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Spherical Analysis Attached to Some *m*-Step Nilpotent Lie Group

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Abstract
We introduce a family of generalized Gelfand pairs (K_m, N_m) where N_m is an $m+2$-step nilpotent Lie group and K_m is isomorphic to the 3-dimensional Heisenberg group. We develop the associated spherical analysis computing the set of the spherical distributions and we obtain some results on the algebra of K_m-invariant and left invariant differential operators on N_m.

Keywords Generalized Gelfand pairs · Lie algebras · Spherical distributions

Mathematics Subject Classification Primary 43A80 · Secondary 22E27

1 Introduction

Let G be a unimodular Lie group and K a compact subgroup of G. We denote by $\widehat{G}$ the set of equivalent classes of irreducible unitary representations of G. We recall that for a wide class of Lie groups which includes nilpotent and semisimple Lie groups, any unitary representation π of G on a separable Hilbert space $\mathcal{H}$ decomposes in a

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unique way into a direct integral of irreducible unitary representations

$$\pi = \int_{\widehat{G}} m_\pi(\tau) d\mu(\tau),$$

where μ is a Borel measure on $\widehat{G}$ and $m_\pi : \widehat{G} \to \mathbb{N} \cup \{\infty\}$ is the multiplicity.

The representation $(\pi, \mathcal{H})$ is called multiplicity free if the ring of continuous endomorphisms commuting with G, $End_G(\mathcal{H})$, is commutative. Equivalently $m_\pi(\tau) \leq 1$ for μ-almost all $\tau \in \widehat{G}$ (see [12]). We denote by $\mathcal{H}^\infty$ the space of C^∞ vectors, equipped with a natural Sobolev topology, and let $\mathcal{H}^{-\infty}$ be its antidual, so $\mathcal{H}^\infty \subset \mathcal{H} \subset \mathcal{H}^{-\infty}$. The restriction of π to $\mathcal{H}^\infty$ gives rise to an action on $\mathcal{H}^{-\infty}$ by duality. The elements of $\mathcal{H}^{-\infty}$ are called distribution vectors.

Let $\mathcal{D}(G/K)$ be the space of C^∞ functions on G/K with compact support and assume that G acts on $\mathcal{D}'(G/K)$ by left traslations. We say that (G, K) is a Gelfand pair if any of the following statements holds:

(i) The convolution algebra of K-bi-invariant integrable functions on G is commutative.
(ii) Any unitary representation of G realized in $\mathcal{D}'(G/K)$ is multiplicity free.
(iii) For any irreducible, unitary representation $(\pi, \mathcal{H})$ of G, the subspace $\mathcal{H}_K$ of vectors fixed by K is at most one dimensional.

In particular the left action of G on $L^2(G/K)$ is multiplicity free.

Well known examples of Gelfand pairs are provided by the symmetric spaces of compact and non compact type, where the set of spherical functions plays a central rolle. More recent works (see [1], [2], [3], [4], [8], [13], [22], among others) deal with Gelfand pairs of the form $(K \ltimes N, K)$ (or (K, N) in short) where N is a nilpotent Lie group and K is a subgroup of automorphisms of N.

The notion of Gelfand pair was extended to the case where K is a non compact unimodular group. In this case, the space of K-invariant integrable functions on G/K is trivial. But in [19], E.G. Thomas introduces the notion that *the pair* $(G, G/K)$ *is multiplicity free or a generalized Gelfand pair when the pair* (G, K) *satisfies the statement (ii) above*. Also from Theorem A in the same work, it is not hard to see that (ii) is equivalent to the fact that for any irreducible representation $(\pi, \mathcal{H})$ of G realized in $\mathcal{D}'(G/K)$, the space $\mathcal{H}_K^{-\infty}$ of distribution vectors fixed by K is one dimensional. Moreover, from Theorem 1.1 in [7], it follows that a unitary representation $(\pi, \mathcal{H})$ admits a cyclic distribution vector fixed by K if and only if π is equivalent to an invariant Hilbert subspace of $\mathcal{D}'(G/K)$. Then the definition of generalized Gelfand pair given in [19] is equivalent to the one introduced by G. Van Dijk (see for example [20]), which we adopt in this paper:

Definition (G, K) *is a generalized Gelfand pair if for any irreducible unitary representation* $(\pi, \mathcal{H})$ *of* G *the space* $\mathcal{H}_K^{-\infty}$ *of distribution vectors fixed by* K *is at most one dimensional.*

One of the fundamental result in [2] states that if (K, N) is a Gelfand pair then N is abelian or two step nilpotent. But in [5], for each $m \in \mathbb{N}$, $m \geq 2$, it is exhibited an $(m+2)$-step nilpotent Lie group N_m and a non compact subgroup H_m of $Aut(N_m)$ such that (H_m, N_m) is a generalized Gelfand pair. One has that the family $\mathfrak{n}_m = Lie(N_m)$

is one of the two families of graded filiform Lie algebras, and H_m is isomorphic to the group $\mathbb{R}^{m+1}$. The case $m = 1$, where $\mathfrak{n}_1$ corresponds to the Engel group, was studied in [9].

The aim of this work is to give new examples of generalized Gelfand pairs (K_m, N_m) where K_m is a subgroup of $Aut(N_m)$ isomorphic to the 3-dimensional Heisenberg group and develop the corresponding spherical analysis.

In order to describe our results, we first introduce some notation: Let N be a nilpotent Lie group and K a subgroup of $Aut(N)$. Let us denote by $\mathfrak{n}$ the Lie algebra of N and by $\mathfrak{n}^*$ the real dual space of $\mathfrak{n}$. From Kirillov's theory there is a one to one correspondence between $\widehat{N}$ and the set of coadjoint orbits. For $\Lambda \in \mathfrak{n}^*$, let ρ_Λ denote the irreducible unitary representation of N associated with the coadjoint orbit $\mathcal{O}_\Lambda$. For $k \in K$, we have a new representation of N defined by $\rho_\Lambda^k(n) := \rho_\Lambda(k \cdot n)$. Let $K^\Lambda := \{k \in K : \rho_\Lambda^k \sim \rho_\Lambda\} = \{k \in K : k \cdot \Lambda \in \mathcal{O}_\Lambda\}$ be the stabilizer of ρ_Λ. Thus for each $k \in K^\Lambda$ there is a unitary operator $\omega_\Lambda(k)$ such that $\rho_\Lambda^k(n) = \omega_\Lambda(k)\rho_\Lambda(n)\omega_\Lambda(k^{-1})$ for all $n \in N$. This defines a projective representation ω_Λ of K^Λ, that is,

$$\omega_\Lambda(k_1k_2) = \sigma(k_1, k_2)\omega(k_1)\omega(k_2).$$

ω_Λ is called *the intertwining representation of* ρ_Λ *or metaplectic representation* and σ *the multiplier* for the projective representation ω_Λ. Denote by $\widehat{K_\Lambda^\sigma}$ the set of (equivalent class) irreducible, unitary projective representations of K^Λ with multiplier σ.

The coadjoint orbits of N_m are described in [5] and they are parametrized by:

- $\Lambda = (\alpha_m, \ldots, \alpha_1, 0, 0, \lambda)$ with $\lambda \neq 0$ and $(\alpha_m, \ldots, \alpha_1) \in \mathbb{R}^m$.
- $\Lambda = (\alpha_m, \ldots, \alpha_1, 0, \nu, 0)$ with $\nu \neq 0$ and $(\alpha_m, \ldots, \alpha_1) \in \mathbb{R}^m$ or $\nu = 0$ and $\alpha_j \neq 0$ for some $j \in \{1, \ldots, m-1\}$ and $\alpha_1 = \cdots = \alpha_{j-1} = 0$.
- $\Lambda = (\alpha_m, 0, \ldots, 0, \mu, 0, 0)$ with $\mu, \alpha_m \in \mathbb{R}$.

In the last case $\mathcal{O}_\Lambda$ consist of a singlet, namely Λ itself.

Our first result is the following

Proposition 1 $K_m^\Lambda = K_m$ *for all* $\Lambda \in \mathfrak{n}_m^*$.

In this situation the Mackey's representation theory states that any irreducible unitary representation of $K_m \ltimes N_m$ is, for the first two cases, of the form

$$\rho_{\tau,\Lambda}(k, n) = \tau(k) \otimes w_\Lambda(k)\,\rho_\Lambda(n),$$

where $\tau \in \widehat{K_m^{\overline{\sigma}}}, \overline{\sigma}$ denote the conjugate of σ, $\rho_\Lambda \in \widehat{N_m}$ and w_Λ is the metaplectic representation. These representations correspond to the infinite dimensional representations ρ_Λ. For the last case $\rho_{\tau,\Lambda}(k, n) = \tau(k) \otimes \chi_\Lambda(n)$, where $\tau \in \widehat{K_m}$ and χ_Λ is a character on N_m.

Theorem 2 (i) *For* $\Lambda = (\alpha_m, ..., \alpha_1, 0, 0, \lambda)$, *with* $\lambda \neq 0$, w_Λ *is an irreducible projective representation of* K_m.

(ii) *For* $\Lambda = (\alpha_m, ..., \alpha_1, 0, \nu, 0)$ *with* $\nu \neq 0$, ω_Λ *is the Schröedinger representation of the* 3*-dimensional Heisenberg group on* $L^2(\mathbb{R})$.

(iii) *For* $m \geq 2$ *and* $\Lambda = (\alpha_m, ..., \alpha_j, 0, \ldots, 0)$ *with* $\alpha_j \neq 0$ *for some* $j \in \{1, \ldots, m-1\}$, ω_Λ *is the left translation on* $L^2(\mathbb{R})$ *of a subgroup of* K_m *isomorphic to* $\mathbb{R}$.

For the proof that given any irreducible unitary representation of $K_m \ltimes N_m$, the space of distribution vectors fixed by K_m is at most one dimensional, a crucial result is a criterion due to Mokni and Thomas, which is an analogous of a Carcano criterion for Gelfand pairs.

Theorem 3 [16] *Let* $(\omega; W)$, $(\gamma; V)$ *be unitary representations of* H *such that* γ *is irreducible. Then* γ *appears in the decomposition of* ω *into irreducible components if and only if* $\gamma^* \otimes \omega$ *has a distribution vector fixed by* H *as* $(H \times H)$*-module.*

Theorems 2 and 3 yield the following

Theorem 4 *The pair* (K_m, N_m) *is a generalized Gelfand pair.*

We recall that when K is compact and (G, K) is a Gelfand pair, the set of spherical functions of positive type is in correspondence with the set of (equivalent classes) irreducible unitary representations (π, H) of G such that the subspace H_K of vectors fixed by K is one dimensional. For the unitary vector $v \in H_K$, the associated spherical function is defined by

$$\zeta(g) = \langle \pi(g)v, v \rangle .$$

Furthermore, in a sharp contrast with the symmetric cases, the spherical functions corresponding to a Gelfand pair of the form (K, N) are of positive type (see [2], Corollary 8.4).

When K is no longer compact and admits a distribution vector $\phi \in H_K^{-\infty}$, then for f smooth on G we have $\pi(f)\phi \in H^\infty$ and so we can associate to ϕ the distribution

$$\langle \Phi_\pi, f \rangle := \langle \phi, \pi(f)\phi \rangle .$$

This is a positive type K-bi-invariant distribution on G, and since is irreducible, it is a extremal point in the cone of positive type K-bi-invariant distributions on G.

Following Molcanov [14, 15] we call Φ_π a *spherical distribution*.

In order to do the spherical analysis associated to our examples, let (K, N) be a generalized Gelfand pair such that $K = K^\Lambda$ for $\Lambda \in \mathfrak{n}^*$. We observe that a K-bi-invariant distributions on G can be identified with a K-invariant distribution on N.

Let us assume that ω_Λ is a *true representation*. It follows from Theorem 3 that ω_Λ is a multiplicity free representation and that the irreducible representation $\rho_{\tau,\Lambda} = \tau \otimes \rho_\Lambda \omega_\Lambda$ of $K \ltimes N$ has a distribution vector fixed by K if and only if the dual representation τ^* of τ appears in the descomposition into irreducible components of ω_Λ. Also, we recall that for $f \in \mathcal{D}(N)$, $\rho_\Lambda(f)$ is an operator of trace class.

Let us asumme that

$$\omega_\Lambda = \int_{\mathcal{J}} \omega_j \, d\mu(j),$$

where $\left(\omega_j, H_j\right)$, $j \in \mathcal{J}$, denotes the irreducible components of ω_Λ. Let $\left\{v_i^j\right\}_{i\in\mathbb{N}}$ be an ortonormal bases of H_j. Then,

Proposition 5 *The spherical distribution corresponding to* $\rho_{j,\Lambda} = \omega_j^* \otimes \rho_\Lambda \omega_\Lambda$ *is* $\Phi_{j,\Lambda} = 1 \otimes \Psi_{j,\Lambda}$, *where*

$$\Psi_{j,\Lambda}(f) = \sum_{i\in\mathbb{N}} \left\langle \rho_\Lambda(f) v_i^j, v_i^j \right\rangle, \tag{1}$$

for $f \in \mathcal{D}(N)$ *(cfr* (1) *with th 8.7 in* [2, p. 114]*).*

The case where ω_Λ is an irreducible projective representation will be considered separately.

Let $\mathcal{U}(\mathfrak{n}_m)$ be the algebra of the left invariant differential operators on N_m, and denote by $\mathcal{U}(\mathfrak{n}_m)^{K_m}$ the subalgebra of $\mathcal{U}(\mathfrak{n}_m)$ of the K-invariant differential operators on N_m. We know that Φ_Λ is an eigendistribution of $\mathcal{U}(\mathfrak{n}_m)^{K_m}$ [[7], Theorem 1.5], but unlike the compact case the set of eigenvalues corresponding to a set of generators do not determine always Φ_Λ (for the compact case see [10], Corollary 2.3, page 402).

More precisely, for $m = 1, 2$ we compute a set of generators of $\mathcal{U}(\mathfrak{n}_m)^{K_m}$ and prove that the corresponding set of eigenvalues do not determine Φ_Λ in the cases $\Lambda = (\alpha, 0, \nu, 0)$, $\nu \neq 0$.

On the other hand, the representations ρ_Λ with $\Lambda = (\alpha, 0, 0, \lambda)$, $\lambda \neq 0$ are the so called generic representations of N_m, i.e., those with nonzero Plancherel measure (see [11], Theorem 10.2), and for the corresponding spherical distributions it holds the following

Theorem 6 *There exists a subset* $\{D_1, \ldots, D_{m+1}\}$ *of* $\mathcal{U}(\mathfrak{n}_m)^{K_m}$ *with* $deg(D_j) = j$ *such that for* $\Lambda = (\overline{\alpha}, \lambda)$ *with* $\overline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ *and* $\lambda \neq 0$,

$$D_1 \Phi_{\overline{\alpha},\lambda} = -i\lambda \Phi_{\overline{\alpha},\lambda}, \quad D_{j+1} \Phi_{\overline{\alpha},\lambda} = \frac{j+1!}{j} (-i)^{j+1} \lambda^j \alpha_j \Phi_{\overline{\alpha},\lambda} \quad \textit{for } j = 1, \ldots, m.$$

The paper is organized as follows: in Sect. 2 we describe $Aut(N_m)$ and prove Proposition 1. Section 3 is devoted to the proofs of Theorems 2 and 4. The computations of Φ_Λ are in Sect. 4. The study of eigenvalues is in Sect. 5.

2 The Automorphism Group of $\mathfrak{n}_m$

Let $\mathfrak{n}_m$ be the Lie algebra introduced in [5]: the underlying vector space has a bases $\mathcal{B} := \{e_m, e_{m-1}, \ldots, e_1, e_x, e_y, e_t\}$ and the Lie bracket is defined by

$$\begin{aligned} [e_j, e_x] &= e_{j-1}, \qquad j \geq 2, \\ [e_1, e_x] &= e_y, \\ [e_x, e_y] &= e_t, \end{aligned}$$

and zero in the other cases. Although $\mathfrak{n}_m$ is $m+2$-step nilpotent it has a one-dimensional center $\mathfrak{z}(\mathfrak{n}_m) = \mathbb{R}e_t$.

Let N_m be the $(m+3)$-dimensional simply connected Lie group with Lie algebra $\mathfrak{n}_m$.

The automorphism group $Aut(\mathfrak{n}_m)$ of $\mathfrak{n}_m$ is characterized by the following

Theorem 7 *Given* $(u_m, \ldots, u_1, 0, u_y, u_t)$, $(h_m, \ldots, h_1, h_x, h_y, h_t) \in \mathbb{R}^{m+3}$ *with* $u_m \neq 0$ *and* $h_x \neq 0$*, there is a uniquely determined* $T \in Aut(\mathfrak{n}_m)$ *such that*

$$[T]_{\mathcal{B}} = \left(\begin{array}{ccccc:ccc} u_m & 0 & 0 & \cdots & 0 & h_m & 0 & 0 \\ u_{m-1} & h_x u_m & 0 & \cdots & 0 & h_{m-1} & 0 & 0 \\ u_{m-2} & h_x u_{m-1} & h_x^2 u_m & \cdots & 0 & h_{m-2} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ u_1 & h_x k_2 & h_x^2 u_3 & \cdots & h_x^{m-1} u_m & h_1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \cdots & 0 & h_x & 0 & 0 \\ u_y & h_x u_1 & h_x^2 u_2 & \cdots & h_x^{m-1} u_{m-1} & h_y & h_x^m u_m & 0 \\ u_t & -h_x u_y & -h_x^2 u_1 & \cdots & -h_x^{m-1} u_{m-2} & h_t & -h_x^m u_{m-1} & h_x^{m+1} u_m \end{array}\right). \tag{2}$$

Reciprocally, for each $T \in Aut(\mathfrak{n}_m)$ *there are* $(m+3)$*-tuple* $(u_m, \ldots, u_1, 0, u_y, u_t)$ *and* $(h_m, \ldots, h_1, h_x, h_y, h_t) \in \mathbb{R}^{m+3}$ *with* $u_m \neq 0$ *and* $h_x \neq 0$ *such that (2) is satisfied.*

Proof $\Rightarrow$) It is easy to see that if $[T]_{\mathcal{B}}$ is given by (2) then $T \in Aut(\mathfrak{n}_m)$.

$\Leftarrow$) Let $T \in Aut(\mathfrak{n}_m)$. Note that T is completely determined by

$$T(e_m) = \sum_{i=1}^{m} u_i e_i + u_x e_x + u_y e_y + u_t e_t \text{ and } T(e_x) = \sum_{i=1}^{m} h_i e_i + h_x e_x + h_y e_y + h_t e_t,$$

since $Te_t = [Te_x, Te_y]$, $Te_y = [Te_1, Te_x]$ and $T(e_j) = [Te_{j+1}, Te_x]$ $\forall j = 1, \ldots, m-1$.

First, we observe that $u_x = 0$. In fact, we have

$$\begin{aligned} Te_{m-1} = [Te_m, Te_x] &= (h_x u_m - u_x h_m) e_{m-1} + \cdots + (h_x u_1 - u_x h_1) e_y \\ &\quad - (h_x u_y - u_x h_y) e_t, \end{aligned} \tag{3}$$

and

$$\begin{aligned} 0 = [T(e_{m-1}), T(e_m)] &= u_x (h_x u_m - u_x h_m) e_{m-2} + \cdots + u_x (h_x u_2 - u_x h_2) e_y \\ &\quad - u_x (h_x u_1 - u_x h_1) e_t. \end{aligned}$$

From this, if $u_x \neq 0$ then $(h_x u_m - u_x h_m) = \cdots = (h_x u_2 - u_x h_2) = (h_x u_1 - u_x h_1) = 0$ and by (3) we get

$$T(e_{m-1}) = -(h_x u_y - u_x h_y)e_t \in \mathfrak{z}(\mathfrak{n}_m),$$

which is impossible because $e_{m-1} \notin \mathfrak{z}(\mathfrak{n}_m)$.
Then,

$$\begin{aligned}
&Te_{m-1} = h_x(u_m e_{m-1} + u_{m-1}e_{m-2} + \cdots + u_1 e_y - u_y e_t),\\
&Te_{m-i} = h_x^i(u_m e_{m-i} + u_{m-1}e_{m-i-1} + \cdots + u_i e_y - u_{i-1}e_t)\\
&\quad \text{for all } i = 2, \ldots, m-1,\\
&Te_y = h_x^m(u_m e_y - u_{m-1}e_t),\\
&Te_t = h_x^{m+1}u_m e_t.
\end{aligned} \tag{4}$$

Since $T(\mathfrak{z}(\mathfrak{n}_m)) = \mathfrak{z}(\mathfrak{n}_m)$, (4) implies that $u_m \neq 0$ and $h_x \neq 0$.

□

Let $\mathcal{D}$ and $Aut_1(\mathfrak{n}_m)$ be the subgroups of $Aut(\mathfrak{n}_m)$ defined by

$$\begin{aligned}
\mathcal{D} = \{T \in Aut(\mathfrak{n}_m) : Te_x = h_x e_x,\ Te_y = h_x^m e_y,\\
Te_t = h_x^{m+1}e_t,\ Te_j = h_x^{m-j}\ e_j\ \forall 1 \leq j \leq m\},\\
Aut_1(\mathfrak{n}_m) = \{T \in Aut(\mathfrak{n}_m) : \langle Te_x, e_x\rangle = 1\}.
\end{aligned}$$

Theorem 8 (i) $Aut(\mathfrak{n}_m) = \mathcal{D} \ltimes Aut_1(\mathfrak{n}_m)$.
(ii) $Aut_1(\mathfrak{n}_m) = \mathcal{H} \ltimes \mathbb{R}^{m+2}$,
where $\mathbb{R}^{m+2} = \{T \in Aut_1(\mathfrak{n}_m) : Te_j = e_j, \forall e_j \neq e_x\}$, $\mathcal{H} = \{T \in Aut_1(\mathfrak{n}_m) : Te_x = e_x\}$ *and* $\mathbb{R}^{m+2}$ *is normal in* $Aut_1(\mathfrak{n}_m)$.

Proof The computations in ii) are straightforward by writing T in the bases $\mathcal{B}' := \{e_m, e_{m-1}, \ldots, e_1, e_y, e_t, e_x\}$. □

We denote by $\Lambda = (\alpha_m, \ldots, \alpha_1, \mu, \nu, \lambda)$ the element of $\mathfrak{n}_m^*$. The pairing between $\mathfrak{n}_m$ and $\mathfrak{n}_m^*$ is given by

$$\langle(\alpha_m, \ldots, \alpha_1, \mu, \nu, \lambda), (s_m, \ldots, s_1, x, y, t)\rangle = \sum_{i=1}^{m} \alpha_i s_i + \mu x + \nu y + \lambda t.$$

For simplicity of notation, we write $\overline{\alpha}$ instead of $(\alpha_m, \ldots, \alpha_1) \in \mathbb{R}^m$. Note that for $k \in Aut(\mathfrak{n}_m)$ and $\Lambda = (\overline{\alpha}, \lambda, \mu, \nu) \in \mathfrak{n}^*$, $k \cdot \Lambda \in \mathcal{O}_\Lambda$ if and only if $k^t(\alpha_m, \ldots, \alpha_1, \mu, \nu, \lambda) \in \mathcal{O}_\Lambda$.

Proposition 9 *Let* $k \in Aut_1(\mathfrak{n}_m)$, $ke_t = e_t$. *Then* $k \cdot \Lambda \in \mathcal{O}_\Lambda$ *for all* $\Lambda \in \mathfrak{n}_m^*$ *if and only if*

$$h_m = 0, \quad u_t = -\frac{x^{m+1}}{m+1!}, \quad u_y = \frac{x^m}{m!}, \quad u_{m-j} = \frac{x^j}{j!} \quad \forall j = 1, 2, \ldots, m-1,$$

for some $x \in \mathbb{R}$.

Proof • For $\Lambda = (\overline{\alpha}, 0, 0, \lambda) \in \mathfrak{n}^*$ with $\lambda \neq 0$, we have

$$\begin{aligned}
&\mathcal{O}_{\overline{\alpha},\lambda} \\
&\quad = \Bigg\{ \Bigg(-\frac{x^{m+1}}{m+1!}\lambda + \sum_{k=0}^{m-1} \frac{x^k}{k!}\alpha_{m-k}, \dots, -\frac{x^{j+1}}{j+1!}\lambda \\
&\qquad + \sum_{k=0}^{j-1} \frac{x^k}{k!}\alpha_{j-k}, \dots, -\frac{x^2}{2!}\lambda + \alpha_1, \mu, x\lambda, \lambda \Bigg) \\
&\qquad : x, \mu \in \mathbb{R} \Big\}.
\end{aligned}$$

So,

$$k^t \Lambda = \begin{pmatrix} \alpha_m + u_{m-1}\alpha_{m-1} + u_{m-2}\alpha_{m-2} + \cdots + u_1\alpha_1 + u_t\lambda \\ \alpha_{m-1} + u_{m-1}\alpha_{m-2} + \cdots + u_2\alpha_1 - u_y\lambda \\ \alpha_{m-2} + \cdots + u_3\alpha_1 - u_1\lambda \\ \vdots \\ \alpha_2 + u_{m-1}\alpha_1 - u_{m-3}\lambda \\ \alpha_1 - u_{m-2}\lambda \\ h_m\alpha_m + h_{m-1}\alpha_{m-1} + h_{m-2}\alpha_{m-2} + \cdots + h_1\alpha_1 + h_t\lambda \\ -u_{m-1}\lambda \\ \lambda \end{pmatrix} \in \mathcal{O}_{\overline{\alpha},\lambda},$$

if and only if

$$u_t = -\frac{x^{m+1}}{m+1!}, \quad u_y = \frac{x^m}{m!}, \quad u_{m-j} = \frac{x^j}{j!} \quad \forall j = 1, 2, \dots, m-1.$$

• For $\Lambda = (\overline{\alpha}, 0, \nu, 0) \in \mathfrak{n}_m^*$ with $\nu \neq 0$ or $\alpha_j \neq 0$ and $\alpha_1 = \cdots = \alpha_{j-1} = 0$ for some $j \in \{1, \dots, m-1\}$, we get

$$\mathcal{O}_{\overline{\alpha},\nu} = \left\{ \left(\sum_{k=0}^{m-1} \frac{x^k}{k!}\alpha_{m-k} + \frac{x^m}{m!}\nu, \dots, \alpha_2 + x\alpha_1 + \frac{x^2}{2!}\nu, \alpha_1 + x\nu, \beta, \nu, 0 \right) : x, \beta \in \mathbb{R} \right\}.$$

Thus,

$$k^t\Lambda = \begin{pmatrix} \alpha_m + u_{m-1}\alpha_{m-1} + u_{m-2}\alpha_{m-2} + \cdots + u_1\alpha_1 + u_y\nu \\ \alpha_{m-1} + u_{m-1}\alpha_{m-2} + \cdots + u_2\alpha_1 + u_1\nu \\ \alpha_{m-2} + \cdots + u_3\alpha_1 + u_2\nu \\ \vdots \\ \alpha_2 + u_{m-1}\alpha_1 + u_{m-2}\nu \\ \alpha_1 + u_{m-1}\nu \\ h_m\alpha_m + h_{m-1}\alpha_{m-1} + h_{m-2}\alpha_{m-2} + \cdots + h_1\alpha_1 + h_y\nu \\ \nu \\ 0 \end{pmatrix} \in \mathcal{O}_{\overline{\alpha},\nu},$$

if and only if

$$u_y = \frac{x^m}{m!}, \quad u_{m-j} = \frac{x^j}{j!}\ \forall j = 1, 2, \ldots, m-1.$$

- For $\Lambda = (\alpha_m, 0, \ldots, 0, \mu, 0, 0) \in \mathfrak{n}_m^*$ is

$$\mathcal{O}_{\alpha_m,\mu} = \{(\alpha_m, 0, \ldots, 0, \mu, 0, 0)\}.$$

Then,

$$k^t\Lambda = \begin{pmatrix} \alpha_m \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ h_m\alpha_m + \mu \\ 0 \\ 0 \end{pmatrix} \in \mathcal{O}_{\alpha_m,\mu} \Leftrightarrow h_m = 0.$$

□

We denote by exp the exponential map of N_m.

Definition 1 From now on, (a, b, c) denotes the automorphism

$$\left(\begin{array}{cccccc:ccc}
1 & 0 & 0 & \cdots & 0 & & 0 & 0 & 0 \\
a & 1 & 0 & \cdots & 0 & & 0 & 0 & 0 \\
\frac{a^2}{2!} & a & 1 & \cdots & 0 & & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots \\
\frac{a^{m-1}}{m-1!} & \frac{a^{m-2}}{m-2!} & \frac{a^{m-3}}{m-3!} & \cdots & 1 & & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & \cdots & 0 & & 1 & 0 & 0 \\
\frac{a^m}{m!} & \frac{a^{m-1}}{m-1!} & \frac{a^{m-2}}{m-2!} & \cdots & a & & b & 1 & 0 \\
-\frac{a^{m+1}}{m+1!} & -\frac{a^m}{m!} & -\frac{a^{m-1}}{m-1!} & \cdots & -\frac{a^2}{2!} & & c & -a & 1
\end{array}\right) \in Aut_1(\mathfrak{n}_m).$$

Let K_m be the subgroup of $Aut(N_m)$ defined by

$$K_m = \{exp \circ (a, b, c) \circ exp^{-1} : a, b, c \in \mathbb{R}\}.$$

It is not difficult to see that the subgroup of automorphisms $\{(a, b, c) \in Aut_1(\mathfrak{n}_m) : a, b, c \in \mathbb{R}\}$ is isomorphic to the tridimentional Heisenberg group

$$H_3 = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} : a \in \mathbb{R} \right\} \ltimes \mathbb{R}^2.$$

Proof of Proposition 1. The result follows immediately from definition of $K_m^\Lambda = \{k \in K_m : k \cdot \Lambda \in \mathcal{O}_\Lambda\}$ and Proposition 9. □

3 Metaplectic Representations

For simplicity, we denote by (a, b, c) the automorphism of N_m corresponding to $(a, b, c) \in Aut_1(\mathfrak{n}_m)$. Note that

$$(a, b, c) = (a, 0, 0)(0, b, 0)(0, 0, c + ab).$$

Also, $(0, b, 0)$ and $(0, 0, c)$ fix the elements $(s_m, \dots, s_1, 0, y, t)$ of N_m and

$$\begin{aligned}
(a, 0, 0)(s_m, \dots, s_1, 0, y, t) &= exp\left[(a, 0, 0)\, exp^{-1}(s_m, \dots, s_1, 0, y, t)\right] \\
&= exp\left[(a, 0, 0)(s_m e_m + \cdots + s_1 e_1 + y e_y + t e_t)\right] \\
&= exp\left[\sum_{j=1}^{m}\left(\sum_{i=0}^{m-j} s_{j+i}\frac{a^i}{i!}\right) e_j + \left(y + \sum_{j=1}^{m} s_j \frac{a^j}{j!}\right) e_y + \left(t - ay - \sum_{j=1}^{m} s_j \frac{a^{j+1}}{j+1!}\right) e_t\right]
\end{aligned}$$

$$= \left(s_m, \ldots, \sum_{i=0}^{m-j} s_{j+i} \frac{a^i}{i!}, \ldots, \sum_{i=0}^{m-1} s_{1+i} \frac{a^i}{i!}, 0, y + \sum_{j=1}^{m} s_j \frac{a^j}{j!}, t - ay - \sum_{j=1}^{m} s_j \frac{a^{j+1}}{j+1!} \right).$$

We denote by $\overline{0}$ any l-tuple $(0, \ldots, 0)$, $l \in \mathbb{N}$. Otherwise, $(a, 0, 0)$ fixes the elements $(\overline{0}, x, 0, 0) \in N_m$ and

$$\begin{aligned}
(0, b, 0)(\overline{0}, x, 0, 0) &= (\overline{0}, x, bx, 0),\\
(0, 0, c)(\overline{0}, x, 0, 0) &= (\overline{0}, x, 0, cx).
\end{aligned}$$

In order to describe the metaplectic representation ω_Λ with $\Lambda \in \mathfrak{n}_m^*$, we take account of the representative of each orbit, the expression of ρ_Λ given in [5] and the action of $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ on N_m to compute $\rho_\Lambda^{(a,0,0)}$, $\rho_\Lambda^{(0,b,0)}$ and $\rho_\Lambda^{(0,0,c)}$.

- Case $\Lambda = (\overline{\alpha}, 0, 0, \lambda)$ with $\lambda \neq 0$:

$$\begin{aligned}
&\left[\rho_{\overline{\alpha},\lambda}^{(a,0,0)} \left(\overline{0}, s_j, \overline{0}\right) f\right](u) = \left[\rho_{\overline{\alpha},\lambda} \left(\overline{0}, s_j, \ldots, s_j \frac{a^{j-1}}{j-1!}, 0, s_j \frac{a^j}{j!}, -s_j \frac{a^{j+1}}{j+1}\right) f\right]\\
&(u) = e^{is_j \sum_{i=1}^{j} \alpha_i \frac{(u+a)^{j-i}}{j-i!} - \lambda \frac{(u+a)^{j+1}}{j+1!}} f(u),\\
&\left[\rho_{\overline{\alpha},\lambda}^{(a,0,0)} (\overline{0}, 0, y, 0) f\right](u) = \left[\rho_{\overline{\alpha},\lambda} \left(\overline{0}, 0, y, -ay\right) f\right](u) = e^{-i\lambda y(u+a)} f(u),\\
&\left[\rho_{\overline{\alpha},\lambda}^{(a,0,0)} (\overline{0}, 0, 0, t) f\right](u) = \left[\rho_{\overline{\alpha},\lambda} \left(\overline{0}, 0, 0, t\right) f\right](u) = e^{i\lambda t} f(u),\\
&\left[\rho_{\overline{\alpha},\lambda}^{(a,0,0)} (\overline{0}, x, 0, 0) f\right](u) = \left[\rho_{\overline{\alpha},\lambda} \left(\overline{0}, x, 0, 0\right) f\right](u) = f(u - x).
\end{aligned}$$

  So,

$$\left[\rho_{\overline{\alpha},\lambda}^{(a,0,0)} (\overline{s}, x, y, t) f\right](u) = \left[\rho_{\overline{\alpha},\lambda} (\overline{s}, x, y, t) \left(u \mapsto f(u - a)\right)\right](u + a). \tag{5}$$

  Similar computations yield

$$\left[\rho_{\overline{\alpha},\lambda}^{(0,b,0)} (\overline{s}, x, y, t) f\right](u) = e^{-i\frac{b\lambda}{2}u^2} \left[\rho_{\overline{\alpha},\lambda} (\overline{s}, x, y, t) \left(u \mapsto e^{i\frac{b\lambda}{2}u^2} f(u)\right)\right](u), \tag{6}$$

$$\left[\rho_{\overline{\alpha},\lambda}^{(0,0,c)} (\overline{s}, x, y, t) f\right](u) = e^{ic\lambda u} \left[\rho_{\overline{\alpha},\lambda} (\overline{s}, x, y, t) \left(u \mapsto e^{-ic\lambda u} f(u)\right)\right](u). \tag{7}$$

- Case $\Lambda = (\overline{\alpha}, 0, \nu, 0)$ with $\nu \neq 0$ or $\alpha_j \neq 0 \wedge \alpha_1 = \cdots = \alpha_{j-1} = 0$ for some $0 \leq j \leq m - 1$ and $2 \leq m$:

$$\begin{aligned}
\left[\rho_{\overline{\alpha},\nu}^{(a,0,0)} (\overline{0}, s_j, \overline{0}) f\right](u) &= e^{is_j \sum_{i=1}^{j} \alpha_i \frac{(u+a)^{j-i}}{j-i!} + \nu \frac{(u+a)^j}{j!}} f(u),\\
\left[\rho_{\overline{\alpha},\nu}^{(a,0,0)} (\overline{0}, 0, y, 0) f\right](u) &= e^{-i\nu y} f(u),
\end{aligned}$$

$$\left[\rho_{\overline{\alpha},\nu}^{(a,0,0)}(\overline{0},0,0,t)f\right](u)=f(u),$$
$$\left[\rho_{\overline{\alpha},\nu}^{(a,0,0)}(\overline{0},x,0,0)f\right](u)=f(u-x).$$

Hence,

$$\left[\rho_{\overline{\alpha},\nu}^{(a,0,0)}(\overline{s},x,y,t)f\right](u)=\left[\rho_{\overline{\alpha},\nu}(\overline{s},x,y,t)\left(u\mapsto f(u-a)\right)\right](u+a). \quad (8)$$

Similarly,

$$\left[\rho_{\overline{\alpha},\nu}^{(0,b,0)}(\overline{s},x,y,t)f\right](u)=e^{i\nu bu}\left[\rho_{\overline{\alpha},\nu}(\overline{s},x,y,t)\left(u\mapsto e^{-i\nu bu}f(u)\right)\right](u), \quad (9)$$

$$\left[\rho_{\overline{\alpha},\nu}^{(0,0,c)}(\overline{s},x,y,t)f\right](u)=e^{i\nu c}\left[\rho_{\overline{\alpha},\nu}(\overline{s},x,y,t)\left(u\mapsto e^{-i\nu c}f(u)\right)\right](u). \quad (10)$$

Proof of Theorem 2 • Case $\Lambda=(\overline{\alpha},0,0,\lambda)$ with $\lambda\neq 0$: Clearly, by (5), (6) and (7) we have

$$\omega_{\overline{\alpha},\lambda}(a,b,c)f(u)=e^{i\lambda(c+ab)(u+a)}e^{-ib\lambda\frac{(u+a)^2}{2}}f(u+a),$$

is the metaplectic representation. Moreover, in order to prove that $\omega_{\overline{\alpha},\lambda}$ is a projective representation we note that

$$\begin{aligned}
&\left[\omega_{\overline{\alpha},\lambda}(a_1,b_1,c_1)\omega_{\overline{\alpha},\lambda}(a_2,b_2,c_2)f\right](u)\\
&\quad=e^{i\lambda(c_1+a_1b_1)(u+a_1)}\\
&\left[\omega_{\overline{\alpha},\lambda}(a_1,0,0)\omega_{\overline{\alpha},\lambda}(0,b_1,0)\omega_{\overline{\alpha},\lambda}(a_2,b_2,c_2)f\right](u)\\
&\quad=e^{i\lambda(c_1+a_1b_1)(u+a_1)}\\
&\left[\omega_{\overline{\alpha},\lambda}(a_1,0,0)\omega_{\overline{\alpha},\lambda}(0,b_1,0)\left(u\mapsto e^{i\lambda(c_2+a_2b_2)(u+a_2)}\omega_{\overline{\alpha},\lambda}(a_2,0,0)\omega_{\overline{\alpha},\lambda}(0,b_2,0)f(u)\right)\right](u)\\
&\quad=e^{i\lambda(c_1+a_1b_1)(u+a_1)}\\
&\left[\omega_{\overline{\alpha},\lambda}(a_1,0,0)\left(u\mapsto e^{i\lambda(c_2+a_2b_2+)(u+a_2)}\omega_{\overline{\alpha},\lambda}(0,b_1,0)\omega_{\overline{\alpha},\lambda}(a_2,0,0)\omega_{\overline{\alpha},\lambda}(0,b_2,0)f(u)\right)\right](u)\\
&\quad=e^{i\lambda(c_1+a_1b_1)(u+a_1)}\\
&\left[\omega_{\overline{\alpha},\lambda}(a_1,0,0)\left(u\mapsto e^{i\lambda(c_2+a_2b_2)(u+a_2)}e^{i\lambda a_2b_1u}e^{i\lambda b_1\frac{a_2^2}{2}}\omega_{\overline{\alpha},\lambda}(a_2,0,0)\omega_{\overline{\alpha},\lambda}\right.\right.\\
&\quad\left.\left.(0,b_1,0)\omega_{\overline{\alpha},\lambda}(0,b_2,0)f(u)\right)\right](u)\\
&\quad=e^{i\lambda(c_1+a_1b_1)(u+a_1)}e^{i\lambda(c_2+a_2b_2)(u+a_1+a_2)}e^{i\lambda a_2b_1(u+a_1)}e^{i\lambda b_1\frac{a_2^2}{2}}\\
&\left[\omega_{\overline{\alpha},\lambda}(a_1,0,0)\omega_{\overline{\alpha},\lambda}(a_2,0,0)\omega_{\overline{\alpha},\lambda}(0,b_1,0)\omega_{\overline{\alpha},\lambda}(0,b_2,0)f\right](u)\\
&\quad=e^{-i\lambda\left[(c_1+a_1b_1)a_2+b_1\frac{a_2^2}{2}\right]}e^{i\lambda[c_1+c_2-a_1b_2+(a_1+a_2)(b_1+b_2)](u+a_1+a_2)}\\
&\left[\omega_{\overline{\alpha},\lambda}(a_1+a_2,0,0)\omega_{\overline{\alpha},\lambda}(0,b_1+b_2,0)f\right](u)\\
&\quad=e^{-i\lambda\left[(c_1+a_1b_1)a_2+b_1\frac{a_2^2}{2}\right]}\left[\omega_{\overline{\alpha},\lambda}(a_1+a_2,b_1+b_2,c_1+c_2-a_1b_2)f\right](u)
\end{aligned}$$

$$= \sigma\left((a_1, b_1, c_1), (a_2, b_2, c_2)\right) \left[\omega_{\overline{\alpha},\lambda}\left((a_1, b_1, c_1)(a_2, b_2, c_2)\right) f\right](u),$$

where σ is defined by

$$\sigma\left((a_1, b_1, c_1), (a_2, b_2, c_2)\right) = e^{-i\lambda\left[(c_1+a_1b_1)a_2+b_1\frac{a_2^2}{2}\right]},$$

and it is easy to check that

$$\begin{aligned}&\sigma\left((a_1, b_1, c_1), (a_2, b_2, c_2)(a_3, b_3, c_3)\right) \sigma\left((a_2, b_2, c_2), (a_3, b_3, c_3)\right)\\&\quad = \sigma\left((a_1, b_1, c_1)(a_2, b_2, c_2), (a_3, b_3, c_3)\right) \sigma\left((a_1, b_1, c_1), (a_2, b_2, c_2)\right).\end{aligned}$$

Therefore, $\omega_{\overline{\alpha},\lambda}$ is a projective representation with multiplier σ. If W is a closed $\omega_{\overline{\alpha},\lambda}$-invariant subspace of $L^2(\mathbb{R})$, then it is invariant by translation and by $e^{i\lambda cu}$ with $c \in \mathbb{R}$. The same lines of Theorem 10.2.1 in [6] shows that $W = L^2(\mathbb{R})$. That is, $\omega_{\overline{\alpha},\lambda}$ is an irreducible projective representation, so $\omega_{\overline{\alpha},\lambda}$ is not equivalent to any true representation of K_m.

- Case $\Lambda = (\overline{\alpha}, 0, \nu, 0)$ with $\nu \neq 0$ or $\alpha_j \neq 0 \wedge \alpha_1 = \cdots = \alpha_{j-1} = 0$ for some $0 \leq j \leq m-1$ and $2 \leq m$: From (8), (9) and (10) we obtain

$$\left[\omega_{\overline{\alpha},\nu}(a, b, c) f\right](u) = e^{i\nu(c+ab)} e^{i\nu b(u+a)} f(u + a).$$

□

We recall that if $\pi \in \widehat{K_m^{\sigma}}$ then the dual representation $\pi^* \in \widehat{K_m^{\overline{\sigma}}}$. Thus, $\pi^* \otimes \pi$ is a true representation of K_m.

It follows from Mackey's theory that, for $\Lambda \in \mathfrak{n}_m^*$ considered in Theorem 2, the irreducible unitary representations of $K_m \ltimes N_m$ are

$$\rho_{\tau,\Lambda}(k, n) = \tau(k) \otimes \rho_{\Lambda}(n)\omega_{\Lambda}(k),$$

with $\tau \in \widehat{K_m^{\overline{\sigma}}}$. And for $\Lambda = (\alpha_m, 0, \ldots, 0, \mu, 0, 0)$,

$$\rho_{\tau,\Lambda}(k, n) = \tau(k) \otimes \chi_{\Lambda}(n),$$

where $\tau \in \widehat{K_m}$ and χ_Λ is a character on N_m. Indeed, a straightforward computation shows that $\rho_{\tau,\Lambda}$ is a representation since $\chi_\Lambda(k\,n) = \chi_\Lambda(n)$ for all $k \in K_m$ and $n \in N_m$.

Proof of Theorem 4 We need to prove that for any irreducible unitary representation $\left(\rho_{\tau,\Lambda}, \mathcal{H}_{\tau,\Lambda}\right)$ of $K_m \ltimes N_m$ the space $\mathcal{H}_{\tau,\Lambda}^{-\infty}$ is at most one dimensional.

- Case $\Lambda = (\overline{\alpha}, 0, \nu, 0)$ with $\nu \neq 0$: we obtain $\omega_{\overline{\alpha},\nu}$ is the irreducible Schröedinger representation of K_m and thus the result by Mokni and Thomas implies that $\mathcal{H}_{\tau,\Lambda}$ has a distribution vector fixed by K_m if and only if τ is equivalent to $\omega_{\overline{\alpha},\nu}^*$ and in this case $\dim \mathcal{H}_{\tau,\Lambda}^{-\infty} = 1$.

Since

$$\left[\omega_{\overline{\alpha},\nu}(a,b,c)F\right](r) = e^{i\nu(c-ba)}e^{i\nu br}F(r+a),$$
$$\left[\omega^*_{\overline{\alpha},\nu}(a,b,c)F\right](r) = e^{-i\nu(c-ba)}e^{-i\nu br}F(r+a).$$

$\omega^*_{\overline{\alpha},\nu} \otimes \omega_{\overline{\alpha},\nu}$ acts on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ by

$$\omega^*_{\overline{\alpha},\nu} \otimes \omega_{\overline{\alpha},\nu}(a,b,c)F_1 \otimes F_2(r,r') = e^{-i\nu br}e^{i\nu br'}F_1(r+a)F_2(r'+a).$$

A distribution vector fixed by K_m is

$$\phi : F_1 \otimes F_2 \to \int_{\mathbb{R}} F_1(r)F_2(r)\,dr. \tag{11}$$

- Case $\Lambda = (\overline{\alpha}, 0, \nu, 0)$ with $\nu = 0$ and $\alpha_j \neq 0$ for some $j \in \{1, \ldots, m-1\}$ and $\alpha_1 = \cdots = \alpha_{j-1} = 0$: $\omega_{\overline{\alpha}}$ is the left action of $\mathbb{R}$ on $L^2(\mathbb{R})$ an thus $\omega_{\overline{\alpha}} = \int \chi_\xi \, d\xi$ is the decomposition of $\omega_{\overline{\alpha}}$ into irreducible components, where χ_ξ is the character defined by $\chi_\xi(t) = e^{i\xi t}$, $\xi \in \mathbb{R}$.
  Since $\omega_{\overline{\alpha}}$ is a multiplicity free representation, [16] implies once again that $\dim \mathcal{H}^{-\infty}_{\tau,\Lambda} = 1$ if and only if τ is equivalent to $\chi_{-\xi}$ for some $\xi \in \mathbb{R}$.
- Case $\Lambda = (\overline{\alpha}, 0, 0, \lambda)$ with $\lambda \neq 0$: a computation shows that

$$\begin{aligned}&\omega^*_{\overline{\alpha},\lambda}(a,b,c) \otimes \omega_{\overline{\alpha},\lambda}(a,b,c)(F_1 \otimes F_2)(r,r')\\ &\quad = e^{-i\lambda(c+ab)r}e^{i\lambda(c+ab)r'}e^{i\lambda b\frac{(r+a)^2}{2}}e^{-i\lambda b\frac{(r'+a)^2}{2}}F_1(r+a)F_2(r'+a),\end{aligned}$$

  for all $F_1 \otimes F_2 \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ and analogously to the case $\nu \neq 0$ we get that

$$\phi : F_1 \otimes F_2 \to \int_{\mathbb{R}} F_1(r)F_2(r)\,dr,$$

  is a distribution vector fixed by K_m.
  Since $\omega_{\overline{\alpha},\lambda}$ is a projective representation, we can not apply Theorem 3 straightforward, but following the same lines of the proof of the sufficient condition there, we see that if $\tau \otimes \omega_{\overline{\alpha},\lambda}$ has a distribution vector fixed by K_m then τ^* is equivalent to $\omega_{\overline{\alpha},\lambda}$.
- Case $\Lambda = (\alpha_m, \overline{0}, \mu, 0, 0)$: we observe that τ has a distribution vector fixed by K_m if and only if τ is the trivial representation of K_m. Indeed it is well known that τ is irreducible if and only if so is $\tau_{-\infty}$ (see [21, p. 136]).

□

4 Spherical Distributions

First of all, we observe that if $G = K \ltimes N$ then there is a correspondence between the set of K-bi-invariant distributions on G and K-invariant distributions on N.

Indeed, a K-invariant distribution Ψ on N gives rise to a K-bi-invariant distribution Φ on G by the rule

$$\langle \Phi, f \rangle_G = \langle \Psi, f_0 \rangle_N \,, \text{ where } f_0(n) = \int_K f(k,n)\, dk.$$

Conversely, let Φ be a K-bi-invariant distribution on G. Since the map $(k,n) \mapsto (e_K, n)(k, e_N)$ is a diffeomorphism, the composition gives a distribution $\tilde{\Phi}$ on $K \times N$, which is right K-invariant. Thus $\tilde{\Phi} = 1 \otimes \Psi$ with Ψ a K-invariant distribution on N. Moreover Φ is of positive type if and only if Ψ is.

Assume $K = K^\Lambda$ for all $\Lambda \in \mathfrak{n}^*$ and that the metaplectic representation decomposes into irreducible component as

$$\omega_\Lambda = \int_J \omega_{j,\Lambda}\, d\mu_J.$$

Let us denote by H_j the representation space of $\omega_{j,\Lambda}$. By Theorem 3, the irreducible representations of $K \ltimes N$ of the form $\tau \otimes \rho_\Lambda \omega_\Lambda$ that have a distribution vector fixed by K are precisely $\rho_{j,\Lambda} = \omega^*_{j,\Lambda} \otimes \rho_\Lambda \omega_\Lambda$.

If ϕ is a distribution vector fixed by K we get $\rho_{j,\Lambda}(k,n)(\phi) = 1(n) \otimes \rho_\Lambda(n)(\phi)$ and for $f \in C_c(K \ltimes N)$ such that $f(k,n) = f_1(k) f_2(n)\ \forall (k,n) \in K \ltimes N$ we have

$$\begin{aligned} \rho_{j,\Lambda}(f)\phi &= \int_K \int_N f(k,n) \rho_{j,\Lambda}(k,n) \phi\, dn\, dk \\ &= \int_K f_1(k) dk \int_N f_2(n)\ 1(n) \otimes \rho_\Lambda(n) \phi\, dn. \end{aligned}$$

Thus, for $\lambda \otimes v \in H^*_j \otimes H_j$,

$$\begin{aligned} \int_N f_2(n)\, \phi\, (1(n) \otimes \rho_\Lambda(n)\, (\lambda \otimes v))\, dn &= \int_N f_2(n)\, \phi\, (\lambda \otimes \rho_\Lambda(n) v)\ dn \\ &= \phi \left(\lambda \otimes \int_N f_2(n) \rho_\Lambda(n) v\, dn \right) \\ &= \phi\, (\lambda \otimes \rho_\Lambda(f_2) v)\,. \end{aligned} \tag{12}$$

Let $\{v_i^j\}_{i \in \mathbb{N}}$ be an ortonormal bases of H_j and $\{\lambda_i^j\}_{i \in \mathbb{N}}$ its dual bases. It is easy to see that the linear functional on $H^*_j \otimes H_j$, $\phi = \sum\limits_{i=1}^{\infty} \lambda_i^j \otimes v_i^j$, given by

$$\phi(\lambda \otimes v) = \sum_{i=1}^{\infty} \left\langle \lambda, \lambda_i^j \right\rangle \left\langle v, v_i^j \right\rangle,$$

is a distribution vector fixed by K. Thus,

$$\phi\,(\lambda \otimes \rho_\Lambda(f_2)v) = \sum_{i=1}^{\infty} \left\langle \lambda, \lambda_i^j \right\rangle \left\langle \rho_\Lambda(f_2)v, v_i^j \right\rangle = \sum_{i=1}^{\infty} \left\langle \lambda, \lambda_i^j \right\rangle \left\langle v, \rho_\Lambda(f_2)^* v_i^j \right\rangle$$
$$= \sum_{i=1}^{\infty} \left\langle \lambda, \lambda_i^j \right\rangle \left\langle v, \rho_\Lambda(f_2^*) v_i^j \right\rangle = \left\langle \lambda \otimes v, \sum_{i=1}^{\infty} \lambda_i^j \otimes \rho_\Lambda(f_2^*) v_i^j \right\rangle,$$

where $f^*(x) = \overline{f(-x)}$. We conclude that

$$\rho_{j,\Lambda}(f)\phi = \int_K f_1(k)dk \, \sum_{i=1}^{\infty} \lambda_i^j \otimes \rho_\Lambda(f_2^*) v_i^j.$$

Note that $\sum_{i=1}^{\infty} \lambda_i^j \otimes \rho_\Lambda(f_2^*) v_i^j$ is a vector in $H_j^* \otimes H_j$ since

$$\left| \left\langle \sum_{i=1}^{\infty} \lambda_i^j \otimes \rho_\Lambda(f) v_i^j, \sum_{k=1}^{\infty} \lambda_k^j \otimes \rho_\Lambda(f) v_k^j \right\rangle \right|^2 = \sum_{i,k=1}^{\infty} \left\langle \lambda_i^j, \lambda_k^j \right\rangle \left\langle \rho_\Lambda(f) v_i^j, \rho_\Lambda(f) v_k^j \right\rangle$$
$$= \sum_{i=1}^{\infty} \left| \lambda_i^j \right|^2 \left\langle \rho_\Lambda(f) v_i^j, \rho_\Lambda(f) v_i^j \right\rangle$$
$$= \sum_{i=1}^{\infty} \left\langle v_i^j, \rho_\Lambda(f)^* \rho_\Lambda(f) v_i^j \right\rangle$$
$$= \sum_{i=1}^{\infty} \left\langle \rho_\Lambda(f * f^*) v_i^j, v_i^j \right\rangle.$$

Moreover, by general theory we know that $\rho_{j,\Lambda}(f)\phi$ is a C^∞ vector and by definition of the spherical distribution we have

$$\Phi_{j,\Lambda}(f) = \phi\left(\rho_{j,\Lambda}(f)\phi\right)$$
$$= \phi\left(\int_K f_1(k)dk \, \sum_{i=1}^{\infty} \lambda_i^j \otimes \rho_\Lambda(f_2^*) v_i^j \right) = \int_K f_1(k)dk \, \phi\left(\sum_{i=1}^{\infty} \lambda_i^j \otimes \rho_\Lambda(f_2^*) v_i^j \right)$$
$$= \int_K f_1(k)dk \, \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \left\langle \lambda_k^j, \lambda_i^j \right\rangle \left\langle v_k^j, \rho_\Lambda(f_2^*) v_i^j \right\rangle = \int_K f_1(k)dk \, \sum_{i=1}^{\infty} \left\langle v_i^j, \rho_\Lambda(f_2^*) v_i^j \right\rangle.$$

Thus,

$$\Phi_{j,\Lambda}(f) = \int_K f_1(k)dk \, \sum_{i=1}^{\infty} \left\langle \rho_\Lambda(f_2) v_i^j, v_i^j \right\rangle. \tag{13}$$

This proves our Proposition 5.

We now determine the spherical distributions corresponding to our cases.

- Case $\nu \neq 0$: let $f \in C_c^\infty(K_m \ltimes N_m)$ be such that $f(k,n) = f_1(k)f_2(n)$ and $F_1 \otimes F_2 \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. By (12), we get

$$\langle \rho(f)\phi, F\rangle = \langle \phi, \rho(f)F\rangle = \int_{K_m} f_1(k)dk \left\langle \phi, F_1 \otimes \rho_{\overline{\alpha},\nu}(f_2)F_2\right\rangle,$$

where $\rho = \omega^*_{\overline{\alpha},\nu} \otimes \omega_{\overline{\alpha},\nu}\rho_{\overline{\alpha},\nu}$. Then, by (11)

$$\begin{aligned}\left\langle \phi, F_1 \otimes \rho_{\overline{\alpha},\nu}(f_2)F_2\right\rangle &= \int_{\mathbb{R}} \left[F_1 \otimes \rho_{\overline{\alpha},\nu}(f_2)F_2\right](r,r)\,dr \\ &= \int_{\mathbb{R}}\int_N f_2(\overline{s},x,y,t)F_1(r)e^{i\nu y}e^{i\sum_{j=2}^{m} s_j \sum_{k=1}^{j} \alpha_k \frac{(r-x)^{j-k}}{j-k!} + \nu\frac{(r-x)^j}{j!}} e^{i(\alpha_1+\nu(r-x))s_1} \\ &\quad F_2(r-x)\,d\overline{s}\,dx\,dy\,dt\,dr \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_2\left(-\sum_{k=1}^{m}\alpha_k\frac{\widehat{(r-x)^{m-k}}}{m-k!} - \nu\frac{(r-x)^m}{m!},\ldots,-\alpha_1\,\widehat{-\nu(r}-x),x,\widehat{-\nu},\hat{0}\right) \\ &\quad F_1(r)F_2(r-x)\,dx\,dr.\end{aligned}$$

We perform the change of variable $(y_1, y_2) = (r, r-x)$ then

$$\begin{aligned}&\left\langle \phi, F_1 \otimes \rho_{\overline{\alpha},\nu}(f_2)F_2\right\rangle \\ &\quad = \int_{\mathbb{R}^2} f_2\left(-\sum_{k=1}^{m}\alpha_k\frac{\widehat{y_2^{m-k}}}{m-k!} - \nu\frac{y_2^m}{m!},\ldots,-\widehat{\alpha_1-\nu}y_2, y_1-y_2,\widehat{-\nu},\hat{0}\right) \\ &\quad F_1(y_1)F_2(y_2)\,dy_1\,dy_2.\end{aligned}$$

Thus,

$$\begin{aligned}\left[\rho(f)\phi\right](y_1,y_2) &= \left(\int_{K_m} f_1(k)\,dk\right) \\ &\quad f_2\left(-\sum_{k=1}^{m}\alpha_k\frac{\widehat{y_2^{m-k}}}{m-k!} - \nu\frac{y_2^m}{m!},\ldots,-\widehat{\alpha_1-\nu}y_2, y_1-y_2,\widehat{-\nu},\hat{0}\right).\end{aligned}$$

The spherical distribution is defined by

$$\begin{aligned}\Phi_{\overline{\alpha},\nu}(f) &= \langle \phi, \rho(f)\phi\rangle \\ &= \int [\rho(f)\phi](r,r)\,dr \\ &= \left(\int_{K_m} f_1(k)dk\right)\int_{\mathbb{R}} f_2\left(-\sum_{k=1}^{m}\alpha_k\frac{\widehat{r^{m-k}}}{m-k!} - \nu\frac{r^m}{m!},\ldots,-\widehat{\alpha_1-\nu}r,0,\widehat{-\nu},\hat{0}\right)\,dr.\end{aligned}$$

That is $\Phi_{\overline{\alpha},\nu} = 1 \otimes \Psi_{\overline{\alpha},\nu}$ where for $f \in \mathcal{D}(N_m)$,

$$\Psi_{\overline{\alpha},\nu}(f) = \int_{\mathbb{R}} f \left(- \sum_{k=1}^{m} \alpha_k \widehat{\frac{r^{m-k}}{m-k!}} - \nu \frac{r^m}{m!}, \ldots, - \sum_{k=1}^{j} \alpha_k \widehat{\frac{r^{j-k}}{j-k!}} - \nu \frac{r^j}{j!}, \ldots, \right. \\ \left. -\widehat{\alpha_1 - \nu r}, 0, \widehat{-\nu}, \hat{0} \right) dr.$$

- Case $\nu = 0$ and $\alpha_j \neq 0 \wedge \alpha_1 = \cdots = \alpha_{j-1} = 0$ for some $0 \leq j \leq m-1$ and $m \geq 2$: $\omega_{\overline{\alpha}}$ is the left representation on

$$L^2(\mathbb{R}) = \int_{\mathbb{R}} \chi_\xi \, d\xi.$$

Then, the spherical distributions are of the form $1 \otimes \Psi_{\xi,\overline{\alpha}}$ where

$$\Psi_{\xi,\overline{\alpha}}(f) = \left\langle \rho_{\overline{\alpha}}(f)\chi_\xi, \chi_\xi \right\rangle \text{ for } f \in \mathcal{D}(N_m).$$

We compute

$$\begin{aligned} \left\langle \rho_{\overline{\alpha}}(f)\chi_\xi, \chi_\xi \right\rangle &= \int_{\mathbb{R}} \rho_{\overline{\alpha}}(f)\chi_\xi(r)\overline{\chi_\xi}(r)\, dr \\ &= \int_{\mathbb{R}} \int_N f(s, x, y, t) e^{i \sum_{j=1}^{m} s_j \sum_{k=1}^{j} \alpha_k \frac{r^{j-k}}{j-k!}} \chi_\xi(r-x)\overline{\chi_\xi(r)}\, ds\, dx\, dy\, dt\, dr \\ &= \int_{\mathbb{R}} \int_N f(s, x, y, t) e^{i \sum_{j=1}^{m} s_j \sum_{k=1}^{j} \alpha_k \frac{r^{j-k}}{j-k!}} e^{-i\xi x}\, ds\, dx\, dy\, dt\, dr \\ &= \int_{\mathbb{R}} f \left(- \sum_{k=1}^{m} \alpha_k \widehat{\frac{r^{m-k}}{m-k!}}, \ldots, - \sum_{k=1}^{j} \alpha_k \widehat{\frac{r^{j-k}}{j-k!}}, \ldots, \widehat{\xi}, \widehat{0}, \widehat{0} \right) dr. \end{aligned}$$

- Case $\lambda \neq 0$: in this case $\omega_{\overline{\alpha},\lambda}$ is a projective representation and by Theorem 4 we obtain that

$$\phi : F_1 \otimes F_2 \mapsto \int_{\mathbb{R}} F_1(r)F_2(r)\, dr,$$

is the distribution vector fixed by K_m for

$$\rho(k, n) = \omega^*_{\overline{\alpha},\lambda}(k) \otimes \rho_{\overline{\alpha},\lambda}(n)\omega_{\overline{\alpha},\lambda}(k).$$

Then, for $f \in C_c^\infty(K_m \ltimes N_m)$ and $F_1 \otimes F_2 \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ we obtain

$$\langle \rho(f)\phi, F \rangle = \langle \phi, \rho(f)F \rangle = \int_{K_m} f_1(k)dk \left\langle \phi, F_1 \otimes \rho_{\overline{\alpha},\lambda}(f_2)F_2 \right\rangle,$$

and

$$
\begin{aligned}
\langle \phi, F_1 \otimes \rho_{\overline{\alpha},\lambda}(f_2)F_2 \rangle &= \int_{\mathbb{R}} \left[F_1 \otimes \rho_{\overline{\alpha},\lambda}(f_2)F_2 \right](r,r)\, dr \\
&= \int_{\mathbb{R}} \int_N f_2(\overline{s}, x, y, t) F_1(r) e^{i \sum_{j=1}^{m} s_j \left(\sum_{i=1}^{j} \alpha_i \frac{(r-x)^{j-i}}{j-i!} - \lambda \frac{(r-x)^{j+1}}{j+1!} \right)} e^{-i\lambda(r-x)y} e^{i\lambda(t - \frac{xy}{2})} \\
&\quad F_2(r-x)\, d\overline{s}\, dx\, dy\, dt\, dr \\
&= \int_{\mathbb{R}^2} f_2 \left(\widehat{\lambda \frac{(r-x)^{m+1}}{m+1!} - \sum_{i=1}^{m} \alpha_i \frac{(r-x)^{m-i}}{m-i!}}, \dots, \widehat{\lambda \frac{(r-x)^2}{2} - \alpha_1}, x, \lambda \widehat{\left(r - \frac{x}{2} \right)}, \widehat{-\lambda} \right) \\
&\quad F_2(r-x) F_1(r)\, dx\, dr \\
&= \int_{\mathbb{R}^2} f_2 \left(\widehat{\lambda \frac{y_2^{m+1}}{m+1!} - \sum_{i=1}^{m} \alpha_i \frac{y_2^{m-i}}{m-i!}}, \dots, \widehat{\lambda \frac{y_2^2}{2} - \alpha_1}, y_1 - y_2, \lambda \widehat{\left(\frac{y_1 + y_2}{2} \right)}, \widehat{-\lambda} \right) \\
&\quad F_2(y_2) F_1(y_1)\, dy_1\, dy_2.
\end{aligned}
$$

Thus, $\Phi_{\overline{\alpha},\lambda} = 1 \otimes \Psi_{\overline{\alpha},\lambda}$ where for $f \in \mathcal{D}(N_m)$,

$$
\begin{aligned}
\Psi_{\overline{\alpha},\lambda}(f) = \int_{\mathbb{R}} f \Bigg(& \widehat{\lambda \frac{r^{m+1}}{m+1!} - \sum_{i=1}^{m} \alpha_i \frac{r^{m-i}}{m-i!}}, \dots, \widehat{\lambda \frac{r^{j+1}}{j+1!} - \sum_{i=1}^{j} \alpha_i \frac{r^{j-i}}{j-i!}}, \dots, \\
& \widehat{\lambda \frac{r^2}{2} - \alpha_1}, 0, \widehat{\lambda r}, \widehat{-\lambda} \Bigg)\, dr.
\end{aligned}
$$

Remark 1 For $f \in \mathcal{D}(N_m)$ let $f_0(s_m, \dots, s_1, y, t) = f(s_m, \dots, s_1, 0, y, t)$. As any element $(0, b, c) \in K_m$ fixes $\Lambda \in \mathfrak{n}_m^*$, we have that $\Psi_\Lambda(f)$ is the integral of the Fourier transform of f_0 along the orbit of the transposed action of K_m on Λ.

5 Eigenvalues of Spherical Distributions

Let N be a nilpotent Lie group with Lie algebra $\mathfrak{n}$ and K a subgroup of automorphisms on N. We denote by $\mathfrak{n}^*$ the dual space of $\mathfrak{n}$, by $\mathcal{P}(\mathfrak{n}^*)$ the polynomial algebra on $\mathfrak{n}^*$ and by $\mathcal{P}(\mathfrak{n}^*)^K$ the subalgebra of $\mathcal{P}(\mathfrak{n}^*)$ of the K-invariant polynomials. The action of K on $\mathfrak{n}^*$ is given by

$$
k \cdot \alpha\, (n) = \alpha(k^{-1} n), \quad \forall n \in \mathfrak{n},
$$

and on $\mathcal{P}(\mathfrak{n}^*)$ by

$$
k \cdot p\, (\alpha) = p(k^{-1} \cdot \alpha), \quad \forall \alpha \in \mathfrak{n}^*.
$$

Let $\mathcal{B} = \{X_1, \ldots, X_l\}$ be a bases of $\mathfrak{n}$. We identify the symmetric algebra $\mathcal{S}(\mathfrak{n})$ with $\mathcal{P}(\mathfrak{n}^*)$ by the map

$$X_1 \cdots X_l \mapsto p_{X_1 \cdots X_l},$$

where $p_{X_1 \cdots X_l}(\alpha) = \alpha(X_1) \cdots \alpha(X_l)$. Even more, if $\mathcal{S}(\mathfrak{n})^K$ denote the K-invariant subalgebra of $\mathcal{S}(\mathfrak{n})$, we identify $\mathcal{S}(\mathfrak{n})^K$ with $\mathcal{P}(\mathfrak{n}^*)^K$.

There is a linear map $\lambda : \mathcal{S}(\mathfrak{n}) \longrightarrow \mathcal{U}(\mathfrak{n})$, called the symmetrization map, defined by

$$\lambda(p)(f)(n) = p\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_l}\right)\Bigg|_{t=0} \left(f\left(n \cdot exp \sum_i t_i X_i\right)\right). \tag{14}$$

λ is a linear bijection that yields a linear isomorphism between $\mathcal{S}(\mathfrak{n})^K$ and $\mathcal{U}(\mathfrak{n})^K$ (see [10]).

Our goal is to study $\mathcal{S}(\mathfrak{n}_m)^{K_m}$.

5.1 Invariant Polynomials

Let $\mathfrak{n}'_m$ be the abelian subalgebra generated by $S_m, \ldots, S_1, Y, T$.

Lemma 10 $\mathcal{S}(\mathfrak{n}_m)^{K_m} = \mathcal{S}(\mathfrak{n}'_m)^{K_m}$.

Proof Let $p \in \mathcal{S}(\mathfrak{n}_m)$, we can write $p(S_m, \ldots, S_1, X, Y, T) = \sum_{i=0}^{N} q_i(S_m, \ldots, S_1, Y, T)X^i$. So, let $k = (0, b, 0) \in \overline{K}_m$ with $b \neq 0$ then

$$\begin{aligned}
k \cdot p = p \Leftrightarrow \sum_{i=0}^{N} k \cdot q_i(S_m, \ldots, S_1, Y, T)\, k \cdot X^i &= \sum_{i=0}^{N} q_i(S_m, \ldots, S_1, Y, T)X^i \\
&\Leftrightarrow \sum_{i=0}^{N} q_i(k \cdot S_m, \ldots, k \cdot S_1, k \cdot Y, k \cdot T)\ (X + bY)^i \\
&= \sum_{i=0}^{N} q_i(S_m, \ldots, S_1, Y, T)X^i \\
&\Leftrightarrow \sum_{i=0}^{N} q_i(S_m, \ldots, S_1, Y, T)\ (X + bY)^i = \sum_{i=0}^{N} q_i(S_m, \ldots, S_1, Y, T)X^i \\
&\Leftrightarrow \sum_{i=1}^{N} q_i(S_m, \ldots, S_1, Y, T) \sum_{j=1}^{i} \binom{i}{j} b^j X^{i-j} Y^j = 0.
\end{aligned}$$

If we see the last equality as a polynomial in the variable b we have

$$0 = q_i(S_m, \ldots, S_1, Y, T) \quad \forall i \geq 1.$$

So,

$$p(S_m, \ldots, S_1, X, Y, T) = q_0(S_m, \ldots, S_1, Y, T).$$

□

Let $(a, 0, 0) \in K_m$ with $a \neq 0$. The action of $(a, 0, 0)$ on $\{S_m, \ldots, S_1, Y, -T\}$ is given by

$$e^{aE} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a & 1 & 0 & \cdots & 0 & 0 \\ \frac{a^2}{2!} & a & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{a^m}{m!} & \frac{a^{m-1}}{m-1!} & \frac{a^{m-2}}{m-2!} & \cdots & 1 & 0 \\ \frac{a^{m+1}}{m+1!} & \frac{a^m}{m!} & \frac{a^{m-1}}{m-1!} & \cdots & a & 1 \end{pmatrix},$$

where

$$E = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Lemma 11 $\mathcal{S}(\mathfrak{n}_m)^{K_m} = ker(E)$.

Proof If we derive $e^{aE}p = p$ with respect to a, we obtain that p is K_m-invariant if and only if $Ep = 0$. □

We use the $\mathfrak{sl}(2, \mathbb{C})$ representation theory in order to solve $Ep = 0$. It is well know that, for each $n \in \mathbb{N}$, $\mathfrak{sl}(2, \mathbb{C})$ has an irreducible representation (ρ_n, V_n) of dimension $n + 1$. The action ρ_n gives rise to an action on $\mathcal{S}(V_n)$ given by

$$g \cdot (v_1 v_2 \cdots v_k) = (g \cdot v_1)v_2 \cdots v_k + v_1(g \cdot v_2) \cdots v_k + \cdots + v_1 v_2 \cdots (g \cdot v_k). \quad (15)$$

We denote by

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

the standard bases of $\mathfrak{sl}(2, \mathbb{C})$. Note that $E = \rho_{m+1}(e)$.

Let $S_j(V_{m+1})$ be the space of homogeneous polynomials of degree j and let $S_j(V_{m+1})^K$ be the K-invariant subspace of $S_j(V_{m+1})$. According to the highest weight theory the dimension of $S_j(V_{m+1})^K$ is equal to the number of $\mathfrak{sl}(2, \mathbb{C})$ irreducible components of $\mathcal{S}_j(V_{m+1})$.

Lemma 12 *For* $l \in \{1, \dots, m\}$*, if* p_l *is given by* $p_l(S_l, \dots, S_1, Y, T) = \frac{l+1!}{l} \sum_{j=0}^{l-1} \frac{1}{j!} S_{l-j} Y^j T^{l-j} + Y^{l+1}$*, then* $p_l \in \mathcal{S}(\mathfrak{n}_m)^{K_m}$.

Proof By (15) we have

$$
\begin{aligned}
Ep_l &= \frac{l+1!}{l} \sum_{j=0}^{l-1} \frac{1}{j!} E\left(S_{l-j} Y^j T^{l-j}\right) + E\left(Y^{l+1}\right) \\
&= \frac{l+1!}{l} \sum_{j=0}^{l-1} \frac{1}{j!} E\left(S_{l-j}\right) Y^j T^{l-j} + \frac{l+1!}{l} \sum_{j=1}^{l-1} \frac{1}{j!} S_{l-j} E\left(Y^j\right) T^{l-j} \\
&\quad + \frac{l+1!}{l} \sum_{j=0}^{l-1} \frac{1}{j!} S_{l-j} Y^j E\left(T^{l-j}\right) + E\left(Y^{l+1}\right) \\
&= \frac{l+1!}{l} \frac{1}{l-1!} Y Y^{l-1} T + \frac{l+1!}{l} \sum_{j=0}^{l-2} \frac{1}{j!} S_{l-j-1} Y^j T^{l-j} \\
&\quad - \frac{l+1!}{l} \sum_{j=1}^{l-1} \frac{1}{j!} S_{l-j}\, j Y^{j-1} T T^{l-j} - (l+1) Y^l T \\
&= \frac{l+1!}{l} \sum_{j=0}^{l-2} \frac{1}{j!} S_{l-j-1} Y^j T^{l-j} - \frac{l+1!}{l} \sum_{j=1}^{l-1} \frac{1}{j-1!} S_{l-j} Y^{j-1} T^{l-j+1} \\
&= 0.
\end{aligned}
$$

The result follows from Lemma 11. □

We determine $\mathcal{S}(\mathfrak{n}_m)^{K_m}$ for the cases $m = 1$ and $m = 2$.

- Case $m = 1$: clearly, $T \in \mathcal{S}(\mathfrak{n}_1)^{K_1}$ and by Lemma 12, $Y^2 + 2S_1T \in \mathcal{S}(\mathfrak{n}_1)^{K_1}$. It is immediate to see that they are algebraically independent.
  We recall that $SO(3)$ acts on V_2 by the natural action on $\mathbb{R}^3$ and it is well known that

$$
\mathcal{S}_k = H_k \oplus \|x\|^2 \mathcal{S}_{k-2},
$$

  where H_k is the space of harmonic polynomials of degree k (see Theorem 2.1 page 139 in [17]). Since H_k is $SO(3)$-irreducible it has, up to a constant, only one highest weight vector. If h_1 denote the highest weight vector of degree 1, we have h_1 and $\|x\|^2$ generate $\mathcal{S}(V_2)$, this is

$$
\mathcal{S}(V_2) = \mathbb{C}[h_1, \|x\|^2].
$$

  So, we have the following

Proposition 13 $\mathcal{S}(\mathfrak{n}_1)^{K_1}$ *is the polynomial algebra generated by* T *and* $Y^2 + 2S_1T$.

- Case $m = 2$: We get

$$q_1(S_2, S_1, Y, T) = T \in \mathcal{S}(\mathfrak{n}_2)^{K_2},$$

and by Lemma 12,

$$q_2(S_2, S_1, Y, T) = Y^2 + 2S_1T \in \mathcal{S}(\mathfrak{n}_2)^{K_2},$$
$$q_3(S_2, S_1, Y, T) = Y^3 + 3S_2T^2 + 3S_1YT \in \mathcal{S}(\mathfrak{n}_2)^{K_2}.$$

Also,

Lemma 14 $q_4(S_2, S_1, Y, T) = 6Y^3S_2 - 3Y^2S_1^2 + 9S_2^2T^2 + 18YS_1S_2T - 8S_1^3T \in \mathcal{S}(\mathfrak{n}_2)^{K_2}$.

We omit the proof since it is straightforward to check that $q_4 \in ker(E)$. However, it should be clarified that, thanks to the representation theory of $\mathfrak{sl}(2,\mathbb{C})$ we know that

$$\mathcal{S}_4(V_3) = V_{12} \oplus V_4 \oplus V_6 \oplus V_8 \oplus V_0,$$

and q_4 corresponds to the highest weight vector of V_0.

We get

$$\mathcal{S}(\mathfrak{n}_2) = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \mathcal{S}_j(\mathfrak{n}_2).$$

From the representation theory of $\mathfrak{sl}(2,\mathbb{C})$, we know that the number of $\mathfrak{sl}(2,\mathbb{C})$ irreducible components of $\mathcal{S}_j(\mathfrak{n}_2)$ is equal to the j-th coefficient in the MacLaurin series expansion of

$$G(t) = \frac{1 - t^6}{(1-t)(1-t^2)(1-t^3)(1-t^4)}, \tag{16}$$

(see [18]). Also, from the highest weight theory, the number of $\mathfrak{sl}(2,\mathbb{C})$ irreducible components of $\mathcal{S}_j(\mathfrak{n}_2)$ is equal to dimension of $ker(E|_{\mathcal{S}_j})$. Then, as

$$\frac{1}{(1-t)}\frac{1}{(1-t^2)}\frac{1}{(1-t^3)}\frac{1}{(1-t^4)} = \sum_{\substack{j=0 \\ j=j_1+2j_2+3j_3+4j_4}}^{\infty} a_j t^j, \tag{17}$$

where

$$a_j = \#\{(j_1, j_2, j_3, j_4) \in \mathbb{N}_0^4 : j_1 + 2j_2 + 3j_3 + 4j_4 = j\},$$

from (16) and (17), we obtain

$$G(t) = \sum_{j=0}^{5} a_j\, t^j + \sum_{j=6}^{\infty} (a_j - a_{j-6})\, t^j.$$

So,

$$dim\left(\mathcal{S}_j(\mathfrak{n}_2)^{K_2}\right) = \begin{cases} a_j, & \text{if } j < 6 \\ a_j - a_{j-6}, & \text{if } j \geq 6. \end{cases} \tag{18}$$

Next, we are devoting to prove that $\{q_1, q_2, q_3, q_4\}$ is a set of generators of $\mathcal{S}(\mathfrak{n}_2)^{K_2}$. It is sufficient to show that

$$\begin{aligned} ev : \mathbb{C}[x, y, z, w] &\longrightarrow \mathcal{S}(\mathfrak{n}_2)^{K_2} \\ f &\longmapsto f(q_1, q_2, q_3, q_4), \end{aligned}$$

is an epimorphism. In fact, we set $F_j := \langle x^{i_1} y^{i_2} z^{i_3} w^{i_4} \mid i_1 + 2i_2 + 3i_3 + 4i_4 = j \rangle$ and thus

$$dim(F_j) = a_j. \tag{19}$$

If $ev_j = ev|_{F_j}$ then we have ev is an epimorphism if and only if ev_j is an epimorphism for all j. So, let us first prove the following

Proposition 15 *Let f_6 be defined by $f_6(x, y, z, w) = z^2 - y^3 - x^2 w$, then*

(i) $ker(ev) = f_6\, \mathbb{C}[x, y, z, w]$,

(ii) $ker(ev_j) = \begin{cases} 0, & \textit{if } j < 6 \\ f_6\, F_{j-6}, & \textit{if } j \geq 6 \end{cases}$.

(iii) *ev_j is an epimorphism.*

Proof (i) On the one hand, it is straightforward to check that $f_6 \in Ker(ev)$.
On the other hand, it easy to see that q_1, q_2 and q_4 are algebraically independent. Also, given $g_0, g_1 \in \mathbb{C}[x, y, w]$ nonzero, by checking the largest exponent of y in

$$g_0(q_1, q_2, q_4) + g_1(q_1, q_2, q_4)\, q_3,$$

we obtain that $g_0 + g_1 z \notin ker(ev)$. So, from the above, we have that given $f \in Ker(ev)$ we can write

$$f(x, y, z, w) = \sum_{j=0}^{n} g_j(x, y, w) z^j,$$

with $n \geq 2$. Now, we prove the statement by induction on n: we set

$$g(x, y, z, w) = f(x, y, z, w) - g_n(x, y, w) z^{n-2} f_6 \in Ker(ev). \tag{20}$$

By induction hypothesis

$$g = p f_6, \tag{21}$$

for some $p \in \mathbb{C}[x, y, z, w]$. Then, from (20) y (21), we obtain that

$$\begin{aligned} f(x, y, z, w) &= g(x, y, z, w) + g_n(x, y, w)z^{n-2} f_6(x, y, z, w) \\ &= [p(x, y, z, w) + g_n(x, y, w)z^{n-2}] f_6(x, y, z, w). \end{aligned}$$

(ii) It follows from the fact that $ker(ev_j) = ker(ev) \cap F_j$.
(iii) Finally,

$$dim(Im(ev_j)) = dim(F_j) - dim(ker(ev_j)) = dim(\mathcal{S}_j^{K_2}).$$

□

Thus, we have proved the following

Theorem 16 *$\mathcal{S}(\mathfrak{n}_2)^{K_2}$ is the algebra generated by q_1, q_2, q_3 and q_4.*

5.2 Eigenvalues

From (14), we have

$$\begin{aligned} S_j f(s_m, \ldots, s_1, x, y, t) &= \frac{d}{dr}\Big|_{r=0} f\left((s_m, \ldots, s_1, x, y, t)exp(re_j)\right) \\ &= \frac{d}{dr}\Big|_{r=0} f\left(s_m, \ldots, s_j + r, \ldots, s_1, x, y, t\right) \\ &= \frac{\partial f}{\partial s_j}(s_m, \ldots, s_1, x, y, t). \\ Y f(s_m, \ldots, s_1, x, y, t) &= \frac{d}{dr}\Big|_{r=0} f\left((s_m, \ldots, s_1, x, y, t)exp(re_y)\right) \\ &= \frac{d}{dr}\Big|_{r=0} f\left(s_m, \ldots, s_1, x, y + r, t + \frac{1}{2}xr\right) \\ &= \frac{\partial f}{\partial y}(s_m, \ldots, s_1, x, y, t) + \frac{x}{2}\frac{\partial f}{\partial t}(s_m, \ldots, s_1, x, y, t). \\ T f(s_m, \ldots, s_1, x, y, t) &= \frac{d}{dr}\Big|_{r=0} f\left((s_m, \ldots, s_1, x, y, t)exp(re_t)\right) \\ &= \frac{d}{dr}\Big|_{r=0} f\left(s_m, \ldots, s_1, x, y, t + r\right) \\ &= \frac{\partial f}{\partial t}(s_m, \ldots, s_1, x, y, t). \end{aligned}$$

Since $\mathfrak{n}'_m$ is an abelian algebra, we have the following invariant operators

$$D_1 = T \text{ and } D_{j+1} = \frac{j+1!}{j}\sum_{k=0}^{j-1}\frac{1}{k!}S_{j-k}Y^kT^{j-k} + Y^{j+1}, \quad \forall j \in \{1,\ldots,m\}.$$

Proof of Theorem 6. For $\Lambda = (\overline{\alpha}, 0, 0, \lambda)$ with $\lambda \neq 0$ and $f \in \mathcal{D}(N_m)$ we have

$$\Psi_{\overline{\alpha},\lambda}(f) = \int_{\mathbb{R}} f\left(\widehat{\lambda\frac{r^{m+1}}{m+1!} - \sum_{i=1}^{m}\alpha_i\frac{r^{m-i}}{m-i!}}, \ldots, \widehat{\lambda\frac{r^{j+1}}{j+1!} - \sum_{i=1}^{j}\alpha_i\frac{r^{j-i}}{j-i!}}, \ldots, \widehat{\lambda\frac{r^2}{2} - \alpha_1}, 0, \widehat{\lambda r}, \widehat{-\lambda}\right) dr.$$

Then,

$$\begin{aligned}
D_1\Psi_{\overline{\alpha},\lambda}(f) &= -\Psi_{\overline{\alpha},\lambda}(D_1 f)\\
&= -\int_{\mathbb{R}} \frac{\partial f}{\partial t}\left(\widehat{\lambda\frac{r^{m+1}}{m+1!} - \sum_{i=1}^{m}\alpha_i\frac{r^{m-i}}{m-i!}}, \ldots, \widehat{\lambda\frac{r^2}{2} - \alpha_1}, 0, \widehat{\lambda r}, \widehat{-\lambda}\right) dr\\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\partial f}{\partial t}\left(\widehat{\lambda\frac{r^{m+1}}{m+1!} - \sum_{i=1}^{m}\alpha_i\frac{r^{m-i}}{m-i!}}, \ldots, \widehat{\lambda\frac{r^2}{2} - \alpha_1}, 0, \widehat{\lambda r}, t\right) e^{it\lambda}\,dt\,dr\\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}} f\left(\widehat{\lambda\frac{r^{m+1}}{m+1!} - \sum_{i=1}^{m}\alpha_i\frac{r^{m-i}}{m-i!}}, \ldots, \widehat{\lambda\frac{r^2}{2} - \alpha_1}, 0, \widehat{\lambda r}, t\right) i\lambda e^{it\lambda}\,dt\,dr\\
&= -i\lambda\int_{\mathbb{R}} f\left(\widehat{\lambda\frac{r^{m+1}}{m+1!} - \sum_{i=1}^{m}\alpha_i\frac{r^{m-i}}{m-i!}}, \ldots, \widehat{\lambda\frac{r^2}{2} - \alpha_1}, 0, \widehat{\lambda r}, \widehat{-\lambda}\right) dr\\
&= -i\lambda\,\Psi_{\overline{\alpha},\lambda}(f),
\end{aligned}$$

and by similar arguments we have

$$\begin{aligned}
&D_{j+1}\Psi_{\overline{\alpha},\lambda}(f) = (-1)^j\Psi_{\overline{\alpha},\lambda}(D_{j+1}f)\\
&= (-1)^j\frac{j+1!}{j}\sum_{k=0}^{j-1}\frac{1}{k!}(i\lambda)^{j-k}\int_{\mathbb{R}}(-i\lambda r)^k\frac{\partial f}{\partial s_{j-k}}\\
&\quad\left(\widehat{\lambda\frac{r^{m+1}}{m+1!} - \sum_{i=1}^{m}\alpha_i\frac{r^{m-i}}{m-i!}}, \ldots, \widehat{\lambda\frac{r^2}{2} - \alpha_1}, 0, \widehat{\lambda r}, \widehat{-\lambda}\right) dr\\
&\quad + (-1)^j\int_{\mathbb{R}}(-i\lambda r)^{j+1} f\left(\widehat{\lambda\frac{r^{m+1}}{m+1!} - \sum_{i=1}^{m}\alpha_i\frac{r^{m-i}}{m-i!}}, \ldots, \widehat{\lambda\frac{r^2}{2} - \alpha_1}, 0, \widehat{\lambda r}, \widehat{-\lambda}\right) dr
\end{aligned}$$

$$= -\frac{j+1!}{j}(-i\lambda)^{j+1}\sum_{k=0}^{j-1}\frac{(-1)^{k+1}}{k!j-k+1!}\int_{\mathbb{R}} r^{j+1}$$

$$f\left(\lambda\frac{r^{m+1}}{m+1!}\widehat{-\sum_{i=1}^{m}}\alpha_i\frac{r^{m-i}}{m-i!},\ldots,\lambda\widehat{\frac{r^2}{2}-}\alpha_1,0,\widehat{\lambda r},\widehat{-\lambda}\right)dr$$

$$-\frac{j+1!}{j}(-i)^{j+1}\lambda^j\sum_{k=0}^{j-1}\sum_{i=1}^{j-k}\alpha_i\frac{(-1)^{k+1}}{k!j-k-i!}\int_{\mathbb{R}} r^{j-i}$$

$$f\left(\ldots,\lambda\frac{r^{j+1}}{j+1!}\widehat{-\sum_{i=1}^{j}}\alpha_i\frac{r^{j-i}}{j-i!},\ldots,\lambda\widehat{\frac{r^2}{2}-}\alpha_1,0,\widehat{\lambda r},\widehat{-\lambda}\right)dr$$

$$-\int_{\mathbb{R}}(i\lambda r)^{j+1}f\left(\lambda\frac{r^{m+1}}{m+1!}\widehat{-\sum_{i=1}^{m}}\alpha_i\frac{r^{m-i}}{m-i!},\ldots,\lambda\widehat{\frac{r^2}{2}-}\alpha_1,0,\widehat{\lambda r},\widehat{-\lambda}\right)dr$$

$$= -\frac{j+1!}{j}(-i\lambda)^{j+1}\left(\frac{(-1)^j}{j!}+\frac{(-1)^{j+1}}{j+1!}\right)\int_{\mathbb{R}} r^{j+1}$$

$$f\left(\lambda\frac{r^{m+1}}{m+1!}\widehat{-\sum_{i=1}^{m}}\alpha_i\frac{r^{m-i}}{m-i!},\ldots,\lambda\widehat{\frac{r^2}{2}-}\alpha_1,0,\widehat{\lambda r},\widehat{-\lambda}\right)dr$$

$$+\frac{j+1!}{j}(-i)^{j+1}\lambda^j\sum_{i=1}^{j}\alpha_i\sum_{k=0}^{j-i}\frac{(-1)^k}{k!j-k-i!}\int_{\mathbb{R}} r^{j-i}$$

$$f\left(\lambda\frac{r^{m+1}}{m+1!}\widehat{-\sum_{i=1}^{m}}\alpha_i\frac{r^{m-i}}{m-i!},\ldots,\lambda\widehat{\frac{r^2}{2}-}\alpha_1,0,\widehat{\lambda r},\widehat{-\lambda}\right)dr$$

$$-\int_{\mathbb{R}}(i\lambda r)^{j+1}f\left(\lambda\frac{r^{m+1}}{m+1!}\widehat{-\sum_{i=1}^{m}}\alpha_i\frac{r^{m-i}}{m-i!},\ldots,\lambda\widehat{\frac{r^2}{2}-}\alpha_1,0,\widehat{\lambda r},\widehat{-\lambda}\right)dr$$

$$= -\frac{j+1!}{j}(-i\lambda)^{j+1}\frac{j}{j+1!}(-1)^j\int_{\mathbb{R}} r^{j+1}$$

$$f\left(\lambda\frac{r^{m+1}}{m+1!}\widehat{-\sum_{i=1}^{m}}\alpha_i\frac{r^{m-i}}{m-i!},\ldots,\lambda\widehat{\frac{r^2}{2}-}\alpha_1,0,\widehat{\lambda r},\widehat{-\lambda}\right)dr$$

$$+\frac{j+1!}{j}(-i)^{j+1}\lambda^j\alpha_j\int_{\mathbb{R}}$$

$$f\left(\lambda\frac{r^{m+1}}{m+1!}\widehat{-\sum_{i=1}^{m}}\alpha_i\frac{r^{m-i}}{m-i!},\ldots,\lambda\widehat{\frac{r^2}{2}-}\alpha_1,0,\widehat{\lambda r},\widehat{-\lambda}\right)dr$$

$$-\int_{\mathbb{R}}(i\lambda r)^{j+1}f\left(\lambda\frac{r^{m+1}}{m+1!}\widehat{-\sum_{i=1}^{m}}\alpha_i\frac{r^{m-i}}{m-i!},\ldots,\lambda\widehat{\frac{r^2}{2}-}\alpha_1,0,\widehat{\lambda r},\widehat{-\lambda}\right)dr$$

$$= \frac{j+1!}{j}(-i)^{j+1}\lambda^j\alpha_j\ \Psi_{\overline{\alpha},\lambda}(f).$$

So,

$$D_1(\Psi_{\overline{\alpha},\lambda}) = -i\lambda\Psi_{\overline{\alpha},\lambda},$$
$$D_{j+1}(\Psi_{\overline{\alpha},\lambda}) = \frac{j+1!}{j}(-i)^{j+1}\lambda^j\alpha_j\ \Psi_{\overline{\alpha},\lambda} \quad \forall j = 1, \dots, m.$$

□

For $m = 2$, let L_j be the differential operator corresponding to q_j for $j = 1, \dots, 4$. So, $\{L_1, L_2, L_3, L_4\}$ is a set of generators of $\mathcal{U}(\mathfrak{n}_2)^{K_2}$ and we will prove that the corresponding set of eigenvalues do not determine Φ_Λ in the cases $\Lambda = (\alpha, 0, \nu, 0)$, $\nu \neq 0$.

In fact,

$$\Psi_{\overline{\alpha},\nu}(f) = \int_{\mathbb{R}} f\left(-\alpha_2 - \widehat{\alpha_1 r} - \nu\frac{r^2}{2}, -\widehat{\alpha_1 - \nu r}, 0, \widehat{-\nu}, \hat{0}\right) dr,$$

and with some similar accounts to the previous case, we have

$$L_1\Psi_{\overline{\alpha},\nu}(f) = 0\ \Psi_{\overline{\alpha},\nu}(f).$$
$$L_2\ \Psi_{\overline{\alpha},\nu}(f) = -\nu^2\ \Psi_{\overline{\alpha},\nu}(f).$$
$$L_3\ \Psi_{\overline{\alpha},\nu}(f) = i\nu^3\ \Psi_{\overline{\alpha},\nu}(f).$$
$$L_4\ \Psi_{\overline{\alpha},\nu}(f) = \left(6\nu^3\alpha_2 - 3\nu^2\alpha_1^2\right)\ \Psi_{\overline{\alpha},\nu}(f).$$

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