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Stable Separation of Orbits for Finite Abelian Group Actions

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Abstract

In this paper we construct two new families of invariant maps that separate the orbits of the action of a finite Abelian group on a finite dimensional complex vector space. One of these families is Lipschitz continuous with respect to the quotient metric on the space of orbits, but involves computing large powers of the components of the vectors which can lead to instabilities. The other family avoids this issue by putting the powers only on the phase of the components, but in turn is not continuous. However, we show that they are Lipschitz continuous on the set of vectors with fixed support, so in particular they are Lipschitz on the set of vectors with no zero entries. Furthermore, the target dimension of these maps is small, i.e., linear in the original dimension.

Keywords Invariant theory · Stable separation · Separating set

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1 Introduction

Suppose a group G acts linearly on a vectors space V, i.e., there is a group homomorphism $\sigma : G \to \mathrm{Gl}(V)$ and for each $x \in V$ and $g \in G$ we have that $g \cdot x = \sigma(g)x$. A function $f : V \to X$ is *invariant* to this action if $f(g \cdot x) = f(x)$ for every x and g, and it is said to be *separating* if it has the property that $f(x) = f(y)$ if and only if $x = g \cdot y$ for some $g \in G$. We denote the *orbit* of a vector x by $[x] = \{g \cdot x : g \in G\}$, and we let V/G denote the set of orbits of this action (this is a slight abuse of notation since this set depends on the specific action of G). If f is invariant then it induces a map $\tilde{f} : V/G \to X$ given by $\tilde{f}([x]) = f(x)$, and it is easy to see that f is separating if and only if $\tilde{f}$ is injective.

Signal representations that are invariant to group actions such as translation and rotation are of immense importance in a variety of classification problems. For example, high resolution radar (HRR) range profiles are often used to classify aircraft however the HRR profile is translated depending on the distance between the radar and the aircraft (see [25]), so in order to effectively use the profiles for the desired classification task the HRR profile needs to be transformed in some way to account for these translations. Another similar problem is galaxy morphology classification where the goal is to assign a morphological category to a galaxy based on images obtained from telescopes. This process has traditionally been carried out by visual inspection by trained experts, however this method cannot scale with the amount of data that is rapidly becoming available. In order to automate this process the authors of [13] construct a representation that incorporates rotation and translation invariance to achieve state of the art results. These are just a couple of specific examples where invariant representations play a critical role, but there are many other such problems. For a more comprehensive overview we refer to the surveys [24] and [18].

The work of Mallat [19] on the construction of a wavelet-based *group invariant scattering transform* has been met with great enthusiasm and has inspired several other works [7, 8, 23]. The original scattering transform in [19] and subsequent modifications in [9, 17] give a family of transforms, one important example of which is a Lipschitz translation invariant function of signals in $L^2(\mathbb{R}^N)$ that unlike the modulus of the Fourier transform, is stable with respect to small diffeomorphisms. The examples in [19], constructed in infinite dimensional settings, are invariant and stable but, to our knowledge, the scattering transforms are not known to be separating and thus these tools do not guarantee perfect classification of signals. On the other hand, for real life applications it is crucial to understand the finite dimensional setting and to have a transform with complete discriminative power. To understand the passage to the limit as the dimension goes to infinity, it is very desirable to have explicit dimension-dependent constants; this limit is known to be problematic for phase retrieval [10].

In [11] we studied this problem, and drawing on algebraic tools, were able to develop a framework that allows one to obtain discriminative invariants with respect to finite group actions under certain hypotheses. Furthermore, our transforms in [11] come with explicit Lipschitz bounds and the dimension of the target space is linear with respect to the dimension of the signal space $\mathbb{C}^N$. As an application we obtained Lipschitz injective translation invariant maps in finite dimensional problems. Our construction uses polynomial invariants [15] to generate a map F into a high dimensional space

$\mathbb{C}^{N(N+1)/2}$ that separates orbits. We reduce the dimension of the target by using a linear transformation without losing injectivity. The last step is to modify this map to make it Lipschitz while keeping injectivity and the dimension of the target. Applications to specific group actions depend on the polynomial invariants used; in particular, on whether or not they satisfy what we called the *non-parallel property*. We prove that some specific choice of monomials in cyclic cases $\mathbb{Z}_m$ satisfy this property and thus yield the desired transform in these situations. The case of general group actions was left open.

In this work we extend our previous results to include all finite Abelian group actions. The main motivation for our present work however, comes from the need to address a much more important problem: the final map in [11] has the form

$$\|x\|\, F\left(\frac{x}{\|x\|}\right).$$

The normalization makes the map above Lipschitz which is therefore stable from a theoretical point of view. However, when implementing this, one still has to compute powers of the components of x at different points and store this information to proceed. The normalization does not avoid this, which is disastrous: two vectors x and y with small entries can be both mapped to 0 while lying on very different orbits, due to rounding errors. Thus, we are led to consider a potential transform that is not purely algebraic. This is a challenge because almost all steps in our construction in [11] relied on the algebraic structure of F. The dimension reduction in [11] relied on an algebraic geometric argument, counting dimensions of intersections of projective varieties.

Here we construct low-dimensional invariant maps for general finite Abelian group actions where we only need to compute powers of phases of entries (whence the map is not algebraic, although the construction relies on our earlier one). Our new maps thus solve the main computational problem at hand. We see that although not globally continuous, the maps are Lipschitz in a generic sense that we specify below (see Theorem 2.2 for more details). In the construction of our transforms, we introduce a family of complete sets of measurements. The new measurements are no longer complex polynomials; this is a step away from purely algebraic methods and their limitations. In our setting we are given a finite Abelian group G acting on $\mathbb{C}^N$ *unitarily* (see (2.1)). Our main result, Theorem 2.2, gives the existence of a map $\Phi : \mathbb{C}^N \to \mathbb{C}^{3N+1}$ which separates G-orbits. The components of the transform Φ grow linearly on the moduli of the entries of $x \in \mathbb{C}^N$, and there is a universal constant C such that if the signals $x, y \in \mathbb{C}^N$ have the same support, then

$$\|\Phi(x) - \Phi(y)\| \leq C \inf_{g\in G} \|x - gy\|.$$

The above is a generic form of stability. A few comments are in order:

(1) The transform in [11] is a good abstract low dimensional discriminative map in the case of the action of finite cyclic groups which is the natural finite dimensional analogue of translation. The hypothesis needed, namely the non-parallel property as we called it, could only be verified for cyclic groups because we used a particular

set of monomials introduced in [15]. In this work, we extend our result in [11] to cover all finite Abelian groups; the non-parallel property now follows from a more general family of monomials that enjoy a very particular structure; these can be obtained from a result in [14]. We illustrate the construction of the set of monomials with the desired structure and derive the non-parallel property in the general case of finite Abelian groups.

(2) Compared to our previous work [11], we go further in that we are concerned with constructing a new kind of map, much more stable from the point of view of computation and much more suitable for implementation.

(3) The reason why a global Lipschitz condition is not possible in our construction is that $\min_{1\le i\le N, |x_i|\neq 0} |x_i|$ is not continuous and this is a factor we need to use as a coefficient at some point to make the map Lipschitz away from signals with zero entries. This is unavoidable in a sense because of the use of phase maps, which are not continuous. We need to add factors vanishing at the origin to tame these discontinuities.

(4) Our transform lives in $\mathbb{C}^{3N+1}$, while in [11] the corresponding map lived in $\mathbb{C}^{2N+1}$. The loss of N in the dimension of the target space comes from the fact that in order to overcome the problem of having to compute high powers of entries, we need to isolate the information coming exclusively from the phases, but to truly separate signals in different G-orbits, we need to store the moduli of the entries which adds the extra N dimensions. This is not a significant loss as the dimension of the target is still linear in the dimension of the space of signals N.

(5) The moduli of entries we need to include in our map correspond, in the contexts of audio and image processing, to the modulus of the Fourier transform - the well-known translation invariant already mentioned.

Since the time this paper was originally written there has been a flurry of activity around the ideas of stable (bilipschitz) separation of orbits of group actions and applications of invariant theory in signal processing and machine learning. For example, a bilipschitz map in the case of certain symmetric group actions is introduced in [1] and similar ideas are further explored in [2, 3]. Another type of bilipschitz map that works for the action of any finite subgroup of the orthogonal group is introduced in [12] and expanded upon in [20, 21]. See [4–6, 16, 22] for some other related ideas.

This paper is organized as follows: Sect. 2 is devoted to introducing some notation and presenting our main results. Section 3 provides some background on separating monomials (see [14]) which will be used in our construction. We also derive some additional properties that will be useful. In Sect. 4 we construct the new transform Φ and prove that it has the desired properties. We also show that with our explicit map, the results in [11] also apply to this case, and we specialize our results (with slightly more explicit constants) to a class of examples in image processing.

2 Preliminaries and Statement of Main Results

Let the finite Abelian group G act on $\mathbb{C}^N$. The actions we consider are *unitary*, that is there is a set of unitary matrices

$$U_1, U_2, \ldots, U_L$$

such that for any $g \in G$, there are exponents $\alpha_1, \alpha_2, \ldots \alpha_k$ such that for any $x \in \mathbb{C}^N$, we have

$$g \cdot x = {U_1}^{\alpha_1} {U_2}^{\alpha_2} \cdots {U_L}^{\alpha_L} x. \tag{2.1}$$

Since G is finite it follows that each U_i is diagonalizable (the minimal polynomial can have only simple roots), and since G is Abelian it follows that all of the U_i's are simultaneously diagonalizable. *Throughout this paper we will assume that we are working in the basis that diagonalizes these matrices*. This means that without loss of generality we can assume that $U_i = \operatorname{diag}(\omega_{i,k})_{k=1}^N$ and each $\omega_{i,k}$ is an m_i-th root of unity where m_i is the smallest positive integer for which $U_i^{m_i} = I$.

We recall the quotient metric on $\mathbb{C}^N/G$,

$$d_G([x],[y]) = \min_{g \in G} \|x - gy\|,$$

where $[x]$ denotes the orbit of x under the action of G. In [11], we studied the problem of constructing Lipschitz low-dimensional representations of G-orbits in $\mathbb{C}^N$ for the case when $G = \mathbb{Z}_m$ is a cyclic group. In particular, we proved the following:

Theorem 2.1 ([11]) *There is a $\mathbb{Z}_m$-invariant map $\Phi : \mathbb{C}^N \to \mathbb{C}^{2N+1}$ that is separating and a constant $C > 0$ depending only on m such that*

$$\|\Phi(x) - \Phi(y)\| \le C d_{\mathbb{Z}_m}([x],[y]),$$

for every $x, y \in \mathbb{C}^N$.

The map constructed in [11] is based on a collection of separating monomials from [15]. More specifically, given a unitary action of $\mathbb{Z}_m$ on $\mathbb{C}^N$ there is a set of monomials of the form

$$F(x) = ((x_i^{m_i})_{i=1}^N, (x_i^{a_{ij}} x_j^{b_{ij}})_{i \neq j})$$

that separate the orbits of this action. Some of these powers can become very large and this induces numerical problems in computations.

One way to overcome this problem is to only put the powers on the phases, that is we can use the measurements

$$\Theta_F(x) = \left(\left(\left(\frac{x_i}{|x_i|} \right)^{m_i} \right)_{i=1}^N, \left(\left(\frac{x_i}{|x_i|} \right)^{a_{ij}} \left(\frac{x_j}{|x_j|} \right)^{b_{ij}} \right)_{i \neq j} \right)$$

where if either $x_i = 0$ or $x_j = 0$ we set the corresponding entries in Θ_F to 0. While this seems to solve the problem of computing large powers of the entries of x this new map Θ_F poses some new problems. First of all, it is not continuous and therefore has no hope of being Lipschitz. While we will not be able to completely overcome this problem, we will construct a map that is Lipschitz almost everywhere. Putting the powers only on the phases poses another challenge: we lose the algebraic structure of F and therefore cannot immediately apply the techniques in Theorem 3.1 of [11] to reduce the dimension of the target space. We will address this problem in this paper and construct a low dimensional G-invariant representation of signals, though we have to add N more measurements to achieve this.

In this paper we prove the following:

Theorem 2.2 *Let G be a finite Abelian group acting on $\mathbb{C}^N$ according to (2.1). Then there is a map $\Phi : \mathbb{C}^N \to \mathbb{C}^{3N+1}$ which separates G-orbits. Moreover, each component of Φ can be chosen to grow linearly on the moduli of the entries of $x \in \mathbb{C}^N$, and there is a constant C such that if $x, y \in \mathbb{C}^N$ are such that $\{k : x_k \neq 0\} = \{k : y_k \neq 0\}$, then*

$$\|\Phi(x) - \Phi(y)\| \leq C d_G([x], [y]).$$

The maps defined in this theorem do not have large powers on the moduli of the components of x, but unfortunately they are not Lipschitz. However, in this paper we find a set of polynomial measurements that allows us to extend Theorem 2.1, which yields a low-dimensional Lipschitz G-invariant map which on the other hand does require the computation of high powers of entries.

To present this result, let $F : \mathbb{C}^N \mapsto \mathbb{C}^M$, be a given map and define $\Phi_F : \mathbb{C}^N \mapsto \mathbb{C}^M$ by

$$\Phi_F(x) := \begin{cases} \|x\| F\left(\frac{x}{\|x\|}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases} \tag{2.2}$$

Theorem 2.3 *Let G be a finite Abelian group acting on $\mathbb{C}^N$. Then there is a polynomial map $F_G : \mathbb{C}^N \to \mathbb{C}^{2N+1}$ such that the map Φ_{F_G} separates G-orbits. Moreover, this map is Lipschitz with respect to the quotient metric, i.e., there is a constant C such that*

$$\|\Phi_{F_G}(x) - \Phi_{F_G}(y)\| \leq C d_G([x], [y])$$

for every $x, y \in \mathbb{C}^N$.

3 Separating Monomials

Our construction of the maps given in Therorems 2.2 and 2.3 rely on a set of separating monomials given in [14]. In this section we will briefly explain this set of monomials and derive some additional properties.

Suppose the Abelian group G acts on $\mathbb{C}^N$ via 2.1. Suppose further that $\{U_i\}_{i=1}^L$ is a minimal generating set for $\sigma(G)$. For a subset $J \subseteq \{1, \ldots, N\}$ and a vector

$v_J = (v_j)_{j \in J}$ of positive integers, we define the monomial $x^{v_J} = \prod_{j \in J} x_j^{v_j}$ where $x = (x_1, ..., x_N)$. It is easy to see that this monomial is invariant for the action of G if and only if $\prod_{j \in J} \omega_{i,j}^{v_j} = 1$ for every $i = 1, ..., L$. If we write $\omega_{i,k} = e^{2\pi i a_{i,k}/m_i}$ then this is equivalent to

$$\sum_{j \in J} a_{i,j} v_j = 0 \mod m_i \text{ for every } i = 1, ..., L.$$

We define the group G_J to be the subgroup of the free abelian group $\mathbb{Z}^J$ satisfying this system of equations. To be more precise, if we write $\hat{G} = \oplus_{i=1}^L \mathbb{Z}_{m_i}$ and define the group homomorphism $\varphi_J : \mathbb{Z}^J \to \hat{G}$ by

$$\varphi_J(v) = \left(\sum_{j \in J} a_{i,j} v_j \right)_{i=1}^L$$

(where $v = (v_j)_{j \in J} \in \mathbb{Z}^J$), then

$$G_J = \ker(\varphi_J).$$

Theorem 2.1 in [14] can now be stated as follows:

Theorem 3.1 ([14]) *Suppose that the minimal number of generators of the finite abelian group G is L and that G acts on $\mathbb{C}^N$ and the groups G_J are defined as above. A set of monomials $\{x^{v_i}\}_{i=1}^M$ is a separating set for this action if and only if G_J is generated by the set $\{v_i : \mathrm{supp}(v_i) \subseteq J\}$ for every $J \subseteq \{1, ..., N\}$ with $|J| \le L + 1$.*

Although this theorem guarantees the existence of a separating set of monomials, it does not tell us how many monomials are required. We will address this issue in the following proposition.

Proposition 3.2 *The separating set in Theorem 3.1 can be constructed to contain at most one monomial for each $J \subseteq \{1, ..., N\}$ with $|J| \le L + 1$.*

Proof If $|J| = 1$, say $J = \{j\}$ then let $v_J = v_{\{j\}} = (0, ..., n_j, ..., 0)$ (where the j-th entry is n_j and all other entries are zero) where n_j is the smallest positive integer such that $\omega_{j,k}^{n_j} = 1$ for every $k = 1, ..., L$. Then $x^{v_{\{j\}}} = x_j^{n_j}$ and clearly $G_{\{j\}}$ is generated by $v_{\{j\}}$.

By way of induction suppose that we have constructed monomials $\{x^{v_L} : L \subseteq \{1, ..., N\},\ |L| \le p - 1\}$ such that $\{v_{L'} : L' \subseteq L\}$ generates G_L for every $L \subseteq \{1, ..., N\}$ with $|L| \le p - 1$ where we have exactly one vector v_L with $\mathrm{supp}(v_L) = L$ for each such L. Choose $J = \{j_1 < \cdots < j_p\} \subseteq \{1, ..., N\}$ with $|J| = p$ and a vector v_J in G_J satisfying the following:

1. $\mathrm{supp}(v_J) = J$ and all entries of v_J indexed by J are positive; and
2. the first nonzero entry $v_J(j_1)$ is minimal among all vectors $v \in G_J$ with $\mathrm{supp}(v) = J$ and $v(j_1) > 0$.

Note that vectors satisfying (1) must exist because by the base case we have the vector whose jth entry is n_j for $j \in J$ and 0 for $j \notin J$. Therefore we take v_J to be any vector that minimizes $v(j_1)$ among the vectors satisfying (1).

We now claim that $\{v_J\} \cup \{v_{J'} : J' \subsetneq J\}$ generates G_J. To see this let $v \in G_J$. If $\mathrm{supp}(v) \neq J$ then $v \in G_{J'}$ for some $J' \subsetneq J$ and we are done by our inductive hypothesis, so assume $\mathrm{supp}(v) = J$. By the same argument as above we can assume that all entries of v indexed by J are positive. By our choice of v_J the first nonzero entry of v must satisfy $v(j_1) \geq v_J(j_1)$. Write $v(j_1) = qv_J(j_1) + r$ and let $v' = v - qv_J$. Then the j_1-st entry of v' is r and by construction we have that $0 \leq r < v_J(j_1)$. Therefore if $r > 0$ and $\mathrm{supp}(v') = J$ then this would contradict our assumption on the minimality of $v_J(j_1)$. This means that $\mathrm{supp}(v') = J'$ for some proper subset $J' \subsetneq J$, and therefore $v' \in G_{J'}$. Since $v = v' + qv_J$ this completes the proof by our inductive hypothesis. □

Note that it is possible that the v_J we constructed in the above proof could already be expressed as a linear combination of $\{v_{J'} : J' \subsetneq J\}$. When this happens we do not really need that v_J. Also observe that by construction $v_J(j) \leq n_j$ for every $J \subseteq \{1, ..., N\}$ and ever $j \in J$. This means that the largest power in any of the monomials in this separating set is no bigger than

$$\max\{n_j\}_{j=1}^N = \max\{|g| : g \in G\} = \mathrm{lcm}\{m_i\}_{i=1}^L.$$

We state the following corollary for later reference.

Corollary 3.3 *If the minimal number of generators of the finite abelian group G is L and G acts on $\mathbb{C}^N$ then there exists a separating set of M monomials where*

$$M \leq \sum_{k=1}^{L+1} \binom{N}{k}.$$

Furthermore, the individual powers in these monomials are bounded by $\mathrm{lcm}\{m_i\}_{i=1}^L$.

4 A New *G*-Invariant, Lipschitz Almost Everywhere, Transform

We can now present the construction of the new transform. To this end, we recall the following dimension reduction result from [11].

Theorem 4.1 (Theorem 3.1 [11]) *Let G act on $\mathbb{C}^N$ and suppose $F : \mathbb{C}^N \to \mathbb{C}^M$ is a polynomial G-invariant map that separates the orbits of this action. Then for $k \geq 2N + 1$, $\ell \circ F$ is also separating for a generic linear map $\ell : \mathbb{C}^M \to \mathbb{C}^k$.*

For any $z \in \mathbb{C}$, define

$$s(z) = \begin{cases} \frac{z}{|z|} & z \neq 0, \\ 0 & z = 0, \end{cases}$$

and for $x = (x_1, ..., x_N) \in \mathbb{C}^N$ define

$$S(x) = (s(x_1), ..., s(x_N)).$$

We now define the following map.

Definition 4.2 Suppose the finite Abelian group G acts unitarily on $\mathbb{C}^N$, and let $F : \mathbb{C}^N \to \mathbb{C}^M$ be a map that evaluates a collection of separating monomials (such as the one described in the previous section. Let $\ell : \mathbb{C}^N \to \mathbb{C}^k$ be a linear map satisfying the conclusion of Theorem 4.1 with respect to this F. We define $\Phi_{\ell,F}(x)$ as:

$$\Phi_{\ell,F}(x) = \begin{cases} (|x_1|, ..., |x_N|, \min_{i \in \mathrm{supp}(x)} |x_i| \ell(F(S(x)))) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Proposition 4.3 *Let G, F, and $\Phi_{\ell,F}$ be defined as in Definition 4.2. If the linear map ℓ satisfies the conclusion of Theorem 4.1 with respect to F, then $\Phi_{\ell,F}$ is separating.*

Proof We need to show that $\Phi_{\ell,F}(x) = \Phi_{\ell,F}(y)$ if and only if $x = \sigma(g)y$ for some $g \in G$. Recall that by our assumptions we can assume that $\sigma(g)$ is diagonal with roots of unity on the diagonal, so let $\sigma(g) = \mathrm{diag}(\omega_{g,i}))_{i=1}^N$.

First suppose that $x = \sigma(g)y$. Then

$$(|x_1|s(x_1), ..., |x_N|s(x_N)) = x = \sigma(g)y = (|y_1|\omega_{g,1}s(y_1), ..., |y_N|\omega_{g,N}s(y_N))$$

from which it follows that $|x_i| = |y_i|$ for every $1 \leq i \leq N$, and that $S(x) = \sigma(g)S(y)$. Since F is invariant to the action of G it follows that $F(S(x)) = F(S(y))$, and so $\Phi_{\ell,F}(x) = \Phi_{\ell,F}(y)$.

Conversely, suppose that $\Phi_{\ell,F}(x) = \Phi_{\ell,F}(y)$. Then $|x_i| = |y_i|$ for every $1 \leq i \leq N$ which implies that $\min_{i \in \mathrm{supp}(x)} |x_i| = \min_{i \in \mathrm{supp}(y)} |y_i|$ and therefore $\ell(F(S(x)) = \ell(F(S(y))$. Since ℓ satisfies the conclusion of Theorem 4.1 and F is separating we have that $S(x) = \sigma(g)S(y)$ for some $g \in G$. It then follows that $x = \sigma(g)y$. □

Proposition 4.4 *Let G, F, ℓ, and $\Phi_{\ell,F}$ be as in Proposition 4.3 with $\mathrm{rank}(\ell) = 2N+1$. Let $\tilde{\Phi}_{\ell,F} : \mathbb{C}^N/G \to \mathbb{C}^{3N+1}$ be the induced map on the quotient space. If $x, y \in \mathbb{C}^N$ are such that supp(x) = supp(y) then*

$$\left\| \tilde{\Phi}_{\ell,F}(x) - \tilde{\Phi}_{\ell,F}(y) \right\| \leq (2 \|\ell\| C + 1) d_G([x], [y]), \tag{4.1}$$

where

$$C = \max \left\{ \left(\sum_{i=1}^{M} \sup_{z \in \mathbb{S}^1 \times \cdots \times \mathbb{S}^1} \|\nabla F_i(z)\|^2 \right)^{\frac{1}{2}}, \sup_{z \in \mathbb{S}^1 \times \cdots \times \mathbb{S}^1} \|F(z)\| \right\}.$$

Proof We will present the proof in the case that $\mathrm{supp}(x) = \mathrm{supp}(y) = \{1, ..., N\}$. The case of any other support set follows from a similar argument. In this case we have that $x_i, y_i \neq 0$ and so

$$\min_{i \in \mathrm{supp}(x)} |x_i| = \min_{1 \leq i \leq N} |x_i| \text{ and } \min_{i \in \mathrm{supp}(y)} |y_i| = \min_{1 \leq i \leq N} |y_i|.$$

We then have that

$$\begin{aligned} \left\| \Phi_{\ell,F}(x) - \Phi_{\ell,F}(y) \right\| &\leq \|(|x_1| - |y_1|, \dots, |x_N| - |y_N|)\| \\ &\quad + \left\| \left(\min_{1 \leq i \leq N} |x_i| \right) \ell(F(S(x))) - \left(\min_{1 \leq i \leq N} |y_i| \right) \ell(F(S(y))) \right\| \\ &= I + II. \end{aligned} \tag{4.2}$$

Suppose without loss of generality that

$$\min_{1 \leq i \leq N} |x_i| \leq \min_{1 \leq i \leq N} |y_i|.$$

Then note that

$$I \leq \|x - y\|, \tag{4.3}$$

and that

$$\begin{aligned} II &\leq \left\| \left(\min_{1 \leq i \leq N} |x_i| \right) \ell(F(S(x))) - \left(\min_{1 \leq i \leq N} |x_i| \right) \ell(F(S(y))) \right\| \\ &\quad + \left\| \left(\min_{1 \leq i \leq N} |x_i| \right) \ell(F(S(y))) - \left(\min_{1 \leq i \leq N} |y_i| \right) \ell(F(S(y))) \right\| \\ &= \left(\min_{1 \leq i \leq N} |x_i| \right) \|\ell(F(S(x))) - \ell(F(S(y)))\| \\ &\quad + \|\ell(F(S(y)))\| \left| \left(\min_{1 \leq i \leq N} |x_i| \right) - \left(\min_{1 \leq i \leq N} |y_i| \right) \right| \\ &\leq \left(\min_{1 \leq i \leq N} |x_i| \right) \|\ell\| \left(\sum_{i=1}^{M} |F_i(S(x)) - F_i(S(y))|^2 \right)^{\frac{1}{2}} \\ &\quad + \|\ell(F(S(y)))\| \left| \left(\min_{1 \leq i \leq N} |x_i| \right) - \left(\min_{1 \leq i \leq N} |y_i| \right) \right| \\ &\leq \left(\min_{1 \leq i \leq N} |x_i| \right) \|\ell\| \left(\sum_{i=1}^{M} \sup_{z \in \mathbb{S}^1 \times \dots \times \mathbb{S}^1} \|\nabla F_i(z)\|^2 \|S(x) - S(y)\|^2 \right)^{\frac{1}{2}} \\ &\quad + \sup_{z \in \mathbb{S}^1 \times \dots \times \mathbb{S}^1} \|\ell(F(z))\| \left| \left(\min_{1 \leq i \leq N} |x_i| \right) - \left(\min_{1 \leq i \leq N} |y_i| \right) \right| \end{aligned}$$

$$
\begin{aligned}
&= \left(\min_{1\le i\le N}|x_i|\right)\|\ell\|\,\|S(x)-S(y)\|\left(\sum_{i=1}^{M}\sup_{z\in\mathbb{S}^1\times\cdots\times\mathbb{S}^1}\|\nabla F_i(z)\|^2\right)^{\frac{1}{2}}\\
&\quad+\sup_{z\in\mathbb{S}^1\times\cdots\times\mathbb{S}^1}\|\ell(F(z))\|\left|\left(\min_{1\le i\le N}|x_i|\right)-\left(\min_{1\le i\le N}|y_i|\right)\right|
\end{aligned}
$$

where the last inequality follows from the Mean Value Inequality and the fact that $\|\nabla F_i(z)\|$ is maximized on $\mathbb{S}^1\times\cdots\times\mathbb{S}^1$.

By the elementary inequality

$$
\min\{|x|,|y|\}\left|\frac{x}{|x|}-\frac{y}{|y|}\right|\le|x-y|,
$$

we obtain

$$
\begin{aligned}
\left(\min_{1\le i\le N}|x_i|\right)\|S(x)-S(y)\| &= \left(\min_{1\le i\le N}|x_i|\right)\left(\sum_{i=1}^{N}\left|\frac{x_i}{|x_i|}-\frac{y_i}{|y_i|}\right|^2\right)^{\frac{1}{2}}\\
&\le\left(\sum_{i=1}^{N}\left(\min_{1\le i\le N}|x_i|\right)^2\left(\frac{|x_i-y_i|}{\min\{|x_i|,|y_i|\}}\right)^2\right)^{\frac{1}{2}}\\
&\le\left(\sum_{i=1}^{N}|x_i-y_i|^2\right)^{\frac{1}{2}}=\|x-y\|
\end{aligned}
$$

Also, if

$$
\left(\min_{1\le i\le N}|x_i|\right)=|x_{j_0}|,\ \left(\min_{1\le i\le N}|y_i|\right)=|y_{k_0}|,
$$

then

$$
\begin{aligned}
\left|\left(\min_{1\le i\le N}|x_i|\right)-\left(\min_{1\le i\le N}|y_i|\right)\right| &= |y_{j_0}|-|x_{k_0}|\le|y_{k_0}|-|x_{k_0}|\\
&\le|y_{k_0}-x_{k_0}|\le\|x-y\|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
II &\le\|x-y\|\,\|\ell\|\left(\sum_{i=1}^{M}\sup_{z\in\mathbb{S}^1\times\cdots\times\mathbb{S}^1}\|\nabla F_i(z)\|^2\right)^{\frac{1}{2}}+\sup_{z\in\mathbb{S}^1\times\cdots\times\mathbb{S}^1}\|\ell(F(z))\|\,\|x-y\|\\
&\le\|x-y\|\,\|\ell\|\left(\sum_{i=1}^{M}\sup_{z\in\mathbb{S}^1\times\cdots\times\mathbb{S}^1}\|\nabla F_i(z)\|^2\right)^{\frac{1}{2}}+\|\ell\|\sup_{z\in\mathbb{S}^1\times\cdots\times\mathbb{S}^1}\|F(z)\|\,\|x-y\|\\
&\le 2\,\|\ell\|\,C\,\|x-y\|\,.
\end{aligned}
$$

Therefore we have

$$\left\|\Phi_{\ell,F}(x) - \Phi_{\ell,F}(y)\right\| \leq I + II \leq (2\,\|\ell\|\,C + 1)\|x - y\|.$$

Since this is true for any x and y we can replace x by gx for any $g \in G$, so by taking the minimum over $g \in G$ and the fact that Φ is invariant, we conclude

$$\left\|\tilde{\Phi}_{\ell,F}(x) - \tilde{\Phi}_{\ell,F}(y)\right\| \leq (2\,\|\ell\|\,C + 1)d_G([x], [y]).$$

□

Theorem 2.2 now follows immediately from Proposition 4.3, Proposition 4.4, and Corollary 3.3.

Proof of Theorem 2.2 Let G, F, ℓ, and $\Phi_{\ell,F}$ be as in Proposition 4.4. Let L be the minimal number of generators of G and let M be the number of monomials as in Corollary 3.3. By Proposition 4.3 we know that $\tilde{\Phi}_{\ell,F}$ is separating, so by Proposition 4.4 it suffices to bound

$$\left(\sum_{i=1}^{M} \sup_{z\in\mathbb{S}^1\times\cdots\times\mathbb{S}^1} \|\nabla F_i(z)\|^2\right)^{\frac{1}{2}}, \sup_{z\in\mathbb{S}^1\times\cdots\times\mathbb{S}^1} \|F(z)\|\,,$$

where F_i denotes the ith component of F, so $F_i(z) = z^{v_J}$ for some subset $J \subseteq \{1, \ldots, N\}$ with $|J| \leq L + 1$.

To estimate the first term note that by Corollary 3.3

$$|\frac{\partial}{\partial z_j} F_i(z)| \leq \mathrm{lcm}\{m_k\}_{k=1}^{L}$$

for $z \in \mathbb{S}^1 \times \cdots \mathbb{S}^1$ and every $j = 1, \ldots, N$. Since F_i depends on at most $L+1$ variables it follows that ∇F_i has at most $L + 1$ nonzero components. This means that

$$\|\nabla F_i\| \leq \sqrt{L+1}\mathrm{lcm}\{m_k\}_{k=1}^{L}$$

and therefore

$$\left(\sum_{i=1}^{M} \sup_{z\in\mathbb{S}^1\times\cdots\times\mathbb{S}^1} \|\nabla F_i(z)\|^2\right)^{\frac{1}{2}} \leq \sqrt{M(L+1)}\mathrm{lcm}\{m_k\}_{k=1}^{L}.$$

To estimate the second term, if $z \in \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, then each component of $F(z)$ has modulus 1, therefore

$$\|F(z)\| \leq \sqrt{M}.$$

□

In our previous construction for the cyclic case in [11] we used the non-parallel property to construct a Lipschitz map starting from a collection of monomials. In order to prove Theorem 2.3 we will need to show that the collection of monomials given in Theorem 3.1 also satisfies this property. We briefly recall the definition of this property.

Definition 4.5 Suppose G acts on $\mathbb{C}^N$ and $F : \mathbb{C}^N \to \mathbb{C}^M$ is G-invariant. We say F has the non-parallel property if the following holds: If $\|x\| = \|y\| = 1$ and $F(x) = \lambda F(y)$ for some $\lambda > 0$, then $x = gy$ for some $g \in G$.

Proposition 4.6 *Suppose the finite Abelian group G acts on $\mathbb{C}^N$ and let $F_G : \mathbb{C}^N \to \mathbb{C}^M$ be given by $f_G(x) = (x^{v_i})_{i=1}^M$ where $\{x_{v_i}\}_{i=1}^M$ is the collection of monomials given in Theorem 3.1. Then F_G satisfies the non-parallel property.*

Proof Suppose that $\|x\| = \|y\| = 1$, $\lambda > 0$, and that $F_G(x) = \lambda F_G(y)$. Recall that each monomial is of the form x^{v_J} where $v_J = (v_j)_{j\in J}$ for some subset $J = \{j_1, ..., j_k\} \subseteq \{1, ..., N\}$. If $J = \{j\}$ then we have $x_j^{m_j} = \lambda y_j^{m_j}$ where $v_{\{j\}} = (m_j)$. It follows that

$$\left|\frac{x_j}{y_j}\right| = \lambda^{1/m_j} \tag{4.4}$$

for every $j = 1, ..., N$. Furthermore, if $|J| > 1$ then we have

$$\prod_{i=1}^{k} x_{j_i}^{v_J(j_i)} = \lambda \prod_{i=1}^{k} y_{j_i}^{v_J(j_i)}. \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\lambda = \prod_{i=1}^{k} \lambda^{v_{j_i}/m_{j_i}}. \tag{4.6}$$

Now define $\tilde{y} = (\lambda^{1/m_j} y_j)_{j=1}^N$. Then (4.6) implies that $F_G(\tilde{y}) = \lambda F_G(y) = F_G(x)$, and since we already know F_G separates orbits, it follows that $x = g\tilde{y}$ for some $g \in G$. But we also know

$$\sum_{1\le j\le N} |y_j|^2 = \|y\|^2 = \|x\|^2 = \|\tilde{y}\|^2 = \sum_{1\le j\le N} \lambda^{2/m_j} |y_j|^2.$$

The last expression is increasing in λ, and so $\lambda = 1$, which implies $\tilde{y} = y$ and therefore $x = gy$ for some $g \in G$. □

Using Proposition 4.6 we can deduce Theorem 2.3 as in [11].

We conclude by applying our previous results to a specific action of $G = \mathbb{Z}_n \times \mathbb{Z}_m$ arising in image processing. Let $\mathbb{Z}_n \times \mathbb{Z}_m$ act on $\mathbb{C}^n \otimes \mathbb{C}^m \simeq \mathbb{C}^{nm}$ via translation. To be more precise, if we think of a vector $A \in \mathbb{C}^{nm}$ as an $n \times m$ matrix $A(i, j)$ then the action is given by linear operators of the form $T_{k,l}A(i,j) = A(i+k \bmod n, j+l \bmod m)$. These operators are diagonalized by the discrete Fourier transform $F_G = F_n \otimes F_m$ where F_n and F_m are the discrete Fourier Transforms on $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. In particular we have $F_G T_{1,0} F_G^* = M_n \otimes I_m$ and $F_G T_{0,1} F_G^* = I_n \otimes M_m$, where

$$M_n = \begin{pmatrix} \omega_n & 0 & \cdots & 0 \\ 0 & \omega_n^2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}, M_m = \begin{pmatrix} \omega_m & 0 & \cdots & 0 \\ 0 & \omega_m^2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix},$$

$\omega_n = e^{2\pi i/n}$, $\omega_m = e^{2\pi i/m}$, and I_n and I_m denote the $n \times n$ and $m \times m$ identity matrices.

Since these matrices are explicit, so will be the map $F_{\mathbb{Z}_n \times \mathbb{Z}_m}$ for this particular action of $\mathbb{Z}_n \times \mathbb{Z}_m$. We can then write the map in Theorem 2.2 explicitly, and we can then get explicit Lipschitz bounds.

Corollary 4.7 *Let $\mathbb{Z}_n \times \mathbb{Z}_m$ act on $\mathbb{C}^{nm}$ via (2.1). Then the map $\Phi_{\ell,F}$ defined in (4.2) has the property that the induced map $\tilde{\Phi}_{\ell,F} : \mathbb{C}^{nm}/\mathbb{Z}_n \times \mathbb{Z}_m \mapsto \mathbb{C}^{3nm+1}$ is separating. Additionally, if* $\operatorname{supp}(x) = \operatorname{supp}(y)$ *we have the Lipschitz bound*

$$\left\| \tilde{\Phi}_{\ell,F}([x]) - \tilde{\Phi}_{\ell,F}([y]) \right\| \le \left(\sqrt{3}(nm)^{\frac{5}{2}} \|\ell\| + 1 \right) d_G([x],[y]).$$

Proof In this case we have $N = mn$ and $L = 2$ so by Corollary 3.3 it follows that $M \le \frac{1}{4}(mn)^3$. Appealing to the proof of Theorem 2.2 we have

$$\|\nabla F_i\|^2 \le 3\operatorname{lcm}(m,n)^2 \le 3(nm)^2$$

for $z \in \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ and therefore

$$\left(\sum_{i=1}^{M} \sup_{z \in \mathbb{S}^1 \times \cdots \times \mathbb{S}^1} \|\nabla F_i(z)\|^2 \right)^{\frac{1}{2}} \le \left(3M(nm)^2 \right)^{\frac{1}{2}} \le \frac{\sqrt{3}}{2}(nm)^{\frac{5}{2}}.$$

For such z we also have

$$\left\| F(z) \right\| = \sqrt{M} \le \frac{1}{2}(nm)^{\frac{3}{2}}.$$

Therefore we can take $C = \frac{\sqrt{3}}{2}(nm)^{\frac{5}{2}}$ and use Proposition 4.4 to get the desired bound. $\square$

In the above proof we used the bound $\operatorname{lcm}(m,n) \le mn$, however when $\operatorname{lcm}(m,n) = mn$ we have that $G \simeq \mathbb{Z}_{mn}$ is a cyclic group, so in this case $L = 1$. This means that $M \le (mn)^2$ and the bound can be improved accordingly. In all other cases we have $\operatorname{lcm}(m,n) \le \frac{1}{2}mn$.

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