Journal of Fourier Analysis and Applications (2024) 30:16
https://doi.org/10.1007/s00041-024-10073-3

A Restriction Estimate with a Log-Concavity Assumption

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Received: 28 September 2022 / Accepted: 30 January 2024 / Published online: 14 March 2024

Abstract

The purpose of this paper is to prove an optimal restriction estimate for a class of flat curves in $\mathbb{R}^d$, $d \geq 3$. Namely, we consider the problem of determining all the pairs (p, q) for which the $L^p - L^q$ estimate holds (or a suitable Lorentz norm substitute at the endpoint, where the $L^p - L^q$ estimate fails) for the extension operator associated to $\gamma(t) = (t, \frac{t^2}{2!}, \ldots, \frac{t^{d-1}}{(d-1)!}, \phi(t))$, $0 \leq t \leq 1$, with respect to the affine arclength measure. In particular, we are interested in the flat case, i.e. when $\phi(t)$ satisfies $\phi^{(d)}(0) = 0$ for all integers $d \geq 1$. A prototypical example is given by $\phi(t) = e^{-1/t}$. The paper (Bak et al., J. Aust. Math. Soc. 85:1–28, 2008) addressed precisely this problem. The examples in Bak et al. (2008) are defined recursively in terms of an integral, and they represent progressively flatter curves. Although these include arbitrarily flat curves, it is not clear if they cover, for instance, the prototypical case $\phi(t) = e^{-1/t}$. We will show that the desired estimate does hold for that example and indeed for a class of examples satisfying some hypotheses involving a log-concavity condition.

Keywords Fourier restriction · Simple curve · Log-concave

Mathematics Subject Classification 42B10 · 42B99

1 Introduction

Let $d \geq 2$. Let $\gamma : I \to \mathbb{R}^d$ be a C^d curve defined on an interval I. The restriction of the Fourier transform of f to γ is given by

$$\hat{f}(\gamma(t)) = \int_{\mathbb{R}^d} e^{-i\langle x, \gamma(t)\rangle} f(x) dx$$

Communicated by Hans G. Feichtinger.

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for Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d)$. We are interested in the $L^p - L^q$ estimate of the restriction of the Fourier transform:

$$\left(\int_I \left| \widehat{f}(\gamma(t)) \right|^q \, dt \right)^{1/q} \leq C \| f \|_{L^p(\mathbb{R}^d)}, \tag{1}$$

and for what $p - q$ range the estimate holds. The trivial estimate is the $L^1 - L^\infty$ estimate. The critical line for the $p - q$ range is $\frac{1}{q} = \frac{d(d+1)}{2} \frac{1}{p'}$, $q > \frac{d^2+d+2}{d^2+d}$, where p' is the Hölder conjugate exponent of p. (See [1].)

We are also interested in the conditions on γ that allows the $L^p - L^q$ estimate to hold on the critical line. The simplest case is $\gamma(t) = (t, \frac{t^2}{2!}, \ldots, \frac{t^d}{d!})$. Zygmund [18] and Hörmander [13] showed that (1) holds on the critical line for $d = 2$ and Drury [11] showed the corresponding result for $d \geq 3$. Christ [8] proved partial results for more general curves, and Bak et al. [4] showed that the estimate (1) holds if γ is nondegenerate. Now consider a curve of simple type of the form $\gamma(t) = (t, \frac{t^2}{2!}, \ldots, \frac{t^{d-1}}{(d-1)!}, \phi(t))$ where ϕ is a C^d function. In this case, (1) may fail if γ is degenerate, unless we replace the Euclidean arclength measure by the affine arclength measure. Let $w(t)$ be a weight function defined by

$$w(t) = |\tau_\gamma(t)|^{\frac{2}{d^2+d}}$$

where $\tau_\gamma = \det(\gamma' \ \gamma'' \ \ldots \ \gamma^{(d)})$ is a torsion of γ. The affine arclength measure is given by $w(t)dt$. Thus, we will replace the estimate (1) by

$$\left(\int_I \left| \hat{f}(\gamma(t)) \right|^q w(t) \, dt \right)^{1/q} \leq C \| f \|_{L^p(\mathbb{R}^d)}. \tag{2}$$

Furthermore, even though (2) fails at the endpoint $p = q = \frac{d^2+d+2}{d^2+d}$, the restricted strong type (p, q) may hold:

$$\left(\int_I \left| \widehat{f}(\gamma(t)) \right|^q w(t) \, dt \right)^{1/q} \leq C \| f \|_{L^{p,1}(\mathbb{R}^d)}. \tag{3}$$

Bak et al. [3] showed that (2) holds for curves satisfying some conditions on the critical line, and in [5], they showed the endpoint estimate (3) holds when ϕ is any polynomial, where $C = C_N$ depends only on the upper bound N on the degree of the polynomial. Also, Bak and Ham [2] showed the corresponding endpoint estimate for certain complex curves $\gamma(z) \in \mathbb{C}^d$ of simple type. For more cases, see also [10], [16] and [17].

In this paper, we extend the result in [3] to the endpoint estimate, i.e., (3) holds for some curves that satisfy some hypotheses involving a certain log-concavity condition.

Theorem 1.1 *Suppose $d \geq 2$. Let $\gamma \in C^d(I)$ be of the form*

$$\gamma(t) = \left(t, \frac{t^2}{2!}, \ldots, \frac{t^{d-1}}{(d-1)!}, \phi(t)\right)$$

defined on $I = (0, 1)$. Suppose that $\phi^{(d)}$ is positive and increasing on I. Suppose that there exists $\delta > 0$ such that $\phi^{(d)}$ is log-concave on $(0, \delta)$, i.e.,

$$\phi^{(d)}(\lambda x_1 + (1-\lambda)x_2) \geq [\phi^{(d)}(x_1)]^{\lambda}[\phi^{(d)}(x_2)]^{1-\lambda} \tag{4}$$

for all $\lambda \in [0, 1]$ and $x_1, x_2 \in (0, \delta)$. Then, for $p_d = (d^2 + d + 2)/(d^2 + d)$, there is a constant $C < \infty$, depending only on d, such that for all $f \in L^{p_d,1}(\mathbb{R}^d)$,

$$\left(\int_I \left|\widehat{f}(\gamma(t))\right|^{p_d} w(t)\, dt\right)^{1/p_d} \leq C\|f\|_{L^{p_d,1}(\mathbb{R}^d)}. \tag{5}$$

The paper is organized as follows. In Sect. 2, we establish a lower bound for a Jacobian related to an offspring curve. In Sect. 3, we collect some useful results on interpolation spaces. Section 4 is devoted to the proof of Theorem 1.1. In Sect. 5, we provide some relevant examples.

We will use the notation $A \lesssim B$ to mean that $A \leq CB$ for some constant C depending only on d. And $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2 A Lower Bound for a Certain Jacobian

In this section, we establish the lower bound for a certain Jacobian, which plays an important role to prove Theorem 1.1. Before formulating this result, we introduce some notation before presenting the crucial proposition needed to prove Theorem 1.1.

For $d \geq 2$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, let $V(x)$ denote the determinant of the Vandermonde matrix:

$$V_d(x) = \prod_{1 \leq i < j \leq d} (x_j - x_i).$$

For $0 \leq t = t_1 \leq \cdots \leq t_d$, let $h_i = t_i - t_1$. Then, $0 = h_1 \leq \cdots \leq h_d$ and $t_i = t + h_i$. Also, define

$$v(h) = V_d(h) = \prod_{1 \leq i < j \leq d} (h_j - h_i).$$

If $\gamma : [0, 1] \to \mathbb{R}^d$ and if $0 < t < 1 - h_d$, define

$$\Gamma(t, h) = \sum_{i=1}^{d} \gamma(t + h_i),$$

which is called an offspring curve of γ for each fixed h. Let $J_\phi(t,h)$ be the Jacobian determinant of Γ:

$$J_\phi(t,h) = \det\left(\frac{\partial \Gamma}{\partial t}, \frac{\partial \Gamma}{\partial h_1}, \dots, \frac{\partial \Gamma}{\partial h_d}\right).$$

Now we formulate the following proposition, which provides the lower bound of Jacobian of the offspring curve. (See also Proposition 2.1 in [3] and Proposition 3.5 in [9].)

Proposition 2.1 *Let $J_\phi(t,h)$ be defined as above, where $\gamma(t) = (t, \frac{t^2}{2!}, \dots, \frac{t^{d-1}}{(d-1)!}, \phi(t))$ satisfies the condition in Theorem 1.1. Then, for $t \in [0,\delta)$, $h \in (0,\delta)^{d-1}$, and $t + h_d < \delta$,*

$$J_\phi(t,h) \geq C_d\ v(h)\left[\prod_{i=1}^{d} \phi^{(d)}(t+h_d)\right]^{1/d} \tag{6}$$

for some constant C_d which depends only on d.

Before embarking on the proof of Proposition 2.1, we need some definitions and lemmas from [3].

Lemma 2.2 (Lemma 2.2 in [3]) *Fix $\lambda \in (0,1)$. Define some intervals (a_i, b_i) by*

$$a_i < b_i \quad for\ i = 1, \dots, N \ and\ b_i \leq a_{i+1} \quad for\ i = 1, \dots, N-1.$$

Suppose also that for $m = 1, \dots, M$, and for $s \in \mathbb{R}^N$, $v_m(s)$ is a function having one of the three following forms:

$$v_m(s) = \begin{cases} s_j - s_i & for\ some\ 1 \leq i < j \leq N, \\ d_i - s_i & for\ some\ d_j \geq b_j, \\ s_i - c_i & for\ some\ c_j \leq a_j. \end{cases}$$

Suppose that $\lambda_n \in (0,1)$ and $\lambda_n \leq \lambda$ for $n = 1, \dots, N$. Let $\mathcal{R}_N(a,b,\lambda)$ be the region of all $s = (s_1, \dots, s_N) \in \mathbb{R}^N$ satisfying $(1-\lambda_n)a_n + \lambda_n b_n \leq s_n \leq b_n$ for $n = 1, \dots, N$. Then

$$\int_{\mathcal{R}_N(a,b,\lambda)} \prod_{m=1}^{M} v_m(s)\, ds_N \dots ds_1 \geq C(M,\lambda)^N \int_{a_1}^{b_1} \dots \int_{a_N}^{b_N} \prod_{m=1}^{M} v_m(s)\, ds_N \dots ds_1. \tag{7}$$

Now, we define a function $\zeta_d(t;h)$ recursively:

$$\zeta_2(t;h_2) = \chi_{[0,h_2]}(t) \tag{8}$$

For $d \geq 3$ and $t \leq h_d$, define

$$\begin{aligned}\mathfrak{R}_{d-1}(t,h) = \{x \in \mathbb{R}^{d-1} : &0 \leq x_1 \leq \min(t, h_2),\\ &h_i \leq x_i \leq h_{i+1},\ i = 2, \dots, d-2\\ &\max(t, h_{d-1}) \leq x_{d-1} \leq h_d\},\end{aligned} \tag{9}$$

and define

$$\zeta_d(t; h) = \int_{\mathfrak{R}_{d-1}(t,h)} \zeta_{d-1}(t - u_1; u_2, \dots, u_{d-1})\, du_1 \dots du_{d-1} \tag{10}$$

if $t \leq h_d$, and $\zeta_d(t; h) = 0$ if $t > h_d$.

Consider a function $\widetilde{J}^d_\phi(s) : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\widetilde{J}^d_\phi(s) = \det(\gamma'(s_1) \dots \gamma'(s_d)). \tag{11}$$

Notice that $\gamma'(s_i) = (1, s_i, \dots, (s_i)^{d-2}/(d-2)!, \phi'(s_i))$.
Observe that by simple calculation,

$$\widetilde{J}^d_\phi(t, t + h_2, \dots, t + h_d) = J_\phi(t, h).$$

Lemma 2.3 (Lemma 2.3 in [3]) *Let ζ_d and $\widetilde{J}^d_\phi(t)$ be defined by (8), (10), and (11) with $s_1 \leq \dots \leq s_d$. Then*

$$\widetilde{J}^d_\phi(s) = \int_{s_1}^{s_d} \zeta_d(u - s_1; s_2 - s_1, \dots, s_d - s_{d-1})\phi^{(d)}(u)\, du.$$

Lemma 2.4 *Suppose that $\phi^{(d)}$ is log-concave on $(0, \delta)$ and $0 = h_1 \leq h_2 \leq \dots \leq h_d$. Then,*

$$\left[\prod_{i=1}^{d} \phi^{(d)}(t + h_i)\right]^{1/d} \leq \phi^{(d)}(H_d(t, h)) \tag{12}$$

where $t + h_i \in (0, \delta)$ for $i = 1, \dots, d$ and $H_d(t, h) = \frac{1}{d}\sum_{i=1}^{d}(t + h_i) \in [t, t + h_d]$.

Proof Let $\beta(t) = -\log[\phi^{(d)}(t)]$. Then, β is convex on $(0, \delta)$. Therefore, by Jensen's inequality,

$$\frac{1}{d}\sum_{i=1}^{d} \beta(t_i) \geq \beta\left(\frac{1}{d}\sum_{i=1}^{d} t_i\right)$$

where $t_i \in (0, \delta)$ for $i = 1, \dots, d$. It follows that

$$\exp\left[\frac{1}{d}\sum_{i=1}^{d} \beta(t_i)\right] \geq \exp\left[\beta\left(\frac{1}{d}\sum_{i=1}^{d} t_i\right)\right],$$

which implies

$$\prod_{i=1}^{d}\Big[\exp\Big(\beta(t_i)\Big)\Big]^{1/d} \geq \exp\Big[\beta\Big(\frac{1}{d}\sum_{i=1}^{d}t_i\Big)\Big].$$

Namely,

$$\prod_{i=1}^{d}\Big[\exp\Big(-\log[\phi^{(d)}(t_i)]\Big)\Big]^{1/d} \geq \exp\Big(-\log\Big[\phi^{(d)}\Big(\frac{1}{d}\sum_{i=1}^{d}t_i\Big)\Big]\Big),$$

which implies

$$\prod_{i=1}^{d}[\phi^{(d)}(t_i)]^{1/d} \leq \phi^{(d)}\Big(\frac{1}{d}\sum_{i=1}^{d}t_i\Big).$$

If we put $t_1 = t$ and $t_i = t + h_i$, we get

$$\Big[\prod_{i=1}^{d}\phi^{(d)}(t+h_i)\Big]^{1/d} \leq \phi^{(d)}(H_d(t,h)).$$

□

Proof of Proposition 2.1 We adapt the proof of Proposition 2.1 in [3].
We will use both notations t_i and $t+h_i$, where $t_i = t+h_i$ for $0 = h_1 \leq h_2 \leq \cdots \leq h_d$.

$$\begin{aligned} J_\phi(t,h) &= \widetilde{J}_\phi^d(t, t+h_2, \dots, t+h_d) \\ &= \int_t^{t+h_d} \zeta_d(u-t;h)\,\phi^{(d)}(u)\,du \\ &\geq \int_{H_d(t,h)}^{t+h_d} \zeta_d(u-t;h)\,\phi^{(d)}(u)\,du. \end{aligned}$$

The equality follows from Lemma 2.3 and the inequality follows from nonnegativity. Since $\phi^{(d)}$ is increasing,

$$J_\phi(t,h) \geq \phi^{(d)}(H_d(t,h)) \int_{H_d(t,h)}^{t+h_d} \zeta_d(u-t;h)\,du. \tag{13}$$

We will show that

$$\int_{H_d(t,h)}^{t+h_d} \zeta_d(u-t;h)\,du \geq c_d\,v(h). \tag{14}$$

To show (14), we will use induction on $d \geq 2$.
It is easy to verify for the case $d = 2$ that the (14) holds with $c_d = 1/2$. Suppose that (14) holds for $d - 1 \geq 2$. Consider a function π such that

$$\pi^{(d)}(u) = \chi_{\{u \geq \bar{t}\}}(u)$$

where $\bar{t} = \frac{1}{d}(t_1 + \cdots + t_d)$. Observe that

$$\begin{aligned} \partial_{t_1} \ldots \partial_{t_{d-1}} \widetilde{J}_\phi^d(t_1, \ldots, t_d) &= (-1)^{d+1} \det(\gamma''(t_1) \ldots \gamma''(t_{d-1})) \\ &= (-1)^{d+1} \widetilde{J}_{\phi'}^{d-1}(t_1, \ldots, t_{d-1}). \end{aligned}$$

Since $\widetilde{J}_\phi^d(t) = 0$ if $t_i = t_{i+1}$, we get

$$\begin{aligned} &\widetilde{J}_\phi^d(t_1, \ldots, t_d) \\ &= (-1)^{d-1} \int_{t_1}^{t_2} \ldots \int_{t_{d-1}}^{t_d} \partial_{s_1} \ldots \partial_{s_{d-1}} \widetilde{J}_\phi^d(s_1, \ldots, s_{d-1}, t_d)\, ds_{d-1} \ldots ds_1 \\ &= \int_{t_1}^{t_2} \ldots \int_{t_{d-1}}^{t_d} \widetilde{J}_{\phi'}^{d-1}(s_1, \ldots, s_{d-1})\, ds_{d-1} \ldots ds_1. \end{aligned} \tag{15}$$

By applying (15) and Lemma 2.3, we get

$$\begin{aligned} &\int_{H_d(t,h)}^{t+h_d} \zeta_d(u - t; h)\, du = \widetilde{J}_\pi^d(t_1, \ldots, t_d) \\ &\quad = \int_{t_1}^{t_2} \ldots \int_{t_{d-1}}^{t_d} \int_{s_1}^{s_{d-1}} \chi_{\{u \geq \bar{t}\}}(u) \\ &\qquad \times \zeta_{d-1}(u - s_1; s_2 - s_1, \ldots, s_{d-1} - s_{d-2})\, du\, ds_{d-1} \ldots ds_1. \end{aligned}$$

Let $\lambda_i = \frac{d-i}{d}$. Note that if $s_i \geq \lambda_i t_i + (1 - \lambda_i) t_{i+1}$, then $\bar{s} = \frac{1}{d-1}(s_1 + \cdots + s_{d-1}) \geq \frac{1}{d}(t_1 + \cdots + t_d) = \bar{t}$, so $\chi_{\{u \geq \bar{t}\}}(u) \geq \chi_{\{u \geq \bar{s}\}}(u)$. Therefore,

$$\begin{aligned} &\int_{H_d(t,h)}^{t+h_d} \zeta_d(u - t; h)\, du \\ &\geq \int_{\lambda_1 t_1 + (1-\lambda_1) t_2}^{t_2} \ldots \int_{\lambda_{d-1} t_{d-1} + (1-\lambda_{d-1}) t_d}^{t_d} \int_{s_1}^{s_{d-1}} \chi_{\{u \geq \bar{s}\}}(u) \\ &\quad \times \zeta_{d-1}(u - s_1; s_2 - s_1, \ldots, s_{d-1} - s_{d-2})\, du\, ds_{d-1} \ldots ds_1. \end{aligned} \tag{16}$$

By the induction hypotheses, we get the inequality

$$\begin{aligned} &\int_{s_1}^{s_{d-1}} \chi_{\{u \geq \bar{s}\}}(u) \zeta_{d-1}(u - s_1; s_2 - s_1, \ldots, s_{d-1} - s_{d-2})\, du \\ &\quad \geq c_{d-1} V_{d-1}(s_1, \ldots, s_{d-1}). \end{aligned} \tag{17}$$

By (16) and (17), we have

$$\int_{H_d(t,h)}^{t+h_d} \zeta_d(u-t;h)\,du \\ \geq c_{d-1}\int_{\lambda_1 t_1+(1-\lambda_1)t_2}^{t_2} \cdots \int_{\lambda_{d-1}t_{d-1}+(1-\lambda_{d-1})t_d}^{t_d} V_{d-1}(s_1,\ldots,s_{d-1})\,ds_1\ldots ds_{d-1}.$$

Using the fact that V_{d-1} is of the form $\prod v_m(t)$ in Lemma 2.2, and

$$V_d(t_1,\ldots,t_d) = (d-1)!\int_{t_1}^{t_2}\cdots\int_{t_{d-1}}^{t_d} V_{d-1}(s_1,\ldots,s_{d-1})\,ds_{d-1}\ldots ds_1,$$

we get the inequality (14) (see [3, p. 9]). If we apply (12) and (14) to (13), we obtain (6). $\square$

3 Preliminaries on Interpolation Space

In this section, we provide some definitions and lemmas established in [5], which are needed to prove Theorem 1.1. Let $\bar{X} = (X_0, X_1)$ be a compatible couple of quasi-normed spaces X_0 and X_1, i.e., both X_0 and X_1 are continuously embedded in the same topological vector space. We can define both the K-functional on $X_0 + X_1$, given by

$$K(f,t,\bar{X}) = \inf_{f=f_0+f_1}(\|f_0\|_{X_0} + t\|f_1\|_{X_1}),$$

and the J-functional on $X_0 \cap X_1$, given by

$$J(f,t,\bar{X}) = \max(\|f\|_{X_0}, t\|f\|_{X_1}).$$

For $0 < \theta < 1$, let the interpolation space $\bar{X}_{\theta,q}$ be a subspace of $X_0 + X_1$, where

$$\|f\|_{\bar{X}_{\theta,q}} = \begin{cases} \left(\sum_{n\in\mathbb{Z}}[2^{-n\theta}K(f,2^n,\bar{X})]^q\right)^{1/q} & 1 \leq q < \infty, \\ \sup_{n\in\mathbb{Z}} 2^{-n\theta}K(f,2^n,\bar{X}) & q = \infty \end{cases}$$

is finite. Then, $X_0 \cap X_1$ is dense in $\bar{X}_{\theta,q}$ when $1 \leq q < \infty$, so we can give an equivalent norm $\|\cdot\|_{\bar{X}_{\theta,q;J}}$ on $\bar{X}_{\theta,q}$ by

$$\|f\|_{\bar{X}_{\theta,q;J}} = \inf\left(\sum_{n\in\mathbb{Z}}[2^{-n\theta}J(f_n,2^n,\bar{X})]^q\right)^{1/q},$$

where the infimum is taken over $f = \sum f_n$ and $f_n \in X_0 \cap X_1$, with convergence in $X_0 + X_1$. Note that $\|\cdot\|_{\bar{X}_{\theta,q}}$ and $\|\cdot\|_{\bar{X}_{\theta,q;J}}$ are equivalent when $0 < \theta < 1$. (For details, see Theorem 3.11.3 in [6].)

To present some lemmas, we introduce some definitions. Let $0 < r \leq 1$. For a quasi-normed space X, its norm is called $r-convex$ if there exists a constant $C > 0$ such that

$$\Big\| \sum_{i=1}^{n} x_i \Big\|_X \leq C \left(\sum_{i=1}^{n} \|x_i\|_X^r \right)^{1/r}$$

for any finite $x_i \in X$. Kalton [14] and Stein et al. [15] showed that the Lorentz space $L^{r,\infty}$ is $r-convex$ for $0 < r < 1$.
For a quasi-normed space X, let $\ell_s^p(X)$ be a sequence space whose element $\{f_n\}$ is X-valued and satisfies

$$\left(\sum_{n\in\mathbb{Z}} 2^{nsp} \|f_n\|_X^p \right)^{1/p} < \infty.$$

We can also define a function space $b_s^p(X; dw)$, where w is a weight function and X is Lorentz space on an interval I, such that $f \in b_s^p(X; dw)$ implies $\{\chi_{\mathcal{W}_{w,n}} f\}_{n\in\mathbb{Z}} \in \ell_s^p(X)$, i.e.,

$$\|f\|_{b_s^p(X;dw)} = \left(\sum_{n\in\mathbb{Z}} 2^{nsp} \|\chi_{\mathcal{W}_{w,n}} f\|_X^p \right)^{1/p} < \infty,$$

where $\mathcal{W}_{w,n} = \{t \in I : 2^n \leq w(t) < 2^{n+1}\}$.
Then, by definition, $b_{1/p}^p(L^p; dw) = L^p(I; dw)$.

Now, we state some lemmas that will be helpful in proving Theorem 1.1.

Lemma 3.1 (Lemma A.3 in [5]) *Let $0 < r \leq 1$ and V be an $r-convex$ space. For $i = 1, \ldots, n$, let*

$$\bar{X}^i = (X_0^i, X_1^i)$$

be couples of compatible quasi-normed spaces and let $\mathcal{M}$ be an n-linear operator defined on $\prod_{i=1}^n (X_0^i \cap X_1^i)$ with values in V. Suppose that

$$\|\mathcal{M}(f_1, \ldots, f_n)\|_V \leq \prod_{i=1}^{n} \|f_i\|_{X_0^i}^{1-\theta_i} \|f_i\|_{X_1^i}^{\theta_i}$$

for $0 < \theta_i < 1$ *for all* i*. Then there is* $C > 0$ *such that for all* $(f_1, \ldots, f_n) \in \prod_{i=1}^n (X_0^i \cap X_1^i)$,

$$\|\mathcal{M}(f_1, \ldots, f_n)\|_V \leq \prod_{i=1}^n \|f_i\|_{\bar{X}^i_{\theta_i, r}}$$

and $\mathcal{M}$ *extends to a bounded operator on* $\prod_{i=1}^n \bar{X}^i_{\theta_i, r}$.

Lemma 3.2 (Theorem 1.3 in [5]) *For* $i = 1, \ldots, n$ *and* $c_1, \ldots, c_n \in \mathbb{R}$, $c_1 \neq c_i$ *for* $i = 2, \ldots, n$*. Let* $0 < r \leq 1$*, and* $\bar{X} = (X_0, X_1)$ *be a couple of compatible complete quasi-normed spaces. Let* V *be an* $r - convex$ *space and* $\mathcal{M}$ *be an* n*-linear operator defined on* $X_0 + X_1$ *and* w *be a weight function. Suppose*

$$\|\mathcal{M}[f_1, \ldots, f_n]\|_V \leq \|f_1\|_{b^r_{c_1}(X_1; dw)} \prod_{i=2}^n \|f_i\|_{b^r_{c_i}(X_0; dw)}.$$

Then,

$$\|\mathcal{M}[f_1, \ldots, f_n]\|_V \lesssim \prod_{i=1}^n \|f_i\|_{b^{nr}_c\left(\bar{X}_{\frac{1}{n}, nr}; dw\right)}$$

where $c = \frac{1}{n}\sum_{i=1}^n c_i$.

Lemma 3.3 (Lemma A.4 in [5]) *Let* $0 < p \leq \infty$, $s_0, s_1 \in \mathbb{R}$*, and* $0 < \theta < 1$*. Let* (X_0, X_1) *be a compatible couple of quasi-normed spaces. If* $p \leq q \leq \infty$*, then there is the continuous embedding*

$$\ell^p_s((X_0, X_1)_{\theta, q}) \hookrightarrow (\ell^p_{s_0}(X_0), \ell^p_{s_1}(X_1))_{\theta, q}$$

for $s = (1 - \theta)s_0 + \theta s_1$.

In fact, $b^p_s(X)$ is a retract of $l^p_s(X)$. Define $r : l^p_s(X) \to b^p_s(X)$ by $r(\{f_n\}) = \sum_{n \in \mathbb{Z}} \mathcal{W}_{w,n} f_n$ and $i : b^p_s(X) \to l^p_s(X)$ by $[i(f)]_n = \mathcal{W}_{w,n} f$. Then, $r \circ i$ is the identity operator on $b^p_s(X)$. Therefore, Lemma 3.3 implies that there is the continuous embedding

$$b^p_s((X_0, X_1)_{\theta, q}) \hookrightarrow (b^p_{s_0}(X_0), b^p_{s_1}(X_1))_{\theta, q}$$

under the hypotheses of Lemma 3.3.

4 Proof of Theorem 1.1

The interval $I = (0, 1)$ can be decomposed into $(0, \delta) \cup [\delta, 1)$. Since $\phi^{(d)}$ is positive and increasing on I, $\gamma(t)$ is nondegenerate if $t \in [\delta, 1)$ for any $0 < \delta < 1$. Then, by Theorem 1.4 in [4], Theorem 1.1 holds on $[\delta, 1)$. Therefore, it is enough to show that Theorem 1.1 holds on $(0, \delta)$, if γ satisfies the log-concavity property (4) for some $\delta > 0$ and $\phi^{(d)}$ is positive and increasing on $(0, \delta)$. Let $q_d = p'_d = \frac{d^2+d+2}{2}$ and $I = (0, \delta)$.

Definition 4.1 Let $\mathfrak{C}$ be a class of $\gamma(t)$, defined on I, given by $\gamma(t) = (t, \frac{t^2}{2!}, \ldots, \frac{t^{d-1}}{(d-1)!}, \phi(t))$, for which $\phi \in C^d(I)$, and $\phi^{(d)}$ is positive, increasing and log-concave on I.

Consider the adjoint operator T_w given by

$$T_w g(x) = \int_I e^{-i\langle x,\gamma(t)\rangle} g(t)w(t)\, dt,$$

and define $\mathcal{C}$ by

$$\mathcal{C} = \sup_{\gamma\in\mathfrak{C}} \sup_{\|g\|_{L^{q_d}(I;dw)}\le 1} \|T_w g\|^{**}_{L^{q_d,\infty}} \tag{18}$$

where $\|f\|^{**}_{L^{q_d,\infty}} = \sup_{t>0} t^{1/q_d} f^{**}(t)$ with f^{**} is the maximal function of nonincreasing rearrangment of f.

The proof is an adaptation of the Proof of Theorem 4.2 in [5]. We will prove an L^2-estimate and an $(L^{q_d},\ L^{q_d,\infty})$-estimate for some d-linear operator $\mathcal{M}$ which will be constructed from T_w, and using a technique introduced in [7] with these two estimates, we will get a suitable estimate for the $L^{q_d/d,\infty}$ norm of $\mathcal{M}$. Then, we can get an estimate for a multi-linear operator $\widetilde{\mathcal{M}}$ using Lemmas 3.1–3.3 and we can show that $\mathcal{C}$ is bounded by some constant depending only on d.

Define a d-linear operator $\mathcal{M}$ by

$$\begin{aligned}\mathcal{M}[g_1, \ldots, g_d](x) &= \prod_{i=1}^{d} T_w g_i(x)\\ &= \int_{I^d} e^{-i\left\langle x, \sum_{i=1}^{d} \gamma(t_i)\right\rangle} \prod_{i=1}^{d} [g_i(t_i)w(t_i)]\, dt_1 \ldots dt_d.\end{aligned}$$

Let $I^d = \bigcup E_\pi$ where

$$E_\pi = \{(t_1, \ldots, t_d) \in I^d : t_{\pi(1)} \le \cdots \le t_{\pi(d)}\}$$

and π is the permutation on d. Then, without loss of generality, we can assume $t_1 \le \cdots \le t_d$ so that the operator $\mathcal{M}$ is defined on $E = E_1 := \{(t_1, \ldots, t_d) \in I^d : t_1 \le \cdots \le t_d\}$. Therefore, redefine the operator $\mathcal{M}$ by

$$\mathcal{M}[g_1, \ldots, g_d](x) = \int_E e^{-i\langle x,\Gamma(t,h)\rangle} G(t, h)W(t, h)\, dtdh$$

where $G(t,h) = \prod_{i=1}^{d} g_i(t+h_i)$, $W(t,h) = \prod_{i=1}^{d} w(t+h_i)$, $h \in I^{d-1}$, and $t+h_d < \delta$. Divide E into F_k, $k \in \mathbb{Z}$, where

$$F_k = \{(t, t+h_2, \dots, t+h_d) \in E : 2^{-(k+1)} < v(h) \le 2^{-k}\},$$

and define

$$\mathcal{M}_k[g_1, \dots, g_d](x) = \int_{F_k} e^{-i\langle x, \Gamma(t,h)\rangle} G(t,h) W(t,h)\, dtdh. \tag{19}$$

We will obtain an upper bound for $\mathcal{M}_k$.

$\mathbf{L^2}$ – estimate By the change of variables $\Gamma(t,h) \to y$, Plancherel's theorem, and the change of variables $y \to \Gamma(t,h)$, we get

$$\|\mathcal{M}_k[g_1, \dots, g_d]\|_2^2 \lesssim \int_{F_k} |G(t,h)W(t,h)|^2 J_\phi(t,h)^{-1}\, dtdh.$$

Observe that $J_\phi(t,h)$ is nonzero on F_k. Then, by [9], the change of variables $\Gamma(t,h) \to y$ is at most d!-to-one, so we can use the change of variables without any problem.

Since $\Gamma \in \mathfrak{C}$, Proposition 2.1 holds, so we get the inequality

$$J_\phi(t,h) \ge C_d\, v(h) \left[\prod_{i=1}^{d} \phi^{(d)}(t+h_d)\right]^{1/d} \tag{20}$$

for some $C_d > 0$, which depends only on d. By (20) and the definition of w, we get

$$\|\mathcal{M}_k[g_1, \dots, g_d]\|_2^2 \lesssim \int_{F_k} |G(t,h)W(t,h)|^2 v(h)^{-1} W(t,h)^{(d+1)/2}\, dt\, dh.$$

It is known (Lemma 1 of [12]) that the sublevel set estimate for $v(h)$ is

$$|\{h \in \mathbb{R}^{d-1} : v(h) \le c\}| \lesssim c^{2/d}.$$

Taking $c = 2^{-k}$, we get

$$\|\mathcal{M}_k[g_1, \dots, g_d]\|_2^2 \lesssim 2^{k\frac{d-2}{d}} \int_{F_k} |G(t,h)[W(t,h)]^{\frac{(3-d)}{4}}|^2\, dt\, dh. \tag{21}$$

Also, we can get the following inequality,

$$\left[\int_{F_k} |G(t,h)[W(t,h)]^{\frac{(3-d)}{4}}|^2\, dtdh\right]^{1/2} \le \|g_j w^{\frac{3-d}{4}}\|_2 \prod_{i \ne j} \|g_i w^{\frac{3-d}{4}}\|_\infty$$

for any $j = 1, \ldots, d$. Complex interpolation and (21) lead to

$$\|\mathcal{M}_k[g_1, \ldots, g_d]\|_2 \lesssim 2^{k\frac{d-2}{2d}} \prod_{i=1}^{d} \|g_i w^{\frac{3-d}{4}}\|_{r_i}$$

with $\sum_{i=1}^{d} r_i^{-1} = \frac{1}{2}$. Finally, putting $r_i = 2d$, we obtain

$$\|\mathcal{M}_k[g_1, \ldots, g_d]\|_2 \lesssim 2^{k\frac{d-2}{2d}} \prod_{i=1}^{d} \|g_i w^{\frac{3-d}{4}}\|_{2d}. \tag{22}$$

$(\mathbf{L^{q_d}}, \mathbf{L^{q_d,\infty}})$ − estimate Fix h and let $I_h = (0, \delta - h_d)$. Observe that $\Gamma(\cdot, h) \in \mathfrak{C}$. Then,

$$\left\| \int_{I_h} e^{-i\langle \cdot, \Gamma_h(t)\rangle} g(t) w_\Gamma(t)\, dt \right\|_{L^{q_d,\infty}} \le \mathcal{C} \|g\|_{L^{q_d}(I_h; dw)}$$

with $w_\Gamma(t) = |\tau_\Gamma(t)|^{\frac{2}{d^2+d}}$. Furthermore, observe that if $w_\epsilon(t) \le w(t)$, then we can write $w_\epsilon(t) = \epsilon(t) w(t)$ with $0 \le \epsilon \le 1$ and

$$\begin{aligned}
\left\| \int_{I_h} e^{-i\langle \cdot, \gamma(t)\rangle} g(t) w_\epsilon(t)\, dt \right\|_{L^{q_d,\infty}} &= \left\| \int_{I_h} e^{-i\langle \cdot, \gamma(t)\rangle} g(t)\epsilon(t) w(t)\, dt \right\|_{L^{q_d,\infty}} \\
&\lesssim \mathcal{C} \left[\int_{I_h} |g(t)\epsilon(t)|^{q_d} w(t)\, dt \right]^{1/q_d} \\
&\le \mathcal{C} \left[\int_{I_h} |g(t)|^{q_d} \epsilon(t) w(t)\, dt \right]^{1/q_d} \\
&= \mathcal{C} \left[\int_{I_h} |g(t)|^{q_d} w_\epsilon(t)\, dt \right]^{1/q_d}.
\end{aligned} \tag{23}$$

Also, for $\sum_{i=1}^{d} \epsilon_i = 1$, let $w_{\epsilon,h}(t) = \prod_{i=1}^{d} w(t + h_i)^{\epsilon_i}$. Then, by the positivity of $\phi^{(d)}$ and Jensen's inequality for a convex function $-\log$,

$$\begin{aligned}
-\log\left(\sum_{i=1}^{d} \phi^{(d)}(t + h_i) \right) &\le -\log\left(\frac{\sum \epsilon_i \phi^{(d)}(t + h_i)}{\sum \epsilon_i} \right) \\
&\le \sum_{i=1}^{d} \epsilon_i \Big(-\log(\phi^{(d)}(t + h_i)) \Big) \\
&= -\log\left(\prod_{i=1}^{d} \phi^{(d)}(t + h_i)^{\epsilon_i} \right),
\end{aligned} \tag{24}$$

so we get $w_{\epsilon,h} \le w_\Gamma$.

By (23) and (24), we get

$$\left\| \int_{I_h} e^{-i\langle\cdot,\Gamma_h(t)\rangle} g(t) w_{\epsilon,h}(t)\, dt \right\|_{L^{q_d,\infty}} \lesssim \mathcal{C}\left[\int_{I_h} |g(t)|^{q_d} w_{\epsilon,h}(t)\, dt \right]^{1/q_d}.$$

If we put $G(t,h)\dfrac{W(t,h)}{w_{\epsilon,h}(t)}$ instead of $g(t)$, then

$$\begin{aligned} &\left\| \int_{I_h} e^{-i\langle\cdot,\Gamma_h(t)\rangle} G(t,h) W(t,h)\, dt \right\|_{L^{q_d,\infty}} \\ &\quad \lesssim \mathcal{C}\left[\int_{I_h} \left| G(t,h)\frac{W(t,h)}{w_{\epsilon,h}(t)} \right|^{q_d} w_{\epsilon,h}(t)\, dt \right]^{1/q_d}. \end{aligned} \tag{25}$$

So we have

$$\begin{aligned} \left\| \mathcal{M}_k[g_1,\ldots,g_d] \right\|_{L^{q_d,\infty}} &= \left\| \int_{F_k} e^{-i\langle x,\Gamma(t,h)\rangle} G(t,h) W(t,h)\, dt dh \right\|_{L^{q_d,\infty}} \\ &\le \int_H \left\| \int_{I_h} e^{-i\langle x,\Gamma(t,h)\rangle} G(t,h) W(t,h)\, dt \right\|_{L^{q_d,\infty}} dh \\ &\lesssim \mathcal{C} \int_H \left[\int_{I_h} \left\| G(t,h) \frac{W(t,h)}{w_{\epsilon,h}(t)} \right|^{q_d} w_{\epsilon,h}(t)\, dt \right]^{1/q_d} dh \end{aligned}$$

where $H = \{(h_1,\ldots,h_d) \in I^d : 0 = h_1 \le h_2 \cdots \le h_d,\ 2^{-(k+1)} < v(h) \le 2^{-k}\}$ and the last expression is bounded by

$$\mathcal{C} \int_H \left[\int_{I_h} \left| g_1(t) w(t)^{1-\frac{\epsilon_i}{p_d}} \prod_{i=2}^{d} g_i(t+h_d) w(t+h_d)^{1-\frac{\epsilon_i}{p_d}} \right|^{q_d} dt \right]^{1/q_d} dh$$

where $p_d' = q_d$. Since H is a subset of F_k, the sublevel set estimate of $v(h)$ gives $|H| \lesssim 2^{-2k/d}$. Since $q_d' = p_d$, we get

$$\begin{aligned} \left\| \mathcal{M}_k[g_1,\ldots,g_d] \right\|_{L^{q_d,\infty}} &\lesssim \mathcal{C} \int_H \left[\int_{I_h} \left| g_1(t) w(t)^{1-\frac{\epsilon_1}{p_d}} \right|^{q_d} dt \right]^{1/q_d} \\ &\qquad \times \prod_{i=2}^{d} \| g_i(\cdot + h_d) w(\cdot + h_d)^{1-\frac{\epsilon_i}{p_d}} \|_\infty\, dh \\ &\lesssim 2^{-2k/d} \mathcal{C} \| g_1 w^{1-\frac{\epsilon_1}{p_d}} \|_{q_d} \prod_{i=2}^{d} \| g_i w^{1-\frac{\epsilon_i}{p_d}} \|_\infty. \end{aligned}$$

By symmetry,

$$\left\|\mathcal{M}_k[g_1,\ldots,g_d]\right\|_{L^{q_d,\infty}} \lesssim 2^{-2k/d}\mathcal{C}\prod_{i=1}^{d}\|g_i w^{1-\frac{\epsilon_i}{p_d}}\|_{s_i} \tag{26}$$

where $\sum_{i=1}^{d}\epsilon_i = 1$ and $\sum_{i=1}^{d}\dfrac{1}{s_i}=\dfrac{1}{q_d}$.

Estimate on the $\mathbf{L^{q_d/d,\infty}}$ norm of $\mathcal{M}$ Fix $y>0$ and define $G_y=\{x : |\mathcal{M}[g_1,\ldots,g_d](x)|>2y\}$. Then, for any constant K,

$$|G_y| \le y^{-2}\Big\|\sum_{2^k\le K}\mathcal{M}_k[g_1,\ldots,g_d]\Big\|_2^2 + y^{-q_d}\Big\|\sum_{2^k>K}\mathcal{M}_k[g_1,\ldots,g_d]\Big\|_{q_d,\infty}^{q_d}.$$

By (22) and (26), we obtain

$$|G_y| \lesssim y^{-2}K^{(d-2)/d}\prod_{i=1}^{d}\|g_i w^{\frac{3-d}{4}}\|_{2d}^2 + y^{-q_d}K^{-2q_d/d}\mathcal{C}^{q_d}\prod_{i=1}^{d}\|g_i w^{1-\frac{\epsilon_i}{p_d}}\|_{s_i}^{q_d}.$$

If we choose K appropriately so that

$$y^{-2}K^{(d-2)/d}\prod_{i=1}^{d}\|g_i w^{\frac{3-d}{4}}\|_{2d}^2 = y^{-q_d}K^{-2q_d/d}\mathcal{C}^{q_d}\prod_{i=1}^{d}\|g_i w^{1-\frac{\epsilon_i}{p_d}}\|_{s_i}^{q_d},$$

which means

$$K^{\frac{d-2+2q_d}{d}} = y^{2-q_d}\mathcal{C}^{q_d}\prod_{i=1}^{d}\|g_i w^{1-\frac{\epsilon_i}{p_d}}\|_{s_i}^{q_d}\prod_{i=1}^{d}\|g_i w^{\frac{3-d}{4}}\|_{2d}^{-2},$$

then we obtain

$$y|G_y|^{\frac{d-2+2q_d}{(d+2)q_d}} \lesssim \mathcal{C}^{\frac{d-2}{d+2}}\prod_{i=1}^{d}\|g_i w^{\frac{3-d}{4}}\|_{2d}^{\frac{4}{d+2}}\prod_{i=1}^{d}\|g_i w^{1-\frac{\epsilon_i}{p_d}}\|_{s_i}^{\frac{d-2}{d+2}}.$$

Since $\frac{d-2+2q_d}{(d+2)q_d}=\frac{d}{q_d}$, we get

$$\begin{aligned}\|\mathcal{M}[g_1,\ldots,g_d]\|_{\frac{q_d}{d},\infty} &\lesssim \mathcal{C}^{\frac{d-2}{d+2}}\prod_{i=1}^{d}\|g_i w^{\frac{3-d}{4}}\|_{2d}^{\frac{4}{d+2}}\prod_{i=1}^{d}\|g_i w^{1-\frac{\epsilon_i}{p_d}}\|_{s_i}^{\frac{d-2}{d+2}}\\ &= \mathcal{C}^{\frac{d-2}{d+2}}\prod_{i=1}^{d}\|g_i w^{\frac{3-d}{4}}\|_{2d}^{\frac{4}{d+2}}\prod_{i=1}^{d}\|g_i w^{1-\frac{\epsilon_i}{p_d}}\|_{s_i}^{\frac{d-2}{d+2}}.\end{aligned}$$

Observe that $\|g_i w^{\frac{3-d}{4}}\|_{2d} \approx \sum_{k\in\mathbb{Z}} 2^{k\frac{3-d}{4}} \|\chi_{\mathcal{W}_{w,k}} g_i\|_{2d}$ and $\|g_i w^{1-\frac{\epsilon_i}{p_d}}\|_{s_i} \approx \sum_{k\in\mathbb{Z}} 2^{k(1-\frac{\epsilon_i}{p_d})} \|\chi_{\mathcal{W}_{w,k}} g_i\|_{s_i}$, so we can write

$$\|\mathcal{M}[g_1,\ldots,g_d]\|_{\frac{q_d}{d},\infty} \lesssim \mathcal{C}^{\frac{d-2}{d+2}} \prod_{i=1}^{d} \|g_i\|^{\frac{4}{d+2}}_{b^1_{\frac{3-d}{4}}(L^{2d};dw)} \prod_{i=1}^{d} \|g_i\|^{\frac{d-2}{d+2}}_{b^1_{1-\frac{\epsilon_i}{p_d}}(L^{s_i};dw)}. \tag{27}$$

Then, by Lemma 3.1 and (27),

$$\|\mathcal{M}[g_1,\ldots,g_d]\|_{\frac{q_d}{d},\infty} \lesssim \mathcal{C}^{\frac{d-2}{d+2}} \prod_{i=1}^{d} \|g_i\|_{\bar{X}^i_{\frac{d-2}{d+2},1}} \tag{28}$$

where $\bar{X}^i_{\frac{d-2}{d+2},1} = \left(b^1_{\frac{3-d}{4}}(L^{2d};dw), b^1_{1-\frac{\epsilon_i}{p_d}}(L^{s_i};dw)\right)_{\frac{d-2}{d+2},1}$.
Also, we can find the continuous embedding

$$\begin{aligned} & b^1_{\frac{4}{d+2}\frac{3-d}{4}+\frac{d-2}{d+2}(1-\frac{\epsilon_i}{p_d})}\left((L^{2d},L^{s_i})_{\frac{d-2}{d+2},1};dw\right) \\ & \hookrightarrow \left(b^1_{\frac{3-d}{4}}(L^{2d};dw), b^1_{1-\frac{\epsilon_i}{p_d}}(L^{s_i};dw)\right)_{\frac{d-2}{d+2},1} \end{aligned}$$

by Lemma 3.3 with b^p_s instead of l^p_s. Therefore, if we define

$$a_i = \frac{3-d}{d+2} + \frac{d-2}{d+2}\left(1-\frac{\epsilon_i}{p_d}\right)$$

and

$$\frac{1}{b_i} = \frac{4}{d+2}\cdot\frac{1}{2d} + \frac{d-2}{d+2}\cdot\frac{1}{s_i},$$

we get $(L^{2d},L^{s_i})_{\frac{d-2}{d+2},1} = L^{b_i,1}$ and

$$\|\mathcal{M}[g_1,\ldots,g_d]\|_{\frac{q_d}{d},\infty} \lesssim \mathcal{C}^{\frac{d-2}{d+2}} \prod_{i=1}^{d} \|g_i\|_{b^1_{a_i}(L^{b_i,1};dw)}, \tag{29}$$

where $\sum_{i=1}^d a_i = \sum_{i=1}^d \frac{1}{b_i} = \frac{d}{q_d}$.

Now, define a multi-linear operator $\widetilde{\mathcal{M}}$ by

$$\widetilde{\mathcal{M}}[g_1,\ldots,g_n] = \prod_{i=1}^{n} T_w g_i(x).$$

for $n > q_d$. Let $r = \frac{q_d}{n} < 1$. Then, as we stated in Sect. 4, $L^{r,\infty}$ is an $r - convex$ space. We may write

$$\widetilde{\mathcal{M}}[g_1, \ldots, g_n] = \mathcal{M}[g_1, \ldots, g_d] \prod_{i=d+1}^{n} T_w g_i(x)$$

and by Hölder's inequality, it follows

$$\|\widetilde{\mathcal{M}}[g_1, \ldots, g_n]\|_{L^{r,\infty}} \lesssim \|\mathcal{M}[g_1, \ldots, g_d]\|_{L^{q_d/d,\infty}} \prod_{i=d+1}^{n} \|T_w g_i(x)\|_{L^{q_d,\infty}}. \tag{30}$$

Observe that if we put $g_i = g$ and $a_i = \frac{1}{b_i} = \frac{1}{q_d}$ for all $i = 1, \ldots, d$ in (29), we get

$$\|T_w g\|_{L^{q_d,\infty}} \lesssim \mathcal{C}^{\frac{d-2}{d^2+2d}} \|g\|_{b^1_{1/q_d}(L^{q_d,1};dw)}. \tag{31}$$

By applying (29) and (31) to (30), and by using the generalized geometric means inequality, we get

$$\begin{aligned}
&\|\widetilde{\mathcal{M}}[g_1, \ldots, g_n]\|_{L^{r,\infty}} \\
&\lesssim \mathcal{C}^{\frac{d-2}{d+2}} \prod_{i=1}^{d} \|g_i\|_{b^1_{a_i}(L^{b_i,1};dw)} \prod_{i=d+1}^{n} \mathcal{C}^{\frac{d-2}{d^2+2d}} \|g_i\|_{b^1_{1/q_d}(L^{q_d,1};dw)} \\
&= \mathcal{C}^{\frac{d-2}{d+2}+\frac{d-2}{d^2+2d}(n-d)} \|g_1\|_{b^1_{a_1}(L^{b_1,1};dw)} \|g_2\|_{b^1_{a_2}(L^{b_2,1};dw)} \\
&\quad \times \prod_{i=3}^{d} \|g_i\|_{b^1_{a_i}(L^{b_i,1};dw)} \prod_{i=d+1}^{n} \|g_i\|_{b^1_{1/q_d}(L^{q_d,1};dw)} \\
&\lesssim \mathcal{C}^{\frac{(d-2)n}{d^2+2d}} \|g_1\|_{b^1_{a_1}(L^{b_1,1};dw)} \|g_2\|_{b^1_{a_2}(L^{b_2,1};dw)} \\
&\quad \times \prod_{i=3}^{n} \|g_i\|^{\frac{d-2}{n-2}}_{b^1_{a_i}(L^{b_i,1};dw)} \|g_i\|^{\frac{n-d}{n-2}}_{b^1_{1/q_d}(L^{q_d,1};dw)},
\end{aligned} \tag{32}$$

where $\sum_{i=1}^{d} a_i = \sum_{i=1}^{d} \dfrac{1}{b_i} = \dfrac{d}{q_d}$.

We will choose a_i and b_i appropriately to get a upper bound of $\widetilde{\mathcal{M}}$. Recall that a_i depends on ϵ_i and b_i depends on s_i. Let $\eta > 0$ be small enough and let

$$\frac{1}{s_i} = \begin{cases} \dfrac{1}{dq_d} - \eta(d+2)\dfrac{n-1}{n-2}, & i = 1, \\ \dfrac{1}{dq_d} + \eta\dfrac{d+2}{n-2}, & i = 2, \\ \dfrac{1}{dq_d} + \eta\dfrac{d+2}{d-2}, & 3 \le i \le d. \end{cases}$$

Then,

$$\frac{1}{b_i} = \begin{cases} \frac{1}{q_d} - \eta(d-2)\frac{n-1}{n-2}, & i = 1, \\ \frac{1}{q_d} + \eta\frac{d-2}{n-2}, & i = 2, \\ \frac{1}{q_d} + \eta, & 3 \le i \le d \end{cases}$$

and it is easy to check that $\sum_{i=1}^{d} \frac{1}{b_i} = \frac{d}{q_d}$. Moreover, we get

$$\frac{1}{b_2} = \frac{d-2}{n-2}\frac{1}{b_3} + \frac{n-d}{n-2}\frac{1}{q_d}.$$

Therefore, applying Lemma 3.1 in (32) allows us to get

$$\|\widetilde{\mathcal{M}}[g_1, \ldots, g_n]\|_{L^{r,\infty}} \lesssim C^{\frac{(d-2)n}{d^2+2d}} \|g_1\|_{b^1_{a_1}(L^{b_1,1};dw)} \|g_2\|_{b^1_{a_2}(L^{b_2,1};dw)} \times \prod_{i=3}^{n} \|g_i\|_{\bar{Y}_{\frac{n-d}{n-2},1}}$$

where $\bar{Y}_{\frac{n-d}{n-2},1} = \big(b^1_{a_3}(L^{b_3,1}; dw), b^1_{1/q_d}(L^{q_d,1}; dw)\big)_{\frac{n-d}{n-2},1}$. By Lemma 3.3, there is a continuous embedding

$$b^1_{c_3}\Big((L^{b_3,1}, L^{q_d,1})_{\frac{n-d}{n-2},1}; dw\Big) = b^1_{c_3}(L^{b_2,1}; dw) \hookrightarrow \bar{Y}_{\frac{n-d}{n-2},1}$$

where $c_3 = \frac{d-2}{n-2}a_3 + \frac{n-d}{n-2}\frac{1}{q_d}$. We put $c_1 = a_1$ and $c_2 = a_2$ and choose ϵ_1, ϵ_2, and ϵ_3 properly so that c_1, c_2, and c_3 are all different. Then,

$$\begin{aligned}\|\widetilde{\mathcal{M}}[g_1, \ldots, g_n]\|_{L^{r,\infty}} &\lesssim C^{\frac{(d-2)n}{d^2+2d}} \|g_1\|_{b^1_{c_1}(L^{b_1,1};dw)} \|g_2\|_{b^1_{c_2}(L^{b_2,1};dw)} \\ &\quad \times \prod_{i=3}^{n} \|g_i\|_{b^1_{c_3}(L^{b_2,1};dw)} \\ &\lesssim C^{\frac{(d-2)n}{d^2+2d}} \|g_1\|_{b^r_{c_1}(L^{b_1,r};dw)} \|g_2\|_{b^r_{c_2}(L^{b_2,r};dw)} \\ &\quad \times \prod_{i=3}^{n} \|g_i\|_{b^r_{c_3}(L^{b_2,r};dw)}.\end{aligned}$$

Note that the last inequality comes from the trivial embedding. If we apply Lemma 3.2 to the last expression, we get

$$\|\widetilde{\mathcal{M}}[g_1, \ldots, g_n]\|_{L^{r,\infty}} \lesssim \mathcal{C}^{\frac{(d-2)n}{d^2+2d}} \prod_{i=1}^{n} \|g_i\|_{b_c^{nr}(\bar{Z}_{\frac{1}{n},nr};dw)}$$

where $c = \frac{1}{n}\sum_{i=1}^{n} c_i$ and $\bar{Z}_{\frac{1}{n},nr} = (L^{b_2,r}, L^{b_1,r})_{\frac{1}{n},nr}$.
By simple calculation, we get $c = \frac{1}{q_d}$ and

$$\bar{Z}_{\frac{1}{n},nr} = (L^{b_2,r}, L^{b_1,r})_{\frac{1}{n},nr} = L^{q_d},$$

since $\frac{1}{n}\frac{1}{b_1} + \frac{n-1}{n}\frac{1}{b_2} = \frac{1}{q_d}$. Therefore, $b_c^{nr}\left(\bar{Z}_{\frac{1}{n},nr}; dw\right) = b_{1/q_d}^{q_d}(L^{q_d}; dw) = L^{q_d}(dw)$ and we obtain

$$\|\widetilde{\mathcal{M}}[g_1, \ldots, g_n]\|_{L^{r,\infty}} \lesssim \mathcal{C}^{\frac{(d-2)n}{d^2+2d}} \prod_{i=1}^{n} \|g_i\|_{L^{q_d}(dw)}.$$

If we put $g = g_i$ for all $i = 1, \ldots, n$, we get

$$\|\widetilde{\mathcal{M}}[g_1, \ldots, g_n]\|_{L^{r,\infty}} \approx \|T_w g\|^n_{L^{q_d,\infty}} \lesssim \mathcal{C}^{\frac{(d-2)n}{d^2+2d}} \|g\|^n_{L^{q_d}(dw)}.$$

By the definition (18) of $\mathcal{C}$, this leads to $\mathcal{C}^{\frac{(d-2)}{d^2+2d}} \lesssim \mathcal{C}$, which implies that $\mathcal{C}$ is bounded by some constant depending only on d.

□

5 Some Examples

Now we provide some examples that satisfy the hypotheses of Theorem 1.1. For a given function $\phi^{(d)} : (0, \delta) \to \mathbb{R}^+$, define $\psi : (\delta^{-1}, \infty) \to \mathbb{R}$ by $\psi(x) = \frac{1}{\phi^{(d)}(1/x)}$. If ψ is log-convex, then $\phi^{(d)}$ is log-concave. The proof is as follows. If we assume that ψ is log-convex,

$$\psi(\lambda x_1 + (1-\lambda)x_2) \leq [\psi(x_1)]^{\lambda}[\psi(x_2)]^{1-\lambda}.$$

It follows that

$$\psi(\lambda/t_1 + (1-\lambda)/t_2)^{-1} \geq [\phi^{(d)}(t_1)]^{\lambda}[\phi^{(d)}(t_2)]^{1-\lambda}$$

where $t_1 = 1/x_1$ and $t_2 = 1/x_2$. Since function $1/x$ is convex and ψ^{-1} is decreasing on $(0, \infty)$, we have

$$\phi^{(d)}(\lambda t_1 + (1-\lambda)t_2) \geq [\phi^{(d)}(t_1)]^{\lambda}[\phi^{(d)}(t_2)]^{1-\lambda}$$

so $\phi^{(d)}$ is log-concave. Therefore, if $\psi(x) = \psi_{\phi^{(d)}}(x) = \frac{1}{\phi^{(d)}(1/x)}$ is positive, increasing, and log-convex on (δ^{-1}, ∞), then $\phi^{(d)}$ satisfies the hypotheses of Theorem 1.1. Also, for following examples, proving ψ is log-convex is easier than proving $\phi^{(d)}$ is log-concave, so we will give a proof that ψ is positive, increasing, and log-convex.

1. Let $\phi(t) = e^{-1/t}$ and $t \in (0, \delta)$, where δ will be chosen later. Then,

$$\phi^{(d)}(t) = e^{-1/t}\left(\frac{a_{1,d}}{t^{d+1}} + \cdots + \frac{a_{d,d}}{t^{2d}}\right)$$

where

$$a_{i,d} = \begin{cases} (-1)^{d+1}d! & i = 1 \\ a_{i-1,d-1} - (d+i-1)a_{i,d-1} & 1 < i < d \\ 1 & i = d \end{cases}.$$

Then, $\psi_{\phi^{(d)}}(x) = e^x\left(\sum_{i=1}^{d} a_{i,d}x^{d+i}\right)^{-1}$. Let $P(x) = \sum_{i=1}^{d} a_{i,d}x^{d+i}$. The leading coefficient of P, P', P'' are 1, $2d$, $2d(2d-1)$, respectively. Therefore, if we take δ small enough, which means x large enough, then $P > 0$ and $PP'' \le (P')^2$, which implies that P is log-concave and P^{-1} is log-convex. So we can check that $\psi_{\phi^{(d)}}(x)$ is log-convex and $\psi_{\phi^{(d)}}(x)$ is positive and increasing for $x \in (\delta^{-1}, \infty)$.

Likewise, for $\phi(t) = e^{-1/t^m}$ with $m \in \mathbb{N}$,

$$\psi_{\phi^{(d)}}(x) = e^{x^m}\left(\sum_{i=0}^{(d-1)m} a_i x^{m+d+i}\right)^{-1},$$

where the leading coefficient $a_{(d-1)m} = 1$ and a_i for $i = 1, \ldots, (d-1)m - 1$ is determined by d and m. Therefore $\psi_{\phi^{(d)}}(x)$ is log-convex, positive, and increasing for $x \in (\delta^{-1}, \infty)$.

2. Let $\phi_2(t) = \exp(-e^{1/t})$. Then,

$$\phi_2^{(d)}(t) = \exp(-e^{1/t})\left[e^{1/t}\frac{P_{d-1}(t)}{t^{2d}} + \cdots + e^{(d-1)/t}\frac{P_1(t)}{t^{2d}} + e^{d/t}\frac{1}{t^{2d}}\right]$$

where the $P_i(t)$ are certain polynomials with degree $\le i$. Therefore,

$$\psi_{\phi_2^{(d)}}(x) = e^{-e^x}\left[e^x\tilde{P}_{d-1}(x) + \cdots + e^{(d-1)x}\tilde{P}_1(x) + e^{dx}x^{2d}\right]^{-1}$$

where degree of $\tilde{P}_i \le 2d$. Let $P(x) = e^x\tilde{P}_{d-1}(x) + \cdots + e^{(d-1)x}\tilde{P}_1(x) + e^{dx}x^{2d}$. If x is large enough, then $P > 0$ and $PP'' \le (P')^2$. (For x large, P acts like $e^{dx}x^{2d}$). Therefore, $\psi_{\phi_2^{(d)}}(x)$ is log-convex, positive, and increasing if x is large enough.

Observe that (Likewise,) for $\phi_n(t) = \exp(-\exp(\ldots(\exp(1/t)\ldots)$, $\psi_{\phi_n^{(d)}}(x)$ satisfies the log-convexity for x large enough too.

Acknowledgements We would like to thank Jong-Guk Bak for suggesting the problem to us and also for giving us many helpful suggestions. We also thank Andreas Seeger for first mentioning the problem to him.

Funding The author has been supported by the NRF of Korea (NRF-2020R1A2C1A01005446).

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