. Note that $\mathrm{Trig}(G)$ is dense in $C(G)$ in the uniform norm. For $f \in L^1(G, m)$ and $[\pi] \in \widehat{G}$, the **Fourier transform** of f is defined in the weak sense as

$$\mathcal{F}_G(f)(\pi) = \widehat{f}(\pi) = \int_G f(t)\pi(t)^* \, dm(t) \in \mathcal{B}(\mathcal{H}_\pi).$$

Given any collection $\{X_i\}_{i\in I}$ of Banach spaces where each X_i is equipped with the norm $\|\cdot\|_i$. The space $\ell^\infty(I; X_i) = \{x \in \prod_{i\in I} X_i : \sup_{i\in I} \|x_i\|_i < \infty\}$ is a Banach space with the norm $\|x\|_\infty = \sup_{i\in I} \|x_i\|_i$. The set $c_0(I; X_i)$ of all $x = (x_i)$ for which $\{i \in I : \|x_i\|_i > \varepsilon\}$ is finite for any $\varepsilon > 0$ is a closed subspace of $\ell^\infty(I; X_i)$. By [12, Theorem 28.40], the Fourier transform operator $\mathcal{F}_G$ is a bounded linear operator from $L^1(G, m)$ into $c_0(\widehat{G}; \mathcal{B}(\mathcal{H}_\pi))$ with $\|\widehat{f}(\pi)\| \le \|f\|_{L^1(G,m)}$. For more details, see [11].

Let H be a closed subgroup of G and G/H the homogeneous space of left cosets equipped with the quotient topology. We denote the quotient map by $q : G \to G/H$. For $\varphi : G/H \to \mathbb{C}$, we write $\varphi_q : G \to \mathbb{C}$ for a function given by $\varphi_q(t) = \varphi(tH)$. Let dh be the normalized Haar measure on H. It is well-known that there is a unique (up to scalar) invariant Radon measure $\widetilde{m}$ on G/H satisfying Weil's formula:

$$\int_{G/H} \int_H f(th)\, dh\, d\widetilde{m}(tH) = \int_G f\, dm \quad (f \in C(G)).$$

In fact, $\widetilde{m}$ is the pushforward measure of m by the quotient map q. Define a bounded operator $T_H : C(G) \to C(G/H)$ by

$$T_H f(tH) = \int_H f(th)\, dh \quad (tH \in G/H,\ f \in C(G)).$$

According to [5], for any $1 \le p < \infty$, the operator T_H can be extended to a norm-decreasing operator from $L^p(G, m)$ onto $L^p(G/H, \widetilde{m})$ (still denoted by T_H) for which the extended Weil's formula holds:

$$\int_{G/H} T_H f\, d\widetilde{m} = \int_G f\, dm \quad \big(f \in L^1(G, m)\big). \tag{1}$$

For more details on Weil's formula, see [16]. The **dual space** of G/H is given by $\widehat{G/H} := \big\{[\pi] \in \widehat{G} : T_H^\pi \neq 0\big\}$ where T_H^π is defined in the weak sense as the operator

$T_H^\pi := \int_H \pi(h)\,dh \in \mathcal{B}(\mathcal{H}_\pi)$. For $\varphi \in L^1(G/H, \widetilde{m})$ and $[\pi] \in \widehat{G/H}$, the **Fourier transform** of φ is defined in the weak sense as

$$\mathcal{F}_{G/H}(\varphi)(\pi) = \widehat{\varphi}(\pi) = \int_{G/H} \varphi(tH)\Gamma_\pi(tH)^*\, d\widetilde{m}(tH) \in \mathcal{B}(\mathcal{H}_\pi),$$

where $\Gamma_\pi(tH) = \pi(t)T_H^\pi$. Then the Fourier transform operator $\mathcal{F}_{G/H}$ is a bounded linear operator from $L^1(G/H, \widetilde{m})$ into $c_0(\widehat{G/H}; \mathcal{B}(\mathcal{H}_\pi))$ with $\|\widehat{\varphi}(\pi)\| \le \|\varphi\|_{L^1(G/H,\widetilde{m})}$, see [5, Theorem 5.5].

2.2 Vector Measures

Let $(\Omega, \mathfrak{B}(\Omega))$ be a Borel measurable space and X a Banach space. The closed unit ball in the dual space X^* is denoted by B_{X^*}. A **(countably additive) vector measure** ν on $(\Omega, \mathfrak{B}(\Omega))$ is an X-valued function $\nu : \mathfrak{B}(\Omega) \to X$ such that $\nu(\cup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \nu(E_n)$ in the norm topology for any sequence (E_n) of pairwise disjoint sets in $\mathfrak{B}(\Omega)$. Given $x^* \in X^*$, let $\langle \nu, x^*\rangle : \mathfrak{B}(\Omega) \to \mathbb{C}$ be the complex measure given by $\langle \nu, x^*\rangle(E) = \langle \nu(E), x^*\rangle$ for $E \in \mathfrak{B}(\Omega)$. The **semivariation** $\|\nu\|$ of ν is the set function defined by $\|\nu\|(E) = \sup_{x^*\in B_{X^*}} |\langle \nu, x^*\rangle|(E)$ for $E \in \mathfrak{B}(\Omega)$. A vector measure ν is said to be **regular** if for each $E \in \mathfrak{B}(\Omega)$ and $\varepsilon > 0$ there exist a compact set K and an open set O such that $K \subset E \subset O$ and $\|\nu\|(O \setminus K) < \varepsilon$. We denote by $\mathcal{M}(\Omega, X)$ the set of all regular X-valued measures on Ω.

A measurable function $f : \Omega \to \mathbb{C}$ is said to be ν-**integrable** if $f \in L^1(\langle \nu, x^*\rangle)$ for every $x^* \in X^*$ and for each $E \in \mathfrak{B}(\Omega)$ there is an $x_E \in X$ such that $\langle x_E, x^*\rangle = \int_E f\, d\langle \nu, x^*\rangle$ for every $x^* \in X^*$. We denote x_E by $\int_E f\, d\nu$. For a measurable function $f : \Omega \to \mathbb{C}$, define

$$\|f\|_\nu = \sup_{x^*\in B_{X^*}} \int_\Omega |f|\, d|\langle \nu, x^*\rangle|$$

and $\|f\|_{\nu,p} := \||f|^p\|_\nu^{1/p}$. The space $L^1(\Omega, \nu)$ of all ν-integrable functions is a Banach space with the norm $\|\cdot\|_\nu$. We say that $f = g$ ν-a.e. if $\|f - g\|_\nu = 0$. For each $1 \le p < \infty$, the space $L^p(\Omega, \nu) := \{f \in L^1(\Omega, \nu) : |f|^p \in L^1(\Omega, \nu)\}$ is a Banach space with the norm $\|\cdot\|_{L^p(\Omega,\nu)} := \|\cdot\|_{\nu,p}$. We denote by $S(\Omega)$ the set of all simple functions on Ω. The **integral operator** $I_\nu : L^1(\Omega, \nu) \to X$ is defined by $I_\nu(f) = \int_\Omega f\, d\nu$ for $f \in L^1(\Omega, \nu)$. Then I_ν is bounded with $\|I_\nu(f)\|_X \le \|f\|_{L^1(\Omega,\nu)}$.

Theorem 2.1 [15] *Let $f : \Omega \to \mathbb{C}$ be a complex function. Then f is ν-integrable if and only if there is a sequence (f_n) of simple functions which converges pointwise to f and for which $(\int_E f_n\, d\nu)$ is Cauchy for any $E \in \mathfrak{B}(\Omega)$.*

Theorem 2.2 [13] *Let $\nu \in \mathcal{M}(G, X)$. Then $C(G)$ is dense in $L^p(G, \nu)$ for all $1 \le p < \infty$.*

For Banach spaces X and Y, a linear operator $T : X \to Y$ is said to be **weakly compact** if $T(B)$ is a relatively weakly compact subset of Y whenever B is a bounded subset of X. By [4, Corollary VI.2.14], we have that on a compact Hausdorff space there is a one-to-one correspondence between the set of all regular vector measures and the set of all weakly compact operators. To be precise, given a regular vector measure $\nu : \mathfrak{B}(\Omega) \to X$, there is a weakly compact operator $T : C(\Omega) \to X$ representing ν, that is, $T(f) = \int_\Omega f \, d\nu$ for all $f \in C(\Omega)$, and vice versa.

A vector measure ν is said to be **absolutely continuous** with respect to a positive scalar measure λ, denoted by $\nu \ll \lambda$, if $\nu(E) \to 0$ in norm as $\lambda(E) \to 0$ where $E \in \mathfrak{B}(\Omega)$. Note that $\nu \ll \lambda$ if and only if ν vanishes on all sets of λ-measure zero, by [4, Theorem I.2.1]. Moreover, ν vanishes on all sets of λ-measure zero if and only if $\|\nu\|$ vanishes on all sets of λ-measure zero. By Rybakov's theorem [4], there is a linear functional $x^* \in X^*$ such that $\nu \ll |\langle \nu, x^* \rangle|$. This functional is called a **Rybakov functional**. For $k \in [0, \infty)$, a vector measure ν is said to be k-**scalarly bounded** by m if for any $x^* \in X^*$ and $E \in \mathfrak{B}(\Omega)$, we have $|\langle \nu, x^* \rangle|(E) \leq km(E)$.

Let $\tau : G \to G$ be a homeomorphism. For a measurable function $f : G \to \mathbb{C}$, we denote $f \circ \tau^{-1}$ by f_τ. For $a \in G$, we define the left translation L_a and the right translation R_a by $L_a(t) = at$ and $R_a(t) = ta^{-1}$ for $t \in G$. In the case that $\tau = L_a$ or R_a, we shall write $L_a f$ or $R_a f$ instead of f_τ. Hence $(L_a f)(t) = f(a^{-1}t)$ and $(R_a f)(t) = f(ta)$ for each $t \in G$.

Definition 1 Let $\tau : G \to G$ be a homeomorphism and ν a vector measure on G. We say that ν is τ-**invariant** if

$$I_\nu(f_\tau) = I_\nu(f) \quad \text{for all } f \in S(G).$$

Given a collection $\mathcal{T}$ of homeomorphisms on G, ν is said to be $\mathcal{T}$-**invariant** if it is τ-invariant for all $\tau \in \mathcal{T}$. In particular, if $\mathcal{T} = \{L_a : a \in G\}$ (or $\mathcal{T} = \{R_a : a \in G\}$), we say that ν is **left** (or **right**) **invariant**.

Definition 2 Let $\tau : G \to G$ be a homeomorphism and ν a vector measure on G. We say that ν is **norm integral** τ-**invariant** if

$$\|I_\nu(f_\tau)\| = \|I_\nu(f)\| \quad \text{for all } f \in S(G).$$

Given a collection $\mathcal{T}$ of homeomorphisms on G, ν is said to be **norm integral** $\mathcal{T}$-**invariant** if it is norm integral τ-invariant for all $\tau \in \mathcal{T}$. In particular, if $\mathcal{T} = \{L_a : a \in G\}$ (or $\mathcal{T} = \{R_a : a \in G\}$), we say that ν is **norm integral left** (or **right**) **invariant**.

Definition 3 Let $\tau : G \to G$ be a homeomorphism and ν a vector measure on G. We say that ν is **semivariation** τ-**invariant** if

$$\|f_\tau\|_{L^1(G,\nu)} = \|f\|_{L^1(G,\nu)} \quad \text{for all } f \in S(G).$$

Given a collection $\mathcal{T}$ of homeomorphisms on G, ν is said to be **semivariation** $\mathcal{T}$-**invariant** if it is semivariation τ-invariant for all $\tau \in \mathcal{T}$. In particular, if $\mathcal{T} = \{L_a : a \in G\}$ (or $\mathcal{T} = \{R_a : a \in G\}$), we say that ν is **semivariation left** (or **right**) **invariant**.

2.3 Tensor Integration

Let X and Y be any Banach spaces. Recall that the space $\mathcal{B}(Y^* \times X^*)$ of bounded bilinear forms on $Y^* \times X^*$ is a Banach space equipped with the norm

$$\|b\| = \sup\{|b(y^*, x^*)| : y^* \in B_{Y^*}, x^* \in B_{X^*}\}.$$

Note that we can realize $Y \otimes X$ as a subspace of $\mathcal{B}(Y^* \times X^*)$ by considering $u = \sum_{i=1}^n y_i \otimes x_i \in Y \otimes X$ as a bilinear form given by $b_u(y^*, x^*) = \sum y^*(y_i)x^*(x_i) = (y^* \otimes x^*)(u)$ for $y^* \in Y^*$ and $x^* \in X^*$. The **injective norm** $\|\cdot\|_\vee$ on $Y \otimes X$ is the norm induced by this embedding, i.e.,

$$\|u\|_\vee = \sup_{y^* \in B_{Y^*}, x^* \in B_{X^*}} |(y^* \otimes x^*)(u)|.$$

Moreover, we have alternative formulas for the injective norm

$$\|u\|_\vee = \sup_{y^* \in B_{Y^*}} \left\| \sum y^*(y_i)x_i \right\|_X = \sup_{x^* \in B_{X^*}} \left\| \sum x^*(x_i)y_i \right\|_Y.$$

The completion of the tensor product space $Y \otimes X$ with the injective norm is called the **injective tensor product** of Y and X, denoted by $Y \check{\otimes} X$. For more details, see [17].

Now we summarize the concept of tensor integration introduced by [18]. Let ν be an X-valued vector measure. A function $f : \Omega \to Y$ is said to be **ν-measurable** if there is a sequence of Y-valued simple functions (f_n) with $\lim_{n\to\infty} \|f_n - f\|_Y = 0$ ν-a.e. We say that a function $f : \Omega \to Y$ is **weakly ν-measurable** if for each $y^* \in Y^*$ the function $y^* f$ is ν-measurable. Note that a function $f : \Omega \to Y$ is ν-measurable if and only if f is $|\langle \nu, x^* \rangle|$-measurable for some Rybakov functional $x^* \in X^*$.

Theorem 2.3 (Pettis's measurability theorem [4]) *Let λ be a finite positive measure. A function $f : \Omega \to Y$ is λ-measurable if and only if f is weakly λ-measurable and λ-essentially separably valued.*

Let $E \in \mathfrak{B}(\Omega)$ and $\phi = \sum_{i=1}^n y_i \chi_{A_i}$ be a Y-valued simple function on Ω, where $y_i \in Y$ and $A_i \in \mathfrak{B}(\Omega)$. We define $\int_E \phi \, d\nu = \sum y_i \otimes \nu(E \cap A_i) \in Y \otimes X$. Then it can be shown that $(y^* \otimes x^*)(\int_E \phi \, d\nu) = \int_E y^* \phi \, d\langle \nu, x^* \rangle$ for $y^* \in Y^*$ and $x^* \in X^*$, hence $\left\| \int_E \phi \, d\nu \right\|_\vee \le \sup_{x^* \in B_{X^*}} \int_E \|\phi\| \, d|\langle \nu, x^* \rangle|$. For a ν-measurable function $f : \Omega \to Y$, we let

$$\mathrm{N}(f) = \sup_{x^* \in B_{X^*}} \int_\Omega \|f\| \, d|\langle \nu, x^* \rangle|.$$

Definition 4 A ν-measurable function $f : \Omega \to Y$ is **$\check{\otimes}$-integrable** if there exists a sequence (f_n) of simple functions such that

$$\lim_{n\to\infty} \mathrm{N}(f - \phi_n) = 0.$$

In this case, the sequence $(\int_E \phi_n \, d\nu)$ is a Cauchy sequence in $Y\check{\otimes}X$ for each $E \in \mathfrak{B}(\Omega)$. By the completeness of $Y\check{\otimes}X$, the limit of $(\int_E \phi_n \, d\nu)$ is denoted by $\int_E f \, d\nu$ and is called the $\check{\otimes}$-**integral** of f over E with respect to ν.

Note that if f is $\check{\otimes}$-integrable, then $(y^* \otimes x^*)(\int_E f \, d\nu) = \int_E y^* f \, d\langle \nu, x^* \rangle$ for $E \in \mathfrak{B}(\Omega)$, $y^* \in Y^*$ and $x^* \in X^*$ and $\| \int_\Omega f \, d\nu \|_\vee \leq N(f)$.

Theorem 2.4 [18] *A ν-measurable function f is $\check{\otimes}$-integrable if and only if $\|f\|$ is ν-integrable.*

3 Extensions of the Operator T_H

In this section, we study extensions of the operator $T_H : C(G) \to C(G/H)$. Given a vector measure $\nu \in \mathcal{M}(G, X)$, we can naturally construct a vector measure on G/H as follows. Let $T_\nu : C(G) \to X$ be the corresponding weakly compact operator for ν, i.e.,

$$T_\nu(f) = \int_G f \, d\nu \quad (f \in C(G)).$$

Define $T_{\tilde{\nu}} : C(G/H) \to X$ by

$$T_{\tilde{\nu}}(\varphi) = T_\nu(\varphi_q) = \int_G \varphi_q \, d\nu \quad (\varphi \in C(G/H)).$$

Then $T_{\tilde{\nu}}$ is weakly compact since $\|\varphi\|_{\sup} = \|\varphi_q\|_{\sup}$ for all $\varphi \in C(G/H)$. Hence there is a representing vector measure $\tilde{\nu} \in \mathcal{M}(G/H, X)$. Moreover, we immediately have that

$$\int_{G/H} \varphi \, d\tilde{\nu} = \int_G \varphi_q \, d\nu \quad (\varphi \in C(G/H)). \tag{2}$$

We shall begin with some basic properties of $\tilde{\nu}$.

Proposition 3.1 *Let $\nu \in \mathcal{M}(G, X)$.*

1. *The vector measure $\tilde{\nu}$ is the pushforward (vector) measure of ν by the quotient map q, i.e., $\tilde{\nu}(E) = \nu(q^{-1}(E))$ for all $E \in \mathfrak{B}(G/H)$. Moreover, the Eq. (2) holds for all $\varphi \in L^1(G/H, \tilde{\nu})$ provided that $\varphi_q \in L^1(G, \nu)$.*
2. *For any $x^* \in X^*$ and $E \in \mathfrak{B}(G/H)$, $|\langle \tilde{\nu}, x^* \rangle|(E) \leq |\langle \nu, x^* \rangle|(q^{-1}(E))$. Then $\|\varphi\|_{L^p(G/H, \tilde{\nu})} \leq \|\varphi_q\|_{\nu, p}$ for any $1 \leq p < \infty$ and $\varphi \in L^p(G/H, \tilde{\nu})$.*

Proof 1. Let λ be the pushforward measure of ν by the quotient map q. It follows from Eq. (2) that

$$\int_{G/H} \varphi \, d\lambda = \int_G \varphi_q \, d\nu = \int_{G/H} \varphi \, d\tilde{\nu}$$

for all $\varphi \in C(G/H)$. Hence $\tilde{\nu} = \lambda$. Next observe that Eq. (2) holds for all $\varphi \in S(G/H)$. Let $0 \leq \varphi \in L^1(G/H, \tilde{\nu})$. Then there is a sequence of positive simple functions $\varphi_n \uparrow \varphi$ pointwise. By the monotone convergence theorem, for each $x^* \in X^*$

$$\begin{aligned}\int_{G/H} \varphi\, d\langle\tilde{\nu}, x^*\rangle &= \lim_{n\to\infty} \int_{G/H} \varphi_n\, d\langle\tilde{\nu}, x^*\rangle \\ &= \lim_{n\to\infty} \int_G (\varphi_n)_q\, d\langle\nu, x^*\rangle \\ &= \int_G \varphi_q\, d\langle\nu, x^*\rangle.\end{aligned}$$

This identity easily extends to $\varphi \in L^1(G/H, \tilde{\nu})$. If we assume that $\varphi_q \in L^1(G, \nu)$, then

$$\left\langle \int_{G/H} \varphi\, d\tilde{\nu}, x^* \right\rangle = \int_{G/H} \varphi\, d\langle\tilde{\nu}, x^*\rangle = \int_G \varphi_q\, d\langle\nu, x^*\rangle = \left\langle \int_G \varphi_q\, d\nu, x^* \right\rangle$$

for all $x^* \in X^*$, which proves the Eq. (2).

2. Let $E \in \mathfrak{B}(G/H)$. Consider any disjoint partition $\{E_n\}_{n=1}^k$ of E where $E_n \in \mathfrak{B}(G/H)$. Since $\{q^{-1}(E_n)\}_{n=1}^k$ forms a disjoint partition of $q^{-1}(E)$,

$$\sum_{n=1}^{k} |\langle\tilde{\nu}, x^*\rangle(E_n)| = \sum_{n=1}^{k} |\langle\nu, x^*\rangle(q^{-1}(E_n))| \leq |\langle\nu, x^*\rangle|(q^{-1}(E)).$$

Hence $|\langle\tilde{\nu}, x^*\rangle|(E) \leq |\langle\nu, x^*\rangle|(q^{-1}(E))$. Consequently,

$$\int_{G/H} \varphi\, d|\langle\tilde{\nu}, x^*\rangle| \leq \int_G \varphi_q\, d|\langle\nu, x^*\rangle|$$

holds for any simple function $\varphi \geq 0$ on G/H. Then for any $\varphi \in L^1(G/H, \mu)$, the monotone convergence theorem implies that

$$\int_{G/H} |\varphi|\, d|\langle\tilde{\nu}, x^*\rangle| \leq \int_G |\varphi_q|\, d|\langle\nu, x^*\rangle|.$$

Therefore, $\|\varphi\|_{L^p(G/H,\tilde{\nu})} \leq \|\varphi_q\|_{\nu,p}$ for any $1 \leq p < \infty$ and $\varphi \in L^p(G/H, \tilde{\nu})$.

□

Example 1 Let $1 \leq p < \infty$ and $S : L^p(G, m) \to X$ be any bounded linear map, where m is the normalized Haar measure on G. Define a vector measure $\nu : \mathfrak{B}(G) \to X$ corresponding to S by $\nu(E) = S(\chi_E)$ for $E \in \mathfrak{B}(G)$. Then by Proposition 3.1.1. the vector measure $\tilde{\nu}$ is given by $\tilde{\nu}(F) = S(\chi_{q^{-1}(F)})$ for $F \in \mathcal{B}(G/H)$.

1. Let $X = \mathbb{C}$ and $S : L^1(G, m) \to \mathbb{C}$ be given by $S(f) = \int_G f\, dm$ for any $f \in L^1(G, m)$. In this case, $\nu = m$. Moreover, $\tilde{\nu} = \widetilde{m}$ since $\tilde{\nu}(F) = \int_G \chi_{q^{-1}(F)}\, dm = \int_{G/H} \chi_F\, d\widetilde{m} = \widetilde{m}(F)$ for all $F \in \mathfrak{B}(G/H)$, where $\widetilde{m}$ is the pushforward measure of m.

2. Let $X = L^1(G, m)$ and $S = \mathrm{Id}_{L^1(G,m)}$. Then $\tilde{\nu}(F) = \chi_{q^{-1}(F)}$ for $F \in \mathfrak{B}(G/H)$.
3. Let λ be a complex regular measure on G and $1 \le p < \infty$. We define $S : L^p(G, m) \to L^p(G, m)$ by $S(f) = f * \lambda$ where $(f * \lambda)(t) = \int_G f(ts^{-1})\, d\lambda(s)$ for $t \in G$. Then $\tilde{\nu}(F) = \chi_{q^{-1}(F)} * \lambda$ for $F \in \mathfrak{B}(G/H)$.
4. Let $1 \le p \le 2$ and $S : L^p(G, m) \to \ell^{p'}(\widehat{G}; \mathcal{B}(\mathcal{H}_\pi))$ be defined by $S(f) = \mathcal{F}_G(f)$. Then $\tilde{\nu}(F) = \mathcal{F}_G(\chi_{q^{-1}(F)}) = \mathcal{F}_{G/H}(\chi_F)$ for $F \in \mathfrak{B}(G/H)$.

Let $\tau : G/H \to G/H$ be a homeomorphism. For example, one can consider a left translation $L_a : G/H \to G/H$ by $a \in G$ given by $L_a(tH) = atH$ for each $tH \in G/H$. For a measurable function $\varphi : G/H \to \mathbb{C}$, we denote $\varphi \circ \tau^{-1}$ by φ_τ. In the case that $\tau = L_a$ where $a \in G$, we shall denote $\varphi \circ (L_a)^{-1}$ by $L_a\varphi$ and by definition we have $(L_a\varphi)(tH) = \varphi(a^{-1}tH)$ for all $tH \in G/H$.

Definition 5 Let $\tau : G/H \to G/H$ be a homeomorphism. For any vector measure μ on G/H, we say that μ is **norm integral τ-invariant** if

$$\|I_\mu(\varphi_\tau)\| = \|I_\mu(\varphi)\| \quad \text{for all } \varphi \in S(G/H).$$

Given a collection $\mathcal{T}$ of homeomorphisms on G/H, μ is said to be **norm integral $\mathcal{T}$-invariant** if it is norm integral τ-invariant for all $\tau \in \mathcal{T}$. In particular, if $\mathcal{T} = \{L_a : a \in G\}$, we say that μ is **norm integral left invariant**.

This proposition is merely a consequence of Proposition 3.1.

Proposition 3.2 *Let $\nu \in \mathcal{M}(G, X)$.*

1. *For $a \in G$, if ν is norm integral L_a-invariant, then so is $\tilde{\nu}$.*
2. *If $\nu \ll m$, then $\tilde{\nu} \ll \widetilde{m}$.*
3. *If ν is k-scalarly bounded by m, then $\tilde{\nu}$ is k-scalarly bounded by $\widetilde{m}$.*

Proof 1. For $\varphi \in S(G/H)$, by Proposition 3.1.1., Eq. (2) holds for simple functions, we have

$$\|I_{\tilde{\nu}}(L_a\varphi)\| = \|I_\nu((L_a\varphi)_q)\| = \|I_\nu(L_a(\varphi_q))\| = \|I_\nu(\varphi_q)\| = \|I_{\tilde{\nu}}(\varphi)\|.$$

2. For any $F \in \mathfrak{B}(G/H)$, $\widetilde{m}(F) = m(q^{-1}(F))$ and $\tilde{\nu}(F) = \nu(q^{-1}(F))$. If $\widetilde{m}(F) \to 0$, then also $m(q^{-1}(F)) \to 0$, and hence $\tilde{\nu}(F) = \nu(q^{-1}(F)) \to 0$ since $\nu \ll m$.
3. It follows immediately from the fact that

$$|\langle \tilde{\nu}, x^* \rangle|(E) \le |\langle \nu, x^* \rangle|(q^{-1}(E)) \le km(q^{-1}(E)) = k\widetilde{m}(E)$$

for any $E \in \mathfrak{B}(G/H)$.

□

Now we prove an existence of an extension of T_H to an operator from $L^p(G, \nu)$ into $L^p(G/H, \tilde{\nu})$ for each $1 \le p < \infty$. This is a generalization of Theorem 3.2 in [5].

Theorem 3.3 *Let* $1 \le p < \infty$ *and* $\mathcal{R} = \{R_h : h \in H\}$. *Suppose that* ν *is semivariation* $\mathcal{R}$*-invariant. Then the operator* $T_H : C(G) \to C(G/H)$ *satisfies*

$$\|T_H f\|_{L^p(G/H,\tilde{\nu})} \le \|f\|_{L^p(G,\nu)} \quad \textit{for all } f \in C(G),$$

hence it has a unique extension to a norm-decreasing operator $T_{H,\nu} : L^p(G,\nu) \to L^p(G/H,\tilde{\nu})$.

Proof Let $f \in C(G)$. By Proposition 3.1.2. and ν being semivariation $\mathcal{R}$-invariant,

$$\begin{aligned}
\|T_H f\|^p_{L^p(G/H,\tilde{\nu})} &\le \|(T_H f)_q\|^p_{L^p(G,\nu)} \\
&= \sup_{x^* \in B_{X^*}} \int_G |(T_H f)(tH)|^p \, d|\langle \nu, x^* \rangle|(t) \\
&\le \sup_{x^* \in B_{X^*}} \int_G \int_H |f(th)|^p \, dh \, d|\langle \nu, x^* \rangle|(t) \\
&= \sup_{x^* \in B_{X^*}} \int_H \int_G |f(th)|^p \, d|\langle \nu, x^* \rangle|(t) \, dh \\
&\le \int_H \left(\sup_{x^* \in B_{X^*}} \int_G |f(th)|^p \, d|\langle \nu, x^* \rangle|(t) \right) dh \\
&= \int_H \|R_h f\|^p_{L^p(G,\nu)} \, dh \\
&= \int_H \|f\|^p_{L^p(G,\nu)} \, dh \\
&= \|f\|^p_{L^p(G,\nu)}.
\end{aligned}$$

By the density of $C(G)$ in $L^p(G,\nu)$, the operator T_H can be extended uniquely to a bounded linear map from $L^p(G,\nu)$ to $L^p(G/H,\tilde{\nu})$. To verify that $T_{H,\nu}$ is norm-decreasing, let $f \in L^p(G,\nu)$ with $f_n \to f$ in $L^p(G,\nu)$ where $f_n \in C(G)$. Then

$$\|T_{H,\nu} f\|_{L^p(G/H,\tilde{\nu})} = \lim_{n\to\infty} \|T_H f_n\|_{L^p(G/H,\tilde{\nu})} \le \lim_{n\to\infty} \|f_n\|_{L^p(G,\nu)} = \|f\|_{L^p(G,\nu)}$$

as desired. □

Remark 1 If there is no ambiguity, we shall denote $T_{H,\nu}$ by T_H. Secondly, it is worth noting that even though the extensions of $T_H : C(G) \to C(G/H)$ to $L^p(G,\nu)$ and $L^q(G,\nu)$ might be different operators if $p \ne q$, they coincide on the intersection of the domains. Suppose that we denote the extension of T_H to $L^p(G,\nu)$ by $T_{H,p}$ for $1 \le p < \infty$. Consider $1 \le p < q < \infty$. Note that it follows from [15, Proposition 3.31(ii)] that for any vector measure μ on Ω, $L^q(\Omega,\mu) \subset L^p(\Omega,\mu)$ with $\|f\|_{L^p(\Omega,\mu)} \le K\|f\|_{L^q(\Omega,\mu)}$ for some constant $K > 0$. Now we show that the extensions $T_{H,p}$ and $T_{H,q}$ coincide on $L^q(G,\nu) \subset L^p(G,\nu)$. Let $f_n \to f$ in $L^q(G,\nu)$ where $f_n \in C(G)$. Then $T_{H,p} f_n \to T_{H,p} f$ in $L^p(G/H,\tilde{\nu})$ and also $T_{H,q} f_n \to T_{H,q} f$ in $L^p(G/H,\tilde{\nu})$. Since $T_{H,p}$ and $T_{H,q}$ agree on $C(G)$, we have that $T_{H,p} f = T_{H,q} f$ in $L^p(G/H,\tilde{\nu})$

which implies $T_{H,p}f = T_{H,q}f$ $\tilde{\nu}$-a.e. Thus there is no ambiguity to denote any extension $T_{H,p}$ for any $1 \le p < \infty$ by T_H.

Now we prove that the extension T_H is norm-decreasing in the sense of the norm in X.

Theorem 3.4 *Let ν be norm integral $\mathcal{R}$-invariant where $\mathcal{R} = \{R_h : h \in H\}$. Then*

$$\left\| \int_{G/H} T_H f \, d\tilde{\nu} \right\|_X \le \left\| \int_G f \, d\nu \right\|_X \quad \big(f \in L^1(G, \nu)\big).$$

Proof Let $f \in C(G)$. For $x^* \in B_{X^*}$, by Eq. (2)

$$\begin{aligned}\int_{G/H} T_H f \, d\langle \tilde{\nu}, x^* \rangle &= \int_G (T_H f)_q \, d\langle \nu, x^* \rangle \\ &= \int_G \int_H f(th) \, dh \, d\langle \nu, x^* \rangle(t) \\ &= \int_H \int_G (R_h f)(t) \, d\langle \nu, x^* \rangle(t) \, dh.\end{aligned}$$

Hence

$$\begin{aligned}\left\| \int_{G/H} T_H f \, d\tilde{\nu} \right\|_X &= \sup_{x^* \in B_{X^*}} \left| \int_{G/H} T_H f \, d\langle \tilde{\nu}, x^* \rangle \right| \\ &= \sup_{x^* \in B_{X^*}} \left| \int_H \int_G (R_h f)(t) \, d\langle \nu, x^* \rangle(t) \, dh \right| \\ &\le \int_H \left(\sup_{x^* \in B_{X^*}} \left| \int_G (R_h f)(t) \, d\langle \nu, x^* \rangle(t) \right| \right) dh \\ &= \int_H \left\| \int_G R_h f \, d\nu \right\|_X dh \\ &= \left\| \int_G f \, d\nu \right\|_X.\end{aligned}$$

For any $f \in L^1(G, \nu)$, let f_n be a sequence of continuous functions converging to f in $L^1(G, \nu)$. Then

$$\|I_{\tilde{\nu}}(T_H f)\|_X = \lim_{n\to\infty} \|I_{\tilde{\nu}}(T_H f_n)\|_X \le \lim_{n\to\infty} \|I_\nu(f_n)\|_X = \|I_\nu(f)\|_X$$

that is $\| \int_{G/H} T_H f \, d\tilde{\nu} \|_X \le \| \int_G f \, d\nu \|_X$ as desired. □

We have investigated the properties of the extension $T_H : L^p(G, \nu) \to L^p(G/H, \tilde{\nu})$ for a given vector measure $\nu \in \mathcal{M}(G, X)$. However, in general, to study Fourier analysis on homogeneous spaces, it is essential to consider the space $L^p(G/H, \mu)$ for a given vector measure μ on G/H instead of the space $L^p(G/H, \tilde{\nu})$.

To deal with this situation, we will define a corresponding measure $\breve{\mu}$ on G and study the extension $T_H : L^p(G, \breve{\mu}) \to L^p(G/H, \mu)$.

Let $\mu \in \mathcal{M}(G/H, X)$ and $T_\mu : C(G/H) \to X$ be the corresponding weakly compact operator given by

$$T_\mu(\varphi) = \int_{G/H} \varphi \, d\mu \quad (\varphi \in C(G/H)).$$

Observe that the operator $T_\mu \circ T_H : C(G) \to X$ is weakly compact since T_H is bounded and T_μ is weakly compact. Then there is a representing regular vector measure on G. Denote the representing vector measure by $\breve{\mu} \in \mathcal{M}(G, X)$ and $T_\mu \circ T_H$ by $T_{\breve{\mu}}$. Hence we immediately have that

$$\int_G f \, d\breve{\mu} = \int_{G/H} T_H f \, d\mu \quad (f \in C(G)). \tag{3}$$

Remark 2 Let $\Phi : \mathcal{M}(G, X) \to \mathcal{M}(G/H, X)$ be defined by $\Phi(\nu) = \tilde{\nu}$ for $\nu \in \mathcal{M}(G, X)$ and $\Psi : \mathcal{M}(G/H, X) \to \mathcal{M}(G, X)$ by $\Psi(\mu) = \breve{\mu}$ for $\mu \in \mathcal{M}(G/H, X)$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}(G/H, X) & \xrightarrow{\ \mathrm{Id}\ } & \mathcal{M}(G/H, X) \\ & \searrow^{\Psi} \qquad \nearrow_{\Phi} & \\ & \mathcal{M}(G, X) & \end{array}$$

In other words, $\Phi \circ \Psi = \mathrm{Id}_{\mathcal{M}(G/H,X)}$ or equivalently $\tilde{\breve{\mu}} = \mu$ for any $\mu \in \mathcal{M}(G/H, X)$. This can be proved by observing that

$$\int_{G/H} \varphi \, d\tilde{\breve{\mu}} = \int_G \varphi_q \, d\breve{\mu} = \int_{G/H} T_H(\varphi_q) \, d\mu = \int_{G/H} \varphi \, d\mu$$

for all $\varphi \in C(G/H)$. Note that the commutativity of the diagram also implies that Φ is surjective and Ψ is injective.

Proposition 3.5 *Let $\mathcal{R} = \{R_h : h \in H\}$ and $x^* \in B_{X^*}$. Then $\breve{\mu}$ and $|\langle \breve{\mu}, x^* \rangle|$ are $\mathcal{R}$-invariant.*

Proof To show that $\breve{\mu}$ is $\mathcal{R}$-invariant, let $h \in H$ and $f \in C(G)$. Observe that

$$T_H(R_h f)(tH) = \int_H (R_h f)(th') \, dh' = \int_H f(th') \, dh' = T_H(f)(tH).$$

Hence

$$T_{\breve{\mu}_{R_h}}(f) = T_{\breve{\mu}}(R_h f) = T_\mu(T_H(R_h f)) = T_\mu(T_H f) = T_{\breve{\mu}}(f).$$

Hence $\breve{\mu}_{R_h} = \breve{\mu}$, that is, $\breve{\mu}$ is R_h-invariant.

Now let $x^* \in B_{X^*}$, $E \in \mathfrak{B}(G)$ and $h \in H$. For any disjoint partition $\{E_n\}_{n=1}^k$ of E where $E_n \in \mathfrak{B}(G)$, note that $\{R_h E_n\}_{n=1}^k$ forms a disjoint partition of $R_h E$ and

$$\sum_{n=1}^{k} |\langle \breve{\mu}, x^* \rangle (E_n)| = \sum_{n=1}^{k} |\langle \breve{\mu}, x^* \rangle (R_h E_n)| \leq |\langle \breve{\mu}, x^* \rangle|(R_h E).$$

Hence $|\langle \breve{\mu}, x^* \rangle|(E) \leq |\langle \breve{\mu}, x^* \rangle|(R_h E)$. Taking E as $R_h E$ and h as h^{-1}, we also get $|\langle \breve{\mu}, x^* \rangle|(R_h E) \leq |\langle \breve{\mu}, x^* \rangle|(E)$. □

This proposition particularly implies that $\breve{\mu}$ is semivariation $\mathcal{R}$-invariant. Hence we can apply Theorem 3.3 to get that the operator T_H has an extension to a norm-decreasing operator from $L^p(G, \breve{\mu})$ to $L^p(G/H, \mu)$ for any $1 \leq p < \infty$. Moreover, Eq. (3) extends to $L^1(G, \breve{\mu})$

$$\int_G f \, d\breve{\mu} = \int_{G/H} T_H f \, d\mu \quad \big(f \in L^1(G, \breve{\mu})\big). \tag{4}$$

Indeed, if $f_n \to f$ in $L^1(G, \breve{\mu})$ where $f_n \in C(G)$, then $I_\mu(T_H f) = \lim_{n\to\infty} T_\mu(T_H(f_n)) = \lim_{n\to\infty} T_{\breve{\mu}}(f_n) = I_{\breve{\mu}}(f)$. Now we prove that the Eq. (4) is also true for the total variation of the associated complex measures.

Lemma 3.6 *For $x^* \in B_{X^*}$ and $f \in L^1(G, \breve{\mu})$,*

$$\int_G f \, d|\langle \breve{\mu}, x^* \rangle| = \int_{G/H} T_H f \, d|\langle \mu, x^* \rangle|.$$

In particular, $\|T_H|f|\|_{L^1(G/H,\mu)} = \|f\|_{L^1(G,\breve{\mu})}$ for any $f \in L^1(G, \breve{\mu})$.

Proof It suffices to prove that for any $f \in L^1(G, \breve{\mu})$ and $x^* \in B_{X^*}$

$$\int_G |f| \, d|\langle \breve{\mu}, x^* \rangle| = \int_{G/H} T_H |f| \, d|\langle \mu, x^* \rangle|.$$

We first claim that for each $E \in \mathfrak{B}(G)$, $T_H(\chi_E) \geq 0$ $|\langle \mu, x^* \rangle|$-a.e that is the set $F = \{tH \in G/H : T_H(\chi_E) < 0\}$ is $|\langle \mu, x^* \rangle|$-null. Let $f_n \to \chi_E$ in $L^1(G, \breve{\mu})$ where $f_n \in C(G)$ is positive (which exists by using Urysohn's lemma together with the regularity of $\breve{\mu}$). Since $T_H(f_n) \geq 0$ for all $n \in \mathbb{N}$,

$$\begin{aligned} \int_F |T_H(\chi_E)| \, d|\langle \mu, x^* \rangle| &\leq \int_F |T_H(\chi_E) - T_H(f_n)| \, d|\langle \mu, x^* \rangle| \\ &\leq \|T_H(\chi_E - f_n)\|_{L^1(G/H,\mu)} \end{aligned}$$

which implies that $|\langle \mu, x^* \rangle|(F) = 0$ as desired. Now fix $E \in \mathfrak{B}(G)$ and consider any disjoint partition $\{E_n\}_{n=1}^k$ of E where $E_n \in \mathfrak{B}(G)$. By Eq. (4) and the claim,

$$\begin{aligned}\sum_{n=1}^{k} |\langle \breve{\mu}, x^* \rangle(E_n)| &= \sum_{n=1}^{k} \left| \int_{G/H} T_H(\chi_{E_n})\, d\langle \mu, x^* \rangle \right| \\ &\leq \sum_{n=1}^{k} \int_{G/H} T_H(\chi_{E_n})\, d|\langle \mu, x^* \rangle| \\ &= \int_{G/H} T_H(\chi_E)\, d|\langle \mu, x^* \rangle|.\end{aligned}$$

Hence $\int_G \chi_E \, d|\langle \breve{\mu}, x^* \rangle| \leq \int_{G/H} T_H(\chi_E)\, d|\langle \mu, x^* \rangle|$. It follows immediately that for any $f \in S(G)$,

$$\int_G |f|\, d|\langle \breve{\mu}, x^* \rangle| \leq \int_{G/H} T_H|f|\, d|\langle \mu, x^* \rangle|$$

which can be extended to any $f \in L^1(G, \breve{\mu})$ by using the density of $S(G)$ in $L^1(G, \breve{\mu})$.

Conversely, by Propositions 3.1.2. and 3.5, for $f \in C(G)$

$$\begin{aligned}\int_{G/H} T_H|f|\, d|\langle \mu, x^* \rangle| &\leq \int_G (T_H|f|)_q \, d|\langle \breve{\mu}, x^* \rangle| \\ &= \int_G \int_H |f(th)|\, dh\, d|\langle \breve{\mu}, x^* \rangle|(t) \\ &= \int_H \int_G |f(th)|\, d|\langle \breve{\mu}, x^* \rangle|(t)\, dh \\ &= \int_H \int_G |f(t)|\, d|\langle \breve{\mu}, x^* \rangle|(t)\, dh \\ &= \int_G |f(t)|\, d|\langle \breve{\mu}, x^* \rangle|(t).\end{aligned}$$

Hence by the density of $C(G)$ in $L^1(G, \breve{\mu})$, for any $f \in L^1(G, \breve{\mu})$

$$\int_{G/H} T_H|f|\, d|\langle \mu, x^* \rangle| \leq \int_G |f|\, d|\langle \breve{\mu}, x^* \rangle|.$$

□

For any $\nu \in \mathcal{M}(G, X)$, we cannot find an example of an operator $T_{H,\nu}$ constructed in the manner of Theorem 3.3 which is not surjective. However, we know that if ν is in the form of $\breve{\mu}$, where $\mu \in \mathcal{M}(G/H, X)$, then the operator $T_{H,\breve{\mu}}$ is certainly surjective as shown in the following theorem.

Theorem 3.7 *Let $1 \leq p < \infty$. The extension $T_H : L^p(G, \breve{\mu}) \to L^p(G/H, \mu)$ satisfies the formula $T_H f(tH) = \int_H f(th)\,dh$ μ-a.e. for all $f \in L^p(G, \breve{\mu})$. Moreover, the extension $T_H : L^p(G, \breve{\mu}) \to L^p(G/H, \mu)$ is surjective.*

Proof Claim that for a lower semicontinuous function $\phi \geq 0$ and $x^* \in B_{X^*}$,

$$\int_G \phi\, d|\langle \breve{\mu}, x^* \rangle| = \int_{G/H} \int_H \phi(th)\, dh\, d|\langle \mu, x^* \rangle|(tH).$$

Let $\Phi = \{g \in C(G) : 0 \leq g \leq \phi\}$. By [10, Proposition 7.12] and Lemma 3.6,

$$\begin{aligned}
\int_G \phi\, d|\langle \breve{\mu}, x^* \rangle| &= \sup_{g \in \Phi} \int_G g\, d|\langle \breve{\mu}, x^* \rangle| \\
&= \sup_{g \in \Phi} \int_{G/H} T_H g\, d|\langle \mu, x^* \rangle| \\
&= \int_{G/H} \left(\sup_{g \in \Phi} T_H g \right) d|\langle \mu, x^* \rangle| \\
&= \int_{G/H} \left(\sup_{g \in \Phi} \int_H g(th)\, dh \right) d|\langle \mu, x^* \rangle|(tH) \\
&= \int_{G/H} \left(\sup_{\tilde{g} \in \Phi(tH)} \int_H \tilde{g}(h)\, dh \right) d|\langle \mu, x^* \rangle|(tH) \\
&= \int_{G/H} \int_H \phi(th)\, dh\, d|\langle \mu, x^* \rangle|(tH)
\end{aligned}$$

where $\Phi(tH) := \{\tilde{g} \in C(H) : 0 \leq \tilde{g}(h) \leq \phi(th) \text{ for } h \in H\}$. Hence for any measurable function F and any lower semicontinuous function $\phi \geq |F|$,

$$\begin{aligned}
\int_{G/H} \int_H |F(th)|\, dh\, d|\langle \mu, x^* \rangle|(tH) &\leq \int_{G/H} \int_H \phi(th)\, dh\, d|\langle \mu, x^* \rangle|(tH) \\
&= \int_G \phi\, d|\langle \breve{\mu}, x^* \rangle|.
\end{aligned}$$

Hence by [10, Proposition 7.14]

$$\int_{G/H} \int_H |F(th)|\, dh\, d|\langle \mu, x^* \rangle|(tH) \leq \int_G |F|\, d|\langle \breve{\mu}, x^* \rangle|. \tag{5}$$

Let $f \in L^p(G, \breve{\mu})$ and $f_n \to f$ in $L^p(G, \breve{\mu})$ where $f_n \in C(G)$. Define a function $\tilde{f} : G/H \to \mathbb{C}$ by $\tilde{f}(tH) = \int_H f(th)\, dh$ for $tH \in G/H$. By taking $F = |f - f_n|^p$

in (5), we have

$$\begin{aligned}\|\tilde{f}-T_H f_n\|^p_{L^p(G/H,\mu)} &= \sup_{x^*\in B_{X^*}}\int_{G/H}\left|\tilde{f}-T_H f_n\right|^p d|\langle\mu,x^*\rangle| \\ &\leq \sup_{x^*\in B_{X^*}}\int_{G/H}\int_H |f-f_n|^p(th)\,dh\,d|\langle\mu,x^*\rangle|(tH) \\ &\leq \sup_{x^*\in B_{X^*}}\int_G |f-f_n|^p\, d|\langle\breve{\mu},x^*\rangle| \\ &= \|f-f_n\|^p_{L^p(G,\breve{\mu})}\end{aligned}$$

which shows that $\tilde{f}$ is well-defined and $\tilde{f}=T_H f$ μ-a.e.

To show that T_H is surjective, we first claim that $\|\phi_q\|_{L^p(G,\breve{\mu})}=\|\phi\|_{L^p(G/H,\mu)}$ for $\phi\in L^p(G/H,\mu)$. Let $\phi_n\uparrow|\phi|$ pointwise where $\phi_n\in S(G/H)$. Then $(\phi_n)_q\uparrow|\phi|_q$ pointwise. Applying the monotone convergence theorem and Lemma 3.6 to each $x^*\in B_{X^*}$, we get

$$\begin{aligned}\int_G |\phi_q|^p\, d|\langle\breve{\mu},x^*\rangle| &= \lim_{n\to\infty}\int_G |(\phi_n)_q|^p\, d|\langle\breve{\mu},x^*\rangle| \\ &= \lim_{n\to\infty}\int_{G/H} |\phi_n|^p\, d|\langle\mu,x^*\rangle| \\ &= \int_G |\phi|^p\, d|\langle\mu,x^*\rangle|\end{aligned}$$

which proves the claim. Now let $\varphi\in L^p(G/H,\mu)$. If we can show that $\varphi_q\in L^p(G,\breve{\mu})$, then by the formula of T_H we have $T_H(\varphi_q)=\varphi$. Let $\varphi_n\to\varphi$ in $L^p(G/H,\mu)$ where $\varphi_n\in S(G/H)$. Then it follows by the claim that $\|\varphi_q-(\varphi_n)_q\|_{L^p(G,\breve{\mu})}=\|\varphi-\varphi_n\|_{L^p(G/H,\mu)}\to 0$. Hence $\varphi_q\in L^p(G,\breve{\mu})$ by the completeness of $L^p(G,\breve{\mu})$. □

Corollary 3.8 *1. Weil's formula holds for all $f\in L^1(G,\breve{\mu})$*

$$\int_{G/H}\int_H f(th)\,dh\,d\mu(tH)=\int_G f\,d\breve{\mu}.$$

Moreover, for all $x^*\in X^*$ and $f\in L^1(G,\breve{\mu})$

$$\int_{G/H}\int_H f(th)\,dh\,d|\langle\mu,x^*\rangle|(tH)=\int_G f\,d|\langle\breve{\mu},x^*\rangle|.$$

2. *For $1\leq p<\infty$, if $\varphi\in L^p(G/H,\mu)$, then $\varphi_q\in L^p(G,\breve{\mu})$ with $\|\varphi_q\|_{L^p(G,\breve{\mu})}=\|\varphi\|_{L^p(G/H,\mu)}$.*

Proof The first two equations follow by applying the formula of T_H to Eq. (4) and Lemma 3.6 while the last assertion is in the proof of the theorem. □

Corollary 3.9 *For $1 \le p < \infty$, $C(G/H)$ is dense in $L^p(G/H, \mu)$.*

Proof Let $\varphi \in L^p(G/H, \mu)$. Then $\varphi_q \in L^p(G, \breve{\mu})$. By the density of $C(G)$ in $L^p(G, \breve{\mu})$, there is a sequence $f_n \to \varphi_q$ in $L^p(G, \breve{\mu})$ with $f_n \in C(G)$. Hence $T_H f_n \to T_H(\varphi_q) = \varphi$ in $L^p(G/H, \mu)$. □

It is straightforward to see that $\breve{\widetilde{m}} = m$. Hence $T_{H,m} = T_{H,\breve{\widetilde{m}}}$ is the same operator T_H given by Farashahi in [5]. Now we provide a relation between the extensions $T_{H,m}$ and $T_{H,\breve{\mu}}$.

Proposition 3.10 *If $\mu \ll \widetilde{m}$, then $T_{H,m} f = T_{H,\breve{\mu}} f$ μ-a.e. for all $f \in L^1(G, m) \cap L^1(G, \breve{\mu})$, and hence $\breve{\mu} \ll m$.*

Proof Let $f \in L^1(G, m) \cap L^1(G, \breve{\mu})$. Then $T_{H,m} f(tH) = \int_H f(th)\, dh$ $\widetilde{m}$-a.e.; in particular, $T_{H,m} f(tH) = \int_H f(th)\, dh$ μ-a.e. since $\mu \ll \widetilde{m}$. By Theorem 3.7, we also have $T_{H,\breve{\mu}} f(tH) = \int_H f(th)\, dh$ μ-a.e., so $T_{H,m} f = T_{H,\breve{\mu}} f$ μ-a.e.

Given $E \in \mathfrak{B}(G)$ with $m(E) = 0$. Then $T_{H,m}\chi_E = 0$ $\widetilde{m}$-a.e. and hence $T_{H,\breve{\mu}}\chi_E = 0$ μ-a.e. By Lemma 3.6, we get $\|\breve{\mu}\|(E) = 0$. We conclude that $\breve{\mu} \ll m$. □

Example 2 Let $1 \le p < \infty$ and $S : L^p(G/H, \widetilde{m}) \to X$ be any bounded linear map. Define a vector measure $\mu : \mathfrak{B}(G/H) \to X$ corresponding to S by $\mu(E) = S(\chi_E)$ for $E \in \mathfrak{B}(G/H)$. Then the vector measure $\breve{\mu}$ is given by $\breve{\mu}(F) = \int_{G/H} T_{H,\breve{\mu}}\chi_F\, d\mu$ for $F \in \mathcal{B}(G)$. Note that for $\varphi \in L^p(G/H, \widetilde{m})$, φ is μ-integrable and $\int_{G/H} \varphi\, d\mu = S(\varphi)$, see [15, Proposition 4.4]. Hence it follows from Proposition 3.10 that $\breve{\mu}(F) = \int_{G/H} T_{H,m}\chi_F\, d\mu = S(T_{H,m}\chi_F)$.

1. Let $X = \mathbb{C}$ and $S : L^1(G/H, \widetilde{m}) \to \mathbb{C}$ be given by $S(\varphi) = \int_{G/H} \varphi\, d\widetilde{m}$ for any $\varphi \in L^1(G/H, \widetilde{m})$. In this case, $\mu = \widetilde{m}$. Moreover, $\breve{\mu} = m$ since $\breve{\mu}(F) = \int_{G/H} T_{H,m}\chi_F\, d\widetilde{m} = \int_G \chi_F\, dm = m(F)$ for all $F \in \mathfrak{B}(G)$.
2. If $X = L^1(G/H, \widetilde{m})$ and $S = Id_{L^1(G/H,\widetilde{m})}$, then $\breve{\mu}(F) = T_{H,m}\chi_F$ for $F \in \mathfrak{B}(G)$.
3. If we let $1 \le p \le 2$ and define $S : L^p(G/H, \widetilde{m}) \to \ell^{p'}(\widehat{G/H}; \mathcal{B}(\mathcal{H}_\pi))$ by $S(\varphi) = \mathcal{F}_{G/H}(\varphi)$, then $\breve{\mu}(F)(\pi) = \mathcal{F}_{G/H}(T_{H,m}\chi_F)(\pi) = T_H^\pi \widehat{\chi_F}(\pi)$ for $F \in \mathfrak{B}(G)$ and $[\pi] \in \widehat{G/H}$, by [5, Proposition 5.3].

Finally, we give relations between μ and $\breve{\mu}$ in terms of invariant properties.

Definition 6 Let μ be a vector measure on G/H. For each $a \in G$, μ is said to be L_a-**invariant** if $\mu(aE) = \mu(E)$ for all $E \in \mathfrak{B}(G/H)$. We say that μ is **left invariant** if it is L_a-invariant for all $a \in G$.

Definition 7 Let $\tau : G/H \to G/H$ be a homeomorphism. For any vector measure μ on G/H, we say that μ is **semivariation τ-invariant** if

$$\|\varphi_\tau\|_{L^1(G/H,\mu)} = \|\varphi\|_{L^1(G/H,\mu)} \quad \text{for all } \varphi \in S(G/H).$$

Given a collection $\mathcal{T}$ of homeomorphisms on G/H, μ is said to be **semivariation $\mathcal{T}$-invariant** if it is semivariation τ-invariant for all $\tau \in \mathcal{T}$. In particular, if $\mathcal{T} = \{L_a : a \in G\}$, we say that μ is **semivariation left invariant**.

Proposition 3.11 *Let $a \in G$.*

1. *μ is L_a-invariant if and only if $\breve{\mu}$ is L_a-invariant.*
2. *μ is norm integral L_a-invariant if and only if $\breve{\mu}$ is norm integral L_a-invariant.*
3. *μ is semivariation L_a-invariant if and only if $\breve{\mu}$ is semivariation L_a-invariant.*

Proof 1. Suppose that μ is L_a-invariant. Then by the Weil formula (4), for any $f \in C(G)$,

$$\int_G L_a f \, d\breve{\mu} = \int_{G/H} T_H(L_a f) \, d\mu = \int_{G/H} L_a(T_H f) \, d\mu$$
$$= \int_{G/H} T_H f \, d\mu = \int_G f \, d\breve{\mu}.$$

Hence $\breve{\mu}$ is L_a-invariant. Conversely, suppose that $\breve{\mu}$ is L_a-invariant. Then for any $\varphi \in S(G/H)$

$$\int_{G/H} L_a \varphi \, d\mu = \int_G L_a \varphi_q \, d\breve{\mu} = \int_G \varphi_q \, d\breve{\mu} = \int_{G/H} \varphi \, d\mu$$

Hence μ is L_a-invariant.

2. Suppose that μ is norm integral L_a-invariant. Then by [2, Theorem 3.3], we have $\|I_\mu(L_a\varphi)\| = \|I_\mu(\varphi)\|$ for all $\varphi \in L^1(G/H, \mu)$. Hence by the Weil formula (4)

$$\|I_{\breve{\mu}}(L_a f)\| = \|I_\mu(T_H(L_a f))\| = \|I_\mu(L_a T_H f)\| = \|I_\mu(T_H f)\| = \|I_{\breve{\mu}} f\|$$

for any $f \in S(G)$. Hence $\breve{\mu}$ is norm integral left invariant. The converse is proved in Proposition 3.2.

3. Suppose that μ is semivariation L_a-invariant. It is routine to check that $\|L_a\varphi\|_{L^1(G/H,\mu)} = \|\varphi\|_{L^1(G/H,\mu)}$ for all $\varphi \in L^1(G/H, \mu)$. So

$$\|L_a f\|_{L^1(G,\breve{\mu})} = \|T_H|L_a f|\|_{L^1(G/H,\mu)} = \|T_H|f|\|_{L^1(G/H,\mu)} = \|f\|_{L^1(G,\breve{\mu})}$$

for any $f \in S(G)$. Conversely, if $\breve{\mu}$ is semivariation L_a-invariant then

$$\|L_a\varphi\|_{L^1(G/H,\mu)} = \|L_a\varphi_q\|_{L^1(G,\breve{\mu})} = \|\varphi_q\|_{L^1(G,\breve{\mu})} = \|\varphi\|_{L^1(G/H,\mu)}$$

for any $\varphi \in S(G/H)$.

□

4 Invariant Measures

In this section, we provide properties of invariant measures on G and their analogies on G/H. The following proposition generalizes Proposition 5.2 in [1].

Proposition 4.1 *Let $\nu \in \mathcal{M}(G, X)$. The following are equivalent:*

1. *ν is left (or right) invariant*
2. *$\langle \nu, x^* \rangle$ is left (or right) invariant for all $x^* \in X^*$*
3. *$\nu = \nu(G)m$.*

Proof We only show that 2 implies 3; the other directions are trivial. Assume that $\langle \nu, x^* \rangle$ is left invariant for all $x^* \in X^*$. Then the real part $\langle \nu, x^* \rangle_r$ is left invariant. Let $G = P \cup N$ be a Hahn decomposition for $\langle \nu, x^* \rangle_r$ where P is positive and N is negative. Note that $G = aP \cup aN$ is also a Hahn decomposition for $\langle \nu, x^* \rangle_r$ for any $a \in G$. Hence $\langle \nu, x^* \rangle_r^+(aE) = \langle \nu, x^* \rangle_r(aE \cap aP) = \langle \nu, x^* \rangle_r(E \cap P) = \langle \nu, x^* \rangle_r^+(E)$ for any $a \in G$ and $E \in \mathfrak{B}(G)$. This shows that $\langle \nu, x^* \rangle_r^+$ is left invariant. By the uniqueness of the left Haar measure, $\langle \nu, x^* \rangle_r^+ = \alpha_r^+(x^*)m$ for some $\alpha_r^+(x^*) \geq 0$. Applying the same argument to all parts of $\langle \nu, x^* \rangle$, we obtain that $\langle \nu, x^* \rangle = \alpha(x^*)m$ for some $\alpha(x^*) \in \mathbb{C}$. Hence $\langle \nu(E), x^* \rangle = \alpha(x^*)m(E) = \langle \nu(G), x^* \rangle m(E) = \langle \nu(G)m(E), x^* \rangle$ for any $E \in \mathfrak{B}(G)$. Since this equation holds for all $x^* \in X^*$, we have that $\nu = \nu(G)m$. A similar argument can be applied to the case of right invariance. □

Proposition 4.2 *Let $\mu \in \mathcal{M}(G/H, X)$. The following are equivalent:*

1. *μ is left invariant*
2. *$\langle \mu, x^* \rangle$ is left invariant for all $x^* \in X^*$*
3. *$\mu = \mu(G/H)\widetilde{m}$.*

Proof The first two assertions follow from the fact that $\langle \breve{\mu}, x^* \rangle = \langle \mu, x^* \rangle^{\smile}$ for all $x^* \in X^*$. Next, assume that μ is left invariant. Then $\breve{\mu}$ is left invariant. By Proposition 4.1, $\breve{\mu} = \breve{\mu}(G)m$. Since μ and $\widetilde{m}$ are the pushforward measures of $\breve{\mu}$ and m, we have $\mu = \breve{\mu}(G)\widetilde{m} = \mu(G/H)\widetilde{m}$. This finishes the proof. □

The following proposition improves Lemma 3.4 in [3].

Proposition 4.3 *Let ν be a vector measure on G. The following are equivalent.*

1. *ν is norm integral left (or right) invariant.*
2. *For each $x^* \in X^*$ and $a \in G$, there exists $x_a^* \in X^*$ such that $\|x_a^*\| \leq \|x^*\|$ and $\langle \nu, x^* \rangle(aE) = \langle \nu, x_a^* \rangle(E)$ (or $\langle \nu, x^* \rangle(Ea) = \langle \nu, x_a^* \rangle(E)$) for all $E \in \mathcal{B}(G)$.*

Moreover, $x_a^* \in X^*$ is unique in the sense that if there is another such functional then they must agree on $I_\nu(S(G))$.

Proof We shall prove only for the case of norm integral left invariance as the other case is similar. The proof of 1. implies 2. follows by the same argument of [3, Lemma 3.4]. For the converse, let $f \in S(G)$ and $a \in G$. Then

$$\|I_\nu(L_a f)\| = \sup_{x^* \in B_{X^*}} |x^* I_\nu(L_a f)| = \sup_{x^* \in B_{X^*}} |x_a^* I_\nu(f)| \leq \|I_\nu(f)\|.$$

This also implies $\|I_\nu(f)\| = \|I_\nu(L_{a^{-1}}(L_a f))\| \leq \|I_\nu(L_a f)\|$.

For the uniqueness, suppose there is another functional $y^* \in X^*$ such that $\|y^*\| \leq \|x^*\|$ and $\langle \nu, x^* \rangle(aE) = \langle \nu, y^* \rangle(E)$ for all $E \in \mathcal{B}(G)$. Then $x_a^*(\nu(E)) = \langle \nu, x_a^* \rangle(E) = \langle \nu, y^* \rangle(E) = y^*(\nu(E))$ for all $E \in \mathcal{B}(G)$. By the linearity of x_a^* and y^*, we have that $x_a^* = y^*$ on $I_\nu(S(G))$. □

Proposition 4.4 *Let μ be a vector measure on G/H. The following are equivalent.*

1. *μ is norm integral left invariant.*
2. *For each $x^* \in X^*$ and $a \in G$, there exists $x_a^* \in X^*$ such that $\|x_a^*\| \leq \|x^*\|$ and $\langle \mu, x^* \rangle(aE) = \langle \mu, x_a^* \rangle(E)$ for all $E \in \mathcal{B}(G/H)$.*

Moreover, $x_a^* \in X^*$ is unique in the sense that if there is another such functional then they must agree on $I_\mu(S(G/H))$.

Proof It can be proven by the same argument as in Proposition 4.3. However, if μ is also assumed to be regular, we can employ Proposition 4.3 with $\breve{\mu}$ and obtain the result immediately. □

The following result can be proved by the same argument as in [13, Theorem 5.6] and [1, Theorem 5.10]. Hence the proof is omitted.

Proposition 4.5 *Let $1 \leq p < \infty$. Suppose that $\nu \in \mathcal{M}(G, X)$ is semivariation left (or right) invariant with $\nu(G) \neq 0$. Then $L^p(G, \nu) \subset L^p(G, m)$ with $\|f\|_{L^p(G,m)} \leq \|\nu(G)\|^{-1/p}\|f\|_{L^p(G,\nu)}$ for $f \in L^p(G, \nu)$.*

Proposition 4.6 *Let $1 \leq p < \infty$. Suppose that $\mu \in \mathcal{M}(G/H, X)$ is semivariation left invariant with $\mu(G/H) \neq 0$. Then $L^p(G/H, \mu) \subset L^p(G/H, \widetilde{m})$ with $\|\varphi\|_{L^p(G/H,\widetilde{m})} \leq \|\mu(G/H)\|^{-1/p}\|\varphi\|_{L^p(G/H,\mu)}$ for $\varphi \in L^p(G/H, \mu)$.*

Proof Since $\breve{\mu}$ is semivariation left invariant, by Proposition 4.5, $L^p(G, \breve{\mu}) \subset L^p(G, m)$ with $\|f\|^p_{L^p(G,m)} \leq \|\breve{\mu}(G)\|^{-1}\|f\|^p_{L^p(G,\breve{\mu})}$ for $f \in L^p(G, \breve{\mu})$. Hence

$$\begin{aligned}\|\varphi\|^p_{L^p(G/H,\widetilde{m})} = \|\varphi_q\|^p_{L^p(G,m)} &\leq \|\breve{\mu}(G)\|^{-1}\|\varphi_q\|^p_{L^p(G,\breve{\mu})} \\ &= \|\mu(G/H)\|^{-1}\|\varphi\|^p_{L^p(G/H,\mu)}\end{aligned}$$

for $\varphi \in L^p(G/H, \mu)$. □

5 Fourier Transforms

In this section, we define a Fourier transform of functions in $L^1(G, \nu)$ and $L^1(G/H, \mu)$. Our definition is motivated by Definition 4.1 in [13]; however, X is not considered as an operator space. Let ν be a vector measure on G.

Definition 8 For $f \in L^1(G, \nu)$ and $[\pi] \in \widehat{G}$, we define the **Fourier transform** of f as

$$\widehat{f}^{\nu}(\pi) = \int_G f(t)\pi(t)^* \, d\nu \in \mathcal{B}(\mathcal{H}_\pi)\check{\otimes}X.$$

To see that the definition is well-defined, we have to show that the function $g : G \to \mathcal{B}(\mathcal{H}_\pi)$ given by $g(t) = f(t)\pi(t)^*$ is ν-measurable and $\check{\otimes}$-integrable. Let $x^* \in X^*$ be a Rybakov functional. Clearly, g is weakly $|\langle \nu, x^* \rangle|$-measurable since

$y^*g(\cdot) = f(\cdot)y^*\pi(\cdot)^*$ is a product of $|\langle \nu, x^*\rangle|$-measurable functions for all $y^* \in \mathcal{B}(\mathcal{H}_\pi)^*$. Moreover, $\mathcal{B}(\mathcal{H}_\pi)$ is separable. Thus, by Pettis's measurability theorem, g is $|\langle \nu, x^*\rangle|$-measurable and hence is ν-measurable. Since the function $\|g\| = |f|$ is ν-integrable, g is $\check{\otimes}$-integrable. This immediately implies the following proposition.

Proposition 5.1 *Define the operator* $\mathcal{F}_G^\nu : L^1(G, \nu) \to \ell^\infty(\widehat{G}; \mathcal{B}(\mathcal{H}_\pi)\check{\otimes}X)$ *by* $\mathcal{F}_G^\nu(f)(\pi) = \widehat{f}^\nu(\pi)$ *for* $f \in L^1(G, \nu)$ *and* $[\pi] \in \widehat{G}$. *Then the Fourier transform operator* $\mathcal{F}_G^\nu$ *is bounded with* $\|\widehat{f}^\nu(\pi)\|_\vee \le \|f\|_{L^1(G,\nu)}$.

Remark 3 If we take ν to be $\langle \nu, x^*\rangle$, then

$$\widehat{f}^{\langle \nu, x^*\rangle}(\pi) = \int_G f(t)\pi(t)^*\, d\langle \nu, x^*\rangle = (Id_{\mathcal{B}(\mathcal{H}_\pi)} \otimes x^*)\big(\widehat{f}^\nu(\pi)\big).$$

This can be considered as a generalization of Definition 4.6 in [13].

Remark 4 If G is abelian, then $\mathcal{B}(\mathcal{H}_\pi) \cong \mathbb{C}$ for any $[\pi] \in \widehat{G}$. In this case, note that $\mathbb{C}\check{\otimes}X \cong X$ isometrically via the map $\alpha \otimes x \mapsto \alpha x$ and $\mathrm{N}(\cdot) = \|\cdot\|_{L^1(G,\nu)}$. Hence our definition generalizes Definition 2.1 in [2].

Definition 9 We say that the Fourier transform $\mathcal{F}_G^\nu$ satisfies the **Riemann–Lebesgue lemma** if $\mathcal{F}_G^\nu(f) \in c_0(\widehat{G}; \mathcal{B}(\mathcal{H}_\pi)\check{\otimes}X)$ for all $f \in L^1(G, \nu)$.

The Fourier transform $\mathcal{F}_G^\nu$ need not satisfy the Riemann–Lebesgue lemma even if G is abelian as shown in [2, Example 2.4]. Now we give a necessary condition for $\mathcal{F}_G^\nu$ to satisfy the Riemann–Lebesgue lemma and also a stronger condition for the sufficiency.

Theorem 5.2 *Let* $\mathcal{M} = \{\pi_{ij} : [\pi] \in \widehat{G}, 1 \le i, j \le d_\pi\}$.

1. *If* $\mathcal{F}_G^\nu$ *satisfies the Riemann–Lebesgue lemma, then the set* $\{\psi \in \mathcal{M} : \|\int_G \phi(t)\overline{\psi(t)}\, d\nu\|_X > \varepsilon\}$ *is finite for any* $\varepsilon > 0$ *and* $\phi \in \mathcal{M}$.
2. *Moreover, if* ν *is regular and* $\{\psi = \pi_{ij} \in \mathcal{M} : d_\pi^2 \|\int_G \phi(t)\overline{\psi(t)}\, d\nu\|_X > \varepsilon\}$ *is finite for any* $\varepsilon > 0$ *and* $\phi \in \mathcal{M}$, *then* $\mathcal{F}_G^\nu$ *satisfies the Riemann–Lebesgue lemma.*

Proof Observe that for $F : G \to Y = \mathcal{B}(\mathcal{H}_\pi)$

$$\left\|\int_G F\, d\nu\right\|_\vee = \sup_{y^* \in B_{Y^*}} \left\|\int_G y^* F\, d\nu\right\|_X.$$

Hence if we write $y^* \in B_{Y^*}$ as $y^* = \sum_{1 \le i,j \le d_\pi} \alpha_{ij} e_{ij}^*$, we have

$$\max_{i,j} \left\|\int_G e_{ij}^* F\, d\nu\right\|_X \le \left\|\int_G F\, d\nu\right\|_\vee \le d_\pi^2 \max_{i,j} \left\|\int_G e_{ij}^* F\, d\nu\right\|_X.$$

1. Let $\varepsilon > 0$ and $\phi \in \mathcal{M} \subset L^1(G, \nu)$. Suppose that $\mathcal{F}_G^\nu$ satisfies the Riemann–Lebesgue lemma. If $\pi_{ij} \in \mathcal{M}$ satisfies $\|\int_G \phi(t)\overline{\pi_{ij}(t)}\, d\nu\|_X > \varepsilon$, by the observation above with $F(t) = \phi(t)\pi(t)^*$, we have $\|\widehat{\phi}^\nu(\pi)\|_\vee > \varepsilon$. Hence if $\{\psi \in \mathcal{M} : \|\int_G \phi(t)\overline{\psi(t)}\, d\nu\|_X > \varepsilon\}$ is infinite, then so does the set $\{[\pi] \in \widehat{G} : \|\widehat{\phi}^\nu(\pi)\|_\vee > \varepsilon\}$, which is a contradiction.

2. Let $\phi \in \mathcal{M}$ and $\varepsilon > 0$. Suppose that $\{\psi \in \mathcal{M} : d_\pi^2 \| \int_G \phi(t)\overline{\psi(t)}\, d\nu\|_X > \varepsilon\}$ is finite. If $[\pi] \in \widehat{G}$ satisfies $\|\widehat{\phi}^\nu(\pi)\|_\vee > \varepsilon$, then $d_\pi^2 \| \int_G \phi(t)\overline{\pi_{ji}(t)}\, d\nu\|_X > \varepsilon$ for some i, j. Hence we must have that $\widehat{\phi}^\nu \in c_0(\widehat{G}; \mathcal{B}(\mathcal{H}_\pi)\check{\otimes}X)$. Note that the linear span of $\mathcal{M}$ is $\mathrm{Trig}(G)$ and $\mathrm{Trig}(G)$ is dense in $L^1(G, \nu)$. By the continuity, $\mathcal{F}_G^\nu$ satisfies the Riemann–Lebesgue lemma.

□

Remark 5 If G is abelian and ν is regular, then $\mathcal{F}_G^\nu$ satisfies the Riemann–Lebesgue lemma if and only if the set $\{\psi \in \mathcal{M} : \| \int_G \phi(t)\overline{\psi(t)}\, d\nu\|_X > \varepsilon\}$ is finite for any $\varepsilon > 0$ and $\phi \in \mathcal{M}$.

We now prove the uniqueness theorem for the Fourier transform $\mathcal{F}_G^\nu$.

Theorem 5.3 *Let $\nu \in \mathcal{M}(G, X)$ and $f \in L^1(G, \nu)$. If $\widehat{f}^\nu(\pi) = 0$ for all $[\pi] \in \widehat{G}$, then $f = 0$ ν-a.e.*

Proof Suppose that $\widehat{f}^\nu(\pi) = 0$ for all $[\pi] \in \widehat{G}$. Fix a Rybakov functional $x^* \in X^*$ and write $d\langle \nu, x^*\rangle = g\, d|\langle \nu, x^*\rangle|$ where $g \in L^1(G, |\langle \nu, x^*\rangle|)$. Then $\int_G f(t)y^*\pi(t)^*\, d\langle \nu, x^*\rangle = 0$ for any $y^* \in \mathcal{B}(\mathcal{H}_\pi)^*$ and $[\pi] \in \widehat{G}$. In particular, $\int_G \overline{\pi_{ij}(t)}(fg)(t)\, d|\langle \nu, x^*\rangle| = 0$ for any $[\pi] \in \widehat{G}$ and $1 \le i, j \le d_\pi$. Since $\overline{\pi_{ij}}$ is a matrix element of the contragradient representation of π, $\int_G \phi(fg)\, d|\langle \nu, x^*\rangle| = 0$ for any $\phi \in \mathrm{Trig}(G)$. By the density of $\mathrm{Trig}(G)$ in $C(G)$ in the uniform norm, $fg\, d|\langle \nu, x^*\rangle| = 0$ as a measure. Hence $fg = 0$ $|\langle \nu, x^*\rangle|$-a.e. However $|g| = 1$ $|\langle \nu, x^*\rangle|$-a.e. Then it must be the case that $f = 0$ $|\langle \nu, x^*\rangle|$-a.e. Therefore $f = 0$ ν-a.e. since $\nu \ll |\langle \nu, x^*\rangle|$. □

Now we give a definition of a Fourier transform of functions on G/H with a vector measure. This definition is motivated by [5]. Let μ be a vector measure on G/H.

Definition 10 For $\varphi \in L^1(G/H, \mu)$ and $[\pi] \in \widehat{G/H}$, we define the **Fourier transform** of φ at $[\pi]$ as

$$\widehat{\varphi}^\mu(\pi) = \int_{G/H} \varphi(tH)\Gamma_\pi(tH)^*\, d\mu(tH) \in \mathcal{B}(\mathcal{H}_\pi)\check{\otimes}X,$$

where $\Gamma_\pi(tH) = \pi(t)T_H^\pi$.

Let $g : G/H \to \mathcal{B}(\mathcal{H}_\pi)$ be defined by $g(tH) = \varphi(tH)\Gamma_\pi(tH)^*$ for $tH \in G/H$. Then the μ-measurability of g can be verified similarly to case of compact groups. Moreover, $\|\Gamma_\pi(tH)\|^2 = \|\Gamma_\pi(tH)^*\Gamma_\pi(tH)\| = \|(T_H^\pi)^*T_H^\pi\| = \|T_H^\pi\|^2 = 1$, so $\|g\|$ is μ-integrable. Hence the definition is well-defined.

Proposition 5.4 *Define the operator $\mathcal{F}_{G/H}^\mu : L^1(G/H, \mu) \to \ell^\infty(\widehat{G/H}; \mathcal{B}(\mathcal{H}_\pi)\check{\otimes}X)$ by $\mathcal{F}_{G/H}^\mu(\varphi)(\pi) = \widehat{\varphi}^\nu(\pi)$ for $\varphi \in L^1(G/H, \mu)$ and $[\pi] \in \widehat{G/H}$. Then the Fourier transform operator $\mathcal{F}_{G/H}^\mu$ is bounded with $\|\mathcal{F}_{G/H}^\mu(\varphi)(\pi)\|_\vee \le \|\varphi\|_{L^1(G/H,\mu)}$.*

Proposition 5.5 *Let $\mu \in \mathcal{M}(G/H, X)$, $\varphi \in L^1(G/H, \mu)$. Then $\widehat{\varphi}^{\mu}(\pi) = \widehat{\varphi_q}^{\check{\mu}}(\pi)$ for each $[\pi] \in \widehat{G/H}$.*

Proof Recall that $T_H^{\pi} = \int_H \pi(h)\, dh$ is a bounded linear operator on $\mathcal{H}_\pi$ defined in the weak sense that is $\langle T_H^{\pi} u, v\rangle = \int_H \langle \pi(h)u, v\rangle\, dh$ for $u, v \in \mathcal{H}_\pi$. For $y^* \in \mathcal{B}(\mathcal{H}_\pi)^*$, write $y^* = \sum_{i,j} \alpha_{ij} e_{ij}^*$. Since $\int_H e_{ij}^* \pi(th)^*\, dh = e_{ij}^* \int_H \pi(th)^*\, dh = e_{ij}^*(T_H^{\pi}\pi(t)^*)$ for any i, j, we have

$$\begin{aligned} T_H(y^*\pi(t)^*) &= \int_H y^*\pi(th)^*\, dh \\ &= \sum_{i,j} \alpha_{ij} \int_H e_{ij}^*\pi(th)^*\, dh \\ &= \sum_{i,j} \alpha_{ij} e_{ij}^*(T_H^{\pi}\pi(t)^*) \\ &= y^*(T_H^{\pi}\pi(t)^*) \end{aligned}$$

for any $t \in G$. Hence $T_H(y^*\pi(\cdot)^*) = y^*(T_H^{\pi}\pi(\cdot)^*)$. Consider for $x^* \in X^*$ and $y^* \in \mathcal{B}(\mathcal{H}_\pi)^*$,

$$\begin{aligned} (y^* \otimes x^*)(\widehat{\varphi}^{\mu}(\pi)) &= \int_{G/H} \varphi(tH) y^*(T_H^{\pi}\pi(t)^*)\, d\langle \mu, x^*\rangle(tH) \\ &= \int_{G/H} T_H(\varphi_q(\cdot) y^*\pi(\cdot)^*)\, d\langle \mu, x^*\rangle(tH) \\ &= \int_G \varphi_q(t) y^*\pi(t)^*\, d\langle \check{\mu}, x^*\rangle(t) \\ &= (y^* \otimes x^*)(\widehat{\varphi_q}^{\check{\mu}}(\pi)). \end{aligned}$$

Hence $\widehat{\varphi}^{\mu}(\pi) = \widehat{\varphi_q}^{\check{\mu}}(\pi)$. □

Definition 11 We say that the Fourier transform $\mathcal{F}_{G/H}^{\mu}$ satisfies the **Riemann–Lebesgue lemma** if $\mathcal{F}_{G/H}^{\mu}(\varphi) \in c_0(\widehat{G/H}; \mathcal{B}(\mathcal{H}_\pi)\check{\otimes} X)$ for all $\varphi \in L^1(G/H, \mu)$.

Corollary 5.6 *If $\mathcal{F}_G^{\check{\mu}}$ satisfies the Riemann–Lebesgue lemma, then so does $\mathcal{F}_{G/H}^{\mu}$.*

The Fourier transform $\mathcal{F}_{G/H}^{\mu}$ also satisfies the uniqueness theorem.

Theorem 5.7 *Let $\mu \in \mathcal{M}(G/H, X)$ and $\varphi \in L^1(G/H, \mu)$. If $\widehat{\varphi}^{\mu}(\pi) = 0$ for all $[\pi] \in \widehat{G/H}$, then $\varphi = 0$ μ-a.e.*

Proof Suppose that $\widehat{\varphi}^{\mu}(\pi) = 0$ for all $[\pi] \in \widehat{G/H}$. Then $\widehat{\varphi_q}^{\check{\mu}}(\pi) = 0$ for all $[\pi] \in \widehat{G/H}$. Moreover, if $[\pi] \in \widehat{G}$ but $[\pi] \notin \widehat{G/H}$, then $\widehat{\varphi_q}^{\check{\mu}}(\pi) = 0$. Indeed, for any

$x^* \in X^*$ and $y^* \in \mathcal{B}(\mathcal{H}_\pi)^*$,

$$(y^* \otimes x^*)(\widehat{\varphi_q}^{\breve{\mu}}(\pi)) = \int_G \varphi_q(t) y^* \pi(t)^* \, d\langle \breve{\mu}, x^* \rangle (t)$$
$$= \int_{G/H} \varphi(tH) y^*(T_H^\pi \pi(t)^*) \, d\langle \mu, x^* \rangle (tH) = 0$$

since $T_H(y^*\pi(\cdot)^*) = y^*(T_H^\pi \pi(\cdot)^*) = 0$. Then one can apply Theorem 5.3 and obtains that $\varphi_q = 0$ $\breve{\mu}$-a.e. Hence $\varphi = T_H(\varphi_q) = 0$ μ-a.e. □

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