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Shape Holomorphy of Boundary Integral Operators on Multiple Open Arcs

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Abstract
We establish shape holomorphy results for general weakly- and hyper-singular boundary integral operators arising from second-order partial differential equations in unbounded two-dimensional domains with multiple finite-length open arcs. After recasting the corresponding boundary value problems as boundary integral equations, we prove that their solutions depend holomorphically upon perturbations of the arcs' parametrizations. These results are key to prove the shape (domain) holomorphy of domain-to-solution maps associated to boundary integral equations appearing in uncertainty quantification, inverse problems and deep learning, to name a few applications.

Keywords Integral operators · Open arcs · Shape regularity · Shape holomorphy

1 Introduction

The efficient approximation of maps with high-dimensional parametric inputs poses major challenges to traditional computational methods. Indeed, as the dimension of the parametric input increases, the computational effort required to construct surrogates of the original map may grow exponentially, thus leading to the *curse of*

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dimensionality. In [8], polynomial surrogates of high-dimensional input maps are shown to converge independently of the dimension. Such results are derived from the well-known approximation properties of one-dimensional analytic functions and the existence of *holomorphic* extensions of the original maps onto tensor products of Bernstein ellipses in the complex plane. We say that a map is holomorphic if it is Fréchet differentiable on open complex subsets, which is equivalent to the existence of derivatives of arbitrary order—analytic maps—in the same open subset. By varying the size of these ellipses on each parameter, namely the *anisotropic* parameter dependence, one can prove convergence rates that do not depend on the dimension of the parametric input, thereby *breaking* the curse of dimensionality in the parametric dimension. Computationally, parametric holomorphy provides rigorous justification and construction bases for a variety of methods such as: Smolyak interpolation and quadrature [37, 44]; high-order Quasi-Monte Carlo integration (HoQMC) [11–15]; deep neural network surrogates [20, 21, 34, 39], together with its implications in Bayesian inverse problems [7, 37, 38].

In this work, we consider a family of boundary value problems (BVPs) set on the complement of a finite collection of open disjoint arcs in two dimensions, with either Dirichlet or Neumann boundary conditions. We study the smoothness properties of the domain-to-solution map in the context of complex variable. Given the lack of Lipschitz regularity of the BVP domain, traditional variational formulations cannot be applied, and hence, the existence of a holomorphic extension of the domain-to-solution map does not follow from volume-based formulations, for instance, as in [10]. Consequently, we recast the volume problems as boundary integral equations (BIEs) posed on the collection of open arcs, as in [2–4, 6, 22, 24, 26, 41–43], and then we extend the analysis of [18, 19] to the corresponding BIEs.

More precisely, we will assume that each arc admits a representation arising from a suitable predefined collection of parametrizations. Our goal is to verify that the solutions of the BIEs depend holomorphically upon perturbations of the boundary shape. In so doing, we prove that the BIOs depend themselves holomorphically on the arcs' shape. By recalling that the inversion of linear isomorphisms defines an analytic map, one can straightforwardly establish shape holomorphy of the domain-to-solution map as in [18, 19]. Therein, the authors extend the solution map to the complex plane, identifying geometries with parametrizations, and prove that the corresponding map is holomorphic for Jordan arcs. Hence, by means of available complex variable results on Banach spaces, one can state that there exists a complex neighborhood of the collection of arc parametrizations for which the domain-to-solution map admits a bounded holomorphic extension. Thus, map derivatives of arbitrary order not only exist: the corresponding Taylor series expansion converges uniformly on an open neighborhood of each of the parametric arcs.

Hence, our work can be seen as an extension of that in [19]. On one hand, our analysis encompasses a collection of open arcs as well as more general BIOs including the possibility of vector-valued ones. The generalization is achieved by assuming the existence of a fundamental solution with a common structure for second-order partial differential operators on two dimensions. Thus, we consider general BIOs whose kernels are given by the assumed structure of the fundamental solution, and obtain their holomorphic extensions by a slightly more abstract version of the result for

BIOs presented in [19, Theorem 3.12]. Also, by assuming a Maue-type representation formula, readily available for many specific operators, one may extend the shape holomorphy of the weakly singular BIO to the hypersingular one. For example, we show explicit results for scalar Helmholtz and Stokes problems, the latter also referred to as the elastic wave equation.

In contrast with [19], we do not only establish the holomorphism result from parametrizations sets to the solution in a fixed energy space, but instead consider the range of the map in a scale of functional spaces defined on the open arcs. Indeed, in the spirit of [13], the presently obtained results allow us to mathematically justify the use of multi-level variants of HoQMC in the context of forward and inverse shape UQ. This provides a functional framework for high-order numerical methods, such as the one presented in [24]. In particular, the families of functional spaces considered here correspond to Sobolev-type spaces defined through Fourier series expansions, mapped back to open arcs by a cosine transformation (cf. [35, Chap. 11]).

As expected, the solutions of the Dirichlet and Neumann problems here considered belong to highly regular spaces provided that both the geometry and right-hand sides are regular enough. In particular, we assume that boundary conditions are given by the restriction of entire functions, thus having arbitrary regularity. Also, the arcs considered here have a limited regularity in a Hölder-continuous spaces. We thoroughly analyze how this limited smoothness restricts the functional spaces wherein solutions of the respective problems are sought.

Outline

The remainder of the article is as follows. After setting notation in Sect. 2, the precise problem under consideration is given in Sect. 3. In Sect. 4.3 we present relevant abstract results to prove the holomorphic extension of BIOs, and also a result on how we obtain parametric holomorphism from the general holomorphic extension. In Sect. 5 we show holomorphic extensions of BIOs and derive the holomorphic extension of domain-to-solution maps. Similarly, in Sect. 5.3 we prove the parametric holomorphism of domain-to-solution maps. To illustrate our findings, in Sect. 6 we apply these results to Helmholtz and time-harmonic elastic wave scattering by showing that the structural assumptions on the corresponding BIOs are fulfilled. Lastly, Sect. 7 presents conclusions and possible extensions.

2 Preliminaries

Set $\imath = \sqrt{-1}$. We define the set of natural numbers $\mathbb{N}$ including zero as $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Vectors and matrices are indicated by boldface symbols, while for general quantities that could be either vectors or scalars we do not use bold fonts. For a pair of vectors $\boldsymbol{v}^1, \boldsymbol{v}^2 \in \mathbb{C}^n$, with $n \in \mathbb{N}$, we define the bilinear form $\boldsymbol{v}^1 \cdot \boldsymbol{v}^2 = \sum_{j=1}^n v_j^1 v_j^2$, and the Euclidean norm $\|\boldsymbol{v}^1\|^2 = \boldsymbol{v}^1 \cdot \overline{\boldsymbol{v}}^1$, where the conjugate of a vector is understood as component-wise conjugation. Also, given real numbers a, b, we say that $a \lesssim b$ if there exists a positive constant c, independent of the variables relevant for the corresponding analysis, such that $a \leq cb$. If $a \lesssim b$ and $b \lesssim a$ we write $a \cong b$.

Let B_1, B_2 be two Banach spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We denote by $\mathcal{L}(B_1, B_2)$ the space of bounded linear operators from B_1 to B_2. As it is customary, we equip it with the standard operator norm, thus rendering it a Banach space itself.

2.1 Hölder Spaces

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$, $d = 1, 2$, be non-empty, open connected sets. Given $m \in \mathbb{N}_0$ and $\alpha \in [0, 1]$, we consider the space $\mathcal{C}^{m,\alpha}(\Omega_1, \Omega_2)$ of functions $f : \Omega_1 \to \Omega_2$ with derivatives up to order m in Ω_1 having a continuous extension to $\overline{\Omega}_1$, and such that the derivatives of order m are α-Hölder continuous. Endowed with the norm

$$\|f\|_{\mathcal{C}^{m,\alpha}(\Omega_1,\Omega_2)} := \sum_{\boldsymbol{k}:|\boldsymbol{k}|\leq m} \sup_{\boldsymbol{x}\in\Omega_1} \left\| \partial^{\boldsymbol{k}} f(\boldsymbol{x}) \right\| + \sum_{\boldsymbol{k}:|\boldsymbol{k}|=m} \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in\Omega_1 \\ \boldsymbol{x}\neq\boldsymbol{y}}} \frac{\left\| \partial^{\boldsymbol{k}} f(\boldsymbol{x}) - \partial^{\boldsymbol{k}} f(\boldsymbol{y}) \right\|}{\|\boldsymbol{x}-\boldsymbol{y}\|^\alpha},$$

where we use the standard multi-index notation [30, p. 61], $\mathcal{C}^{m,\alpha}(\Omega_1, \Omega_2)$ becomes a Banach space. The case $\alpha = 0$, $\mathcal{C}^m(\Omega_1, \Omega_2)$, corresponds to functions with m continuous derivatives in $\overline{\Omega}_1$ with norm

$$\|f\|_{\mathcal{C}^{m,0}(\Omega_1,\Omega_2)} := \sum_{\boldsymbol{k}:|\boldsymbol{k}|\leq m} \sup_{\boldsymbol{x}\in\Omega_1} \left\| \partial^{\boldsymbol{k}} f(\boldsymbol{x}) \right\|.$$

On the other hand, the case $\alpha = 1$ corresponds to the one where the m-derivatives are Lipschitz continuous, and thus the derivatives of order $m+1$ exist and are bounded almost everywhere (see [17, p. 280]). Notice that for $m_1, m_2 \in \mathbb{N}_0$ and $\alpha_1, \alpha_2 \in [0, 1]$, such that $m_1 + \alpha_1 < m_2 + \alpha_2$, one has the inclusion $\mathcal{C}^{m_2,\alpha_2}(\Omega_1, \Omega_2) \subset \mathcal{C}^{m_1,\alpha_1}(\Omega_1, \Omega_2)$.

2.2 Chebyshev Polynomials and Periodic Sobolev Spaces

Next, we recall definitions and properties of Chebyshev polynomials that will be employed to define functional spaces. For $|t| \leq 1$, set $w(t) := \sqrt{1-t^2}$, and denote by $T_n(t)$ the nth first-kind Chebyshev polynomial normalized according to

$$\int_{-1}^{1} T_n(t) T_m(t) w^{-1}(t) \mathrm{d}t = \delta_{n,m}, \quad \forall\, n, m \in \mathbb{N}_0,$$

being $\delta_{n,m}$ the Kronecker delta. Additionally, let U_n denote the nth Chebyshev polynomial of the second kind normalized as follows

$$\int_{-1}^{1} U_n(t) U_m(t) w(t) \mathrm{d}t = \delta_{n,m}, \quad \forall\, n, m \in \mathbb{N}_0.$$

Furthermore, we define $\widehat{e}_n(\theta):=\frac{\exp(\imath nt)}{\sqrt{2\pi}}$ as the nth Fourier polynomial normalized in the $L^2(-\pi,\pi)$-norm. For any smooth, periodic function $u:[-\pi,\pi]\to\mathbb{C}$, its Fourier coefficients are defined as

$$\widetilde{u}_n=\int_{-\pi}^{\pi}u(\theta)\widehat{e}_{-n}(\theta)\mathrm{d}\theta.$$

Similarly, given $u:[-1,1]\to\mathbb{C}$, we define two families of first-kind Chebyshev coefficients:

$$u_n:=\int_{-1}^{1}u(t)T_n(t)\mathrm{d}t,\quad \text{and}\quad \widehat{u}_n:=\int_{-1}^{1}u(t)T_nw^{-1}(t)\mathrm{d}t,$$

and two families of second-kind Chebyshev coefficients:

$$\ddot{u}_n:=\int_{-1}^{1}u(t)U_n(t)\mathrm{d}t,\quad \text{and}\quad \check{u}_n:=\int_{-1}^{1}u(t)U_nw(t)\mathrm{d}t.$$

Fourier coefficients of a bi-periodic function $u:[-\pi,\pi]\times[-\pi,\pi]\to\mathbb{C}$ are defined as

$$\widetilde{u}_{n,l}:=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}u(\theta,\phi)\widehat{e}_{-n}(\theta)\widehat{e}_{-l}(\phi)\mathrm{d}\theta\mathrm{d}\phi, \tag{2.1}$$

and similarly for Chebyshev coefficients of bi-variate functions on $[-1,1]\times[-1,1]$. We remark that the above coefficients' definitions may be extended to those of distributions by duality respect to the bases [35, Sect. 5.2].

Throughout, we will make extensive use of periodic Sobolev spaces over $[-\pi,\pi]$, defined for $s\in\mathbb{R}$ as

$$H^s[-\pi,\pi]:=\left\{u:\|u\|_{H^s}^2=\sum_{n=-\infty}^{\infty}(1+n^2)^s|\widetilde{u}_n|^2<\infty\right\}.$$

We refer to [35, Chap. 5] for a more rigorous definition. Following [2, 16, 40], we introduce for $s\in\mathbb{R}$ the following spaces defined over $(-1,1)$:

$$T^s:=\left\{u:\|u\|_{T^s}^2=\sum_{n=0}^{\infty}(1+n^2)^s\,|u_n|^2<\infty\right\},$$
$$W^s:=\left\{u:\|u\|_{W^s}^2=\sum_{n=0}^{\infty}(1+n^2)^s\,|\widehat{u}_n|^2<\infty\right\}.$$

These two functional spaces can be defined rigorously by recalling the definition of H^s and by defining two periodic lifting operators as

$$(\mathcal{N}u)(\theta) := u(\cos\theta)|\sin\theta|, \quad \text{and,} \quad (\widehat{\mathcal{N}}u)(\theta) := u(\cos\theta),$$

which again are extended to distributions by duality along with the equivalences

$$\begin{aligned} u \in T^s &\Leftrightarrow \mathcal{N}u \in H^s[-\pi,\pi], \quad \text{with,} \quad \|u\|_{T^s} \cong \|\mathcal{N}u\|_{H^s[-\pi,\pi]} \\ u \in W^s &\Leftrightarrow \widehat{\mathcal{N}}u \in H^s[-\pi,\pi], \quad \text{with,} \quad \|u\|_{W^s} \cong \|\widehat{\mathcal{N}}u\|_{H^s[-\pi,\pi]}. \end{aligned} \tag{2.2}$$

In addition, for $s \in \mathbb{R}$, we define second-kind spaces over $(-1, 1)$ as

$$U^s := \left\{ u : \|u\|^2_{U^s} = \sum_{n=0}^{\infty} (1+n^2)^s |\ddot{u}_n|^2 < \infty \right\},$$
$$Y^s := \left\{ u : \|u\|^2_{M^s} = \sum_{n=0}^{\infty} (1+n^2)^s |\check{u}_n|^2 < \infty \right\}.$$

These spaces may also be defined from periodic Sobolev spaces via the next odd-periodic liftings

$$(\mathcal{Z}u)(\theta) := \mathcal{N}u(\theta)\,\mathrm{sign}(\sin\theta), \quad \text{and,} \quad (\widehat{\mathcal{Z}}u)(\theta) := \widehat{\mathcal{N}}u(\theta)\mathrm{sign}(\sin\theta),$$

where the sign function is defined with the convention $\mathrm{sign}(0) = 0$. In addition, we have the equivalences

$$\begin{aligned} u \in U^s &\Leftrightarrow \widehat{\mathcal{Z}}u \in H^s[-\pi,\pi], \quad \text{with,} \quad \|u\|_{U^s} \cong \|\widehat{\mathcal{Z}}u\|_{H^s[-\pi,\pi]} \\ u \in Y^s &\Leftrightarrow \mathcal{Z}u \in H^s[-\pi,\pi], \quad \text{with,} \quad \|u\|_{Y^s} \cong \|\mathcal{Z}u\|_{H^s[-\pi,\pi]}. \end{aligned}$$

Using the previous characterizations along with the inequality, $|(\widetilde{u\sin(\cdot)})_n| \lesssim |\widetilde{u}_{n+1}| + |\widetilde{u}_{n-1}|$ (for u an even function), one can readily observe that, for all $s \in \mathbb{R}$, it holds that

$$W^s \subset Y^s. \tag{2.3}$$

The dual space of H^s can be identified with H^{-s} in the $L^2(-\pi, \pi)$ duality pairing. Thus, by using the lifting operators, one can identify the dual space of T^s with W^{-s} and the dual of U^s with Y^{-s}, now with respect to the $L^2(-1, 1)$ duality pairing. Furthermore, these duality identifications and (2.3) imply that

$$U^s \subset T^s, \quad \forall\, s \in \mathbb{R}. \tag{2.4}$$

From the density of the Fourier basis in H^s using the inverse of the lifting operators, it is possible to deduce that the functions $\{wU_n\}_{n\in\mathbb{N}}$ are dense in U^s.

A mayor role in the analysis of the hyper-singular BIO is played by the mapping properties of the derivative operator. Specifically, using the density of the Chebyshev basis one can easily see that

$$\frac{d}{dt} : U^s \to T^{s-1}, \quad \text{and,} \quad \frac{d}{dt} : W^s \to Y^{s-1}. \tag{2.5}$$

Also we recall from [35, Lemma 5.3.2] that the periodic spaces $H^s[-\pi, \pi]$ are compactly embedded for increasing values of s. Hence, the same holds for the spaces T^s, W^s, U^s, Y^s.

Finally, depending on whether the underlying differential operator $\mathcal{P}$ (see Sect. 3.1) is scalar or vectorial, we use set either

$$\mathbb{T}^s = T^s,\ \mathbb{U}^s = U^s,\ \mathbb{W}^s = W^s,\ \mathbb{Y}^s = Y^s,$$

or

$$\mathbb{T}^s = T^s \times T^s,\ \mathbb{U}^s = U^s \times U^s,\ \mathbb{W}^s = W^s \times W^s,\ \mathbb{Y}^s = Y^s \times Y^s,$$

respectively. We will also consider the Cartesian product spaces

$$\prod_{j=1}^{M} \mathbb{T}^s, \quad \prod_{j=1}^{M} \mathbb{U}^s, \quad \prod_{j=1}^{M} \mathbb{W}^s, \quad \prod_{j=1}^{M} \mathbb{Y}^s,$$

provided with standard norms.

2.3 Bi-periodic Sobolev and Hölder Spaces

Along with the previous spaces, the forthcoming analysis will require the use of Sobolev spaces of bi-periodic functions along with an immersion result of Hölder spaces in their Sobolev counterparts. The latter will be employed in Sect. 4.4.

Given s_1, s_2 non-negative real numbers we recall the Sobolev norm for bi-periodic functions [35, Chap. 6]:

$$\|g\|^2_{s_1,s_2} = \sum_{n=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} (1+n^2)^{s_1}(1+\ell^2)^{s_2} |\widetilde{g}_{n,l}|^2,$$

where $\widetilde{g}_{n,l}$, are Fourier coefficients of the bi-periodic function g defined in (2.1). Notice that in contrast to standard Sobolev spaces we could have different levels of regularity s_1,s_2 on each variable.

Lemma 2.1 *Let* $g \in \mathcal{C}^{m,\alpha}([-\pi,\pi] \times [-\pi,\pi], \mathbb{C})$ *be a bi-periodic function with* $m \in \mathbb{N}_0$ *and* $\alpha \in [0, 1]$. *Provided non-negative real numbers* s_1, s_2 *satisfying* $s_1 + s_2 <$

$m+\alpha$, one has that

$$\|g\|^2_{s_1,s_2} \lesssim \|g\|^2_{\mathcal{C}^{m,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})},$$

where the implied constant is independent of g.

We relegate the proof of the previous lemma to Appendix A.

Remark 2.2 A similar result to Lemma 2.1 can be established by noticing that the smoothness of g implies a certain decay speed for the Fourier coefficients. In fact, for a pair of non-negative real values s_1, s_2 and $g \in \mathcal{C}^m([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})$ one recovers the bound of Lemma 2.1, under the more restrictive condition $s_1+s_2+1<m$. This requirement can be relaxed to $s_1+s_2<m$ when considering functions in $\mathcal{C}^m([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})$ with derivatives of order $m+1$ integrables or of bounded variation. We also notice that, for $s_1, s_2 \in \mathbb{N}_0$, one requires the slightly less restrictive condition $s_1+s_2\leq m+\alpha$.

2.4 Arc Parametrizations

Let $(-1,1)$ be the canonical interval. Throughout, we will define an open arc as the image of a continuously differentiable, globally invertible function $\boldsymbol{r}:(-1,1)\to\mathbb{R}^2$. We further assume that the tangent vector is nowhere null and fix the normal vector to have the same direction of $(r_2', -r_1')$. Rigorously speaking, this definition identifies open arcs with their corresponding parametrization instead of the set of points that describe the arc in $\mathbb{R}^2$. The function $\boldsymbol{r}$ is also called parametrization of the arc.

Remark 2.3 The parametrization of the open arcs will be taken as elements of $\mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$. The ensuing shape holomorphy analysis considers *complex* Banach spaces in contrast to these real-valued. We overcome this issue by considering $\mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{C}^2\right)$ and will only use those elements with global inverse and non-null tangent vector at every point. The subset of such functions is denoted by $\mathcal{C}_b^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$. In some instances, we will use the Cartesian product space, $\prod_{j=1}^M \mathcal{C}_b^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$. This is a subset of the product space $\prod_{j=1}^M \mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$, equipped with the standard norm:

$$\|\boldsymbol{g}\|_{\prod_{j=1}^M \mathcal{C}^{m,\alpha}((-1,1),\mathbb{R}^2)} = \max_{j=1,\ldots,M} \|g_j\|_{\mathcal{C}^{m,\alpha}((-1,1),\mathbb{R}^2)}.$$

3 Boundary Value Problems and Boundary Integral Formulation

3.1 Boundary Value Problems on Open Arcs

Let us denote by Γ the set of M disjoint open finite-length arcs $\{\Gamma_1,\ldots,\Gamma_M\}$, where each arc is characterized by a parametrization $\boldsymbol{r}_j:[-1,1]\to\Gamma_j\subset\mathbb{R}^2$. In addition, we also refer to Γ as the geometric configuration of the associated problem.

Let us consider a second-order partial differential operator of the following general form

$$\mathcal{P} = -\sum_{j=1}^{2}\sum_{k=1}^{2} \partial_{x_j} A_{j,k} \partial_k + A,$$

where $A_{j,k}$ and A can be constant complex-valued scalars or 2×2 matrices. In the latter case, we further assume that $\overline{\mathbf{A}}_{j,k}^{\top} = \mathbf{A}_{k,j}^{\top}$ and $\overline{\mathbf{A}}^{\top} = \mathbf{A}$. We define the co-normal trace operator over the boundary Γ_p as

$$\mathcal{B}_p u := \sum_{j=1}^{2} (\boldsymbol{n}_p)_j \sum_{k=1}^{2} A_{j,k} \partial_k u \big|_{\Gamma_p}, \quad p = 1, \ldots, M,$$

for any smooth function u and where $\boldsymbol{n}_p$ denotes the unitary normal vector of Γ_p. Equipped with these definitions, we consider the following BVPs:

Problem 3.1 (*Dirichlet and Neumann BVPs*) Seek u such that

$$\mathcal{P}u = 0, \quad \text{on } \mathbb{R}^2 \setminus \overline{\Gamma}, \tag{3.1}$$

$$\text{condition at infinity}(\mathcal{P}), \tag{3.2}$$

complemented with either boundary conditions:

$$u = f^D \quad \text{on } \Gamma_p, \quad p = 1, \ldots, M, \quad \text{(Dirichlet)},$$
$$\mathcal{B}_p u = f^N \quad \text{on } \Gamma_p, \quad p = 1, \ldots, M, \quad \text{(Neumann)}.$$

Condition (3.2) specifies the behavior of u far away from Γ and it is crucial to show uniqueness. Its particular form depends on the specific partial differential operator $\mathcal{P}$. Boundary data f^D and f^N correspond to the right-hand sides of the Dirichlet and Neumann boundary value problems, respectively. Throughout, we assume that these are the restriction to Γ of *entire* functions in each coordinate in $\mathbb{R}^2$. Still, analytic functions on bounded domains can be used without fundamentally changing any result.

In what follows, we assume uniqueness for both Dirichlet and Neumann BVPs, while existence results will be a consequence of the boundary integral formulation presented next.

Remark 3.2 Certain assumptions on the operator $\mathcal{P}$ are worth further comments. Specifically,

(i) The coefficients of $\mathcal{P}$ are assumed to be constants. Though this assumption is not entirely necessary, one would require the coefficients to be at least analytic in the spatial variable. Otherwise, structural assumptions on the fundamental solutions of $\mathcal{P}$ will not hold as detailed in the upcoming section.

(ii) We have also assumed that $\overline{\mathbf{A}}_{j,k}^{\top} = \mathbf{A}_{k,j}^{\top}$ and $\overline{\mathbf{A}}^{\top} = \mathbf{A}$, and also that $\mathcal{P}$ lacks any first order derivative term. These conditions ensure that the co-normal trace is self-adjoint, thus rendering the analysis simpler. Yet, our results still hold without this restriction by suitably modifying the associated integral operators [30, Chap. 7].

3.2 Boundary Integral Formulation

We now recall the boundary integral formulation of BVPs for the partial differential operator $\mathcal{P}$ introduced in Sect. 3.1. To this end, we assume the existence of a fundamental solution associated to $\mathcal{P}$, which we denote by $G(\boldsymbol{x}, \boldsymbol{y})$. For further details we refer to [30, Chap. 6] and references therein. In addition, we assume that the fundamental solution admits a decomposition of the form

$$G(\boldsymbol{x}, \boldsymbol{y}) = F_1\left(\|\boldsymbol{x} - \boldsymbol{y}\|^2\right) \log \|\boldsymbol{x} - \boldsymbol{y}\|^2 + F_2\left(\|\boldsymbol{x} - \boldsymbol{y}\|^2\right), \tag{3.3}$$

where the functions F_1 and F_2 are assumed to be entire complex-valued scalars or 2×2 matrices. Furthermore, whenever F_1 is a scalar we assume that $F_1(0) \neq 0$ while if matrix-valued then $\mathbf{F_1}(0)$ should admit an inverse.

Remark 3.3 One could lessen the restrictions for F_1, F_2 and impose that they are only analytic on an open connected set of $\mathbb{C}$. If so, our results would still hold inside the analyticity domain of F_1, F_2.

Next, we introduce the single and double layer potentials, respectively, on a generic open arc γ as

$$\left(\widehat{\mathrm{SL}}_\gamma \widehat{\lambda}\right)(\boldsymbol{x}) := \int_\gamma G(\boldsymbol{x}, \boldsymbol{y}) \widehat{\lambda}(\boldsymbol{y}) \mathrm{ds}_{\boldsymbol{y}}, \ \left(\widehat{\mathrm{DL}}_\gamma \widehat{\mu}\right)(\boldsymbol{x}) := \int_\gamma \left(\mathcal{B}_{\boldsymbol{n},\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y})\right)^\top \widehat{\mu}(\boldsymbol{y}) \mathrm{ds}_{\boldsymbol{y}},$$

where $\mathcal{B}_{\boldsymbol{n},\boldsymbol{y}}$ denotes the co-normal trace[1] in the $\boldsymbol{y}$ variable. The densities $\widehat{\lambda}$ and $\widehat{\mu}$ are defined on γ, and are scalar-valued or two-dimensional vectors depending on the nature of $\mathcal{P}$. The fundamental solution definition ensures that both potentials are homogeneous solutions of (3.1) in $\mathbb{R}^2 \setminus \overline{\gamma}$. Furthermore, we assume that both potentials satisfy the radiation condition (3.2).

Let $\boldsymbol{r} : (-1, 1) \to \mathbb{R}^2$ be a parametrization of the open arc γ. We introduce the transformed densities:

$$\lambda(\tau) := \widehat{\lambda} \circ \boldsymbol{r}(\tau) \|\boldsymbol{r}'(\tau)\|, \quad \mu(\tau) := \widehat{\mu} \circ \boldsymbol{r}(\tau), \quad \tau \in (-1, 1),$$

[1] The general definition of the double layer potential involves the adjoint of the co-normal trace operator but under our assumptions on $\mathcal{P}$ this operator is self-adjoint.

and the pulled-back potentials

$$(\mathrm{SL}_{\boldsymbol{r}}\varrho)\,(\boldsymbol{x}) := \int_{-1}^{1} G(\boldsymbol{x}, \boldsymbol{r}(\tau))\varrho(\tau)\mathrm{d}\tau,$$
$$(\mathrm{DL}_{\boldsymbol{r}}\varrho)\,(\boldsymbol{x}) := \int_{-1}^{1} \left(\mathcal{B}_{\boldsymbol{n},\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{r}(\tau))\right)^{\top} \varrho(\tau)\|\boldsymbol{r}'(\tau)\|\mathrm{d}\tau,$$

defined for $\varrho : [-1, 1] \to \mathbb{C}$. From these definitions it is direct that $\widehat{\mathrm{SL}}_{\gamma}\widehat{\lambda} = \mathrm{SL}_{\boldsymbol{r}}\lambda$, and also that $\widehat{\mathrm{DL}}_{\gamma}\widehat{\mu} = \mathrm{DL}_{\boldsymbol{r}}\mu$.

With these elements, one can now reformulate the BVPs presented in the previous section as a set of BIEs. We do so by imposing boundary conditions on indirect integral representations built via the above boundary layer potentials.

Problem 3.4 (*Dirichlet and Neumann BIEs*) We seek densities $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_M)$, with each λ_i and μ_i defined over $[-1, 1]$ for $i = 1, \ldots, M$, such that

$$\sum_{j=1}^{M} (\mathrm{SL}_{\boldsymbol{r}_j}\lambda_j) \circ \boldsymbol{r}_i = f^D \circ \boldsymbol{r}_i, \quad i = 1, \ldots, M, \text{ (Dirichlet BIE)},$$
$$\sum_{j=1}^{M} (\mathcal{B}_{\boldsymbol{n},\boldsymbol{x}}\mathrm{DL}_{\boldsymbol{r}_j}\mu_j) \circ \boldsymbol{r}_i = f^N \circ \boldsymbol{r}_i, \quad i = 1, \ldots, M, \text{ (Neumann BIE)}.$$

With these, we derive the following solutions for BVPs (Problem 3.1):

$$u = \sum_{j=1}^{M} \mathrm{SL}_{\boldsymbol{r}_j}\lambda_j \quad \text{(Dirichlet)}, \quad u = \sum_{j=1}^{M} \mathcal{B}_{\boldsymbol{n},\boldsymbol{x}}\mathrm{DL}_{\boldsymbol{r}_j}\mu_j \quad \text{(Neumann)}.$$

We can rewrite the BIEs in matrix form:

$$\mathbf{V}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M}\boldsymbol{\lambda}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M} = \mathbf{f}^D_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M}, \quad \mathbf{W}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M}\boldsymbol{\mu}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M} = \mathbf{f}^N_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M}, \tag{3.4}$$

where

$$(V_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})_{i,j} := (\mathrm{SL}_{\boldsymbol{r}_j}) \circ \boldsymbol{r}_i \quad \text{and} \quad (W_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})_{i,j} := (\mathcal{B}_{\boldsymbol{n},\boldsymbol{x}}\mathrm{DL}_{\boldsymbol{r}_j}) \circ \boldsymbol{r}_i$$

are weakly- and hyper-singular BIOs, and

$$(f^D_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})_i = f^D \circ \boldsymbol{r}_i \quad \text{and} \quad (f^N_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})_i := f^N \circ \boldsymbol{r}_i.$$

The weakly singular operators can be represented as a Lebesgue integral as follows

$$(V_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})_{i,j}\varrho(t) = \int_{-1}^{1} G(\boldsymbol{r}_i(t), \boldsymbol{r}_j(\tau))\varrho(\tau)\mathrm{d}s, \quad t \in (-1, 1),$$

for a function $\varrho : [-1, 1] \to \mathbb{R}$. On the other hand, the hyper-singular operator can only be expressed as a Hadamard's finite-part integral. However, for every $s \in \mathbb{R}$, and $\varrho \in U^s$, we will assume the existence of a Maue-type representation formula of the form

$$(W_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})_{i,j}\varrho = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} G(\boldsymbol{r}_i(t), \boldsymbol{r}_j(\tau))\frac{\mathrm{d}}{\mathrm{d}s}\varrho(\tau)\mathrm{d}s + \int_{-1}^{1} \widetilde{G}(\boldsymbol{r}_i(t), \boldsymbol{r}_j(\tau))\varrho(\tau)\mathrm{d}s, \tag{3.5}$$

where $\widetilde{G}$ is a function with the same structure of the fundamental solution, i.e. as in (3.3) . Such expressions for the hyper-singular operators are well known for particular cases of $\mathcal{P}$ on closed boundaries; see for example [29, 33] or the general result for scalar operators in [36, Chap. 3.3.4]. Yet, to the best of our knowledge, there is no known result for the general case. Similar results to the case of open arcs are derived by zero-extensions of boundary densities on the arc onto closed curves containing the arc (cf. [23]).

To conclude this section, we remark that since we construct solutions of the BVPs upon boundary potentials acting on the resulting densities for the BIEs above, we in fact show existence results of these BVPs by showing that of the BIEs. Moreover, as it was pointed out in [24, Remark 3.11], the uniqueness of the boundary integral formulations can be inferred from the uniqueness of the boundary value problems, which was assumed to hold in the previous section.

4 Holomorphic Extensions

We now introduce the main tools to prove the sought shape holomorphy property of the BIOs on open arcs. This result is stated ahead in Theorems 5.7 and 5.11. The main ingredient is the holomorphic extension of BIOs to complex-valued parametrizations. In view of this, in Sect. 4.1 we introduce holomorphic maps in general Banach spaces. In Sect. 4.2, we consider subsets of Banach spaces that are characterized by countably-many parameters (parametric subsets), and introduce the associated notion of parametric holomorphy in Definition 4.3. We show that the general holomorphy property is inherited as parametric holomorphism when a map is restricted to a parametric subset.

Based on previous work [18, 19], in Sect. 4.3 we present a general framework for establishing the holomorphic extension of some general class of integral operators on arbitrary Banach spaces. Finally, in Sects. 4.4, 4.5, and 4.6 we present the tools that enable us to consider the extension of the BIOs in Sect. 3.1 to complex parametrizations in the framework of Sect. 4.3.

4.1 Holomorphic Maps in Banach Spaces

Let B_1, B_2 be Banach spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. When the underlying field of either B_1 or B_2 is $\mathbb{R}$, we say that B_1 or B_2 is a real Banach space, otherwise we refer to them as complex Banach spaces. One can construct a complex Banach space by taking as starting point a real one. For instance, suppose that B_1 is a real Banach space, we define the space $B_1^{\mathbb{C}}$ as the set of elements of the form $b + \imath d$, with $b, d \in B_1$, and we refer to it as the *complexification* of B_1. The complexification of B_1 is also a Banach space with the norm $\|b+\imath d\|_{B_1^{\mathbb{C}}} := \sup_{\theta\in[0,2\pi]} \|b\cos\theta + d\sin\theta\|_{B_1}$, and $\mathbb{C}$ as the underlying field. If B_1 is a real Banach space, given an arbitrary subset $K \subset B_1$, and δ a positive real number, we define

$$K_\delta = \left\{k \in B_1^{\mathbb{C}} \text{ such that } \exists\, b \in K \;:\; \|b-k\|_{B_1^{\mathbb{C}}} < \delta\right\}.$$

We now introduce the notion of holomorphy in Banach spaces (cf. [32] for a more details on complex analysis in Banach spaces).

Definition 4.1 Let $K \subset B_1$. Assume that there exists an open set $\mathcal{O} \subset B_1^{\mathbb{C}}$ such that $K \subset \mathcal{O}$. We say that the map $f : K \subset B_1 \to B_2$ is holomorphic in K if there exists an extension of f to $\mathcal{O}$, still denoted by f, such that $f : \mathcal{O} \to B_2$ is Fréchet differentiable.

The next result states that if an invertible operator admits a bounded holomorphic extension so does its inverse. The proof is based on well-known results from complex variable theory.

Theorem 4.2 ([19, Proposition 4.20]) *For $K \subset B_1$, let $(\mathsf{A}_k)_{k\in K}$ be a family of operators in $\mathcal{L}(X, Y)$ such that:*

(i) For every $k \in K$, A_k has a bounded inverse, i.e. $\mathsf{A}_k^{-1} \in \mathcal{L}(Y, X)$.
(ii) There exists $\delta > 0$ such that the map $K \ni k \mapsto \mathsf{A}_k \in \mathcal{L}(X, Y)$ admits a bounded holomorphic extension into K_δ.

Then, there exists η, depending of K and δ, such that the map

$$K \ni k \mapsto \mathsf{A}_k^{-1} \in \mathcal{L}(Y, X)$$

admits a bounded holomorphic extension into K_η.

4.2 Parametric Holomorphy

In concrete applications, such as the ones arising in forward and inverse computational UQ, one is usually interested in using a parametrically defined compact set that in turn defines the set of admissible parametric representations. A particular example of this construction consists in considering an *affine*-parametric set of the form

$$K = \left\{k_{\boldsymbol{y}} \in B_1 : k_{\boldsymbol{y}} = k_0 + \sum_{n=1}^{\infty} y_n k_n,\ \ \boldsymbol{y} = \{y_n\}_{n\in\mathbb{N}} \in \mathrm{U}\right\}, \tag{4.1}$$

where $k_0 \in B_1$ is fixed, $\mathrm{U}:=[-1, 1]^{\mathbb{N}}$ and $\{k_n\}_{n\in\mathbb{N}} \subset B_1$ is a fixed sequence, usually referred to as *perturbation basis*, as one can interpret the elements $k_{\boldsymbol{y}}$ as perturbations of the *nominal* value k_0 modulated by the parameter sequence $\boldsymbol{y} \in \mathrm{U}$. By assuming $\{\|k_n\|_{B_1}\} \in \ell^1(\mathbb{N})$, one can prove that K is compact in B_1, as shown in [9, Lemma 2.7]. Within this framework, we consider maps of the form $F : \mathrm{U} \to B_2 : \boldsymbol{y} \mapsto f(k_{\boldsymbol{y}})$ where $f : B_1 \to B_2$ denotes a holomorphic map in the sense of Definition 4.1. This construction renders F a Banach-space-valued map with a high-dimensional input.

For a rigorous study of F, we make use of the so-called $(\boldsymbol{b}, p, \varepsilon)$-holomorphic maps, originally introduced in [8], a key mathematical property to break the curse of dimensionality in the approximation of parametric maps with high-dimensional inputs. Specifically, for $\varrho > 1$, we consider the Bernstein ellipse in the complex plane

$$\mathcal{E}_\varrho := \left\{ \frac{z + z^{-1}}{2} : z \in \mathbb{C} \text{ with } 1 \leq |z| \leq \varrho \right\} \subset \mathbb{C}.$$

This ellipse has foci at $z = \pm 1$ and semi-axes of length $a := \frac{1}{2}(\varrho + \varrho^{-1})$ and $b := \frac{1}{2}(\varrho - \varrho^{-1})$. Let us consider the tensorized poly-ellipse

$$\mathcal{E}_{\boldsymbol{\rho}} := \bigotimes_{j \geq 1} \mathcal{E}_{\rho_j} \subset \mathbb{C}^{\mathbb{N}},$$

where $\boldsymbol{\rho} := \{\rho_j\}_{j \geq 1}$ is such that $\rho_j > 1$, for $j \in \mathbb{N}$. We adopt the convention $\mathcal{E}_1 := [-1, 1]$ to include the case $\rho_j = 1$.

Definition 4.3 ([8, *Definition 2.1]*) Let X be a complex Banach space equipped with the norm $\|\cdot\|_X$. For $\varepsilon > 0$ and $p \in (0, 1)$, we say that the map

$$\mathrm{U} \ni \boldsymbol{y} \mapsto u_{\boldsymbol{y}} \in X$$

is $(\boldsymbol{b}, p, \varepsilon)$-holomorphic if and only if

(i) The map $\mathrm{U} \ni \boldsymbol{y} \mapsto u_{\boldsymbol{y}} \in X$ is uniformly bounded, i.e.

$$\sup_{\boldsymbol{y} \in \mathrm{U}} \|u_{\boldsymbol{y}}\|_X \leq C_0,$$

for some finite constant $C_0 > 0$.

(ii) There exists a positive sequence $\boldsymbol{b} := \{b_j\}_{j \geq 1} \in \ell^p(\mathbb{N})$ and a constant $C_\varepsilon > 0$ such that for any sequence $\boldsymbol{\rho} := \{\rho_j\}_{j \geq 1}$ of numbers strictly larger than one that is $(\boldsymbol{b}, \varepsilon)$-admissible, i.e. satisfying

$$\sum_{j \geq 1} (\rho_j - 1) b_j \leq \varepsilon,$$

the map $\boldsymbol{y} \mapsto u_{\boldsymbol{y}}$ admits a complex extension $\boldsymbol{z} \mapsto u_{\boldsymbol{z}}$ that is holomorphic with respect to each variable z_j on a set of the form

$$\mathcal{O}_{\boldsymbol{\rho}} := \bigotimes_{j \geq 1} \mathcal{O}_{\rho_j},$$

where $\mathcal{O}_{\rho_j} \subset \mathbb{C}$ is an open set containing $\mathcal{E}_{\rho_j}$. This extension is bounded on $\mathcal{E}_{\boldsymbol{\rho}}$ according to

$$\sup_{z \in \mathcal{E}_{\boldsymbol{\rho}}} \|u_{\boldsymbol{z}}\|_X \leq C_\varepsilon.$$

In the context of the multiple open arcs problem (Problem 3.1) and its boundary integral formulation (cf. Sect. 3.2), we will consider a *nominal* geometric configuration parametrized by functions $\boldsymbol{r}_1^0, \ldots, \boldsymbol{r}_M^0$, and perturbations defined for each individual arc in an affine manner as in (4.1). We show that under some assumptions in the perturbation basis, the map from the parameter space to the boundary is holomorphic in the sense of Definition 4.3.

We conclude this section by a introducing a result that allows us to establish parametric holomorphy in the sense of Definition 4.3.

Theorem 4.4 *Assume that the sequence $\{k_n\}_{n\in\mathbb{N}} \subset B_1$ in (4.1) is such that $\|k_n\|_{B_1} \in \ell^p(\mathbb{N})$ for some $p \in (0, 1)$. Assume that there exists $\delta > 0$ such that the map $f : K \subset B_1 \to B_2$ admits a bounded holomorphic extension onto $K_\delta \subset B_1^{\mathbb{C}}$. Then, there exists $\varepsilon > 0$ such that the map*

$$\mathrm{U} \ni \boldsymbol{y} \mapsto f(k_{\boldsymbol{y}}) \in B_2,$$

is $(\boldsymbol{b}, p, \varepsilon)$-holomorphic with $\boldsymbol{b} = \{\|k_n\|_{B_1}\}_{n\in\mathbb{N}}$ and the same $p \in (0, 1)$, and it is continuous in the product topology.

Proof The map $\mathrm{U} \ni \boldsymbol{y} \mapsto k_{\boldsymbol{y}} \in B_1$ is $(\boldsymbol{b}, p, \varepsilon)$-holomorphic. The proof follows the same steps as that of [10, Lemma 5.8], and we skip it for the sake of brevity. Being $f : K \subset B_1 \to B_2$ holomorphic itself in the sense of Definition 4.1, the composition of these two maps preserves this property with the same $\boldsymbol{b} \in \ell^p(\mathbb{N})$ and $p \in (0, 1)$. The continuity statement follows by using the exact same technique used in the proof of [10, Lemma 5.7]. □

4.3 Holomorphic Integral Operators

We continue by following the framework introduced in [19] so as to establish the holomorphic dependence of certain classes of BIOs. For the sake of completeness, we revisit the most important results presented therein and elaborate on their extension to the BIOs for two-dimensional screens or cracks.

As in Sect. 4.2, we consider a real-valued Banach space B_1, its corresponding complexification $B_1^{\mathbb{C}}$ as introduced in Sect. 4.1, and a compact set $K \subset B_1$. For each $k \in K$, we consider the integral operator P_k defined as

$$(\mathsf{P}_k u)(t) := \int_{-1}^{1} S(t-\tau) p_k(t,\tau) u(\tau) \mathrm{d}s, \tag{4.2}$$

where the function S does not depend on the parameter $k \in K$. We further assume that for each $k \in K$ the integral operator P_k introduced in (4.2) defines a bounded linear operator between two Banach spaces X and Y, i.e. for each $k \in K$ we have that $\mathsf{P}_k \in \mathcal{L}(X, Y)$. Furthermore, we assume that the continuous functions are dense in X. The following result enables us to construct the holomorphic extension of the map $K \ni k \mapsto \mathsf{P}_k \in \mathcal{L}(X, Y)$.

Theorem 4.5 ([19, Theorem 3.12]) *Assume that*

(i) The function S is continuous everywhere except possibly at the origin. In addition, in a neighborhood of $t = 0$ we assume that

$$|S(t)| \lesssim |t|^{-\beta}$$

for some $\beta \in [0, 1)$, and that $p_k \in \mathcal{C}^0((-1, 1) \times (-1, 1), \mathbb{C})$ for each $k \in K$.

(ii) There exists a $\delta > 0$ such that the map

$$K \ni k \mapsto p_k \in \mathcal{C}^0((-1, 1) \times (-1, 1), \mathbb{C})$$

admits a bounded holomorphic extension onto K_δ denoted by

$$K_\delta \ni k \mapsto p_{k,\mathbb{C}} \in \mathcal{C}^0((-1, 1) \times (-1, 1), \mathbb{C}).$$

(iii) For δ as in item (ii), the corresponding extension of the integral operator P_k to K_δ defined as

$$(\mathsf{P}_{k,\mathbb{C}} u)(t) := \int_{-1}^{1} S(t - \tau) p_{k,\mathbb{C}}(t, \tau) u(\tau) \mathrm{d}s$$

is uniformly bounded upon K_δ, i.e. there exists a positive constant $C(K, \delta)$, depending on K and δ only, such that

$$\sup_{k \in K_\delta} \|\mathsf{P}_{k,\mathbb{C}}\|_{\mathcal{L}(X,Y)} < C(K, \delta).$$

Then, the map

$$K \ni k \mapsto \mathsf{P}_k \in \mathcal{L}(X, Y)$$

admits a bounded holomorphic extension into K_δ.

Remark 4.6 In [19], the above result is explicitly proved for continuous functions and then extended by density to an appropriate scale of Sobolev spaces. A close inspection of the proof reveals that this hypothesis could be further relaxed. Indeed, it is enough to consider functions in $L^1(-1, 1)$ that are denses in B_1 so as to apply Fubini's theorem.

4.4 BIOs on the Canonical Arc

Based on the functional spaces defined on Sect. 2.1, we proceed to study the mapping properties of the following three types of BIOs:

$$
\begin{aligned}
(R_f u)(t) &:= \int_{-1}^{1} f(t,\tau)u(\tau)\mathrm{d}s,\\
(L_f u)(t) &:= \int_{-1}^{1} \log|t-\tau| f(t,\tau)u(\tau)\mathrm{d}s,\\
(S_f u)(t) &:= \int_{-1}^{1} \log|t-\tau|(t-\tau)^2 f(t,\tau)u(\tau)\mathrm{d}s,
\end{aligned}
$$

where $f \in \mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})$, for some $m \in \mathbb{N}_0$ and $\alpha \in [0,1]$. We will use the results of this section to establish the mapping properties of the weakly- and hyper-singular operators, as this will enable us to invoke Theorem 4.5. The analysis follows closely that of [35, Chaps. 6 and 11]. In particular, we consider periodizations of the three types of integral operators, and then apply [35, Theorem 6.1.3] to obtain the mapping properties of the operators. However, and in contrast to [35, Chap. 6], we consider kernels of limited regularity.

Remark 4.7 Let $f \in \mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})$ be given. Set

$$
\sigma(\theta,\phi) := f(\cos\theta,\cos\phi), \quad \text{and} \quad \varphi(\theta,\phi) := f(\cos\theta,\cos\phi)\sin\theta\sin\phi.
$$

One can readily observe that both functions are bi-periodic and belong to $\mathcal{C}^{m,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})$. Moreover, since trigonometric functions and their derivatives are trivially bounded, we have that

$$
\|\sigma\|_{\mathcal{C}^{m,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})} \cong \|\varphi\|_{\mathcal{C}^{m,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})} \cong \|f\|_{\mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})},
$$

where the implicit constants are independent of f.

4.4.1 Operator R_f

Let us recall the periodic lifting operators $\widehat{\mathcal{N}}, \mathcal{N}, \mathcal{Z}, \widehat{\mathcal{Z}}$ defined as in Sect. 2.1. When applied to the operator R_f, we obtain the following periodic operators:

$$
\begin{aligned}
(\widehat{\mathcal{N}} R_f u)(\theta) &= \frac{1}{2}\int_{-\pi}^{\pi} f(\cos\theta,\cos\phi)\mathcal{N}u(\phi)\mathrm{d}\phi,\\
(\mathcal{Z} R_f u)(\theta) &= \frac{1}{2}\int_{-\pi}^{\pi} f(\cos\theta,\cos\phi)\sin\theta\sin\phi\widehat{\mathcal{Z}}u(\phi)\mathrm{d}\phi.
\end{aligned}
$$

We will also make use of the following operators:

$$R_f^{\mathcal{N}} u(\theta) := \frac{1}{2} \int_{-\pi}^{\pi} f(\cos\theta, \cos\phi) u(\phi) d\phi,$$
$$R_f^{\mathcal{Z}} u(\theta) := \frac{1}{2} \int_{-\pi}^{\pi} f(\cos\theta, \cos\phi) \sin\theta \sin\phi u(\phi) \mathrm{d}\phi,$$

related to $\widehat{\mathcal{N}} R_f$, and $\mathcal{Z} R_f$ as follows

$$\widehat{\mathcal{N}} R_f u = R_f^{\mathcal{N}} \mathcal{N} u, \quad \mathcal{Z} R_f u = R_f^{\mathcal{Z}} \widehat{\mathcal{Z}} u.$$

The following results follows directly from [35, Theorem 6.1.1] and Lemma 2.1.

Lemma 4.8 *Let $s \in \mathbb{R}$, and $f \in \mathcal{C}^{m,\alpha}((-1,1) \times (-1,1), \mathbb{C})$ the kernel function of R_f. If one of the following conditions is satisfied*
(i) $s > -\frac{1}{2}$ and $s + \frac{3}{2} < m + \alpha$, or
(ii) $s \leq -\frac{1}{2}$ and $-s + \frac{1}{2} < m + \alpha$,
we have that $R_f^{\mathcal{N}} \in \mathcal{L}(H^s, H^{s+1})$ and $R_f^{\mathcal{Z}} \in \mathcal{L}(H^s, H^{s+1})$. Furthermore, they are compact operators in the corresponding spaces, and satisfy

$$\|R_f^{\mathcal{N}}\|_{\mathcal{L}(H^s, H^{s+1})} \lesssim \|\sigma\|_{\mathcal{C}^{m,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})},$$
$$\|R_f^{\mathcal{Z}}\|_{\mathcal{L}(H^s, H^{s+1})} \lesssim \|\varphi\|_{\mathcal{C}^{m,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})},$$

where σ, φ are defined as in Remark 4.7 and unspecified constants do not depend on f.

Proof Let us focus on the operator $R_f^{\mathcal{N}}$ as for $R_f^{\mathcal{Z}}$ the proof is equivalent when changing the kernel σ with φ. We can directly see that the kernel has no singularity, and consequently, the integral operator is of arbitrary order, in particular, of order -1. Thus, by [35, Theorem 6.1.3], for ν arbitrary close to $\frac{1}{2}$, we have that

$$\|R_f^{\mathcal{N}}\|^2_{\mathcal{L}(H^s, H^{s+1})} \lesssim \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (1+n^2)^a (1+l^2)^b |\widetilde{\sigma}_{n,l}|^2 + (1+n^2)^c (1+l^2)^d |\widetilde{\sigma}_{n,l}|^2,$$

where $a = s+1$, $b = \max\{\nu, |\nu - 1|\}$, $c = \nu$, $d = \max\{|s|, \nu\}$ for $s > -\frac{1}{2}$, and $a = s+1$, $b = \max\{\nu, |\nu - 1|\}$, $c = \nu$, $d = \max\{|s|, \nu\}$ for $s < -\frac{1}{2}$.

For $s = \frac{1}{2}$, we use [35, Theorem 6.1.1] so as to get the same bound with $a = \max\{|s+1|, \nu\}$, $b = \max\{|s|, \nu\}$, with ν as before, and parameters c, d not specified as the second term of the right-hand side of the above inequality does not appear in this case. From Lemma 2.1, if one of the conditions specified in the hypothesis holds, we obtain the bound

$$\|R_f^{\mathcal{N}}\|_{\mathcal{L}(H^s, H^{s+1})} \lesssim \|\sigma\|_{\mathcal{C}^{m,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})} < \infty,$$

and therefore, $R_f^{\mathcal{N}} \in \mathcal{L}(H^s, H^{s+1})$.

Compactness follows from similar arguments as we can consider that the operator is of order $-1-\epsilon$, for arbitrary small $\epsilon > 0$, and show that $R_f^{\mathcal{N}} \in \mathcal{L}\left(H^s, H^{s+1+\epsilon}\right)$. Then, by the compact embedding of H^s spaces we obtain the stated result. □

Using the properties of lifting operators we derive those of the operator R_f.

Corollary 4.9 *Let $s \in \mathbb{R}$ be such that the hypotheses of Lemma 4.8 are fulfilled. Then, we have that $R_f \in \mathcal{L}(T^s, W^{s+1})$ and $R_f \in \mathcal{L}(U^s, Y^{s+1})$. Furthermore, they are compact operators satisfying*

$$\|R_f\|_{\mathcal{L}(T^s, W^{s+1})} \lesssim \|f\|_{\mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})},$$
$$\|R_f\|_{\mathcal{L}(U^s, Y^{s+1})} \lesssim \|f\|_{\mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})},$$

with implicit constants independent of f.

Proof From the properties of the periodic lifting operator (2.2), and the previous Lemma, one deduces that

$$\|R_f u\|_{W^{s+1}} \cong \|\widehat{\mathcal{N}} R_f u\|_{H^{s+1}} = \|R_f^{\mathcal{N}} \mathcal{N} u\|_{H^{s+1}} \leq \|R_f^{\mathcal{N}}\|_{\mathcal{L}(H^s, H^{s+1})} \|\mathcal{N} u\|_{H^s}.$$

We recall that $\|\mathcal{N} u\|_{H^s} \cong \|u\|_{T^s}$ (cf. Eq. (2.2)), and also by the previous Lemma and Remark 4.7, we get

$$\|R_f^{\mathcal{N}}\|_{\mathcal{L}(H^s, H^{s+1})} \lesssim \|f\|_{\mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})},$$

as stated. The proof is analogous for U^s and Y^{s+1} by using $R_f^{\mathcal{Z}}$ instead of $R_f^{\mathcal{N}}$. □

4.4.2 Operator L_f

As in the previous case, we consider two lifting versions of L_f:

$$(\widehat{\mathcal{N}} L_f u)(\theta) = \frac{\log 2}{2} \int_{-\pi}^{\pi} f(\cos\theta, \cos\phi) \mathcal{N} u(\phi) \mathrm{d}\phi + \int_{-\pi}^{\pi} f(\cos\theta, \cos\phi) \log \left| \sin\left(\frac{\theta-\phi}{2}\right) \right| \mathcal{N} u(\phi) \mathrm{d}\phi,$$

and

$$(\mathcal{Z} L_f u)(\theta) = \frac{\log 2}{2} \int_{-\pi}^{\pi} f(\cos\theta, \cos\phi) \sin\theta \sin\phi \widehat{\mathcal{Z}} u(\phi) \mathrm{d}\phi + \int_{-\pi}^{\pi} f(\cos\theta, \cos\phi) \sin\theta \sin\phi \log \left| \sin\left(\frac{\theta-\phi}{2}\right) \right| \widehat{\mathcal{Z}} u(\phi) \mathrm{d}\phi.$$

We see that these two operators can be characterized as the sum of a regular operator plus a logarithmic one. The logarithmic part gives rise to an operator of order -1.

Thus, by the same arguments used in the analysis of R_f, we arrive at the following result:

Corollary 4.10 *For $s \in \mathbb{R}$, let the hypotheses of Lemma 4.8 hold. Then, $L_f \in \mathcal{L}(T^s, W^{s+1})$ and $L_f \in \mathcal{L}(U^s, Y^{s+1})$. Furthermore, the bounds*

$$\|L_f\|_{\mathcal{L}(T^s,W^{s+1})} \lesssim \|f\|_{\mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})},$$
$$\|L_f\|_{\mathcal{L}(U^s,Y^{s+1})} \lesssim \|f\|_{\mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})},$$

hold with unspecified constants independent of f.

4.4.3 Operator S_f

Finally, consider the S_f operator, whose periodic liftings are

$$(\widehat{\mathcal{N}} S_f u)(\theta) = \frac{\log 2}{2}\int_{-\pi}^{\pi} (\cos\theta - \cos\phi)^2 f(\cos\theta, \cos\phi)\mathcal{N}u(\phi)\mathrm{d}\phi$$
$$+ 4\int_{-\pi}^{\pi} \log\left|\sin\left(\frac{\theta-\phi}{2}\right)\right| \sin^2\left(\frac{\theta-\phi}{2}\right)\sin^2\left(\frac{\theta+\phi}{2}\right) f(\cos\theta, \cos\phi)\mathcal{N}u(\phi)\mathrm{d}\phi,$$

and

$$(\mathcal{Z} S_f u)(\theta) = \frac{\log 2}{2}\int_{-\pi}^{\pi} (\cos\theta - \cos\phi)^2 f(\cos\theta, \cos\phi)\sin\theta\sin\phi\widehat{\mathcal{Z}}u(\phi)\mathrm{d}\phi$$
$$+ 4\int_{-\pi}^{\pi} \log\left|\sin\left(\frac{\theta-\phi}{2}\right)\right| \sin^2\left(\frac{\theta-\phi}{2}\right)\sin^2\left(\frac{\theta+\phi}{2}\right)$$
$$\times f(\cos\theta, \cos\phi)\sin\theta\sin\phi\widehat{\mathcal{Z}}u(\phi)\mathrm{d}\phi.$$

While these operators are of order -3, we will consider them as a compact operator of order -1. This can be done by analyzing the mapping properties from T^s (resp. U^s) to $W^{s+1+\epsilon}$ (resp. $Y^{s+1+\epsilon}$). In particular, we can select ϵ small enough such that the same conditions of Lemma 4.8 apply to deduce the following result.

Corollary 4.11 *Let $s \in \mathbb{R}$ be such that the same conditions of Corollary 4.10 are satisfied. Then, $S_f \in \mathcal{L}(T^s, W^{s+1})$ and $S_f \in \mathcal{L}(U^s, Y^{s+1})$ booth been compact operators. Moreover, we have the bounds:*

$$\|S_f\|_{\mathcal{L}(T^s,W^{s+1})} \lesssim \|f\|_{\mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})},$$
$$\|S_f\|_{\mathcal{L}(U^s,Y^{s+1})} \lesssim \|f\|_{\mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})},$$

with unspecified constants independent of f.

4.5 Holomorphic Functions

In the ensuing analysis, we show the existence of holomorphic extensions for certain recurrently appearing functions (cf. [19, Sect. 4.1]). However, as we are working with open arcs, the functions considered herein are not periodic. The analysis provided in this section lies in the context of spaces of the form $\mathcal{C}^{m,\alpha}((-1,1),\mathbb{R}^2)$, with $m \in \mathbb{N}$ and $\alpha \in [0,1]$, with at least $m+\alpha > 2$, instead of twice continuously differentiable, periodic functions. Consequently, the holomorphic extension of functions for multiples arcs requires suitable sets such as the ones below.

Definition 4.12 Let $m \in \mathbb{N}$ and $\alpha \in [0,1]$. We say that K is an (m,α)-admissible set of arc parametrizations if $K \subset \mathcal{C}_b^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$ and if K is a compact subset of $\mathcal{C}^{m,\alpha}((-1,1),\mathbb{R}^2)$.

When dealing with multiple arcs we further need to impose that two pair of arcs intersect or touch each other. In the following, we work under the assumption stated below.

Assumption 4.13 Let $K^1,\dots,K^M$ be a collection of $M \in \mathbb{N}$ (m,α)-admissible set of arc parametrizations, in the sense of Defintion 4.12, for some $m \in \mathbb{N}$ and $\alpha \in [0,1]$. For each $i,j \in \{1,\dots,n\}$ with $i \neq j$ it holds

$$\inf_{(\boldsymbol{r},\boldsymbol{p})\in K^i\times K^j}\ \inf_{(t,\tau)\in(-1,1)\times(-1,1)} \|\boldsymbol{r}(t)-\boldsymbol{p}(\tau)\| > 0.$$

Due to the structure of the $G(\cdot,\cdot)$, previously introduced in (3.3), we extensively make use of the logarithmic function, which admits an holomorphic extension in $\mathbb{C}\setminus(-\infty,0]$. Similarly, we also use the holomorphic extension of the squared distance function between two points located in two—not necessarily different—arcs.

For $\boldsymbol{r},\boldsymbol{p} : (-1,1)\to\mathbb{R}^2$, the squared distance is defined as $d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau) = \|\boldsymbol{r}(t)-\boldsymbol{p}(\tau)\|^2$, and its extension to complex parametrizations takes the form (cp. [19, Sect. 4.1])

$$d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau) = (\boldsymbol{r}(t)-\boldsymbol{p}(\tau))\cdot(\boldsymbol{r}(t)-\boldsymbol{p}(\tau)),$$

where we have used the Euclidean inner product in the bilinear sense, as the inclusion of the complex conjugation prevents the existence of any holomorphic extension. Whenever $\boldsymbol{r}=\boldsymbol{p}$ we use the notation $d^2_{\boldsymbol{r}} = d^2_{\boldsymbol{r},\boldsymbol{r}}$.

Another relevant function required to establish holomorphic extensions of our integral operators is the following. For each arc parametrization $\boldsymbol{r} : (-1,1)\to\mathbb{R}^2$ we define $Q_{\boldsymbol{r}} : (-1,1)\times(-1,1)\to\mathbb{R}$ as

$$Q_{\boldsymbol{r}}(t,\tau) := \begin{cases} \dfrac{d^2_{\boldsymbol{r}}(t,\tau)}{(t-\tau)^2}, & t\neq\tau, \\ \boldsymbol{r}'(t)\cdot\boldsymbol{r}'(t), & t = s, \end{cases} \qquad (t,\tau)\in(-1,1)\times(-1,1).$$

The next conditions will be later required to establish bounds on how large are the regions where the BIOs admit holomorphic extensions.

Condition 4.14 *Let $m \in \mathbb{N}$ and $\alpha \in [0, 1]$, and let K^1, K^2 be two (m, α)-admissible sets of arc parametrizations satisfying Assumption 4.13. The values $\delta_1, \delta_2 > 0$ satisfy*

$$\delta_1 + \delta_2 < \sqrt{\mathcal{I}_d^2 + \mathcal{S}_d^2} - \mathcal{S}_d,$$

where

$$\mathcal{I}_d := \inf_{(\boldsymbol{r},\boldsymbol{p})\in K^1\times K^2} \inf_{(t,\tau)\in(-1,1)\times(-1,1)} \|\boldsymbol{r}(t) - \boldsymbol{p}(\tau)\|,$$
$$\mathcal{S}_d := \sup_{\boldsymbol{r}\in K^1_{\delta_1}} \sup_{t\in(-1,1)} \|\boldsymbol{r}(t)\| + \sup_{\boldsymbol{p}\in K^2_{\delta_2}} \sup_{t\in(-1,1)} \|\boldsymbol{p}(t)\|.$$

Condition 4.15 *Let $m \in \mathbb{N}$ and $\alpha \in [0, 1]$, and let K be an (m, α)-admissible set of arc parametrizations. The value $\delta > 0$ satisfies*

$$\delta < \sqrt{\mathcal{I}_Q^2 + \mathcal{S}_Q^2} - \mathcal{S}_Q,$$

where

$$\mathcal{I}_Q := \inf_{\boldsymbol{r}\in K} \inf_{t\in(-1,1)} \|\boldsymbol{r}'(t)\|, \quad \text{and} \quad \mathcal{S}_Q := \sup_{\boldsymbol{r}\in K} \sup_{t\in(-1,1)} \|\boldsymbol{r}'(t)\|.$$

The following result ensures that the square of the distance function admits a bounded holomorphic extension to a set of the form K_δ for some $\delta > 0$.

Lemma 4.16 *Let $m \in \mathbb{N}$ and $\alpha \in [0, 1]$, and let K^1, K^2 be two (m, α)-admissible sets of arc parametrizations.*

(i) For any pair $\tau_1, \tau_2 > 0$, the map

$$K^1 \times K^2 \ni (\boldsymbol{r}, \boldsymbol{p}) \mapsto d^2_{\boldsymbol{r},\boldsymbol{p}} \in \mathcal{C}^{m,\alpha}((-1, 1) \times (-1, 1), \mathbb{C})$$

admits a bounded holomorphic extension into $K^1_{\tau_1} \times K^2_{\tau_2}$.

(ii) For $\delta_1 > 0$ and $\delta_2 > 0$ satisfying Condition 4.14 there exists $\eta > 0$ such that

$$\inf_{(\boldsymbol{r},\boldsymbol{p})\in K^1_{\delta_1}\times K^2_{\delta_2}} \inf_{(t,\tau)\in(-1,1)\times(-1,1)} \mathfrak{Re}\{d^2_{\boldsymbol{r},\boldsymbol{p}}(t, \tau)\} \geq \eta > 0.$$

Proof See Appendix B.1. □

Lemma 4.17 *Let $m \in \mathbb{N}$ and $\alpha \in [0, 1]$, and let K be an (m, α)-admissible set of arc parametrizations.*

(i) For δ as in Condition 4.15, the map

$$K \ni \boldsymbol{r} \mapsto Q_{\boldsymbol{r}} \in \mathcal{C}^{m-1,\alpha}((-1, 1) \times (-1, 1), \mathbb{C})$$

admits a bounded holomorphic extension into K_δ.

(ii) There exists a constant $\zeta = \zeta(K, \delta) > 0$*, depending upon* K *and* δ *only, such that*

$$\inf_{\boldsymbol{r}\in K_\delta}\ \inf_{(t,\tau)\in(-1,1)\times(-1,1)} \mathfrak{Re}\{Q_{\boldsymbol{r}}(t,\tau)\} \geq \zeta,$$
$$\inf_{\boldsymbol{r}\in K_\delta}\ \inf_{(t,\tau)\in(-1,1)\times(-1,1)} \mathfrak{Re}\{Q_{\boldsymbol{r}}^{-1}(t,\tau)\} \geq \zeta,$$

where $Q_{\boldsymbol{r}}^{-1}$ *represents the multiplicative inverse* $1/Q_{\boldsymbol{r}}$*.*

(iii) The map

$$K \ni \boldsymbol{r} \mapsto Q_{\boldsymbol{r}}^{-1} \in \mathcal{C}^{m-1,\alpha}\left((-1,1)\times(-1,1),\mathbb{C}\right)$$

admits a bounded holomorphic extension into K_δ*.*

Proof See Appendix B.2. □

4.6 Holomorphic Extension of Integral Kernels

We now show that the kernels of the weakly- and hyper-singular BIOs—according to the representations in (3.3)—have holomorphic extensions. We do so by extending the integral kernels using our previous results on the functions $d^2_{\boldsymbol{r},\boldsymbol{p}}$ and $Q_{\boldsymbol{r}}$.

By (3.3), for two—not necessarily different—arc parametrizations $\boldsymbol{r}, \boldsymbol{p} : (-1, 1) \to \mathbb{R}^2$, one can write

$$G(\boldsymbol{r}(t), \boldsymbol{p}(\tau)) = \log(d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau))F_1(d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau)) + F_2(d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau)). \tag{4.3}$$

The next result follows straightforwardly from Lemma 4.16 and the structure assumed for $G(\cdot,\cdot)$ in (3.3).

Lemma 4.18 *Let* $m \in \mathbb{N}$ *and* $\alpha \in [0, 1]$*, and let* K^1, K^2 *be two* (m,α)*-admissible sets of arc parametrizations satisfying Assumption 4.13. Then, there exist* $\delta_1, \delta_2 > 0$ *satisfying Condition 4.14 such that the map*

$$K^1 \times K^2 \ni (\boldsymbol{r}, \boldsymbol{p}) \mapsto G(\boldsymbol{r}(t), \boldsymbol{p}(\tau)) \in \mathcal{C}^{m,\alpha}\left((-1,1)\times(-1,1),\mathbb{C}\right) \tag{4.4}$$

admits a bounded holomorphic extension into $K^1_{\delta_1} \times K^2_{\delta_2}$*.*

Proof By Lemma 4.16 (ii), there exist $\delta_1, \delta_2 > 0$ such that the real part of the logarithm argument in (4.3) is bounded from below away from zero. Hence, the function $G(\boldsymbol{r}(t), \boldsymbol{p}(\tau))$ is well defined for all $(t, \tau) \in (-1, 1) \times (-1, 1)$, and for any non-intersecting arc parametrizations $\boldsymbol{r}, \boldsymbol{p} : (-1, 1) \to \mathbb{R}^2$. Furthermore, by Lemma 4.16 (i) along with the fact that the logarithm is analytic in the branch cut $\mathbb{C}\backslash(-\infty, 0]$, one concludes that the map in (4.4) admits a bounded holomorphic extension in $K^1_{\delta_1} \times K^2_{\delta_2}$. □

For the self-interaction case, i.e. $\boldsymbol{r} = \boldsymbol{p}$, the result does not follow from the arguments used in the proof of Lemma 4.18, as a logarithmic singularity inevitably occurs at $d_{\boldsymbol{r}}(t,t) = 0$, thus breaking the analyticity of the logarithmic. In this case, we consider the following decomposition:

$$G(\boldsymbol{r}(t), \boldsymbol{r}(\tau)) = G^R_{\boldsymbol{r}}(t,\tau) + G^S_{\boldsymbol{r}}(t,\tau), \tag{4.5}$$

where

$$\begin{aligned} G^R_{\boldsymbol{r}}(t,\tau) &:= (\log Q_{\boldsymbol{r}}(t,\tau))\, F_1(d^2_{\boldsymbol{r}}(t,\tau)) + F_2(d^2_{\boldsymbol{r}}(t,\tau)), \\ G^S_{\boldsymbol{r}}(t,\tau) &:= 2\log|t-\tau| F_1(d^2_{\boldsymbol{r}}(t,\tau)). \end{aligned} \tag{4.6}$$

Notice that now the logarithmic singularity has been isolated in the term $G^S_{\boldsymbol{r}}$ defined in (4.6). Furthermore, it does not depend on any arc parametrization, and we have following result.

Lemma 4.19 *Let $m \in \mathbb{N}$ and $\alpha \in [0,1]$, and let K be an (m,α)-admissible set of arc parametrizations. Then, there exists $\delta > 0$ satisfying Condition 4.15 such that*

$$\begin{aligned} &K \ni \boldsymbol{r} \mapsto G^R_{\boldsymbol{r}} \in \mathcal{C}^{m-1,\alpha}((-1,1)\times(-1,1),\mathbb{C}) \text{ and} \\ &K \ni \boldsymbol{r} \mapsto F_1(d^2_{\boldsymbol{r}}) \in \mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C}) \end{aligned}$$

admit bounded holomorphic extensions onto K_δ.

Proof The only part that does not follow directly is the logarithmic term of $G^R_{\boldsymbol{r}}$. However, by Lemma 4.17, we are again in the holomorphic domain of the logarithmic and one retrieves the above statements. □

Remark 4.20 We have assumed that the functions F_1 and F_2 in the decomposition of $G(\boldsymbol{x}, \boldsymbol{y})$ stated in (3.3) are entire, and that they depend solely on the square of the distance between points $\boldsymbol{x}$ and $\boldsymbol{y}$. However, less restrictive cases are to be considered. For example, we will consider cases where F_1, F_2 are replaced by the functions G_1, G_2 that take the following form:

$$G_j(t,\tau) = f_j\left(\boldsymbol{r}'(t), \boldsymbol{p}'(\tau)) F_j(d^2_{\boldsymbol{r},\boldsymbol{p}}(t-\tau)\right), \quad j = 1,2,$$

where f_j is entire in both coordinates, and F_j as before for $j = 1,2$. Under this assumption, both Lemmas 4.18 and 4.19 still hold true but the space $\mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})$ has to be replaced by $\mathcal{C}^{m-1,\alpha}((-1,1)\times(-1,1),\mathbb{C})$, as the functions f_1, and f_2 now depend on the derivative of the arc parametrizations. This loss of one order of regularity has no effect, as one needs to consider the holomorphic extension of the full kernel function which, by the first map in Lemma 4.19, needs to lie in $\mathcal{C}^{m-1,\alpha}((-1,1)\times(-1,1),\mathbb{C})$.

5 Shape Holomorphy of Domain-to-Solution Maps

We now study the holomorphic properties of the boundary-to-solution maps:

$$(\boldsymbol{r}_1, \dots, \boldsymbol{r}_M) \mapsto \boldsymbol{\lambda}_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M} \quad \text{and} \quad (\boldsymbol{r}_1, \dots, \boldsymbol{r}_M) \mapsto \boldsymbol{\mu}_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M},$$

where $\boldsymbol{\lambda}_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M}$, $\boldsymbol{\mu}_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M}$ are the solutions of the Dirichlet and Neumann boundary integral formulations introduced in (3.4). The study is carried out in three main steps:

(i) By Theorem 4.5 and results from Sects. 4.4, 4.5, and 4.6, we show that the following maps have holomorphic extensions

$$(\boldsymbol{r}_1, \dots, \boldsymbol{r}_M) \mapsto \mathbf{V}_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M}, \quad (\boldsymbol{r}_1, \dots, \boldsymbol{r}_M) \mapsto \mathbf{W}_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M},$$

on proper compact subsets of $\prod_{j=1}^{M} \mathcal{C}^{m,\alpha}\left((-1, 1), \mathbb{R}^2\right)$, where the boundary integral operatos $\mathbf{V}_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M}$, $\mathbf{W}_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M}$ are those from Sect. 3.1.

(ii) We prove that the previous operators have inverses, and then use Theorem 4.2 to obtain the holomorphic extensions of the boundary to solution map.

(iii) We consider arc parametrizations determined by a countable set of parameters and study the parametric holomorphism of the domain-to-solution map. We do so by combining the above results and the abstract ones in Sect. 4.2.

Steps (i) and (ii) are carried out in Sects. 5.1 (single arc) and 5.2 (multiples arcs). The final step is presented in Sect. 5.3.

5.1 Single Interaction

Firstly, let us study the weakly singular BIO between two arc parametrizations. Let $\boldsymbol{r}, \boldsymbol{p} \in \mathcal{C}_b^{m,\alpha}\left((-1, 1), \mathbb{R}^2\right)$ be the parametrization of two open arcs. For u defined in $[-1, 1]$ we set

$$\left(V_{\boldsymbol{r},\boldsymbol{p}} u\right)(t) = \int_{-1}^{1} G(\boldsymbol{r}(t), \boldsymbol{p}(\tau)) u(\tau) \mathrm{d}\tau, \quad t \in (-1, 1). \tag{5.1}$$

Following the notation of Sect. 3.2, we have $V_{\boldsymbol{r}_i,\boldsymbol{r}_j} = (V_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M})_{i,j}$, for $i, j = 1, \dots, M$.

Due to the fundamental solution structure (3.3) as well as Lemmas 4.18 and 4.19, the operator $V_{\boldsymbol{r},\boldsymbol{p}}$ (5.1) can be expressed in terms of R_f, L_f and S_f introduced in Sect. 4.4. The function $G(\boldsymbol{r}(t), \boldsymbol{p}(\tau))$ and its suitable decomposition will play the role of f in the aforementioned operators on the canonical arc. This analysis in performed thoroughly in Lemma 5.2 ahead. For the hyper-singular BIO, we assume the existence of a suitable Maue's formula so as to reuse the shape holomorphy result for BIOs resembling weakly singular BIOs.

Based on results of Sect. 4.4, we introduce the following condition that will enable us to ensure the continuity of the integral operators on appropriate spaces.

Condition 5.1 *The triple* (m, α, s) *with* $m \in \mathbb{N}$, $\alpha \in [0, 1]$, $s \in \mathbb{R}$ *satisfies either one of the following conditions:*

(i) $s > -\frac{1}{2}$, *and* $s + \frac{5}{2} < m + \alpha$,
(ii) $s \leq -\frac{1}{2}$, *and* $\frac{3}{2} - s < m + \alpha$.

Observe that these conditions are exactly as those required in Lemma 4.8 but with $m-1$ instead of m. This is due to the loss of regularity in the kernel with respect to the parametrization. Equipped with these results, we can state the main result concerning the holomorphic dependence of the operator $V_{\boldsymbol{r},\boldsymbol{p}}$ upon a set of arc parametrizations.

Lemma 5.2 *Assume that Condition 5.1 holds for some* $m \in \mathbb{N}$, $\alpha \in [0, 1]$, *and* $s \in \mathbb{R}$.

(i) Let K *be an* (m, α)*-admissible set of arc parametrizations. Then, there exists* $\delta > 0$, *depending only on* K *and satisfying Condition 4.15, such that*

$$K \ni \boldsymbol{r} \mapsto V_{\boldsymbol{r},\boldsymbol{r}} \in \mathcal{L}(\mathbb{T}^s, \mathbb{W}^{s+1}) \quad \text{and} \quad K \ni \boldsymbol{r} \mapsto V_{\boldsymbol{r},\boldsymbol{r}} \in \mathcal{L}(\mathbb{U}^s, \mathbb{Y}^{s+1})$$

admit bounded holomorphic extensions into K_δ. *Furthermore, for each* $\boldsymbol{r} \in K_\delta$ *it holds that* $V_{\boldsymbol{r},\boldsymbol{r}} \in \mathcal{L}(\mathbb{T}^s, \mathbb{W}^{s+1})$ *is a Fredholm operator of index zero.*

(ii) Let K^1, K^2 *be two* (m, α)*-admissible sets of arc parametrizations satisfying Assumption 4.13. Then, there exist* δ_1, $\delta_2 > 0$, *depending on* K^1 *and* K^2 *and satisfying Condition 4.14, such that the maps*

$$K^1 \times K^2 \ni (\boldsymbol{r}, \boldsymbol{p}) \mapsto V_{\boldsymbol{r},\boldsymbol{p}} \in \mathcal{L}(\mathbb{T}^s, \mathbb{W}^{s+1}) \quad \text{and}$$
$$K^1 \times K^2 \ni (\boldsymbol{r}, \boldsymbol{p}) \mapsto V_{\boldsymbol{r},\boldsymbol{p}} \in \mathcal{L}(\mathbb{U}^s, \mathbb{Y}^{s+1})$$

admit bounded holomorphic extensions into $K^1_{\delta_1} \times K^2_{\delta_2}$. *Moreover, for any* $(\boldsymbol{r}, \boldsymbol{p}) \in K^1_{\delta_1} \times K^2_{\delta_2}$ *the maps* $V_{\boldsymbol{r},\boldsymbol{p}} \in \mathcal{L}(\mathbb{T}^s, \mathbb{W}^{s+1})$, $V_{\boldsymbol{r},\boldsymbol{p}} \in \mathcal{L}(\mathbb{U}^s, \mathbb{Y}^{s+1})$ *define compact operators.*

Proof For the sake of brevity, we assume that $\mathcal{P}$ is scalar and consider only spaces T^s, W^{s+1}, as either vector $\mathcal{P}$ or the case of spaces U^s, Y^{s+1} follow verbatim.

We start by proving item (i), i.e. when $\boldsymbol{p} = \boldsymbol{r}$ for $V_{\boldsymbol{r},\boldsymbol{r}}$. To this end, let us recall the decomposition of the fundamental solution (4.5) and define $V^R_{\boldsymbol{r},\boldsymbol{r}}$ (resp. $V^S_{\boldsymbol{r},\boldsymbol{r}}$) for the integral operator with kernel $G^R_{\boldsymbol{r}}$ (resp. $G^S_{\boldsymbol{r}}$). Hence, we first proceed to show that $V^R_{\boldsymbol{r},\boldsymbol{r}}$ fulfills the assumptions of Theorem 4.5.

(i) The operator $V^R_{\boldsymbol{r},\boldsymbol{r}}$ satisfies Theorem 4.5 with $S \equiv 1$ and $p_k = G^R_{\boldsymbol{r}}$.
(ii) Thus, it follows from Lemma 4.19 that there exists $\delta > 0$ such that

$$G^R_{\boldsymbol{r}} \in \mathcal{C}^{m-1,\alpha}\left((-1, 1) \times (-1, 1), \mathbb{C}\right),$$

for each $\boldsymbol{r} \in K_\delta$. Furthermore, Lemma 4.19 ensures that the map

$$K \ni \boldsymbol{r} \mapsto G^R_{\boldsymbol{r}} \in \mathcal{C}^{m-1,\alpha}\left((-1, 1) \times (-1, 1), \mathbb{C}\right)$$

admits a bounded holomorphic extension into K_δ.

(iii) By assuming that Condition 5.1 holds for a triple (m, α, s), it follows from Corollary 4.9 that for each $\boldsymbol{r} \in K_\delta$ one has $V^R_{\boldsymbol{r},\boldsymbol{r}} \in \mathcal{L}\left(T^s, W^{s+1}\right)$, furthermore, it defines a compact operator, satisfying

$$\left\| V^R_{\boldsymbol{r},\boldsymbol{r}} \right\|_{\mathcal{L}(T^s, W^{s+1})} \lesssim \left\| G^R_{\boldsymbol{r}} \right\|_{\mathcal{C}^{m-1,\alpha}((-1,1)\times(-1,1),\mathbb{C})}, \tag{5.2}$$

where the implied constant is independent of the parametrization $\boldsymbol{r} : (-1, 1) \to \mathbb{R}^2$. By (5.2) and, again, Lemma 4.19 the quantity $\|V^R_{\boldsymbol{r},\boldsymbol{r}}\|_{\mathcal{L}(T^s, W^{s+1})}$ is uniformly bounded over $\boldsymbol{r} \in K_\delta$.

By Theorem 4.5 the map $K \ni \boldsymbol{r} \mapsto V^R_{\boldsymbol{r},\boldsymbol{r}} \in \mathcal{L}(T^s, W^{s+1})$ admits a bounded holomorphic extension onto K_δ.

Now, let us consider $G^S_{\boldsymbol{r}}$ and decompose it as follows: $G^S_{\boldsymbol{r}} = G^{S,1}_{\boldsymbol{r}} + G^{S,2}_{\boldsymbol{r}}$ with:

$$G^{S,1}_{\boldsymbol{r}}(t, \tau) := 2F_1(d^2_{\boldsymbol{r}}(t, t)) \log |t - \tau|,$$
$$G^{S,2}_{\boldsymbol{r}}(t, \tau) := 2(F_1(d^2_{\boldsymbol{r}}(t, \tau)) - F_1(d^2_{\boldsymbol{r}}(t, t))) \log |t - \tau|.$$

The integral operator with kernel $G^{S,1}_{\boldsymbol{r}}$ (resp. $G^{S,2}_{\boldsymbol{r}}$) is denoted by $V^{S,1}_{\boldsymbol{r},\boldsymbol{r}}$ (resp. $V^{S,2}_{\boldsymbol{r},\boldsymbol{r}}$). Observe that for each $\boldsymbol{r}$ it holds that $d^2_{\boldsymbol{r}}(t, t) = 0$ for all $t \in (-1, 1)$, and thus, $F_1(d^2_{\boldsymbol{r}}(t, t)) = F_1(0)$. Consequently, $V^{S,1}_{\boldsymbol{r},\boldsymbol{r}}$ is independent of the parametrization and it follows from Corollary 4.10 that $V^{S,1}_{\boldsymbol{r},\boldsymbol{r}} \in \mathcal{L}(T^s, W^{s+1})$. The map $\boldsymbol{r} \mapsto V^{S,1}_{\boldsymbol{r},\boldsymbol{r}}$ is trivially holomorphic as it is constant with respect to the parametrization $\boldsymbol{r}$. Moreover, in the representation of the fundamental solution we have assumed that $F_1(0) \neq 0$. Hence, $V^{S,1}_{\boldsymbol{r},\boldsymbol{r}}$ is invertible from T^s into W^{s+1} (cf. [22]). Notice that when $\mathcal{P}$ is a vector-valued operator the conclusion still holds as $F_1(0)$ is assumed to be an invertible matrix. However, for the pair U^s, Y^{s+1} this does not hold as the associated integral operator to the logarithmic term is not an invertible operator on the mentioned spaces.

We proceed to show that the operator $V^{S,2}_{\boldsymbol{r},\boldsymbol{r}}$ fulfils the assumptions of Theorem 4.5.

(i) Taylor's theorem yields

$$F_1(d_{\boldsymbol{r}}(t, \tau)^2) - F_1(0) = d_{\boldsymbol{r}}(t, \tau)^2 \int_0^1 F_1'(\eta d^2_{\boldsymbol{r}}(t, \tau))\mathrm{d}\eta,$$

where we have used $d_{\boldsymbol{r}}(t, t) = 0$ for all $t \in (-1, 1)$. By expanding the distance function, for each $(t, \tau) \in (-1, 1) \times (-1, 1)$ we obtain

$$F_1(d_{\boldsymbol{r}}(t, \tau)^2) - F_1(0) = (t - \tau)^2 f_{\boldsymbol{r}}(t, \tau),$$

where

$$f_{\boldsymbol{r}}(t, \tau) := \left(\int_0^1 \boldsymbol{r}'(t + \eta(\tau - t))\mathrm{d}\eta\right) \cdot \left(\int_0^1 \boldsymbol{r}'(t + \eta(\tau - t))\mathrm{d}\eta\right)$$
$$\times \int_0^1 F_1'(\eta d^2_{\boldsymbol{r}}(t, \tau))\mathrm{d}\eta.$$

We obtain the following representation of $G_{\boldsymbol{r}}^{S,2}$:

$$G_{\boldsymbol{r}}^{S,2}(t,\tau) = (t-\tau)^2 \log|t-\tau| f_{\boldsymbol{r}}(t,\tau).$$

Consequently, for each $\boldsymbol{r} \in K$, the operator $V_{\boldsymbol{r},\boldsymbol{r}}^{S,2}$ fits the framework of Theorem 4.5 with $S(t) = t^2 \log|t|$ and $p_k(t,\tau) = f_{\boldsymbol{r}}(t,\tau)$.

(ii) It follows from Lemma 4.19 that the map

$$K \ni \boldsymbol{r} \mapsto f_{\boldsymbol{r}} \in \mathcal{C}^{m-1,\alpha}\left((-1,1)\times(-1,1),\mathbb{C}\right)$$

admits a bounded holomorphic extension into K_δ for some $\delta > 0$.

(iii) Assume that the triple (m,α,s) satisfies Condition 5.1. Then, by Corollary 4.11, for each $\boldsymbol{r} \in K$ we have that $V_{\boldsymbol{r},\boldsymbol{r}}^{S,2} \in \mathcal{L}\left(T^s, W^{s+1}\right)$, and, furthermore, $V_{\boldsymbol{r},\boldsymbol{r}}^{S,2}$ defines a compact operator satisfying the bound

$$\left\| V_{\boldsymbol{r},\boldsymbol{r}}^{S,2} \right\|_{\mathcal{L}(T^s,W^{s+1})} \lesssim \| f_{\boldsymbol{r}} \|_{\mathcal{C}^{m-1,\alpha}((-1,1)\times(-1,1),\mathbb{C})}. \tag{5.3}$$

The right-hand side of (5.3) is uniformly bounded on K_δ as a consequence of Lemma 4.19.

It follows from Theorem 4.5 that the map $K \ni \boldsymbol{r} \mapsto V_{\boldsymbol{r},\boldsymbol{r}}^{S,2} \in \mathcal{L}(T^s, W^{s+1})$ admits a bounded holomorphic extension onto K_δ.

Lastly, we obtain the holomorphic extension of $V_{\boldsymbol{r},\boldsymbol{r}}$ by acknowledging that

$$V_{\boldsymbol{r},\boldsymbol{r}} = V_{\boldsymbol{r},\boldsymbol{r}}^{R} + V_{\boldsymbol{r},\boldsymbol{r}}^{S,1} + V_{\boldsymbol{r},\boldsymbol{r}}^{S,2},$$

since the three operators on the right-hand side have holomorphic extension at least in K_ξ, for $0 < \xi < \delta$. On the other hand $V_{\boldsymbol{r},\boldsymbol{r}}^{R}$, $V_{\boldsymbol{r},\boldsymbol{r}}^{S,2}$ are compact operators, and since $V_{\boldsymbol{r},\boldsymbol{r}}^{S,1}$ is invertible, $V_{\boldsymbol{r},\boldsymbol{r}}$ is Fredholm of index zero.

The proof of the second part of the lemma is proved as with the part involving $V_{\boldsymbol{r},\boldsymbol{r}}^{R}$, but by using Lemma 4.18 instead of Lemma 4.19. We skip it for the sake of brevity. □

We also consider the generic hyper-singular operator interaction and establish the sought shape holomorphy property for this type of BIOs. To do so, we employ the previously assumed Maue's representation formula (3.5), which takes the form

$$\left(W_{\boldsymbol{r},\boldsymbol{p}}u\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} G(\boldsymbol{r}(t),\boldsymbol{p}(\tau))\frac{\mathrm{d}}{\mathrm{d}\tau}u(\tau)\mathrm{d}\tau + \int_{-1}^{1} \widetilde{G}(\boldsymbol{r}(t),\boldsymbol{p}(\tau))u(\tau)\mathrm{d}\tau,$$

where again we have $W_{\boldsymbol{r}_i,\boldsymbol{r}_j} = (W_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})_{i,j}$, and also the next result.

Lemma 5.3 *Assume that Condition 5.1 holds for some $m \in \mathbb{N}$, $\alpha \in [0,1]$, and $s \in \mathbb{R}$.*

(i) Let K be an (m,α)-admissible set of arc parametrizations. Then, there exists $\delta > 0$, depending only on K and satisfying Condition 4.15, such that

$$K \ni \boldsymbol{r} \mapsto W_{\boldsymbol{r},\boldsymbol{r}} \in \mathcal{L}(\mathbb{U}^s, \mathbb{Y}^{s-1}),$$

admits a bounded holomorphic extensions into K_δ. Furthermore, for any $\boldsymbol{r} \in K_\delta$ one has that $W_{\boldsymbol{r},\boldsymbol{r}} \in \mathcal{L}(\mathbb{U}^s, \mathbb{Y}^{s-1})$ is a Fredholm operator of index zero.

(ii) Let K^1, K^2 be two (m,α)-admissible sets of arc parametrizations satisfying Assumption 4.13. Then, there exist $\delta_1, \delta_2 > 0$, satisfying Condition 4.14, such that the map

$$K^1 \times K^2 \ni (\boldsymbol{r}, \boldsymbol{p}) \mapsto W_{\boldsymbol{r},\boldsymbol{p}} \in \mathcal{L}(\mathbb{U}^s, \mathbb{Y}^{s-1}),$$

admits a bounded holomorphic extensions into $K_{\delta_1} \times K_{\delta_2}$. Moreover, for any $(\boldsymbol{r}, \boldsymbol{p}) \in K^1_{\delta_1} \times K^2_{\delta_2}$, the map $W_{\boldsymbol{r},\boldsymbol{p}} \in \mathcal{L}(\mathbb{U}^s, \mathbb{Y}^{s-1})$ is compact.

Proof Again, we restrict ourselves to the scalar case. Except for the Fredholm order, the proof follows directly from Maue's representation formula, the mapping properties of the derivative operators (2.5)—also independent of the parametrizations—and the arguments of Lemma 5.2.

To show the Fredholm order, we use the decomposition of the hyper-singular operator $W_{\boldsymbol{r},\boldsymbol{p}} = W^1_{\boldsymbol{r},\boldsymbol{p}} + W^2_{\boldsymbol{r},\boldsymbol{p}}$, with

$$\begin{aligned}\left(W^1_{\boldsymbol{r},\boldsymbol{p}} u\right)(t) &:= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} G(\boldsymbol{r}(t), \boldsymbol{p}(\tau))\frac{\mathrm{d}}{\mathrm{d}\tau}u(\tau)\mathrm{d}\tau,\\ \left(W^2_{\boldsymbol{r},\boldsymbol{p}} u\right)(t) &:= \int_{-1}^{1} \widetilde{G}(\boldsymbol{r}(t), \boldsymbol{p}(\tau))u(\tau)\mathrm{d}\tau.\end{aligned}$$

For $W^1_{\boldsymbol{r},\boldsymbol{p}}$, we argue as in Lemma 5.2. We decompose this operator into three parts: two of them are compact by the previous lemma, and the remaining part is

$$2F_1(0)\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} \log|t-s|\frac{\mathrm{d}}{\mathrm{d}s}u(\tau)\mathrm{d}s. \tag{5.4}$$

The factor $2F_1(0)$ is assumed to be invertible, and the integral operator is the standard hyper-singular one for the Laplace equation. Hence, the operator in (5.4) is invertible as a map in $\mathcal{L}(\mathbb{U}^s, \mathbb{Y}^{s-1})$ (cf. [22]).

The operator $W^2_{\boldsymbol{r},\boldsymbol{p}}$ can be analyzed as in Lemma 5.2, so as to find that $W^2_{\boldsymbol{r},\boldsymbol{p}} \in \mathcal{L}\left(\mathbb{U}^s, \mathbb{Y}^{s+1}\right)$. Therefore, by the compact embedding of the corresponding spaces (see Sect. 2.1) we have that $W^2_{\boldsymbol{r},\boldsymbol{p}} \in \mathcal{L}\left(\mathbb{U}^s, \mathbb{Y}^{s-1}\right)$ is a compact operator. □

Remark 5.4 In practice, the term $\widetilde{G}$ of Maue's representation formula includes a factor involving normal vectors. This implies that the corresponding functions F_1, F_2 have the structure described in Remark 4.20. Consequently, none of the results needs to be modified.

We can further generalize the structure of the functions G_1, G_2 in Remark 4.20 by considering the form:

$$G_j(t,\tau) = f_j\left((\boldsymbol{r}'(t), \boldsymbol{r}''(t), \ldots, \boldsymbol{r}^{(n)}(t), \boldsymbol{p}'(\tau), \boldsymbol{p}''(\tau), \ldots, \boldsymbol{p}^{(n)}(\tau)\right) F_j(d^2_{\boldsymbol{r},\boldsymbol{p}}(t-\tau)),$$

where f_j is entire in each coordinate. With the above representation, Condition 5.1 is changed to:

(i) $s > -\frac{1}{2}$, and $s + \frac{3}{2} + n < m + \alpha$,
(ii) $s \leq -\frac{1}{2}$, and $\frac{1}{2} + n - s < m + \alpha$.

5.2 Multiple Arcs ($M > 1$)

Lemmas 5.2 and 5.3 ensure that for every pair of arcs $\boldsymbol{r}, \boldsymbol{p}$ there exists a region—depending on the arcs—such that weakly- and hyper-singular BIOs have holomorphic extensions. Now we return to the original problem (Sects. 3.1 and 3.2) to prove the BIOs' holomorphic extension for the interaction of $M > 1$ arcs (cf. Theorem 5.6). With this, we obtain the holomorphic extension of the so-called domain-to-solution map for the problem presented in Sect. 3.1.

Condition 5.5 *Consider $M \in \mathbb{N}$ different (m, α)-admissible sets of parametrizations $K^1, \ldots, K^M$ satisfying Assumption 4.13. Let $\delta_1, \ldots, \delta_M$ be M strictly positive real numbers, such that*

(i) Each δ_j satisfies Condition 4.15 in the compact set K^j, for $j = 1, \ldots, M$.
(ii) For each (δ_i, δ_j) with $i, j \in \{1, \ldots, M\}$ and $i \neq j$, Condition 4.14 is fulfilled in $K^i \times K^j$.

Theorem 5.6 *Let $s \in \mathbb{R}$, $m \in \mathbb{N}$ and $\alpha \in [0, 1]$ be such that Condition 5.1 is fulfilled. Let $K^1, \ldots, K^M$ be M (m, α)-admissible sets of parametrizations satisfying Assumption 4.13. Then there exist $\delta_1, \ldots, \delta_M > 0$ satisfying Condition 5.5 such that*

$$K^1 \times \cdots \times K^M \ni (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \mapsto \mathbf{V}_{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M} \in \mathcal{L}\left(\prod_{j=1}^{M} \mathbb{T}^s, \prod_{j=1}^{M} \mathbb{W}^{s+1} \right),$$

$$K^1 \times \cdots \times K^M \ni (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \mapsto \mathbf{W}_{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M} \in \mathcal{L}\left(\prod_{j=1}^{M} \mathbb{U}^s, \prod_{j=1}^{M} \mathbb{Y}^{s-1} \right),$$

admit bounded holomorphic extensions into $K^1_{\delta_1} \times \cdots \times K^M_{\delta_M}$.

Proof We prove only the result for the weakly singular BIO as the hyper-singular case follows similarly. Our first observation is that one can write

$$\mathbf{V}_{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M} = \begin{pmatrix} V_{\boldsymbol{r}_1, \boldsymbol{r}_1} & V_{\boldsymbol{r}_1, \boldsymbol{r}_2} & \cdots & V_{\boldsymbol{r}_1, \boldsymbol{r}_M} \\ V_{\boldsymbol{r}_2, \boldsymbol{r}_1} & V_{\boldsymbol{r}_2, \boldsymbol{r}_2} & \cdots & V_{\boldsymbol{r}_2, \boldsymbol{r}_M} \\ \vdots & \ddots & \cdots & \vdots \\ V_{\boldsymbol{r}_M, \boldsymbol{r}_1} & V_{\boldsymbol{r}_M, \boldsymbol{r}_2} & \cdots & V_{\boldsymbol{r}_M, \boldsymbol{r}_M} \end{pmatrix}.$$

By Lemma 5.2, there exist $\delta_1, \ldots, \delta_M > 0$ satisfying Condition 5.5 such that the maps

$$K^j \ni \boldsymbol{r}_j \mapsto V_{\boldsymbol{r}_j, \boldsymbol{r}_j} \in \mathcal{L}\left(\mathbb{T}^s, \mathbb{W}^{s+1}\right),$$
$$K^i \times K^j \ni (\boldsymbol{r}_i, \boldsymbol{r}_j) \mapsto V_{\boldsymbol{r}_i, \boldsymbol{r}_j} \in \mathcal{L}\left(\mathbb{T}^s, \mathbb{W}^{s+1}\right), \quad i \neq j,$$

admit bounded holomorphic extension into $K^j_{\delta_j}$ and $K^i_{\delta_i} \times K^j_{\delta_j}$, respectively. Since each component has a holomorphic extension, by defining the norms for $\prod_{j=1}^M \mathbb{T}^s$ and $\prod_{j=1}^M \mathbb{W}^{s+1}$ as the standard Euclidean norm of a Cartesian product space, we directly deduce that

$$K^1 \times \cdots K^M \ni (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \mapsto \mathbf{V}_{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M} \in \mathcal{L}\left(\prod_{j=1}^M \mathbb{T}^s, \prod_{j=1}^M \mathbb{W}^{s+1}\right)$$

admits a bounded holomorphic extension into $K^1_{\delta_1} \cdots \times K^M_{\delta_M}$. □

From this last result, by Theorem 4.2, and assuming that the right-hand sides of the BVPs are given by entire functions (Sect. 3.1), we conclude that $\boldsymbol{\lambda}_{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M}$ and $\boldsymbol{\mu}_{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M}$, solutions to the Dirichlet and Neumann problems, respectively, depend holomorphically upon perturbations of arc parametrizations $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M$.

Theorem 5.7 *Under the same hypothesis of Theorem 5.6, there exists $\eta > 0$ such that the maps*

$$K_1 \times \cdots \times K_M \ni (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \mapsto \boldsymbol{\lambda}_{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M} \in \prod_{j=1}^M \mathbb{T}^s \tag{5.5}$$

and

$$K_1 \times \cdots \times K_M \ni (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \mapsto \boldsymbol{\mu}_{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M} \in \prod_{j=1}^M \mathbb{U}^s \tag{5.6}$$

admit bounded holomorphic extensions into $K^1_\eta \times \cdots K^M_\eta$, where for each $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \in K^1 \times \cdots \times K^M$ we have that $\boldsymbol{\lambda}_{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M}$ and $\boldsymbol{\mu}_{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M}$ are the boundary solutions of the Dirichlet and Neumann problems stated in (3.4).

Proof As before, we prove only the result for the weakly singular BIO and provide remarks whenever the proof differs for the hyper-singular case.

The proof relies on Theorem 4.2. Firstly, we need to ensure that for each $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \in K^1 \times \cdots K^M$ the maps introduced in (5.5) and (5.6) are well defined. To this end, we use the block-wise decomposition of the weakly singular BIO defined on multiple disjoint arcs into diagonal and off-diagonal components, i.e.

$$\mathbf{V}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M} = \begin{pmatrix} V_{\boldsymbol{r}_1,\boldsymbol{r}_1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & V_{\boldsymbol{r}_M,\boldsymbol{r}_M} \end{pmatrix} + \begin{pmatrix} 0 & \ldots & V_{\boldsymbol{r}_1,\boldsymbol{r}_M} \\ \vdots & \ddots & \vdots \\ V_{\boldsymbol{r}_M,\boldsymbol{r}_1} & \ldots & 0 \end{pmatrix} \tag{5.7}$$

An equivalent decomposition can be stated for the hyper-singular BIO. It follows from Lemma 5.2—Lemma 5.3 for the hyper-singular BIO—that the diagonal part, i.e. the first summand in (5.7), is composed of index zero Fredholm operators, while the off-diagonal one includes compact operators. Recalling the definition of a Cartesian product space, the block operators are Fredholm of index zero. From the Fredholm property, we obtain that $\mathbf{V}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M} \in \mathcal{L}\left(\prod_{j=1}^M \mathbb{T}^s, \prod_{j=1}^M \mathbb{W}^{s+1}\right)$—also $\mathbf{W}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M} \in \mathcal{L}\left(\prod_{j=1}^M U^s, \prod_{j=1}^M Y^{s-1}\right)$—is invertible provided that is injective. This is equivalent to the unisolvence of the corresponding volume problem presented in Sect. 3.1 (cf. [24]).

Hence, for each $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \in K^1 \times \cdots \times K^M$, the operators $\mathbf{V}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M}$, $\mathbf{W}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M}$ are invertible mapping as follows

$$(\mathbf{V}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})^{-1} \in \mathcal{L}\left(\prod_{j=1}^M \mathbb{W}^{s+1}, \prod_{j=1}^M \mathbb{T}^s\right),$$
$$(\mathbf{W}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})^{-1} \in \mathcal{L}\left(\prod_{j=1}^M \mathbb{Y}^{s-1}, \prod_{j=1}^M \mathbb{U}^s\right).$$

Thus, we have proved that the maps introduced in (5.5) and (5.6) are well defined. It follows straightforwardly from Theorems 4.2 and 5.6 the there exists $\eta > 0$ such the maps

$$K^1 \times \cdots \times K^M \ni (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \mapsto (\mathbf{V}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})^{-1} \in \mathcal{L}\left(\prod_{j=1}^M \mathbb{W}^{s+1}, \prod_{j=1}^M \mathbb{T}^s\right)$$
$$K^1 \times \cdots \times K^M \ni (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M) \mapsto \left(\mathbf{W}_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M}\right)^{-1} \in \mathcal{L}\left(\prod_{j=1}^M \mathbb{Y}^{s-1}, \prod_{j=1}^M \mathbb{U}^s\right)$$

admit bounded holomorphic extensions into K_η. Recall that the right-hand sides of the boundary integral formulations considered are of the form:

$$(f^D_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})_j = f^D \circ \boldsymbol{r}_j \quad \text{and} \quad (f^N_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M})_j = f^N \circ \boldsymbol{r}_j,$$

with f^D, f^N assumed to be entire functions. From this representation, arguing as in Lemma 2.1, but for univariate functions, one can check that Condition 5.1 ensures that

$$f^D_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M} \in \prod_{j=1}^{M} \mathbb{W}^{s+1}, \quad f^N_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M} \in \prod_{j=1}^{M} \mathbb{Y}^{s+1} \subset \prod_{j=1}^{M} \mathbb{Y}^{s-1}.$$

Furthermore, once again using that f^D, f^N are entire functions, it is direct to see that the maps $(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M) \mapsto \mathbf{f}^D_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M}$, and $(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M) \mapsto \mathbf{f}^N_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_M}$ admit bounded holomorphic extensions to any region. The final result follows by composition of maps with holomorphic extensions. □

Remark 5.8 Theorem 5.7 enables us to obtain holomorphic extensions for some linear functionals. In particular, if we consider linear functionals of the form

$$(L\boldsymbol{u})\,(\boldsymbol{x}) = \int_{-1}^{1} \boldsymbol{\vartheta}(\boldsymbol{x}, \boldsymbol{r}_1(t), \ldots, \boldsymbol{r}_M(t)) \cdot \boldsymbol{u}(t)dt, \quad \boldsymbol{x} \in \mathbb{R}^2,$$

where $\boldsymbol{u} = \boldsymbol{\lambda}$ or $\boldsymbol{u} = \boldsymbol{\mu}$, solutions of the corresponding BIEs (Problem 3.4), and $\boldsymbol{\vartheta}$ is entire on each coordinate, except possibly on the first one. The holomorphic extension of this functional is proven by showing that it is a composition of holomorphic functions. First, we notice that by Theorem 4.5, $(\boldsymbol{r}_1,\ldots \boldsymbol{r}_M) \mapsto L\cdot$, has a bounded holomorphic extension, and secondly, by the previous theorem, $\boldsymbol{u}$ also have a bounded holomorphic extension. Hence, Lu has a bounded holomorphic extension.

5.3 Parametric Holomorphy of the Domain-to-Solution Map

Throughout this section we denote by $\boldsymbol{r}^0_1, \boldsymbol{r}^0_2, \ldots, \boldsymbol{r}^0_M$ a collection of M arc parametrizations, each of them contained in $\mathcal{C}^{m,\alpha}_b\left((-1,1),\mathbb{R}^2\right)$, and such that no crossing among them occurs. Specifically, we consider the next affine-parametric arc parametrizations:

$$\boldsymbol{r}_{j,\boldsymbol{y}_j} = \boldsymbol{r}^0_j + \sum_{n=1}^{\infty} y^n_j \boldsymbol{r}^n_j, \quad j = 1,\ldots,M, \quad \boldsymbol{y}_j := (y^n_j)_{n\in\mathbb{N}} \in \mathrm{U}, \tag{5.8}$$

where, for each $j \in \{1,\ldots,M\}$, the sequence $\{\boldsymbol{r}^n_j\}_{n\in\mathbb{N}} \subset \mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$. For each $n \in \mathbb{N}$ and $j = 1,\ldots,M$, let us set

$$b^n_j := \left\| \boldsymbol{r}^n_j \right\|_{\mathcal{C}^{m,\alpha}((-1,1),\mathbb{R}^2)} \quad \text{and} \quad \boldsymbol{b}_j = \{b^n_j\}_{n\in\mathbb{N}}. \tag{5.9}$$

In order to restrict ourselves to admissible geometric configurations—arc parametrizations satisfying $\boldsymbol{r}_{j,\boldsymbol{y}_j} \in \mathcal{C}^{m,\alpha}_b\left((-1,1),\mathbb{R}^2\right)$ for $j = 1,\ldots,M$ and each $\boldsymbol{y} \in \mathrm{U}$, such that no crossings or intersections occur—and adopt the framework of Sect. 4.2, we work under the following assumptions.

Assumption 5.9 We assume that

(i) For each $j \in \mathbb{N}$ the sequence $\boldsymbol{b}_j \in \ell^p(\mathbb{N})$ for some $p \in (0,1)$.

(ii) There exists a single $\zeta \in (0,1)$ such that, for each $j \in \{1,\dots,M\}$, it holds that

$$\sup_{t\in(-1,1)} \sum_{n=1}^{\infty} \|(\boldsymbol{r}_j^n)'(t)\| \le \zeta \inf_{t\in(-1,1)} \|(r_j^0)'(t)\|.$$

(iii) There exists $\eta \in (0,1)$ such that, for any $i, j \in \{1,\dots,M\}$ and for each $\boldsymbol{y}_i, \boldsymbol{y}_j \in \mathrm{U}$, one has

$$\left\| \sum_{n=1}^{\infty} y_i^n \boldsymbol{r}_i^n(t) - y_j^n \boldsymbol{r}_j^n(\tau) \right\| \le \eta \inf_{(t,\tau)\in(-1,1)\times(-1,1)} \left\| \boldsymbol{r}_i^0(t) - \boldsymbol{r}_j^0(\tau) \right\|.$$

In the following, for $j \in \{1,\dots,M\}$ we set

$$K_j := \left\{ \boldsymbol{r}_{j,\boldsymbol{y}} \in \mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{R}^2\right) : \boldsymbol{r}_{j,\boldsymbol{y}} = \boldsymbol{r}_j^0 + \sum_{n=1}^{\infty} y^n \boldsymbol{r}_j^n,\ \ \boldsymbol{y} = \{y^n\}_{n\in\mathbb{N}} \in \mathrm{U} \right\}. \tag{5.10}$$

Observe that due to item (i) in Assumption 5.9 the series in (5.8) converges absolutely and uniformly with respect to $\boldsymbol{y} \in \mathrm{U}$. Moreover, one can now obtain a proper set of arc parametrizations as shown below.

Lemma 5.10 *Let $K_1,\dots,K_M \subset \mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$ be as in (5.10) for some $m \in \mathbb{N}$ and $\alpha \in [0,1]$, and let Assumption 5.9 be satisfied. Then $K_1,\dots,K_M$ are (m,α)-admissible arc parametrizations in the sense of Definition 4.13 satisfying Assumption 4.13.*

Proof We start by proving that for each $\boldsymbol{y} \in \mathrm{U}$ the arc parametrization $\boldsymbol{r}_{j,\boldsymbol{y}} : (-1,1) \to \mathbb{R}^2$ defined as in (5.8) renders an element of $\mathcal{C}_b^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$. The proof follows substantially that of [18, Lemma 6.2], however we include the details for the sake of completeness.

For $t, \tau \in (-1,1)$, we directly have that

$$\left\|\boldsymbol{r}_{j,\boldsymbol{y}}(t) - \boldsymbol{r}_{j,\boldsymbol{y}}(\tau)\right\| = \left\| \boldsymbol{r}_j^0(t) - \boldsymbol{r}_j^0(\tau) + \sum_{n=1}^{\infty} y^n (\boldsymbol{r}_j^n(t) - \boldsymbol{r}_j^n(\tau)) \right\|.$$

Hence, by the Taylor expansion of $\boldsymbol{r}_{j,\boldsymbol{y}}$, for every $j = 0,\dots M$, there exist $\xi \in (-1,1)$ such that

$$\begin{aligned}\left\|\boldsymbol{r}_{j,\boldsymbol{y}}(t) - \boldsymbol{r}_{j,\boldsymbol{y}}(\tau)\right\| &= (t-\tau) \left\| (\boldsymbol{r}_j^0)'(\xi) + \sum_{n=1}^{\infty} y^n (\boldsymbol{r}_j^n)'(\xi) \right\| \\ &\ge (t-\tau) \left(\left\|(\boldsymbol{r}_j^0)'(\xi)\right\| - \left\| \sum_{n=1}^{\infty} y^n (\boldsymbol{r}_j^n)'(\xi) \right\| \right).\end{aligned}$$

By Assumption 5.9 (ii), we obtain

$$\left\|(\boldsymbol{r}_j^0)'(\xi)\right\| - \left\|\sum_{n=1}^{\infty} y^n (\boldsymbol{r}_j^n)'(\xi)\right\| \geq (1-\zeta) \inf_{t\in(-1,1)} \|(r_j^0)'(t)\| > 0.$$

We deduce that $\boldsymbol{r}_{j,\boldsymbol{y}}$ is injective for every $\boldsymbol{y} \in \mathrm{U}$, thus it has a global inverse. Furthermore, if we make the same analysis for the tangent vector we have that

$$\left\|(\boldsymbol{r}_{j,\boldsymbol{y}})'(t)\right\| \geq \left(\left\|(\boldsymbol{r}_j^0)'(t)\right\| - \left\|\sum_{n=1}^{\infty} y^n (\boldsymbol{r}_j^n)'(t)\right\|\right) \geq (1-\zeta) \inf_{t\in(-1,1)} \|(r_j^0)'(t)\| > 0.$$

Thus, the tangent vector is nowhere null, and we have that $K_j \subset \mathcal{C}_b^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$. As explained in Sect. 4.2, it follows from [9, Lemma 2.7] that each K_i is a compact subset of $\mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$, thus rendering each of those sets (m,α)-admissible arc parametrizations.

To conclude, we verify that under Assumption 5.9 the sets $K_1,\ldots,K_M$ fulfill Assumption 4.13. Using item (iii) in Assumption 5.9 For any $i, j \in \{1,\ldots,M\}$ and for each $\boldsymbol{y}_i, \boldsymbol{y}_j \in \mathrm{U}$, one has that

$$\begin{aligned}
\left\|\boldsymbol{r}_{i,\boldsymbol{y}_i}(t) - \boldsymbol{r}_{j,\boldsymbol{y}_j}(\tau)\right\| &= \left\|\boldsymbol{r}_i^0(t) + \sum_{n=1}^{\infty} y_n \boldsymbol{r}_i^n(t) - \boldsymbol{r}_j^0(\tau) - \sum_{n=1}^{\infty} y_n \boldsymbol{r}_j^n(\tau)\right\| \\
&\geq \left|\left\|\boldsymbol{r}_i^0(t) - \boldsymbol{r}_j^0(\tau)\right\| - \left\|\sum_{n=1}^{\infty} y_i^n \boldsymbol{r}_i^n(t) - y_j^n \boldsymbol{r}_j^n(\tau)\right\|\right| \\
&\geq (1-\eta) > 0,
\end{aligned}$$

as stated. □

Let us set for each $\boldsymbol{y} \in \mathrm{U}$

$$\boldsymbol{\lambda}_{\boldsymbol{y}} := \boldsymbol{\lambda}_{\boldsymbol{r}_{1,\boldsymbol{y}_1},\ldots,\boldsymbol{r}_{M,\boldsymbol{y}_M}} \quad \text{and} \quad \boldsymbol{\mu}_{\boldsymbol{y}} := \boldsymbol{\mu}_{\boldsymbol{r}_{1,\boldsymbol{y}_1},\ldots,\boldsymbol{r}_{M,\boldsymbol{y}_M}},$$

where the elements $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots \boldsymbol{y}_M \in \mathrm{U}$ are defined as

$$(y_j)_n = y_{j+nM}, \quad j \in \{1,\ldots,M\},\ n \in \mathbb{N}.$$

We will make use of the set $\boldsymbol{K} := K_1 \times \cdots \times K_M \subset \prod_{j=1}^M \mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$, which can be written as

$$\boldsymbol{K} = \left\{\mathbf{k}_{\boldsymbol{y}} = \begin{pmatrix} \boldsymbol{r}_{1,\boldsymbol{y}_1} \\ \boldsymbol{r}_{2,\boldsymbol{y}_2} \\ \vdots \\ \boldsymbol{r}_{M,\boldsymbol{y}_M} \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_1^0 \\ \boldsymbol{r}_2^0 \\ \vdots \\ \boldsymbol{r}_M^0 \end{pmatrix} + \sum_{n=1}^{\infty} y_n \mathbf{k}_n,\ \ \boldsymbol{y} = \{y_n\}_{n\in\mathbb{N}} \in \mathrm{U}\right\},$$

wherein,

$$\{\mathbf{k}_n\}_{n\in\mathbb{N}} = \left\{ \begin{pmatrix} \boldsymbol{r}_1^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \boldsymbol{r}_2^1 \\ \vdots \\ 0 \end{pmatrix}, \dots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \boldsymbol{r}_M^1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{r}_1^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \boldsymbol{r}_2^2 \\ \vdots \\ 0 \end{pmatrix}, \dots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \boldsymbol{r}_M^2 \end{pmatrix}, \dots \right\}$$

or more rigorously $\mathbf{k}_n = \boldsymbol{r}_{j(n)}^{\lceil \frac{n}{M} \rceil} \mathbf{e}_{j(n)}$, with $\lceil \frac{n}{M} \rceil$ the upper integer part of $\frac{n}{M}$, and $j(n)$ is one plus the residual of the integer division $\frac{n-1}{M}$. We also define the parameter norm set:

$$\boldsymbol{b} := \left\{ \|\mathbf{k}_n\|_{\prod_{j=1}^M \mathcal{C}^{m,\alpha}((-1,1),\mathbb{R}^2)} \right\}_{n\in\mathbb{N}}. \tag{5.11}$$

Theorem 5.11 *Let Condition 5.1 hold for some* $m \in \mathbb{N}$, $\alpha \in [0, 1]$, *and* $s \in \mathbb{R}$. *Let Assumption 5.9 be satisfied with* $\boldsymbol{b}_j$ *for* $j = 1, \dots, M$ *as in (5.9) and for some* $p \in (0, 1)$. *Then the maps*

$$\mathrm{U} \ni \boldsymbol{y} \mapsto \boldsymbol{\lambda}_{\boldsymbol{y}} \in \prod_{j=1}^M \mathbb{T}^s, \quad \textit{and} \quad \mathrm{U} \ni \boldsymbol{y} \mapsto \boldsymbol{\mu}_{\boldsymbol{y}} \in \prod_{j=1}^M \mathbb{U}^s$$

are $(\boldsymbol{b}, p, \varepsilon)$*-holomorphic for some* $\varepsilon > 0$, $p \in (0, 1)$ *as in Assumption 5.9, and* $\boldsymbol{b}$ *as in (5.11). Also, these maps are continuous when* U *is equipped with the product topology.*

Proof Being a direct consequence of Theorem 4.4, we only need to verify that the hypotheses are satisfied. The role of the compact set K of Theorem 4.4 is played by $\boldsymbol{K}$, which has the desired form and is compact due to the previous Lemma. Our definition of $\boldsymbol{b}$ (5.11) coincides with that of Theorem 4.4, and we also have that $\boldsymbol{b} \in \ell^p(\mathbb{N})$ by Assumption 5.9 (i). Finally, we observe that by Theorem 5.7, the maps

$$\boldsymbol{K} \ni (\boldsymbol{r}_1, \dots, \boldsymbol{r}_M) \mapsto \boldsymbol{\lambda}_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M} \in \prod_{j=1}^M \mathbb{T}^s,$$

$$\boldsymbol{K} \ni (\boldsymbol{r}_1, \dots, \boldsymbol{r}_M) \mapsto \boldsymbol{\mu}_{\boldsymbol{r}_1,\dots,\boldsymbol{r}_M} \in \prod_{j=1}^M \mathbb{U}^s$$

have bounded holomorphic extensions. Moreover, by definition $\boldsymbol{\lambda}_{\boldsymbol{y}}, \boldsymbol{\mu}_{\boldsymbol{y}}$, we have that $\boldsymbol{\lambda}_{\boldsymbol{y}} = \boldsymbol{\lambda}_{\mathbf{k}_{\boldsymbol{y}}}$, $\boldsymbol{\mu}_{\boldsymbol{y}} = \boldsymbol{\mu}_{\mathbf{k}_{\boldsymbol{y}}}$, where $\boldsymbol{\lambda}_{\mathbf{k}_{\boldsymbol{y}}}, \boldsymbol{\mu}_{\mathbf{k}_{\boldsymbol{y}}}$ are the corresponding elements $f(k_{\boldsymbol{y}})$ in Theorem 4.4. The results then follow from Theorem 4.4 as stated. □

Remark 5.12 The result stated in Theorem 5.11 enables us to conclude the parametric holomorphy of linear functionals acting on $\boldsymbol{\lambda}_{\boldsymbol{y}}$ and $\boldsymbol{\mu}_{\boldsymbol{y}}$ (cf. Remark 5.8).

6 Applications: Time-Harmonic Acoustic and Elastic Wave Scattering

In the following, we consider two particular instances of the operator $\mathcal{P}$, namely, Helmholtz and elastic wave operators, and check whether the assumptions to guarantee holomorphic extensions of their corresponding BIOs are satisfied.

6.1 Helmholtz Equation

The scalar Helmholtz operator with wavenumber $\kappa \in \mathbb{R}_+$ is given by $\mathcal{P} = -\Delta - \kappa^2$. In order to ensure well-posedness, one prescribes the following behavior at infinity

$$\lim_{\|\boldsymbol{x}\|\to\infty} \|\boldsymbol{x}\|^{\frac{1}{2}} \left(\frac{\partial u}{\partial \|\boldsymbol{x}\|} - \imath\kappa u \right) = 0,$$

known as the Sommerfeld radiation condition. We refer to [25, 43] for uniqueness of the Dirichlet problem and to [31] for the Neumann one. For the latter, the co-normal trace becomes the standard Neumann trace, i.e. $\mathcal{B}_{\boldsymbol{n}} u = \boldsymbol{n} \cdot \nabla u = \partial_{\boldsymbol{n}} u$ for u smooth enough, with continuous extensions to Sobolev spaces [30, Lemma 4.3]. The fundamental solution of the operator $\mathcal{P} = -\Delta - k^2$ in two-dimensional space is

$$G_\kappa(\boldsymbol{x}, \boldsymbol{y}) = \frac{i}{4} H_0^{(1)}(\kappa \|\boldsymbol{x} - \boldsymbol{y}\|), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2, \tag{6.1}$$

where $H_0^{(1)}$ denotes the Hankel function of first kind and order zero, defined as $H_0^{(1)}(z) = J_0(z) + i Y_0(z)$ for $z \neq 0$, where J_0, Y_0 are the zeroth order Bessel functions of first and second kind, respectively.

Corollary 6.1 (Helmholtz case) *Consider the Dirichlet and Neumann BIEs (Problem 3.4) for the Helmholtz kernel (6.1) with* $\kappa > 0$. *Then, the arising BIOs and domain-to-solution maps are shape holomorphic.*

Proof From [1, 9.1.12 and 9.1.13], one has that

$$G_\kappa(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{4\pi} \log \|\boldsymbol{x} - \boldsymbol{y}\|^2 J_0(\kappa\|\boldsymbol{x} - \boldsymbol{y}\|) + R(\kappa\|\boldsymbol{x} - \boldsymbol{y}\|^2),$$

where the first kind Bessel function J_0, and R are entire functions. Furthermore, from [1, 9.1.12], one has that $J_0(0) = 1$, and also $J_0(z) = j_0(z^2)$, with j_0 being an entire function. Hence, the representation of the form of (3.3) holds with

$$F_1(z) = -\frac{1}{2\pi} j_0(\kappa z), \quad \text{and} \quad F_2(z) = R(\kappa z).$$

Therefore, shape holomorphy results for the Dirichlet problem follow directly from Theorems 5.7 and 5.11.

For the Neumann case, we make use of the corresponding Maue's representation formula [36, Corollary 3.3.24]:

$$(W_{\Gamma_1,\ldots,\Gamma_M})_{i,j}u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} G(\boldsymbol{r}_i(t),\boldsymbol{r}_j(\tau))\frac{\mathrm{d}}{\mathrm{d}\tau}u(\tau)\mathrm{d}\tau \\ - k^2\int_{-1}^{1}(\boldsymbol{r}_i'(t)\cdot\boldsymbol{r}_j'(\tau))G(\boldsymbol{r}_i(t),\boldsymbol{r}_j(\tau))u(\tau)\mathrm{d}\tau.$$

Thus, following the notation of Sect. 3.2 we have that

$$\widetilde{G}(\boldsymbol{r}_i(t),\boldsymbol{r}_j(\tau)) = -k^2(\boldsymbol{r}_i'(t)\cdot\boldsymbol{r}_j'(\tau))G(\boldsymbol{r}_i(t),\boldsymbol{r}_j(\tau)).$$

Notice that this function has almost the same structure of $G(\boldsymbol{r}_i(t),\boldsymbol{r}_j(\tau))$ except for the loss of one degree of regularity because of the factors $\boldsymbol{r}_i'$, and $\boldsymbol{r}_j'$. However, as mentioned in Remark 4.20, this does not have any impact, and we obtain the corresponding results for the Neumann problem. □

Remark 6.2 For $\kappa = 0$ (Laplace operator), the fundamental solution becomes $G(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{4\pi}\log\|\boldsymbol{x}-\boldsymbol{y}\|^2$. Thus, well-posedness requires a suitable condition at infinity. One particular alternative is to impose solutions to decay at infinity, which for the Dirichlet problem implies that we have to change the space T^s for the subspace of functions such that $\langle u, 1\rangle = 0$ (cf. [24, 43]).

6.2 Elastic Wave Operators

In this case, one has that $\mathcal{P} = \alpha\Delta + (\alpha+\beta)\nabla\nabla\cdot + \omega^2$. The parameters α, β are called Lamé parameters[2], with $\alpha > 0$, $\alpha+\beta > 0$, and $\omega > 0$ is the angular frequency. We also define the two standard pressure and shear wavenumbers

$$k_p^2 := \frac{\omega^2}{\alpha+2\beta}, \quad k_s^2 := \frac{\omega^2}{\beta}.$$

The co-normal trace corresponds to the traction operator, defined as:

$$\mathcal{B}_{\boldsymbol{n}}\mathbf{u} = \alpha\mathbf{n}(\mathrm{div}(\mathbf{u})) + 2\beta\partial_{\mathbf{n}}\mathbf{u} + \beta\mathbf{n}^\perp(\mathrm{div}(\mathbf{u}^\perp)),$$

where for $\mathbf{v} = (v_1, v_2)$ we set $\mathbf{v}^\perp = (v_2, -v_1)$. The standard condition at infinity is called the *Kupradze radiation condition* [28]. We refer to [26] for uniqueness of the related BVPs. In this case, the fundamental solution is given by

$$\mathbf{G}(\boldsymbol{x},\boldsymbol{y}) := \frac{i}{4\beta}H_0^{(1)}(k_s d)\mathbf{I} + \frac{i}{4\omega^2}\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}\cdot\left(H_0^{(1)}(k_s d) - H_0^{(1)}(k_p d)\right), \tag{6.2}$$

[2] Typically, these are denoted μ, λ but we have changed this convention to α, β so as to avoid any confusion with solutions of Dirichlet and Neumann problems, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, respectively.

where $d := \|\boldsymbol{x} - \boldsymbol{y}\|$, and $\mathbf{I}$ denotes the identity matrix. Alternatively, following [26] this can be expressed as

$$\mathbf{G}(\mathbf{x}, \mathbf{y}) = G^1(d)\mathbf{I} + G^2(d)\mathbf{D}(\mathbf{x} - \mathbf{y}),$$

where $D(\mathbf{d}) = \dfrac{\mathbf{d}\mathbf{d}^t}{\|\mathbf{d}\|^2}$, and

$$G^1(d) := \frac{i}{4\beta} H_0^{(1)}(k_s d) - \frac{i}{4\omega^2 d}\left(k_s H_1^{(1)}(k_s d) - k_p H_1^{(1)}(k_p d)\right),$$
$$G^2(d) := \frac{i}{4\omega^2}\left(\frac{2k_s H_1^{(1)}(k_s d) - 2k_p H_1^{(1)}(k_p d)}{d} + k_p^2 H_0^{(1)}(k_p d) - k_s^2 H_0^{(1)}(k_s d)\right)$$

Using the expansion of Hankel functions [1, 9.1.10 and 9.1.11], we can express G^1, G^2 as

$$G^j(d) = R^j(d) + (\log d^2) J^j(d), \quad j = 1, 2.$$

where R^1, R^2 are entire functions on the variable d^2, and

$$J^1(d) := -\frac{J_0(k_s d)}{4\pi\beta} + \frac{1}{4\pi\omega^2 d}\left(k_s J_1(k_s d) - k_p J_1(k_p d)\right),$$
$$J^2(d) := -\frac{1}{4\pi\omega^2}\left(\frac{2k_s J_1(k_s d) - 2k_p J_1(k_p d)}{d} + k_p^2 J_0(k_p d) - k_s^2 J_0(k_s d)\right).$$

Hence, from the series expansion of Bessel functions of zeroth and first order, we have that J^1, J^2 are entire functions in the d^2 variable, and also that

$$J^1(0) = -\frac{1}{4\pi\beta} - \frac{1}{8\pi\omega^2}(k_p^2 - k_s^2), \quad J^2(0) = 0.$$

Thus, we can express the fundamental solution as

$$\mathbf{G}(\boldsymbol{x}, \boldsymbol{y}) = (\log d^2)\mathbf{J} + \mathbf{R}, \tag{6.3}$$

where $\mathbf{J} = J^1(d)\mathbf{I} + J^2(d)\mathbf{D}(\boldsymbol{x} - \boldsymbol{y})$, and $\mathbf{R} = R^1(d)\mathbf{I} + R^2(d)\mathbf{D}(\boldsymbol{x} - \boldsymbol{y})$. The only difference with the canonical expression (3.3) is the presence of a factor $\mathbf{D}(\boldsymbol{x} - \boldsymbol{y})$. Let us study the properties of this factor.

Lemma 6.3 *Consider two arcs* $\boldsymbol{r}$, *and* $\boldsymbol{p}$ *and define the matrix function* $\mathbf{D}_{\boldsymbol{r},\boldsymbol{p}}$ *as*

$$(\mathbf{D}_{\boldsymbol{r},\boldsymbol{p}}(t,\tau))_{j,k} := \begin{cases} \dfrac{(r_j(t) - p_j(\tau)) \cdot (r_k(t) - p_k(\tau))}{d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau)}, & \boldsymbol{r} \neq \boldsymbol{p}, \text{ or, } t \neq s \\ \dfrac{r_j'(t) r_k'(\tau)}{\boldsymbol{r}'(t) \cdot \boldsymbol{r}'(\tau)}, & \text{otherwise} \end{cases},$$

where $j, k \in 1, 2$, and also two compact sets $K^1, K^2 \subset \mathcal{C}_b^{m,\alpha}\left((-1,1), \mathbb{R}^2\right)$ for some $m \in \mathbb{N}$, and $\alpha \in [0, 1]$.

(i) For $K^1 = K^2$, if we select δ as in Condition 4.15, then it holds that

$$\boldsymbol{r} \in K^1 \mapsto (D_{\boldsymbol{r},\boldsymbol{r}})_{j,k} \in \mathcal{C}^{m-1,\alpha}\left((-1,1)\times(-1,1), \mathbb{C}\right), \quad j,k = 1,2,$$

has a holomorphic extension in K^1_δ.

(ii) For K^1, K^2 disjoint sets, if δ_1, δ_2 as in Condition 4.14, then the map

$$(\boldsymbol{r}, \boldsymbol{p}) \in K^1 \times K^2 \mapsto (D_{\boldsymbol{r},\boldsymbol{p}})_{j,k} \in \mathcal{C}^{m,\alpha}\left((-1,1)\times(-1,1), \mathbb{C}\right), \quad j,k = 1,2,$$

has a holomorphic extension in $K^1_{\delta_1} \times K^2_{\delta_2}$.

Proof For the first part, if $t \neq \tau$ we have that

$$(D_{\boldsymbol{r},\boldsymbol{r}}(t,\tau))_{j,k} = Q_{\boldsymbol{r}}^{-1}(t,\tau)\left(\frac{r_j(t) - r_j(\tau)}{t-\tau}\right)\cdot\left(\frac{r_k(t) - r_k(\tau)}{t-\tau}\right),$$

where $Q_{\boldsymbol{r}}^{-1}(t,\tau) = 1/Q_{\boldsymbol{r}}(t,\tau)$, and the results follow as in the proof of Lemma 4.17. The second part is direct from Lemma 4.16 and elementary results of complex variable. □

Corollary 6.4 (Elastic case) *Consider the Dirichlet and Neumann BIEs (Problem 3.4) for the time-harmonic elastic kernel (6.2) for $\alpha > 0$, $\alpha + \beta > 0$, and $\omega > 0$. Then, the arising BIOs and domain-to-solution maps are shape holomorphic.*

Proof We only need to ensure that the integral kernels can be expressed as in 4.3. The result for the Dirichlet problem follows directly by the decomposition of the fundamental solution (6.3), Lemma 6.3 and also Remark 4.20 for the $D_{\boldsymbol{r},\boldsymbol{r}}$ factor.

For the Neumann problem, we use the following formula [5, Eq. 3.9],

$$\begin{aligned}(W_{\Gamma_1,\ldots,\Gamma_M})_{i,j}\boldsymbol{u} = &\int_{-1}^{1}\mathbf{G}_1(\boldsymbol{r}_i(t),\boldsymbol{r}_j(\tau))u_j(\tau)\mathrm{d}s + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\mathbf{G}_2(\boldsymbol{r}_i(t),\boldsymbol{r}_j(\tau))\frac{\mathrm{d}u_j(\tau)}{\mathrm{d}s}\mathrm{d}s\\ &+\int_{-1}^{1}\mathbf{G}_3(\boldsymbol{r}_i(t),\boldsymbol{r}_j(\tau))\frac{\mathrm{d}u_j(\tau)}{\mathrm{d}s}\mathrm{d}s\\ &+\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\mathbf{G}_2(\boldsymbol{r}_i(t),\boldsymbol{r}_j(\tau))u_j(\tau)\mathrm{d}s,\end{aligned} \tag{6.4}$$

where the first kernel function is

$$\begin{aligned}\mathbf{G}_1(\boldsymbol{r}_i(t),\boldsymbol{r}_j(\tau)) := \frac{i}{4}\Big(&\rho\omega^2(\boldsymbol{r}_i'^{\perp}(t)\boldsymbol{r}_j'^{\perp}(\tau)^T - \boldsymbol{r}_i'(t)\cdot\boldsymbol{r}_j'(\tau)\mathbf{I})H_0^{(1)}(k_s d)\\ &-\beta k_s^s(\boldsymbol{r}_j'^{\perp}(\tau)\boldsymbol{r}_i'^{\perp}(t)^T - \boldsymbol{r}_i'^{\perp}(t)\boldsymbol{r}_j'^{\perp}(\tau)^T)H_0^{(1)}(k_s d)\\ &-\rho\omega^2(\boldsymbol{r}_i'^{\perp}(t)\boldsymbol{r}_j'^{\perp}(\tau)^T)H_0^{(1)}(k_p d)\Big),\end{aligned}$$

which only has logarithmic singularities as it is composed of zeroth-order Hankel functions. Hence, it corresponds to the term $\widetilde{G}$ in Maue's formula (3.5). The holomorphic extension of the corresponding BIO is direct since the kernel function is of the form described in Remark 4.20.

The second kernel in (6.4) is

$$\mathbf{G}_2(\boldsymbol{r}_i(t), \boldsymbol{r}_j(\tau)) = 4\beta^2 \mathbf{A}\mathbf{G}(\boldsymbol{r}_i(t), \boldsymbol{r}_j(\tau))\mathbf{A} + i\beta H_0^{(1)}(k_s d)\mathbf{I},$$

where $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From the analysis of the weakly singular BIO, we obtain the holomorphic extension of the corresponding integral operator.

The third term in (6.4) is

$$\frac{i\beta \boldsymbol{r}_i'^{\perp}(t)(\boldsymbol{r}_i(t) - \boldsymbol{r}_j(\tau))}{2d} \left(k_s H_1^{(1)}(k_s d) - k_p H_1^{(1)}(k_p d) \right) \mathbf{A},$$

which can be shown not to have any singularities, but the structure still is the one described in Remark 4.20, only with $G_1 = 0$. The associated integral BIO in (6.4) can be seen as a map in $\mathcal{L}(\mathbb{U}^s, \mathbb{W}^s)$, where the range is in $\mathbb{W}^s$ instead of $\mathbb{Y}^s$. This is due to the fact that if we start in $\mathbb{U}^s$, by (2.5) the derivative changes the argument in the operator to a function in $\mathbb{T}^{s-1}$. Thus, by Corollary 4.9, we obtain the mentioned mapping property. Though operators with the mentioned mapping properties were not studied, by (2.3), we can still consider that this operator lies in $\mathcal{L}(\mathbb{U}^s, \mathbb{Y}^s)$, and hence it is compact in $\mathcal{L}\left(\mathbb{U}^s, \mathbb{Y}^{s-1}\right)$. The corresponding holomorphic extension then follows arguing as in the regular part of Theorem 5.2. The final kernel function is

$$\mathbf{G}_4(\boldsymbol{r}_i(t), \boldsymbol{r}_j(\tau)) = \frac{i\beta(\boldsymbol{r}_i(t) - \boldsymbol{r}^{\perp}(\tau))\boldsymbol{r}_j'^{\perp}(\tau)^T}{2d} \left(k_s H_1^{(1)}(k_s d) - k_p H_1^{(1)}(k_p d) \right),$$

which is also a regular kernel, whose structure is described in Remark 4.20, with $G_1 = 0$. The mapping properties of the associated operator are not easily derived since, if $u_j \in \mathbb{U}^s$, the evaluation of the BIO, discarding the derivative, would lie in $\mathbb{Y}^{s+1}$. Yet, we do not have a characterization of the derivative map in $\mathbb{Y}^{s+1}$. We circumvent this by using (2.4), and so if $u_j \in \mathbb{U}^s$ then $u_j \in \mathbb{T}^s$, and hence the evaluation of the integral operator is in $\mathbb{W}^{s+1}$. Thus, by (2.5) the full operator is in $\mathcal{L}(\mathbb{U}^s, \mathbb{Y}^s)$ and compact in $\mathcal{L}\left(\mathbb{U}^s, \mathbb{Y}^{s-1}\right)$. Then, the holomorphic extension follows as in the previous case. □

7 Conclusions and Future Work

We have shown a general framework for establishing parametric shape holomorphy of BIEs in two-dimensional space with multiple arcs. Though we have limited our findings to the case of homogeneous media, heterogeneous coefficients and non-explicit fundamental solutions could also be addressed. Indeed, as long as the kernel can be decomposed as (3.3) all results hold. Future work involves the application of these results in UQ and deep learning.

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Appendix A Immersion of Hölder Spaces

Herein, the symbol $\mathbf{n}$ will be used to denote a vector of two integers, not the normal of an arc. Also, we denote the non-normalized Fourier basis by $e_n(t) = \exp(int)$, for $n \in \mathbb{Z}$, and the bi-periodic basis as $e_{n,l}(t,\tau) = e_n(t)e_l(\tau)$, $n, l \in \mathbb{Z}$.

We begin by introducing the Sobolev–Slobodeckij norm for bi-periodic functions with domain $[-\pi, \pi] \times [-\pi, \pi]$. Let $\mathbf{n} = (n_1, n_2) \in \mathbb{N}_0^2$, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in [0, 1]^2$, we define

$$\|g\|^2_{\boldsymbol{n},\boldsymbol{\gamma}} = \sum_{p\leq n_1}\sum_{q\leq n_2} \|\partial_t^p \partial_s^q g(t,\tau)\|^2_{L^2([-\pi,\pi]\times[-\pi,\pi])} + |\partial_t^{n_1}\partial_s^{n_2} g(t,\tau)|^2_{\boldsymbol{\gamma}},$$

where the Sobolev–Slobodeckij semi-norm is defined as

$$|u|^2_{\boldsymbol{\gamma}} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|u(x,y) - u(t,y) + u(t,\tau) - u(x,s)|^2}{\left(\sin\frac{|x-t|}{2}\right)^{1+2\gamma_1}\left(\sin\frac{|y-s|}{2}\right)^{1+2\gamma_2}} \mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s.$$

The semi-norm is generated by the following inner product:

$$\langle u, v\rangle_{\boldsymbol{\gamma}} := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\widetilde{\Delta} u(x,t,y,s)\widetilde{\Delta}\overline{v}(x,t,y,s)}{\left(\sin\frac{|x-t|}{2}\right)^{1+2\gamma_1}\left(\sin\frac{|y-s|}{2}\right)^{1+2\gamma_2}} \mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s,$$

where, the difference operator $\widetilde{\Delta}$ is given by

$$\widetilde{\Delta} u(x,t,y,s) := u(x,y) - u(t,y) + u(t,s) - u(x,s).$$

Following the uni-variate case [27, Theorem 8.6], we derive the following two results.

Lemma A.1 *For $(n_1, l_1), (n_2, l_2) \in \mathbb{Z}^2$ we have that:*

$$\langle e_{n_1,l_1}, e_{n_2,l_2}\rangle_{\boldsymbol{\gamma}} = 16\widetilde{\delta}_{n_1,n_2}\widetilde{\delta}_{l_1,l_2} S_{\gamma_1}(n_2) S_{\gamma_2}(l_2),$$

where $\widetilde{\delta}_{n,m} = 2\pi\delta_{n,m}$, for $n, m \in \mathbb{Z}$, and

$$S_a(n) = \int_0^{\pi} \left(\sin\frac{n}{2}u\right)^2 \left(\sin\frac{|u|}{2}\right)^{-1-2a} du.$$

Proof We compute the inner product between two basis, e_{n_1,l_1} and e_{n_2,l_2}. By permuting integration variables, one has that

$$\langle e_{n_1,l_1}, e_{n_2,l_2}\rangle \boldsymbol{\gamma} = 4\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e_{n_1,l_1}(x,y)\widetilde{\Delta}e_{-n_2,-l_2}(x,t,y,s)}{\left(\sin\frac{|x-t|}{2}\right)^{1+2\gamma_1}\left(\sin\frac{|y-s|}{2}\right)^{1+2\gamma_2}}\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s.$$

The right-hand side term is decomposed into the sum of four integrals defined as

$$I_1 := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e_{n_1,l_1}(x,y)e_{-n_2,-l_2}(x,y)}{\left(\sin\frac{|x-t|}{2}\right)^{1+2\gamma_1}\left(\sin\frac{|y-s|}{2}\right)^{1+2\gamma_2}}\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s,$$

$$I_2 := -\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e_{n_1,l_1}(x,y)e_{-n_2,-l_2}(t,y)}{\left(\sin\frac{|x-t|}{2}\right)^{1+2\gamma_1}\left(\sin\frac{|y-s|}{2}\right)^{1+2\gamma_2}}\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s,$$

$$I_3 := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e_{n_1,l_1}(x,y)e_{-n_2,-l_2}(t,\tau)}{\left(\sin\frac{|x-t|}{2}\right)^{1+2\gamma_1}\left(\sin\frac{|y-s|}{2}\right)^{1+2\gamma_2}}\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s,$$

$$I_4 := -\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e_{n_1,l_1}(x,y)e_{-n_2,-l_2}(x,s)}{\left(\sin\frac{|x-t|}{2}\right)^{1+2\gamma_1}\left(\sin\frac{|y-s|}{2}\right)^{1+2\gamma_2}}\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s.$$

We perform the change of variables $u = t - x$, and $v = s - y$ in the four integrals, and by the periodicity of the involving factors, we get

$$I_1 := \widetilde{\delta}_{n_1,n_2}\widetilde{\delta}_{l_1,l_2}\int_{-\pi}^{\pi}\left(\sin\frac{|u|}{2}\right)^{-1-2\gamma_1}\mathrm{d}u\int_{-\pi}^{\pi}\left(\sin\frac{|v|}{2}\right)^{-1-2\gamma_2}\mathrm{d}v,$$

$$I_2 := -\widetilde{\delta}_{n_1,n_2}\widetilde{\delta}_{l_1,l_2}\int_{-\pi}^{\pi}e_{-n_2}(u)\left(\sin\frac{|u|}{2}\right)^{-1-2\gamma_1}\mathrm{d}u\int_{-\pi}^{\pi}\left(\sin\frac{|v|}{2}\right)^{-1-2\gamma_2}\mathrm{d}v,$$

$$I_3 := \widetilde{\delta}_{n_1,n_2}\widetilde{\delta}_{l_1,l_2}\int_{-\pi}^{\pi}e_{-n_2}(u)\left(\sin\frac{|u|}{2}\right)^{-1-2\gamma_1}\mathrm{d}u\int_{-\pi}^{\pi}e_{-l_2}(v)\left(\sin\frac{|v|}{2}\right)^{-1-2\gamma_2}\mathrm{d}v,$$

$$I_4 := -\widetilde{\delta}_{n_1,n_2}\widetilde{\delta}_{l_1,l_2}\int_{-\pi}^{\pi}\left(\sin\frac{|u|}{2}\right)^{-1-2\gamma_1}\mathrm{d}u\int_{-\pi}^{\pi}e_{-l_2}(v)\left(\sin\frac{|v|}{2}\right)^{-1-2\gamma_2}\mathrm{d}v.$$

Since the Fourier basis elements are $e_n(t) = \cos(nt) + i\sin(nt)$, onw can use the symmetries of cosine and sine functions to obtain

$$\int_{-\pi}^{\pi}(1-e_{-n_2}(u))\left(\sin\frac{|u|}{2}\right)^{-1-2\gamma_1}\mathrm{d}u = 2\int_{0}^{\pi}(1-\cos(n_2u))\left(\sin\frac{|u|}{2}\right)^{-1-2\gamma_1}\mathrm{d}u.$$

Furthermore, we invoke the double-angle formula for the cosine so as to arrive at

$$\int_{-\pi}^{\pi}(1-e_{-n_2}(u))\left(\sin\frac{|u|}{2}\right)^{-1-2\gamma_1}\mathrm{d}u = 4\int_{0}^{\pi}\left(\sin\frac{n_2u}{2}\right)^{2}\left(\sin\frac{|u|}{2}\right)^{-1-2\gamma_1}\mathrm{d}u.$$

Using this, we find

$$\langle e_{n_1,n_2}, e_{l_1,l_2}\rangle_{\boldsymbol{\gamma}} = 16\widetilde{\delta}_{n_1,n_2}\widetilde{\delta}_{l_1,l_2}$$
$$\int_0^{\pi}\left(\sin\frac{n_2 u}{2}\right)^2\left(\sin\frac{|u|}{2}\right)^{-1-2\gamma_1}\mathrm{d}u\int_0^{\pi}\left(\sin\frac{l_2 v}{2}\right)^2\left(\sin\frac{|v|}{2}\right)^{-1-2\gamma_2}\mathrm{d}u.$$

which is equivalent to the statement of the lemma. □

Lemma A.2 *Let* $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in [0, 1]^2$*, and* ϱ *a bi-periodic function in* $[-\pi, \pi] \times [-\pi, \pi]$*, then*

$$\|\varrho\|^2_{\gamma_1,\gamma_2} \lesssim \|\varrho\|^2_{L^2([-\pi,\pi]\times[-\pi,\pi])} + |\varrho|^2_{\boldsymbol{\gamma}},$$

i.e. the Sobolev norm for two pure fractional orders γ_1, γ_2 *of a bi-periodic function is bounded by the Sobolev–Slobodeckij norm of order* $(0, 0)$, (γ_1, γ_2).

Proof We consider a function ϱ expanded in terms of bi-periodic Fourier basis functions. By Lemma A.1, it holds that

$$|\varrho|^2_{\boldsymbol{\gamma}} = \langle \varrho, \varrho\rangle_{\boldsymbol{\gamma}} \cong \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}|\widetilde{\varrho}_{n,l}|^2 S_{\gamma_1}(n)S_{\gamma_2}(l).$$

From [27, (8.8)], we have that the function $S_a(n)$ from Lemma A.1, behave as $S_a(n) \cong (n^2)^a$, for $a \in [0, 1]$. Consequently,

$$|\varrho|^2_{\boldsymbol{\gamma}} \cong \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}|\widetilde{\varrho}_{n,l}|^2 (n^2)^{\gamma_1}(l^2)^{\gamma_2}.$$

We can now use the well-known inequality $(1+n^2)^{\gamma_1} \le (n^2)^{\gamma_1} + 1$—analogously for l and γ_2—, so that

$$\|\varrho\|^2_{\gamma_1,\gamma_2} = \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(1+n^2)^{\gamma_1}(1+l^2)^{\gamma_2}|\widetilde{\varrho}_{n,l}|^2 \lesssim |\varrho|^2_{\boldsymbol{\gamma}} + \|\varrho\|^2_{L^2([-\pi,\pi]\times[-\pi,\pi])}$$

as stated. □

A.1 Proof of Lemma 2.1

We will first show that $\|g\|^2_{s_1,s_2}$ can be bounded by the Sobolev–Slobodeckij norm $\|g\|^2_{\mathbf{n},\boldsymbol{\gamma}}$, with $\mathbf{n} = ([\mathbf{s_1}], [\mathbf{s_2}])$, and $\boldsymbol{\gamma} = (\{s_1\}, \{s_2\})$, where $[s_1], [s_2]$ denote the integer parts of s_1, s_2 respectively and $\{s_1\}, \{s_2\}$ are the corresponding fractional parts. By definition, one has that

$$\|g\|^2_{s_1,s_2} = \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(1+n^2)^{\{s_1\}}(1+l^2)^{\{s_2\}}(1+n^2)^{[s_1]}(1+l^2)^{[s_2]}\left|\widetilde{g}_{n,l}\right|^2$$

Using the inequality $(1+n^2)^s \lesssim (n^2)^s + 1$, we obtain

$$\begin{aligned}\|g\|^2_{s_1,s_2} &\lesssim \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(1+n^2)^{\{s_1\}}(1+l^2)^{\{s_2\}}(1+n^{2[s_1]})(1+l^{2[s_2]})\left|\widetilde{g}_{n,l}\right|^2\\ &\lesssim \sum_{p=0}^{[s_1]}\sum_{q=0}^{[s_2]}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(1+n^2)^{\{s_1\}}(1+l^2)^{\{s_2\}}(1+n^{2p})(1+l^{2q})\left|\widetilde{g}_{n,l}\right|^2\\ &\lesssim \sum_{p=0}^{[s_1]}\sum_{q=0}^{[s_2]}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(1+n^2)^{\{s_1\}}(1+l^2)^{\{s_2\}}(1+n^{2p}l^{2q})\left|\widetilde{g}_{n,l}\right|^2.\end{aligned}$$

We notice that the Fourier coefficients of $\partial_t^p\partial_s^q g$ are in fact $(in)^p(il)^q\widetilde{g}_{n,l}$. Hence, we obtain that

$$\begin{aligned}\|g\|^2_{s_1,s_2} \lesssim& \sum_{p=0}^{[s_1]}\sum_{q=0}^{[s_2]}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(1+n^2)^{\{s_1\}}(1+l^2)^{\{s_2\}}\left|\left(\widetilde{\partial_t^{[p]}\partial_s^{[q]}g(t,\tau)}\right)_{n,l}\right|^2\\ &+\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(1+n^2)^{\{s_1\}}(1+l^2)^{\{s_2\}}\left|\widetilde{g}_{n,l}\right|^2.\end{aligned} \tag{A1}$$

From the last inequality, we notice that for $\{s_1\}=\{s_2\}=0$, we have that

$$\|g\|^2_{s_1,s_2} \lesssim \sum_{p=0}^{[s_1]}\sum_{q=0}^{[s_2]}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left|\left(\widetilde{\partial_t^{[p]}\partial_s^{[q]}g(t,\tau)}\right)_{n,l}\right|^2+\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left|\widetilde{g}_{n,l}\right|^2.$$

Then, by Parseval's identity we obtain

$$\|g\|^2_{s_1,s_2} \lesssim \|g\|^2_{([s_1],[s_2]),(0,0)}.$$

In any other case, $\{s_1\} > 0$ or $\{s_2\} > 0$, for every $p \in \{0,\dots,[s_1]\}$ and $q \in \{0,\dots,[s_2]\}$ we define $\varrho^{p,q} = \partial_t^p\partial_s^q g(t,\tau)$, and from (A1) we see that we only need to show that

$$\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(1+n)^{\{s_1\}}(1+l^2)^{\{s_2\}}\left|\widetilde{\varrho^{p,q}}_{n,l}\right|^2 \lesssim \|\varrho\|^2_{(0,0),(\{s_1\},\{s_2\})},$$

which holds by Lemma A.2 with $\varrho := \varrho^{[s_1],[s_2]}$. We conclude that

$$\|g\|^2_{s_1,s_2} \lesssim \|g\|^2_{([s_1],[s_2]),(\{s_1\},\{s_2\})}. \tag{A2}$$

To finish the proof we will bound the general norm $\|g\|^2_{\mathbf{n},\boldsymbol{\gamma}}$, in terms of a $\mathcal{C}^{m,\alpha}\left([-\pi,\pi]\times[-\pi,\pi],\mathbb{C}\right)$-norm with appropriate $m\in\mathbb{N}_0$, and $\alpha\in[0,1]$. First, from the definition of Hölder norms, it holds that

$$\|g\|_{\mathbf{n},(0,0)}^2 \lesssim \|g\|_{\mathcal{C}^{m,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})},$$

for $m+\alpha \geq n_1+n_2$. For the purely fractional case, we have that for any $a, b > 0$ such that $a+b=1$

$$\begin{aligned}|g|_{(\gamma_1,\gamma_2)}^2 \lesssim \|g\|_{\mathcal{C}^{0,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})}^2 \\ \times\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|x-t|^{2a\alpha}}{\left(\sin\frac{|x-t|}{2}\right)^{1+2\gamma_1}}\mathrm{d}t\mathrm{d}x\right)\left(\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|y-s|^{2b\alpha}}{\left(\sin\frac{|y-s|}{2}\right)^{1+2\gamma_2}}\mathrm{d}s\mathrm{d}y\right),\end{aligned}$$

and thus, the right-hand side is finite only if $\alpha > \gamma_1+\gamma_2$. However, this condition cannot be used whenever $\gamma_1+\gamma_2 \geq 1$. For the latter, we assume that $m \geq 1$, and by the mean value theorem, we see that

$$\begin{aligned}|g(x,y)-g(t,y)+g(t,\tau)-g(x,s)| &= \left|\int_t^x \partial_1 g(\lambda,y)-\partial_1 g(\lambda,s)\mathrm{d}\lambda\right| \\ &= |x-t|\,|\partial_1 g(\xi,y)-\partial_t g(\xi,s)| \\ &\leq |x-t||y-s|^{\alpha}\|g\|_{\mathcal{C}^{1,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})}.\end{aligned}$$

Similarly, by reordering terms we conclude that

$$|g(x,y)-g(t,y)+g(t,\tau)-g(x,s)| \leq |y-s||x-t|^{\alpha}\|g\|_{\mathcal{C}^{1,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})}.$$

Thus, for every $a \geq 0$, $b \geq 0$ such that $a+b=1$ we have

$$\begin{aligned}|g|_{\boldsymbol{\gamma}}^2 \lesssim \|g\|_{\mathcal{C}^{1,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})}^2 \\ \times\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{(|x-t|^{a+\alpha b}|y-s|^{b+\alpha a})^2}{\left(\sin\frac{|x-t|}{2}\right)^{1+2\gamma_1}\left(\sin\frac{|y-s|}{2}\right)^{1+2\gamma_2}}\mathrm{d}x\mathrm{d}y\mathrm{d}t\mathrm{d}s\text{'}\end{aligned}$$

and one can easily see that integrals in the right-hand side are finite if $\alpha > \gamma_1+\gamma_2-1$. Finally, we conclude that

$$\begin{aligned}\|g\|_{\mathbf{n},\boldsymbol{\gamma}}^2 \lesssim \|g\|_{\mathcal{C}^{m,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})}^2 \\ \times\begin{cases}\text{for } n_1+n_2+\gamma_1+\gamma_2 < m+\alpha \text{ if } \alpha_1+\alpha_2 < 1, \\ \text{for } (n_1+n_2+1)+(\gamma_1+\gamma_2-1) < m+\alpha, \text{ if } \gamma_1+\gamma_2 \geq 1.\end{cases}\end{aligned}$$

which is equivalent to

$$\|g\|_{\mathbf{n},\boldsymbol{\gamma}}^2 \lesssim \|g\|_{\mathcal{C}^{m,\alpha}([-\pi,\pi]\times[-\pi,\pi],\mathbb{C})}^2,$$

if $n_1 + n_2 + \gamma_1 + \gamma_2 < m + \alpha$. The statement of the lemma then follows from this last bound and (A2). □

Appendix B Proofs Lemmas 4.16 and 4.17

We present proofs for Lemmas 4.16 and 4.17. For the latter, we will need the following auxiliary result.

Lemma B.1 *Let* $m \in \mathbb{N}$, $\alpha \in [0,1]$ *and* $K \subset \mathcal{C}_b^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$ *be a compact set of* $\mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$.

(i) It holds that

$$\inf_{\boldsymbol{r}\in K}\inf_{t\in(-1,1)} \|\boldsymbol{r}'(t)\| > 0 \quad \textit{and} \quad \sup_{\boldsymbol{r}\in K}\sup_{t\in(-1,1)} \|\boldsymbol{r}'(t)\| < \infty.$$

(ii) There exists $\delta > 0$ *such that*

$$\inf_{\boldsymbol{r}\in K_\delta}\inf_{t\in(-1,1)} \mathfrak{Re}(\{(\boldsymbol{r}'(t)\cdot\boldsymbol{r}'(t))\} > 0.$$

i.e. there exists $\delta > 0$ *fulfilling Condition 4.15.*

Proof The first part follows from the continuity of

$$\mathcal{I}(\boldsymbol{r}) := \inf_{t\in(-1,1)} \|\boldsymbol{r}'(t)\| \quad \text{and} \quad \mathcal{S}(\boldsymbol{r}) := \sup_{t\in(-1,1)} \|\boldsymbol{r}'(t)\|$$

in $\mathcal{C}_b^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$. Indeed, if for each $\boldsymbol{r} \in \mathcal{C}_b^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$ we consider any $\boldsymbol{p} \in \mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{R}^2\right)$ such that $\|\boldsymbol{r}-\boldsymbol{p}\|_{\mathcal{C}^{m,\alpha}((-1,1),\mathbb{R}^2)} < \epsilon$, one has that

$$\begin{aligned}|\mathcal{I}(\boldsymbol{r}) - \mathcal{I}(\boldsymbol{p})| &\leq \inf_{t\in(-1,1)} \left|\left\|\boldsymbol{r}'(t)\right\| - \left\|\boldsymbol{p}'(t)\right\|\right| \\ &\leq \inf_{t\in(-1,1)} \left\|\boldsymbol{r}'(t) - \boldsymbol{p}'(t)\right\| \leq \|\boldsymbol{r}-\boldsymbol{p}\|_{\mathcal{C}^{m,\alpha}((-1,1),\mathbb{R}^2)} < \epsilon.\end{aligned}$$

Thus, one concludes that the map $\boldsymbol{r} \mapsto \mathcal{I}(\boldsymbol{r})$ is continuous, and then the infimum in K is achieved since K is compact. The supremum case follows similarly.

For the second part we set

$$\mathcal{I}_1 = \inf_{\boldsymbol{r}\in K} \mathcal{I}(\boldsymbol{r}) \quad \text{and} \quad \mathcal{S}_1 = \sup_{\boldsymbol{r}\in K} \mathcal{S}(\boldsymbol{r}),$$

and consider any element $\boldsymbol{r} \in K_\delta$. Then, there is $\boldsymbol{p} \in K$ such that $\|\boldsymbol{r} - \boldsymbol{p}\|_{\mathcal{C}^{m,\alpha}((-1,1),\mathbb{R}^2)} < \delta$, and it holds that

$$\boldsymbol{r}'\cdot\boldsymbol{r}' = \|\boldsymbol{r}'\|^2 + 2(\boldsymbol{r}'-\boldsymbol{p}')\cdot\boldsymbol{p}' + (\boldsymbol{r}'-\boldsymbol{p}')\cdot(\boldsymbol{r}'-\boldsymbol{p}').$$

Therefore,

$$\mathfrak{Re}(\boldsymbol{r}'\cdot\boldsymbol{r}') \geq \mathcal{I}_1^2 - 2\mathcal{S}_1\delta - \delta^2,$$

and the result then follows by selecting $\delta < \sqrt{\mathcal{I}_1^2 + \mathcal{S}_1^2} - \mathcal{S}_1$. □

B.1 Proof of Lemma 4.16

First, we prove that the map

$$\boldsymbol{r}, \boldsymbol{p} \in \mathcal{C}^{m,\alpha}\left((-1,1), \mathbb{C}^2\right) \times \mathcal{C}^{m,\alpha}\left((-1,1), \mathbb{C}^2\right) \to d^2_{\boldsymbol{r},\boldsymbol{p}} \in \mathcal{C}^{m,\alpha}\left((-1,1), \mathbb{C}\right)$$

has a Fréchet derivative at every point. Consequently, the holomorphic extension in compact sets of $\mathcal{C}_b^{m,\alpha}\left((-1,1), \mathbb{R}^2\right) \times \mathcal{C}_b^{m,\alpha}\left((-1,1), \mathbb{R}^2\right)$ follows directly. We limit ourselves to the case $m \geq 1$ since the result is formulated in terms of $\mathcal{C}_b^{m,\alpha}\left((-1,1), \mathbb{R}^2\right)$ and by convention $\mathcal{C}_b^{0,\alpha}\left((-1,1), \mathbb{R}^2\right) = \emptyset$.

Choose two arbitrary functions $\boldsymbol{r}, \boldsymbol{p} \in \mathcal{C}^{m,\alpha}\left((-1,1), \mathbb{C}^2\right)$. Since, $d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau) = (r_1(t) - p_1(\tau))^2 + (r_2(t) - p_2(\tau))^2$, we have that

$$d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau) \in \mathcal{C}^{m,\alpha}\left((-1,1) \times (-1,1), \mathbb{C}\right).$$

For a pair $\boldsymbol{v}^1, \boldsymbol{v}^2 \in \mathcal{C}^{m,\alpha}\left((-1,1), \mathbb{C}^2\right)$, we define

$$Dd^2_{\boldsymbol{r},\boldsymbol{p}}[\boldsymbol{v}^1, \boldsymbol{v}^2](t,\tau) := 2(\boldsymbol{r}(t) - \boldsymbol{p}(\tau)) \cdot \left(\boldsymbol{v}^1(t) - \boldsymbol{v}^2(\tau)\right),$$

which is linear in $\boldsymbol{v}^1, \boldsymbol{v}^2$ and lies in $\mathcal{C}^{m,\alpha}\left((-1,1) \times (-1,1), \mathbb{C}\right)$. Finally, we notice that

$$d^2_{\boldsymbol{r}+\boldsymbol{v}^1,\boldsymbol{p}+\boldsymbol{v}^2}(t,\tau) - d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau) - Dd^2_{\boldsymbol{r},\boldsymbol{p}}[\boldsymbol{v}^1, \boldsymbol{v}^2](t,\tau) = d^2_{\boldsymbol{v}^1,\boldsymbol{v}^2}(t,\tau).$$

by the product derivation rule, one can directly show that

$$\|d^2_{\boldsymbol{v}^1,\boldsymbol{v}^2}(t,\tau)\|_{\mathcal{C}^{m,\alpha}((-1,1)\times(-1,1),\mathbb{C})} \lesssim \|\boldsymbol{v}^1\|^2_{\mathcal{C}^{m,\alpha}((-1,1),\mathbb{C}^2)} + \|\boldsymbol{v}^2\|^2_{\mathcal{C}^{m,\alpha}((-1,1),\mathbb{C}^2)}.$$

Hence, $Dd^2_{\boldsymbol{r},\boldsymbol{p}}[\boldsymbol{v}^1, \boldsymbol{v}^2]$ is in fact the Fréchet derivative of $d^2_{\boldsymbol{r},\boldsymbol{p}}$. Since $\boldsymbol{r}, \boldsymbol{p}$ are arbitrary, the distance function is holomorphic in an arbitrary open set of $\mathcal{C}^{m,\alpha}\left((-1,1), \mathbb{C}^2\right) \times \mathcal{C}^{m,\alpha}\left((-1,1), \mathbb{C}^2\right)$.

For the final part, since the sets are admissible we have by Assumption 4.13, it holds that

$$I_d = \inf_{(\boldsymbol{r},\boldsymbol{p}) \in K^1 \times K^2} \inf_{(t,\tau) \in (-1,1)\times(-1,1)} \|\boldsymbol{r}(t) - \boldsymbol{p}(\tau)\| > 0.$$

Then, for any $\delta_1, \delta_2 > 0$ $(\boldsymbol{r}, \boldsymbol{p}) \in K^1_{\delta_1} \times K^2_{\delta_2}$, and $(\widetilde{\boldsymbol{r}}, \widetilde{\boldsymbol{p}}) \in K^1 \times K^2$ we have that

$$\begin{aligned} d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau) = (\boldsymbol{r}(t) - \boldsymbol{p}) \cdot (\boldsymbol{r}(t) - \boldsymbol{p}) &= ((\boldsymbol{r} - \widetilde{\boldsymbol{r}})(t) + (\widetilde{\boldsymbol{p}} - \boldsymbol{p})(\tau) + \widetilde{\boldsymbol{r}}(t) - \widetilde{\boldsymbol{p}}(\tau)) \\ &\cdot ((\boldsymbol{r} - \widetilde{\boldsymbol{r}})(t) + (\widetilde{\boldsymbol{p}} - \boldsymbol{p})(\tau) + \widetilde{\boldsymbol{r}}(t) - \widetilde{\boldsymbol{p}}(\tau)). \end{aligned}$$

Aassuming that $\widetilde{\boldsymbol{r}}$, $\widetilde{\boldsymbol{p}}$ are such that

$$\|\boldsymbol{r}-\widetilde{\boldsymbol{r}}\|_{\mathcal{C}^{m,\alpha}((-1,1),\mathbb{C}^2)}<\delta_1, \quad \|\boldsymbol{p}-\widetilde{\boldsymbol{p}}\|_{\mathcal{C}^{m,\alpha}((-1,1),\mathbb{C}^2)}<\delta_2$$

we get the bound

$$\Re\mathfrak{e}(d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau)) \geq \mathcal{I}_d^2 - 2\mathcal{S}_d(\delta_1+\delta_2)-(\delta_1+\delta_2)^2,$$

where $\mathcal{I}_d$, and $\mathcal{S}_d$ are defined as in Condition 4.14. Hence, it follows directly that if δ_1, δ_2 satisfying Condition 4.14 we have that

$$\Re\mathfrak{e}(d^2_{\boldsymbol{r},\boldsymbol{p}}(t,\tau)) > 0.$$

B.2 Proof of Lemma 4.17

Consider $\boldsymbol{r} \in \mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{C}^2\right)$, with $m \geq 1$, as in Lemma B.1. By Taylor expansion, it is immediate that

$$Q_{\boldsymbol{r}} \in \mathcal{C}^{m-1,\alpha}\left((-1,1)\times(-1,1),\mathbb{C}\right).$$

For $\boldsymbol{v} \in \mathcal{C}^{m,\alpha}\left((-1,1),\mathbb{C}^2\right)$, let us define

$$DQ_{\boldsymbol{r}}[\boldsymbol{v}](t,\tau) = \frac{2(\boldsymbol{r}(t)-\boldsymbol{r}(\tau))\cdot(\boldsymbol{v}(t)-\boldsymbol{v}(\tau))}{(t-s)^2},$$

with the continuous extension for $t=s$, given by the Taylor expansions of $\boldsymbol{r}$ and $\boldsymbol{v}$. It is clear that $DQ_{\boldsymbol{r}}[\boldsymbol{v}]$ is linear in the $\boldsymbol{v}$ variable and also $DQ_{\boldsymbol{r}}[\boldsymbol{v}] \in \mathcal{C}^{m-1,\alpha}\left((-1,1)\times(-1,1),\mathbb{C}\right)$. We also have that

$$d^2_{\boldsymbol{r}+\boldsymbol{v}}(t,\tau) = d^2_{\boldsymbol{r}}(t,\tau) + 2(\boldsymbol{r}(t)-\boldsymbol{r}(\tau))\cdot(\boldsymbol{v}(t)-\boldsymbol{v}(\tau)) + d^2_{\boldsymbol{v}}(t,\tau).$$

Therefore, one has that

$$Q_{\boldsymbol{r}+\boldsymbol{v}}(t,\tau) = Q_{\boldsymbol{r}}(t,\tau) + DQ_{\boldsymbol{r}}[v](t,\tau) + Q_{\boldsymbol{v}}(t,\tau).$$

Arguing as in the proof of Lemma 4.16, we have that

$$\|Q_{\boldsymbol{v}}\|_{\mathcal{C}^{m-1,\alpha}((-1,1)\times(-1,1),\mathbb{C}^2)} \lesssim \|\boldsymbol{v}\|^2_{\mathcal{C}^{m,\alpha}((-1,1),\mathbb{C}^2)},$$

and one concludes that there is a Fréchet derivative everywhere. Thus, the function has a holomorphic extension in $\mathcal{C}^{m,\alpha}_b\left((-1,1),\mathbb{R}^2\right)$.

Now, we show that the real part is strictly positive. Using the Taylor expansion once again, we have that

$$d_{\boldsymbol{r}}^2(t,\tau) = (t-s)^2 \left(\int_0^1 \boldsymbol{r}'(t+\delta(s-t))\mathrm{d}\delta \right) \cdot \left(\int_0^1 \boldsymbol{r}'(t+\delta(s-t))\mathrm{d}\delta \right).$$

Consequently, we can write

$$Q_{\boldsymbol{r}}(t,\tau) = \left(\int_0^1 \boldsymbol{r}'(t+\delta(s-t))\mathrm{d}\delta \right) \cdot \left(\int_0^1 \boldsymbol{r}'(t+\delta(s-t))\mathrm{d}\delta \right).$$

The result then follows from the mean value theorem and selecting δ as in Lemma B.1. The results for $Q_{\boldsymbol{r}}^{-1}$ follows using the fact that the function z^{-1} is holomorphic away from zero.

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