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The Turán Problem and Its Dual for Positive Definite Functions Supported on a Ball in $\mathbb{R}^d$

Jean-Pierre Gabardo[1]

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Abstract

The Turán problem for an open ball of radius r centered at the origin in $\mathbb{R}^d$ consists in computing the supremum of the integrals of positive definite functions compactly supported on that ball and taking the value 1 at the origin. Siegel proved, in the 1930s that this supremum is equal to 2^{-d} mutiplied by the Lebesgue measure of the ball and is reached by a multiple of the self-convolution of the indicator function of the ball of radius $r/2$. Several proofs of this result are known and, in this paper, we will provide a new proof of it based on the notion of "dual Turán problem", a related maximization problem involving positive definite distributions. We provide, in particular, an explicit construction of the Fourier transform of a maximizer for the dual Turán problem. This approach to the problem provides a direct link between certain aspects of the theory of frames in Fourier analysis and the Turán problem. In particular, as an intermediary step needed for our main result, we construct new families of Parseval frames, involving Bessel functions, on the interval [0, 1].

Keywords Positive definite functions and distributions · Fourier frames · Bessel functions

Mathematics Subject Classification 43A45 · 42C15

1 Introduction

Consider a *symmetric* open set $U \subset \mathbb{R}^d$, i.e. $0 \in U$ and $-x \in U$ whenever $x \in U$. The Turán problem associated with U consists in computing the supremum of the

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✉ Jean-Pierre Gabardo
gabardo@mcmaster.ca

1 Department of Mathematics & Statistics, McMaster University, 1280 Main Street West, Hamilton, ON L8S 4K1, Canada

integrals $\int_U h(x)\,dx$, where h is a continuous, positive definite function with support contained in U and satisfying $h(0) = 1$. The corresponding supremum is called the *Turán constant* of U. The name of the problem originates from a discussion between Turán and Stečkin [24] in the 1970s, but, already in the 1930s, Siegel [23] had solved the particular case of the problem where U is a ball, by showing that its Turán constant is $2^{-d}\,|U|$, where $|\cdot|$ denotes the Lebesgue measure. If, in addition to being symmetric, U is also assumed to be convex, there is a natural candidate for a maximizer for the Turán, namely the function

$$f = |V|^{-1}\,\chi_V * \tilde{\chi}_V, \tag{1}$$

where $V = \frac{1}{2}U = \{x/2,\ x \in U\}$ and where $\tilde{g}$ is defined by $\tilde{g}(x) = \overline{g(-x)}$ for any function g on $\mathbb{R}^d$. Since the Fourier transform of f is $\hat{f} = |V|^{-1}\,|\hat{\chi}_V|^2 \geq 0$, f is positive definite and satisfies $f(0) = 1$. Note that f is not compactly supported in U, but it is a limit of positive definite, compactly supported functions in U with value 1 at the origin (see Definition 1). A symmetric, convex open set U is called a *Turán domain* if its Turán constant is $2^{-d}\,|U| = |V|$ and if it is achieved by the function f defined above. As far as we know, no example of symmetric, convex open set which is not a Turán domain is known. More recently, the Turán problem has been investigated for particular domains in $\mathbb{R}^d$ [2, 3, 12, 17] and also in the setting of other l.c.a. groups [13, 15, 16, 18, 21]. We refer the reader to Révész's paper [21] for an historical perspective on the Turán problem and its extensions to various settings.

In this paper, we will be mostly interested in the problem where U is a ball. In addition to the proof given by Siegel in [23], Gorbachev [12], as well as Kolountzakis and Révész [17], provided alternate proofs for the case of the ball. Our main goal in this paper is to provide yet a different proof for this result, which involves the concept of "dual Turán problem". To define this last problem in the case of $B(0,r)$, the ball of radius r centered at 0 in $\mathbb{R}^d$, we need to consider the class of positive definite distributions on $\mathbb{R}^d$. Note that the distributional Fourier transform of a positive definite distribution S in $\mathbb{R}^d$, $\mathcal{F}(S)$, is a positive tempered measure μ by the Bochner-Schwartz theorem (see [22]). The dual Turán problem consists then in maximizing the quantity $\mathcal{D}(S) := \mu(\{0\})$, where $\mu = \mathcal{F}(S)$, over the collection of positive definite distributions on $\mathbb{R}^d$ equal to the Dirac mass δ_0 on $B(0,r)$. It turns out that a maximizer for this problem, T_r, exists and we will give an explicit formula for its Fourier transform $\mathcal{F}(T_r)$. If f_r is the function given in (1) with $V = B(0, r/2)$, we will show that the convolution equation $f_r * T_r = 1$ holds on $\mathbb{R}^d$. This last equation essentially characterizes maximizers for both the Turán problem and its dual and Siegel's result will easily follow from it once we find the explicit form of $\mathcal{F}(T_r)$.

The paper is organized as follows. In Sect. 2, we define the notion of Turán maximizer and that of dual Turán maximizer for an arbitrary bounded symmetric open subset of $\mathbb{R}^d$ and discuss some of their properties. In Sect. 3, we make a connection between the problem of constructing a dual Turán maximizer for U and the problem of constructing Parseval Fourier frames for the space $L^2(V)$ if $U = V - V$ and V is open. We specialize to the case where U is a ball centered at the origin in Sect. 4 and find a possible candidate for a dual Turán maximizer. Using spherical harmonics, we show that proving that our candidate is actually a Turán maximizer is equivalent to proving that certain collections of functions built using Bessel functions are Parseval frames for some L^2-spaces. This last result is proved in Sect. 5 and, in Sect. 6, we prove in particular, that, for $d \geq 2$, the dual Turán maximizer for the ball that we constructed cannot be a locally bounded, complex measure on $\mathbb{R}^d$.

2 The Turán Problem and Its Dual

We will start by introducing some notations and some basic definitions and facts. If $h \in L^1(\mathbb{R}^d)$, the Lebesgue space of integrable functions on $\mathbb{R}^d$, we define its *Fourier transform* by the formula

$$\mathcal{F}(h)(\xi) = \hat{h}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x}\, h(x)\, dx, \quad \xi \in \mathbb{R}^d.$$

This definition can be extended to the whole space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions with the mapping $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ being bijective (see [22]). Recall that a continuous function $f : \mathbb{R}^d \mapsto \mathbb{C}$ is *positive definite* (abbr. “p.d.”) if, for any $x_1, \ldots, x_m \in \mathbb{R}^d$ and any $\xi_1, \ldots, \xi_m \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{m} f(x_i - x_j)\, \xi_i\, \overline{\xi_j} \geq 0.$$

Bochner’s theorem states that if f is a continuous p.d. function on $\mathbb{R}^d$, then f can be represented as the integral

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi}\, \mathrm{d}\mu(\xi), \quad x \in \mathbb{R}^d,$$

for some bounded, positive Borel measure μ on $\mathbb{R}^d$ (i.e. f is the inverse Fourier transform of the measure μ). Note that this implies, in particular, that

$$|f(x)| \leq f(0), \quad x \in \mathbb{R}^d.$$

For the remaining part of this section, we will assume that U is an open symmetric subset of $\mathbb{R}^d$ and that U is bounded. We will denote by $\mathcal{A}(U)$ the collection of continuous p.d. functions on $\mathbb{R}^d$ compactly supported in U and satisfying $f(0) = 1$. The Turán problem for U consists thus in computing the number

$$\mathcal{T}_{\mathbb{R}^d}(U) := \sup_{h \in \mathcal{A}(U)} \int_{\mathbb{R}^d} h(x)\, dx,$$

which is called the *Turán constant* of U.

Definition 1 A continuous function $f : \mathbb{R}^d \to \mathbb{C}$ is called a Turán maximizer for U if f is p.d. with $f(0) = 1$, $\int_{\mathbb{R}^d} f(x)\,dx = \mathcal{T}_{\mathbb{R}^d}(U)$ and there exists a sequence (f_n) of functions in $\mathcal{A}(U)$ such that $f_n \to f$ in $\mathcal{S}'(\mathbb{R}^d)$.

Note that the sequence (f_n) in the previous definition satisfies

$$\int_{\mathbb{R}^d} f_n(x)\,dx \to \int_{\mathbb{R}^d} f(x)\,dx = \mathcal{T}_{\mathbb{R}^d}(U),$$

by applying the convergence of the sequence (f_n) in $\mathcal{S}'(\mathbb{R}^d)$ to a test function identically equal to 1 on a neighborhood of U.

If $V \subset \mathbb{R}^d$ is open, let $C_0^\infty(V)$ denote the space of complex valued, infinitely differentiable functions defined on $\mathbb{R}^d$ and compactly supported in V. Recall that a distribution S on $\mathbb{R}^d$ is called positive definite if, for every $\varphi \in C_0^\infty(\mathbb{R}^d)$, we have the inequality

$$\langle S, \varphi * \tilde{\varphi} \rangle \geq 0.$$

A positive Borel measure on $\mathbb{R}^d$ is called *tempered* if it defines a tempered distribution. This will be the case if and only if

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^m}\,\mathrm{d}\mu(\xi) < \infty,$$

for some integer $m \geq 0$. By the Bochner-Schwartz theorem, a distribution on $\mathbb{R}^d$ is positive definite if and only if it is tempered and its distributional Fourier transform is a positive tempered measure. (See [8, 20, 22] for more details.)

If S is a positive definite distribution on $\mathbb{R}^d$, we define the *density* of S to be the number

$$\mathcal{D}(S) := \lim_{\epsilon \to 0^+} \left\langle S, \epsilon^{d/2} e^{-\epsilon\pi|\cdot|^2} \right\rangle.$$

Note that, if $\mathcal{F}(S) = \mu \geq 0$, we have

$$\lim_{\epsilon \to 0^+} \left\langle S, \epsilon^{d/2} e^{-\epsilon\pi|\cdot|^2} \right\rangle = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^d} e^{-\pi|\xi|^2/\epsilon}\,\mathrm{d}\mu(\xi) = \mu(\{0\}),$$

using the Lebesgue dominated convergence theorem, showing that $\mathcal{D}(S) = \mu(\{0\})$. We will denote by $\tilde{\mathcal{A}}(U)$ the collection of positive definite distributions on $\mathbb{R}^d$ which are equal to the Dirac mass at the origin, δ_0, on U. The dual Turán problem for U consists then in computing the number

$$\tilde{\mathcal{T}}_{\mathbb{R}^d}(U) := \sup_{S \in \tilde{\mathcal{A}}(U)} \mathcal{D}(S).$$

An element T of $\tilde{\mathcal{A}}(U)$ such that $\mathcal{D}(T) = \tilde{\mathcal{T}}_{\mathbb{R}^d}(U)$ is call a *dual Turán maximizer* for U. We point out that, if $h \in \mathcal{A}(U)$ and $S \in \tilde{\mathcal{A}}(U)$, the convolution product $h * S$

is well-defined as a tempered distribution since h has compact support. Furthermore, $h * S$ is positive definite since, letting $\mu = \mathcal{F}(S)$, we have

$$\mathcal{F}(h * S) = \hat{h}\,\hat{S} = \hat{h}\,\mathrm{d}\mu,$$

and $\hat{h}\,\mathrm{d}\mu$ is a positive, tempered measure. The following result shows that $h * S$ is actually a continuous positive definite function bounded by 1 and, in particular, that $\hat{h}\,\mathrm{d}\mu$ is a bounded measure.

Proposition 1 *Suppose that $U \subset \mathbb{R}^d$ is a bounded symmetric set. Let $h \in \mathcal{A}(U)$ and $S \in \tilde{\mathcal{A}}(U)$, then the convolution product $h*S$ is a continuous positive definite function satisfying*

$$|(h * S)(x)| \leq 1, \quad x \in \mathbb{R}^d.$$

Furthermore, we have the inequality

$$\left(\int_U h(x)\,dx\right)\mathcal{D}(S) \leq 1, \quad h \in \mathcal{A}(U), S \in \tilde{\mathcal{A}}(U). \tag{2}$$

Proof If $S \in \tilde{\mathcal{A}}(U)$, we can write S in the form $S = \delta_0 + R$, where R is supported in the set $\mathbb{R}^d \setminus U$. Since h is compactly supported in U, there exists $\epsilon > 0$ such that $\mathrm{supp}(h) + B(0, 2\,\epsilon) \subset U$ and, in particular, $h * R = 0$ on the ball $B(0, \epsilon)$. Thus,

$$h * S = h + h * R = h \quad \text{on } B(0, \epsilon).$$

Let $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\psi \geq 0$, supported in the ball $B(0, 1)$ and satisfying $\int_{\mathbb{R}^d} \psi(x)\,dx = 1$. Since $\hat{\psi}$ is a continuous p.d. function, we have $|\hat{\psi}(\xi)| \leq \hat{\psi}(0) = 1$ for $\xi \in \mathbb{R}^d$. Define

$$\psi_n(x) = n^d\,\psi(n\,x), \quad n \geq 1.$$

Then,

$$\int_{\mathbb{R}^d} (\psi_n * \tilde{\psi}_n)(x)\,dx = 1 \quad \text{and} \quad \mathrm{supp}\left(\psi_n * \tilde{\psi}_n\right) \subset B(0, 2/n).$$

If $N > 0$ is fixed, the Lebesgue dominated convergence theorem shows that

$$\begin{aligned}
\int_{|\xi|\leq N} \hat{h}(\xi)\,\mathrm{d}\mu(\xi) &= \int_{|\xi|\leq N} |\hat{\psi}(0)|^2\,\hat{h}(\xi)\,\mathrm{d}\mu(\xi)\\
&= \lim_{n\to\infty} \int_{|\xi|\leq N} |\hat{\psi}(\xi/n)|^2\,\hat{h}(\xi)\,\mathrm{d}\mu(\xi)\\
&\leq \lim_{n\to\infty} \int_{\mathbb{R}^d} |\hat{\psi}(\xi/n)|^2\,\hat{h}(\xi)\,\mathrm{d}\mu(\xi) = \lim_{n\to\infty} \langle h * S,\ \psi_n * \tilde{\psi}_n\rangle\\
&= \lim_{n\to\infty} \langle h,\ \psi_n * \tilde{\psi}_n\rangle = h(0) = 1,
\end{aligned}$$

using the fact that the sequence $(\psi_n * \tilde{\psi}_n)$ is an approximate identity in the last line. Letting $N \to \infty$ and applying the Lebesgue monotone convergence theorem, we obtain that

$$\int_{\mathbb{R}^d} \hat{h}(\xi)\, \mathrm{d}\mu(\xi) \le 1.$$

This implies that $h * S$ is a continuous p.d. function with $(h * S)(0) \le 1$, which, in turn, yields the inequality $|h * S| \le 1$ on $\mathbb{R}^d$. We have also

$$\mathcal{D}(h * S) = \int_{\{0\}} \hat{h}(\xi)\, \mathrm{d}\mu(\xi) = \hat{h}(0)\, \mu(\{0\}) \le \int_{\mathbb{R}^d} \hat{h}(\xi)\, \mathrm{d}\mu(\xi) \le 1,$$

yielding the inequality (2). □

Corollary 2 *Let $U \subset \mathbb{R}^d$ be a bounded symmetric set. Suppose that f is a continuous p.d. function on $\mathbb{R}^d$ satisfying $f(0) = 1$ and that there exists a sequence (h_n) in $\mathcal{A}(U)$ such that $h_n \to f$ in $\mathcal{S}'(\mathbb{R}^d)$. If there exists $T \in \tilde{\mathcal{A}}(U)$ such that $f * T = 1$ on $\mathbb{R}^d$, then f is a Turán maximizer and T is a dual Turán maximizer for U. Furthermore, we have the identity*

$$\mathcal{T}_{\mathbb{R}^d}(U)\, \tilde{\mathcal{T}}_{\mathbb{R}^d}(U) = 1. \tag{3}$$

Proof We have $\mathcal{F}(f * T) = \hat{f}\mathrm{d}\mu = \delta_0$, where $\mu = \mathcal{F}(T)$. Hence,

$$\mathcal{D}(f * T) = \hat{f}(0)\, \mu(\{0\}) = 1,$$

yielding

$$\left(\int_U f(x)\, dx\right) \mathcal{D}(T) = 1, \tag{4}$$

Choosing $\psi \in \mathcal{S}(\mathbb{R}^d)$ identically equal to 1 on a neigborhood of U, we have

$$\int_U h_n(x)\, dx = \langle h_n, \psi\rangle \to \langle f, \psi\rangle = \int_U f(x)\, dx.$$

which shows, in particular, that $\int_U f(x)\, dx \le \mathcal{T}_{\mathbb{R}^d}(U)$. If $h \in \mathcal{A}(U)$, we have

$$\left(\int_U h(x)\, dx\right) \mathcal{D}(T) \le 1$$

by Proposition 1, which shows that $\mathcal{T}_{\mathbb{R}^d}(U) \le \mathcal{D}(T)^{-1} = \int_U f(x)\, dx$. Hence, $\int_U f(x)\, dx = \mathcal{T}_{\mathbb{R}^d}(U)$ and f is a Turán maximizer for U. Finally, by Proposition 1 again, we have

$$\left(\int_U f(x)\, dx\right) \mathcal{D}(S) \le 1$$

for all $S \in \tilde{\mathcal{A}}(U)$, with equality when $S = T$. We conclude that T must be a dual Turán maximizer for U and thus also, using (4), that the identity (3) holds. □

When dealing with the particular case where U is a ball centered at 0, we can assume, for simplicity, that $U = B(0, 2)$. Consider now the function f defined by the formula

$$f = |B|^{-1} \chi_B * \tilde{\chi}_B, \tag{5}$$

where $B = B(0, 1) = \{x \in \mathbb{R}^d, \ |x| < 1\}$, and the functions f_n defined by a similar formula with B replaced by $B_n = B(0, 1 - 1/n)$ for $n \geq 2$. Then, each f_n is a p.d. function compactly supported in $U = B(0, 2)$ with $f_n(0) = 1$, i.e. each f_n belongs to $\mathcal{A}(U)$. Furthermore, it is easily checked that $f_n \to f$ uniformly on $\mathbb{R}^d$ and thus also in $\mathcal{S}'(\mathbb{R}^d)$. Using Corollary 2, we obtain thus the following.

Corollary 3 *If f is the function defined by (5) and, if there exists $T \in \tilde{\mathcal{A}}(B(0, 2))$ such that $f * T = 1$ on $\mathbb{R}^d$, then f is a Turán maximizer and T is a dual Turán maximizer for $B(0, 2)$.*

We now state the main result of this paper which involves a measure constructed using certain Bessel functions. (See (14) for the definition of the Bessel function J_ν, if $\nu \geq 0$.)

Theorem 4 *Suppose that $d \geq 2$ and let f be the p.d. function defined in (5). Let μ be the positive measure defined on $\mathbb{R}^d$ by the formula*

$$\mu = \frac{1}{|B|}\,\delta_0 + \sum_{n \geq 1} \frac{1}{\pi\,\gamma_n\,|J_{d/2-1}(\gamma_n)|^2}\,\sigma_{\gamma_n/2\pi}, \tag{6}$$

where, if $t > 0$, σ_t denotes the $(d-1)$-dimensional surface measure on the sphere of radius t and $(\gamma_n)_{n\geq 1}$ denotes the sequence of positive zeros of the Bessel function $J_{d/2}$ written in increasing order. Then, μ defines a tempered distribution. Furthermore, $T := \mathcal{F}^{-1}(\mu)$ belongs to $\tilde{\mathcal{A}}(B(0, 2))$ and satisfies $f * T = 1$ on $\mathbb{R}^d$. In particular, T is a dual Turán maximizer for $B(0, 2)$ and $\tilde{\mathcal{T}}_{\mathbb{R}^d}(B(0, 2)) = \mu(\{0\}) = \frac{1}{|B|}$.

As an immediate consequence of the previous theorem, of Corollary 2 and Corollary 3, we obtain the value of the Turán constant for the ball $B(0, 2)$, first obtained by Siegel [23].

Corollary 5 *Under the same assumptions as in the previous theorem, f is a Turán maximizer for $B(0, 2)$ and $\mathcal{T}_{\mathbb{R}^d}(B(0, 2)) = |B|$.*

Because of the statements given in Corollary 2 and Corollary 3, in order to prove Theorem 4, we only need to show that the measure μ in the previous theorem is tempered and that $T := \mathcal{F}^{-1}(\mu)$ belongs to $\tilde{\mathcal{A}}(B(0, 2))$ and satisfies $f * T = 1$ on $\mathbb{R}^d$. This is the main difficulty in the proof of this theorem and it will be our main focus in the next two sections.

Before getting to that, we should mention an interesting property of any measure which is the Fourier transform of an element in the class $\tilde{\mathcal{A}}(U)$, (where U is any symmetric bounded open set) and thus also of the measure associated with any dual Turán maximizer for U. This property is related to the notion of Beurling density of a positive Borel measure on $\mathbb{R}^d$ which we now define. If $z \in \mathbb{R}^d$ and $R > 0$, let $I_R(z)$ denote the (closed) cube centered a z with side length R. If μ is a positive Borel measure on $\mathbb{R}^d$, we define the upper and lower Beurling density of μ, denoted by $D^+(\mu)$ and $D^-(\mu)$ respectively, to be the quantities

$$D^+(\mu) = \limsup_{R\to\infty} \sup_{z\in\mathbb{R}^d} \frac{\mu(I_R(z))}{R^d} \quad \text{and} \quad D^-(\mu) = \liminf_{R\to\infty} \inf_{z\in\mathbb{R}^d} \frac{\mu(I_R(z))}{R^d}.$$

The Beurling density of μ, denoted by $D(\mu)$, is say to exist if the two densities above are equal and, in that case, we let $D(\mu) = D^+(\mu) = D^-(\mu)$. We will need the following result which can be found in [9, Proposition 6.2]. Note that the measures of the form $\mathcal{F}(S)$, where $S \in \tilde{\mathcal{A}}(U)$, must be translation-bounded (see (9)) as we will show in Proposition 9.

Proposition 6 *Let μ be a positive, translation-bounded measure on $\mathbb{R}^d$ such that, for some $r > 0$, we have $\mathcal{F}^{-1}(\mu) = \delta_0$ on the ball $B(0, r)$. Then, $D(\mu)$ exists and is equal to* 1.

By definition, any distribution in $\tilde{\mathcal{A}}(U)$ must be equal to δ_0 on some neighborhood of 0 and the following result follows immediately.

Corollary 7 *If $T \in \tilde{\mathcal{A}}(U)$ and $\mu = \mathcal{F}(T)$, then $D(\mu)$ exists and is equal to* 1.

We now go back to the problem of constructing the distribution T in Theorem 4 and proving the properties of T mentioned there.

Before studying more specifically the case where U is a ball, we consider the more general case where the set U has the form $U = V - V$, where V is a bounded, open subset of $\mathbb{R}^d$. We explore some connections between the collection $\tilde{A}(U)$ and frame theory in the next section. These will used when proving our main result, Theorem 4, by applying them to the set $U = B(0, 2) = B - B$, where $B = B(0, 1)$.

3 The Connection with Fourier Frames

A countable collection of vectors $\{h_k\}$ in a separable Hilbert $\mathcal{H}$ is called a *frame* if there exist two positive constants C, D such that

$$C\,\|h\|^2 \le \sum_k |(h, h_k)|^2 \le D\,\|h\|^2, \quad h \in \mathcal{H}.$$

The frame is called a *Parseval frame* if $C = D = 1$. In that case, any $h \in \mathcal{H}$ admits the expansion

$$h = \sum_k (h, h_k)\, h_k$$

in terms of the Parseval frame system, which generalizes the corresponding expansion in terms of an orthonormal basis for $\mathcal{H}$. We refer the reader to [6] for an overview of frames and their properties in various settings. Let $V \subset \mathbb{R}^d$ be a bounded, open set. The Hilbert space of interest for us will be $L^2(V)$. A collection of exponentials $\{e^{2\pi i\lambda\cdot x}\}_{\lambda\in\Lambda}$, where $\Lambda \subset \mathbb{R}^d$ is a discrete set, is called a *Fourier frame* for $L^2(V)$ if their restrictions to V form a frame for $L^2(V)$. This is thus equivalent to having

$$C\,\|h\|_2^2 \le \sum_{\lambda\in\Lambda} |\hat{h}(\lambda)|^2 \le D\,\|h\|_2^2, \quad f \in L^2(V),$$

for some positive constants C, D, where $\|h\|_2^2 = \int |h(x)|^2\,dx$. Introducing the measure $\mu = \sum_{\lambda\in\Lambda} \delta_\lambda$, where δ_λ is the Dirac mass at λ, we can rewrite the frame inequalities in the form

$$C\,\|h\|_2^2 \le \int_{\mathbb{R}^d} |\hat{h}(\xi)|^2\,\mathrm{d}\mu(\xi) \le D\,\|h\|_2^2, \quad h \in L^2(V). \tag{7}$$

This last formulation of the frame inequalities suggests to extend the definition of Fourier frames to arbitrary Borel measures μ on $\mathbb{R}^d$ by using the inequalities in (7). Thus, we will say that the collection of exponentials $\{e^{2\pi i\lambda\cdot x}\}_{\lambda\in\mathbb{R}^d}$ forms a *continuous Fourier frame* with respect to μ for $L^2(V)$ if the inequalities in (7) hold for some $C, D > 0$. In the case $C = D = 1$, we use the term *continuous Parseval Fourier frame* (with respect to μ). (See [1, 11] for more details about continuous frames and frames associated with measures.) The main result of this section shows that the Fourier tranforms of the elements of $\tilde{\mathcal{A}}(U)$, where $U = V - V$, are exactly the positive Borel measures on $\mathbb{R}^d$ associated with continuous Parseval Fourier frames for $L^2(V)$. Before stating it, we need some preliminary results. Note that we use the notation $A - B$ for the set $\{a - b,\ a \in A,\ , b \in B\}$ and $A - b$ for $\{a - b,\ a \in A\}$, if $A, B \subset \mathbb{R}^d$ and $b \in \mathbb{R}^d$. The sets $A + B$ and $A + b$ are defined in a similar way.

Lemma 8 *Let $V \subset \mathbb{R}^d$ be a bounded, open set and consider the bounded, symmetric open set $U = V - V$. Then, given a test function $\phi \in C_0^\infty(U)$, there exist finitely many sequences $(\varphi_k^i)_{k\ge1}$ and $(\psi_k^i)_{k\ge1}$ in $C_0^\infty(V)$, where $1 \le i \le m$, such that $\sum_{i=1}^m \varphi_k^i * \tilde{\psi}_k^i \to \phi$ in $C_0^\infty(U)$ as $k \to \infty$.*

Proof We first show that given any $z \in U$, there exists $\epsilon > 0$ such that our claim is true for any $\phi \in C_0^\infty(U)$ whose support is contained in $B(z,\epsilon)$. Indeed, since $U = V - V$, $z \in V - x_0$ for some $x_0 \in V$ and since $V - x_0$ is open, there exists $\epsilon > 0$ such that $B(z,\epsilon) \subset V - x_0$. If ϕ is supported in $B(z,\epsilon)$, then $\delta_{x_0} * \phi$ is supported in V. Let $\rho \in C_0^\infty\,(B(0,1)))$ with $\rho \ge 0$ and $\int_B \rho(x)\,dx = 1$. Define the approximation of the identity $\rho_k \in C_0^\infty(B(0,1/k))$ for $k \ge 1$ by

$$\rho_k(x) = k^d\,\rho(x\,k), \quad x \in \mathbb{R}^d.$$

Then standard arguments show that $\delta_{x_0} * \phi * \rho_k \to \delta_{x_0} * \phi$ in $C_0^\infty(V)$ as $k \to \infty$ and thus

$$\phi = \lim_{k\to\infty} (\delta_{x_0} * \phi) * (\delta_{-x_0} * \rho_k) = \lim_{k\to\infty} (\delta_{x_0} * \phi) * (\widetilde{\delta_{x_0} * \tilde{\rho}_k})$$

in $C_0^\infty(U)$ since $\delta_{x_0} * \phi$ and $\delta_{x_0} * \tilde{\rho}_k$, for k large enough, both belong to $C_0^\infty(V)$. The case of a general function $\phi \in C_0^\infty(U)$ follows from the above using a standard partition of unity argument. □

If a positive Borel measure μ satisfies the second inequality in (7), i.e. the so-called Bessel inequality

$$\int_{\mathbb{R}^d} |\hat{h}(\xi)|^2 \,\mathrm{d}\mu(\xi) \le D\,\|h\|_2^2, \quad h \in L^2(V), \tag{8}$$

then μ must be *translation-bounded*, i.e. there exists a constant $C > 0$ such that

$$\mu(B + \xi) \le C \ \text{ for all } \xi \in \mathbb{R}^d, \tag{9}$$

which implies, in particular, that μ must be tempered. We will prove this in the next proposition as well as other useful facts deduced from the assumption (8). Before doing so, we remark that the Bessel condition (8) always holds if μ is a bounded measure since

$$\int_{\mathbb{R}^d} |\hat{h}(\xi)|^2 \,\mathrm{d}\mu(\xi) \le \|\hat{h}\|_\infty^2\, \mu(\mathbb{R}^d) \le \|h\|_2^2\, |V|\, \mu(\mathbb{R}^d), \quad h \in L^2(V),$$

a fact which we will be using in the next section.

Proposition 9 *Let μ be a positive Borel measure satisfying the Bessel condition (8) in $L^2(V)$. Then,*

(a) μ is translation-bounded and, in particular, μ is tempered.
(b) Let $R = \mathcal{F}^{-1}(\mu)$. Then, for any $h \in L^2(V)$, we have $R * h \in L^2_{loc}(\mathbb{R}^d)$ and, if $K \subset \mathbb{R}^d$ is any compact set, the mapping $L^2(V) \to L^2(K) : h \mapsto R * h|_K$ is bounded.
(c) We have the identity

$$\int_B (R * h)(x)\,\overline{g(x)}\,dx = \int_{\mathbb{R}^d} \hat{h}(\xi)\,\overline{\hat{g}(\xi)}\,\mathrm{d}\mu(\xi), \quad \textit{for any } h, g \in L^2(V).$$

Proof To prove (a), choose $h \neq 0$ in $L^2(V)$ as well as a point $\xi_0 \in \mathbb{R}^d$ with $2\,a := |\hat{h}(\xi_0)|^2 > 0$ and $r > 0$ such that $|\hat{h}(\xi)|^2 \ge a$ if $|\xi - \xi_0| \le r$. Then, applying the inequality (8) to the function $e^{2\pi i(\eta-\xi_0)\cdot x}\, h(x)$, where $\eta \in \mathbb{R}^d$ is arbitrary, we have

$$a\,\mu(B(\eta, r)) = a \int_{|\xi-\eta|\le r} 1 \,\mathrm{d}\mu(\xi)$$

$$\leq \int_{\mathbb{R}^d} |\hat{h}(\xi_0 + \xi - \eta)|^2 \, d\mu(\xi) \leq D \, \|h\|_2^2, \quad \eta \in \mathbb{R}^d,$$

from which (9) easily follows.

To prove (b), note that if the inequality (8) holds for all functions in $L^2(V)$, it also does for all functions in $L^2(V+a)$, if $a \in \mathbb{R}^d$, since translation of a function by a does not affect the modulus of its Fourier transform. Furthermore, the inequality means that the linear mapping $L^2(V+a)) \to L^2(\mu) : h \mapsto \hat{h}$ is bounded and it is easily checked that its adjoint applied to a function $G \in L^2(\mu)$ is the restriction of the distribution $\mathcal{F}^{-1}(G \, d\mu)$ to the set $V + a$. Since the adjoint operator is also bounded, it follows that $\mathcal{F}^{-1}(G \, d\mu)$ belongs to $L^2(V + a)$ for any $a \in \mathbb{R}^d$ and thus to $L^2_{\text{loc}}(\mathbb{R}^d)$. (See [10] for more details and extensions of these ideas to other spaces of functions and distributions.) In particular, if R is the distribution $\mathcal{F}^{-1}(\mu)$ and $h \in L^2(V)$, we have $R * h = \mathcal{F}^{-1}(\hat{h} \, d\mu) \in L^2_{\text{loc}}(\mathbb{R}^d)$ with the mapping $L^2(V) \to L^2(K) : h \mapsto R * h|_K$ being bounded for any compact set $K \subset \mathbb{R}^d$. This proves (b). Finally, if $\varphi \in C_0^\infty(V)$, we have the identity

$$\int_B (R * \varphi)(x) \, \overline{\varphi(x)} \, dx = \langle R * \varphi, \overline{\varphi} \rangle = \langle \hat{\varphi} \, d\mu, \overline{\hat{\varphi}} \rangle = \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^2 \, d\mu(\xi), \quad (10)$$

by definition of the Fourier transform of a tempered distribution. If $h \in L^2(V)$, consider a sequence (φ_n) in $C_0^\infty(V)$ with $\varphi_n \to h$ in $L^2(V)$. Standard approximation arguments show that we can replace φ by h in the Eq. (10) and the identity in (c) easily follows. □

We now prove the main result in this section.

Theorem 10 *Let $V \subset \mathbb{R}^d$ be a bounded, open set and consider the bounded, symmetric open set $U = V - V$. Let μ be a positive Borel measure on $\mathbb{R}^d$. Then, the following are equivalent.*

(a) μ is tempered and $R = \mathcal{F}^{-1}(\mu) \in \tilde{\mathcal{A}}(U)$.
(b) We have the identity

$$\int_V |h(x)|^2 \, dx = \int_{\mathbb{R}^d} |\hat{h}(\xi)|^2 \, d\mu(\xi), \quad h \in L^2(V).$$

(c) μ is a tempered measure and, letting $R = \mathcal{F}^{-1}(\mu)$, we have the identity $R * h = h$ on V for any $h \in L^2(V)$.
(d) μ is a tempered measure and, letting $R = \mathcal{F}^{-1}(\mu)$, we have the identity

$$\int_V |h(x)|^2 \, dx = \int_V (R * h)(x) \, \overline{h(x)} \, dx, \quad h \in L^2(V).$$

Proof Suppose that (a) holds and let $\varphi \in C_0^\infty(V)$. Using the fact that the test function $\varphi * \tilde{\varphi}$ is supported in U and that $R = \delta_0$ on U, we have

$$\int_V |\varphi(x)|^2 \, dx = (\varphi * \tilde{\varphi})(0) = \langle \delta_0, \varphi * \tilde{\varphi} \rangle$$

$$= \langle \overline{R}, \varphi * \tilde{\varphi} \rangle = \langle \mu, |\hat{\varphi}|^2 \rangle = \int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^2 \, \mathrm{d}\mu(\xi). \tag{11}$$

This shows that the identity in (b) holds if $h = \varphi \in C_0^\infty(V)$ and the general case follows by a standard approximation argument. Conversely, if (b) holds, we have also

$$\int_V g(x)\, \overline{h(x)}\, dx = \int_{\mathbb{R}^d} \hat{g}(\xi)\, \overline{\hat{h}(\xi)}\, \mathrm{d}\mu(\xi), \quad g, h \in L^2(V), \tag{12}$$

using the polarization identity. By Proposition 9, μ is tempered and, if $R = \mathcal{F}^{-1}(\mu)$ and $\varphi, \psi \in C_0^\infty(V)$, we have thus

$$(\varphi * \tilde{\psi})(0) = \int_V \varphi(x)\, \overline{\psi(x)}\, dx = \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\, \overline{\hat{\psi}(\xi)}\, \mathrm{d}\mu(\xi) = \langle \overline{R},\, \varphi * \tilde{\psi} \rangle.$$

If $\phi \in C_0^\infty(U)$, the previous identity together with Lemma 8 imply that $\langle \delta_0,\, \phi \rangle = \langle \overline{R},\, \phi \rangle$,

which shows that $\overline{R} = \delta_0$ on U. Hence, $R = \delta_0$ on U, which yields (a). If (b) holds, μ is tempered by part (a) of Proposition 9 and we can let $R = \mathcal{F}^{-1}(\mu)$. Let $h \in L^2(V)$ and let $\psi \in C_0^\infty(V)$. Using the identity (12), we have then

$$\langle R * h, \overline{\psi} \rangle = \langle \hat{h}\, \mathrm{d}\mu, \overline{\hat{\psi}} \rangle = \int_{\mathbb{R}^d} \hat{h}(\xi)\, \overline{\hat{\psi}(\xi)}\, \mathrm{d}\mu(\xi) = \int_V h(x)\, \overline{\psi(x)}\, dx$$

which shows that (c) holds. Conversely, if (c) holds and $\psi \in C_0^\infty(V)$, we have

$$\begin{aligned} \int_V |\psi(x)|^2\, dx &= \int_V (R * \psi)(x)\, \overline{\psi(x)}\, dx = \langle R * \psi, \overline{\psi} \rangle \\ &= \langle \hat{\psi}\, \mathrm{d}\mu, \overline{\hat{\psi}} \rangle = \int_{\mathbb{R}^d} |\hat{\psi}(\xi)|^2\, \mathrm{d}\mu(\xi), \end{aligned}$$

showing that (11) holds. This yields (b) using an approximation argument . Finally, (c) clearly implies (d) and, if (d) holds, the operator $S : L^2(V) \to L^2(V) : h \mapsto R * h|_V - h$, which is bounded and self-adjoint, satisfies $(Sh, h) = 0$ for all $h \in L^2(V)$. Hence $S = 0$ and (c) holds. This proves our claim. □

4 Construction of a Dual Turán Maximizer for the Ball

We now go back to the problem of constructing a p.d. distribution $T \in \tilde{\mathcal{A}}(B(0, 2))$ satisfying $f * T = 1$ on $\mathbb{R}^d$, where $f = |B|^{-1} \chi_B * \tilde{\chi}_B$ and $B = B(0, 1)$. In one dimension, this problem is easily solved. Indeed, in that case,

$$f(x) = \frac{1}{2}\, (\chi_{(-1,1)} * \chi_{(-1,1)})(x) = (1 - |x/2|)\, \chi_{(-2,2)}(x), \quad x \in \mathbb{R},$$

and we can take T to be the "Dirac train" $T = \sum_{k\in\mathbb{Z}} \delta_{2k}$ which belongs to $\tilde{\mathcal{A}}((-2,2))$ since $T = \delta_0$ on $(-2,2)$ and $\hat{T} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \delta_{k/2} \geq 0$. This yields the well-known fact that $\mathcal{T}_{\mathbb{R}}((-2,2)) = 2$ and, of course, that $\tilde{\mathcal{T}}_{\mathbb{R}}((-2,2)) = 1/2$ for the dual problem.

In the following, we will thus assume that $d \geq 2$. Letting $\mu = \mathcal{F}(T)$ and taking Fourier transforms in the equation $f * T = 1$, we obtain

$$|B|^{-1}\,|\hat{\chi}_B|^2\,\mathrm{d}\mu = \delta_0 \quad \text{on } \mathbb{R}^d, \tag{13}$$

in the Fourier domain. Since $\hat{\chi}_B(0) = |B|$, this last identity is equivalent to the properties that 0 is an isolated point of the support of μ with $\mu = |B|^{-1}\,\delta_0$ on a neighborhood of 0 and that the set $\mathrm{supp}(\mu) \setminus \{0\}$ is contained in the set $\{\xi \in \mathbb{R}^d,\ \hat{\chi}_B(\xi) = 0\}$. Now, in dimension d, the function $\hat{\chi}_B$ is given explicitly by the formula

$$\hat{\chi}_B(\xi) = |\xi|^{-d/2}\,J_{d/2}(2\pi|\xi|), \quad \xi \in \mathbb{R}^d,$$

(see [25]), where, for $\nu \geq 0$, J_ν is the Bessel function defined by

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\,\left(\frac{x}{2}\right)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}. \tag{14}$$

We refer the reader to the books [7, 19, 26] for detailed accounts of classical properties and various applications of Bessel functions. In particular, it is well-known that the function $J_{d/2}$ admits an infinite number of positive zeros $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n < \cdots$. Let S^{d-1} denote the unit sphere in $\mathbb{R}^d$ and σ_1 the usual surface measure on S^{d-1}. Since f and thus also $\hat{f}$ are radial functions, it is clear that if a measure μ satisfies the identity (13), so is the radial measure $\mathring{\mu}$ defined by

$$\langle \mathring{\mu}, \varphi \rangle = \int_{\mathbb{R}^d} \mathring{\varphi}\,\mathrm{d}\mu, \quad \varphi \in C_c(\mathbb{R}^d),$$

where $C_c(\mathbb{R}^d)$ denotes the space of continuous functions with compact support on $\mathbb{R}^d$ and

$$\mathring{\varphi}(\xi) = \frac{1}{\sigma_1(S^{d-1})} \int_{S^{d-1}} \varphi(|\xi|\,\tau)\,\mathrm{d}\sigma_1(\tau), \quad \xi \in \mathbb{R}^d.$$

Assuming thus that μ is radial and supported in the set $\{0\} \cup \{\xi \in \mathbb{R}^d,\ \hat{\chi}_B(\xi) = 0\}$, it follows that, for some coefficients $c_n \geq 0$, μ must have the form

$$\mu = c_0\,\delta_0 + \sum_{n\geq 1} c_n\,\sigma_{\gamma_n/2\pi}, \tag{15}$$

where $\{\gamma_n\}_{n\geq 1}$ is the sequence of positive zeros of the function $J_{d/2}$ written in increasing order and σ_t denotes the standard $(d-1)$-dimensional surface measure on the sphere of radius t. The inverse Fourier transform of the measure σ_t also has an explicit expression in term of a Bessel function which is given by the formula:

$$\mathcal{F}^{-1}(\sigma_t)(x) = 2\pi\, t^{d/2}\, |x|^{-d/2+1}\, J_{d/2-1}(2\pi t|x|), \quad x \in \mathbb{R}^d.$$

Taking (formally) the inverse Fourier transform of both sides of (15), we obtain an expansion of the form

$$\mathcal{F}^{-1}(\mu) = d_0 + \sum_{n\geq 1} d_n\, |x|^{-d/2+1}\, J_{d/2-1}(|x|\gamma_n) \quad \text{on } \mathbb{R}^d, \tag{16}$$

for some constants d_n, $n \geq 0$. In order to determine the values of these constants, we notice that, if we define $\psi_0(x) = 1$ and

$$\psi_n(x) = |x|^{-d/2+1}\, J_{d/2-1}(|x|\gamma_n), \quad n \geq 1,$$

we have

$$-\Delta\psi_0 = 0, \quad \text{and} \quad -\Delta\psi_n = \gamma_n^2\, \psi_n, \quad \text{for } n \geq 1,$$

on $\mathbb{R}^d$ and thus also on B. Furthermore, If $z \in \mathbb{R}^d$ and $|z| = 1$, we clearly have $\frac{d}{dr}\,\{\psi_0(rz)\}\big|_{r=1} = 0$. Also, using the identity

$$\frac{d}{dx}\left\{x^{-\nu}\, J_\nu(x)\right\} = -x^{-\nu}\, J_{\nu+1}(x), \tag{17}$$

where $\nu \geq 0$, we have, for $n \geq 1$,

$$\begin{aligned}
\frac{d}{dr}\,\{\psi_n(rz)\}\Big|_{r=1} &= \frac{d}{dr}\left\{r^{-d/2+1}\, J_{d/2-1}(\gamma_n r)\right\}\Big|_{r=1}\\
&= \gamma_n^{d/2-1}\, \frac{d}{dr}\left\{(\gamma_n r)^{-d/2+1}\, J_{d/2-1}(\gamma_n r)\right\}\Big|_{r=1}\\
&= -\gamma_n^{d/2}\, (\gamma_n r)^{-d/2+1}\, J_{d/2}(\gamma_n r)\Big|_{r=1} = 0.
\end{aligned}$$

Let $L^2_{\text{rad}}(B)$ denote the subspace of $L^2(B)$ consisting of radial functions. It is well known that the collections of radial eigenfunctions of the operator $-\Delta$ on B satisfying the Neumann boundary condition $\frac{\partial\psi}{\partial n} = 0$ on the boundary ∂B form a complete orthogonal system for $L^2_{\text{rad}}(B)$, and that collection coincides precisely with the restrictions of the functions in the collection $\{\psi_n\}_{n\geq 0}$ defined above to the ball B. Another way to obtain this result is to reduce it to a one dimensional problem. The fact that the system defined above is an orthonormal basis for $L^2_{\text{rad}}(B)$ is easily seen to be equivalent to the one-dimensional system $\{\phi_n\}_{n\geq 0}$ being an orthonormal basis for $L^2((0, 1))$, where

$$\phi_0(r) = r^{\frac{d-1}{2}} \quad \text{and} \quad \phi_n(r) = \sqrt{r}\, J_{d/2-1}(\gamma_n r) \text{ for } n \geq 1.$$

A proof that this last collection forms an orthonormal basis for $L^2((0,1))$ can be found in [14].

Thus, for any $g \in L^2_{\text{rad}}(B)$, we have the expansion

$$g = \frac{1}{|B|}\left(\int_B g(x)\,dx\right) + \sum_{n\geq 1}\left(\int_B g(x)\,|x|^{-d/2+1}\,J_{d/2-1}(|x|\gamma_n)dx\right)\frac{|x|^{-d/2+1}\,J_{d/2-1}(|x|\gamma_n)}{\|\psi_n\|_2^2}, \quad (18)$$

with

$$\|\psi_n\|_2^2 = \int_B |x|^{-d+2}\,J_{d/2-1}(|x|\gamma_n)^2\,dx = \hat{\sigma}_1(0)\int_0^1 J_{d/2-1}(r\gamma_n)^2\,r\,dr = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}\int_0^1 J_{d/2-1}(r\gamma_n)^2\,r\,dr, \quad n \geq 1.$$

The integral in the last identity can be expressed in terms of the values of the function $J_{d/2-1}$ at the number γ_n using the following lemma. (This lemma might be a known result in the theory of Bessel functions, but, as we could not find it in the litterature, we provide a proof here for completeness.)

Lemma 11 *If* $\nu \geq 0$, $\alpha > 0$ *and* $J_{\nu+1}(\alpha) = 0$, *we have the identity*

$$\int_0^1 |J_\nu(\alpha r)|^2\,r\,dr = \frac{1}{2}\,|J_\nu(\alpha)|^2\,.$$

Proof Note that the function $y(r) = J_\nu(\alpha r)$ is solution of the differential equation

$$(r\,y')' + \left(\alpha^2\,r - \frac{\nu}{r}\right)\,y = 0, \quad r > 0,$$

and, multiplying both sides of this equation by $2\,r\,y'$ yields

$$\frac{d}{dr}\left\{(r\,y')^2\right\} + (\alpha^2\,r^2 - \nu^2)\,\frac{d}{dr}\left\{y^2\right\} = 0, \quad r > 0.$$

Integrating both sides on the interval $[0, 1]$, we obtain

$$\left[(r\,y')^2 + (\alpha^2\,r^2 - \nu^2)\,y^2\right]_0^1 - \int_0^1 2\,\alpha^2\,y^2(r)\,r\,dr = 0.$$

Note that $(r\,y')^2 + (\alpha^2\,r^2 - \nu^2)\,y^2$ vanishes at $r = 0$. Indeed this is clear if $\nu = 0$ and, if $\nu > 0$, this follows from the fact that $y(0) = 0$. Hence, the previous identity can be written as

$$\int_0^1 2\,\alpha^2\,y^2(r)\,r\,dr = y'(1)^2 + (\alpha^2 - \nu^2)\,y(1)^2.$$

Since $y(r) = J_\nu(\alpha r)$, we have thus

$$\int_0^1 2\,\alpha^2\, J_\nu(\alpha r)^2\, r\, dr = \alpha^2\, J_\nu'(\alpha)^2 + (\alpha^2 - \nu^2)\, J_\nu(\alpha)^2. \tag{19}$$

Now using the identity,

$$\frac{d}{dr}\left\{r^{-\nu}\, J_\nu(r)\right\} = -r^{-\nu}\, J_{\nu+1}(r),$$

we have thus

$$-\nu\, r^{-\nu-1}\, J_\nu(r) + r^{-\nu}\, J_\nu'(r) = -r^{-\nu}\, J_{\nu+1}(r)$$

which implies that

$$0 = -J_{\nu+1}(\alpha) = -\nu\, J_\nu(\alpha) + \alpha\, J_\nu'(\alpha).$$

Substituting into (19) yields

$$2\int_0^1 |J_\nu(\alpha r)|^2\, r\, dr = |J_\nu(\alpha)|^2,$$

proving our claim. □

We can thus write (18) in the form

$$g = \frac{1}{|B|}\left(\int_B g(x)\, dx\right) + \frac{\Gamma(d/2)}{\pi^{d/2}} \sum_{n\geq 1}\left(\int_B g(x)\, |x|^{-d/2+1}\, J_{d/2-1}(|x|\gamma_n) dx\right) \frac{|x|^{-d/2+1}\, J_{d/2-1}(|x|\gamma_n)}{\left|J_{d/2-1}(\gamma_n)\right|^2}, \tag{20}$$

for any $g \in L^2_{\text{rad}}(B)$, where the series converges in $L^2(B)$. Since $\hat{\sigma}_1(0) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ and $|B| = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$, this is, of course, equivalent to having

$$h(s) = d\left(\int_0^1 h(r)\, r^{d-1}\, dr\right) + 2\sum_{n\geq 1}\left(\int_0^1 h(r)\, r^{-d/2+1}\, J_{d/2-1}(\gamma_n r)\, r^{d-1}\, dr\right) \frac{s^{-d/2+1}\, J_{d/2-1}(s\gamma_n)}{\left|J_{d/2-1}(\gamma_n)\right|^2}, \tag{21}$$

for any h satisfying

$$\int_0^1 |h(r)|^2 r^{d-1}\, dr < \infty. \tag{22}$$

where the convergence is in the weighted L^2-space $L^2((0,1), r^{d-1})$ defined using the integral in (22). Replacing $h(s)$ by $p(s) := h(s)\, s^{\frac{d-1}{2}}$ in (21), we can easily see that this last expansion formula is also equivalent to the expansion

$$\begin{aligned} p(s) &= d \left(\int_0^1 p(r)\, r^{\frac{d-1}{2}}\, dr \right) s^{\frac{d-1}{2}} \\ &\quad + 2 \sum_{n\geq 1} \left(\int_0^1 p(r)\, r^{1/2}\, J_{d/2-1}(\gamma_n r)\, dr \right) \frac{s^{1/2}\, J_{d/2-1}(s\gamma_n)}{\left| J_{d/2-1}(\gamma_n)\right|^2}, \end{aligned} \tag{23}$$

valid for p in the unweighted $L^2((0,1))$ space. Going back to the problem of identifying the coefficients d_n, $n \geq 0$, in (16), we note that, if the expansion (16) holds on $\mathbb{R}^d$, it must also hold on B, and this suggests to formally expand the radial distribution δ_0 in term of the orthogonal system $\{\psi_n\}_{n\geq 0}$ on B to obtain these coefficients.

Since we have, for $\alpha > 0$,

$$x^{-\nu}\, J_\nu(\alpha x)\big|_{x=0} = \frac{\alpha^\nu}{\Gamma(\nu+1)\, 2^\nu},$$

a formal application of formula (20) yields the expansion

$$\begin{aligned} \delta_0 &= \frac{1}{|B|}\, \psi_0(0) + \frac{\Gamma(d/2)}{\pi^{d/2}} \sum_{n\geq 1} \psi_n(0)\, \frac{\psi_n(x)}{\left| J_{d/2-1}(\gamma_n)\right|^2} \\ &= \frac{1}{|B|} + \frac{\Gamma(d/2)}{\pi^{d/2}} \sum_{n\geq 1} \left(\frac{\gamma_n^{d/2-1}}{\Gamma(d/2)\, 2^{d/2-1}} \right) \frac{|x|^{-d/2+1}\, J_{d/2-1}(|x|\gamma_n)}{\left| J_{d/2-1}(\gamma_n)\right|^2} \\ &= \frac{1}{|B|} + \frac{1}{(2\pi)^{d/2-1}} \sum_{n\geq 1} \left(\frac{\gamma_n^{d/2-1}}{\pi} \right) \frac{|x|^{-d/2+1}\, J_{d/2-1}(|x|\gamma_n)}{\left| J_{d/2-1}(\gamma_n)\right|^2} \end{aligned}$$

on $B(0,1)$ and thus, taking Fourier tranforms,

$$\begin{aligned} \mu &= \frac{1}{|B|}\, \delta_0 + \frac{1}{(2\pi)^{d/2-1}} \sum_{n\geq 1} \left(\frac{\gamma_n^{d/2-1}}{\pi} \right) \frac{\gamma_n^{-d/2}\, (2\pi)^{d/2-1}}{|J_{d/2-1}(\gamma_n)|^2}\, \sigma_{\gamma_n/2\pi} \\ &= \frac{1}{|B|}\, \delta_0 + \sum_{n\geq 1} \frac{1}{\pi\, \gamma_n\, |J_{d/2-1}(\gamma_n)|^2}\, \sigma_{\gamma_n/2\pi}. \end{aligned} \tag{24}$$

Let μ be the positive Borel measure defined by formula (24). Our next goal will be to show that μ is a tempered measure and that $T = \mathcal{F}^{-1}(\mu)$ belongs to $\tilde{\mathcal{A}}(B(0,2))$).

We will show this by proving that the statement in part (b) of Theorem 10 holds if μ is the measure defined in (24) and $V = B$, using the fact that $B(0,2) = B - B$.

In order to achieve this goal, we will need to use the theory of spherical harmonics. We briefly recall some basic aspects of the theory of these functions. We refer the reader to the books [4, 25] for more details. Recall that if $t \geq 0$ and F is a function on $\mathbb{R}^d$, F is called *homogeneous of degree* t, if $F(ax) = a^t\, F(x)$ whenever $a > 0$ and $x \in \mathbb{R}^d$. Homogeneous polynomials of degree n which satisfies the Laplace equation $\Delta u = 0$ on $\mathbb{R}^d$ are called *solid spherical harmonics of degree* m. A *spherical harmonic of degree* m is the restriction of a solid spherical harmonics of degree m to the unit sphere S^{d-1}. It is well known that spherical harmonics of different degrees are orthogonal to each other with respect to the usual surface measure on S^{d-1}. Furthermore, if $\mathcal{H}_m$ denotes the space of spherical harmonic of degree m, then the dimension of $\mathcal{H}_m$ is given by

$$\binom{d+m-1}{m} - \binom{d+m-3}{m-2}$$

if $n \geq 2$ while $\dim(\mathcal{H}_0) = 1$ and $\dim(\mathcal{H}_1) = d$. In the following, we will denote by $a(m)$ the dimension of $\mathcal{H}_m$ and choose, for each m, an orthogonal basis

$$Y_{m,1}, Y_{m,2}, \dots, Y_{m,a(m)},$$

for $\mathcal{H}_m$. We can assume that the functions $Y_{m,k}$ are real-valued without any loss of generality. We have thus

$$\int_{S^{d-1}} Y_{n,k}(x)\, Y_{m,l}(x)\, \mathrm{d}\sigma_1(x) = \delta_{m,n}\, \delta_{k,l} \quad m, n \geq 0, \quad 1 \leq k \leq a(n), \quad 1 \leq l \leq a(m).$$

We will need the following result (see [5, p. 58]).

Proposition 12 *Let $Y_m \in \mathcal{H}_m$ be any spherical harmonic of degree m. Then*

$$\int_{S^{d-1}} e^{2\pi i x\cdot\xi}\, Y_m(\xi)\, \mathrm{d}\sigma_1(\xi) = i^m\, 2\pi\, |x|^{-d/2+1}\, J_{d/2+m-1}(2\pi|x|)\, Y_m(x/|x|), \quad x \in \mathbb{R}^d.$$

To simplify the notations, we define the functions

$$\Psi_{m,k}(x) = i^m\, 2\pi\, |x|^{-d/2+1}\, J_{d/2+m-1}(2\pi|x|)\, Y_{m,k}(x/|x|), \quad x \in \mathbb{R}^d, \tag{25}$$

for $m \geq 0$ and $1 \leq k \leq a(m)$. We will also need the following proposition in which, as before, σ_t denotes the $(d-1)$-dimensional surface measure on the set $t\, S^{d-1} = \{\xi \in \mathbb{R}^d,\ |\xi| = t\}$.

Proposition 13 *Suppose that $F \in L^2(\mathbb{R}^d)$ is compactly supported and consider the function*

$$G(x) = \int_{t\, S^{d-1}} \hat{F}(\xi)\, e^{2\pi i x\cdot\xi}\, \mathrm{d}\sigma_t(\xi), \quad x \in \mathbb{R}^d.$$

Then,

$$G(x) = t^{d-1} \sum_{m\geq 0} \sum_{1\leq k\leq a(m)} c_{m,k}(t)\, \Psi_{m,k}(t\,x) \tag{26}$$

where

$$c_{m,k}(t) = \int_{\mathbb{R}^d} F(x)\, \overline{\Psi_{m,k}(t\,x)}\, dx.$$

Furthermore, the series in (26) converges uniformly on $\mathbb{R}^d$*.*

Proof First note that, for fixed t, the continuous function $\xi \mapsto \hat{F}(t\xi)$ defined on S^{d-1} can be expanded as the series

$$\hat{F}(t\,\xi) = \sum_{m\geq 0} \sum_{1\leq k\leq a(m)} c_{m,k}(t)\, Y_{m,k}(\xi), \quad \xi \in S^{d-1},$$

which converges in $L^2(\sigma_1)$ and where

$$c_{m,k}(t) = \int_{S^{d-1}} \hat{F}(t\xi)\, Y_{m,k}(\xi)\, \mathrm{d}\sigma_1(\xi), \quad m \geq 0,\ 1 \leq k \leq a(m).$$

It follows that

$$\begin{aligned} G(x) &= t^{d-1} \int_{S^{d-1}} \hat{F}(t\,\xi)\, e^{2\pi i x\cdot t\xi}\, \mathrm{d}\sigma_1(\xi) \\ &= t^{d-1} \sum_{m\geq 0} \sum_{1\leq k\leq a(m)} c_{m,k}(t) \int_{S^{d-1}} Y_{m,k}(\xi)\, e^{2\pi i t x\cdot\xi}\, \mathrm{d}\sigma_1(\xi). \end{aligned}$$

To show the uniform convergence of this last series, we note that, if $M \geq 1$, we have

$$\begin{aligned} &\left| G(x) - t^{d-1} \sum_{m=0}^{M} \sum_{1\leq k\leq a(m)} c_{m,k}(t) \int_{S^{d-1}} Y_{m,k}(\xi)\, e^{2\pi i t x\cdot\xi}\, \mathrm{d}\sigma_1(\xi) \right| \\ &= \left| t^{d-1} \sum_{m\geq M+1} \sum_{1\leq k\leq a(m)} c_{m,k}(t) \int_{S^{d-1}} Y_{m,k}(\xi)\, e^{2\pi i t x\cdot\xi}\, \mathrm{d}\sigma_1(\xi) \right| \\ &\leq t^{d-1} \left(\sum_{m\geq M+1} \sum_{1\leq k\leq a(m)} |c_{m,k}(t)|^2 \right)^{1/2} \\ &\times \left(\sum_{m\geq M+1} \sum_{1\leq k\leq a(m)} \left| \int_{S^{d-1}} Y_{m,k}(\xi)\, e^{2\pi i t x\cdot\xi}\, \mathrm{d}\sigma_1(\xi) \right|^2 \right)^{1/2}, \end{aligned}$$

and this last quantity goes to 0 as $N \to \infty$ since

$$\sum_{m\geq 0} \sum_{1\leq k\leq a(m)} |c_{m,k}(t)|^2 = \int_{S^{d-1}} |\hat{F}(t\,\xi)|^2 \,\mathrm{d}\sigma_1(\xi) < \infty$$

and

$$\sum_{m\geq 0} \sum_{1\leq k\leq a(m)} \left| \int_{S^{d-1}} Y_{m,k}(\xi)\, e^{2\pi i t x\cdot\xi} \,\mathrm{d}\sigma_1(\xi) \right|^2$$
$$= \int_{S^{d-1}} |e^{2\pi i t x\cdot\xi}|^2 \,\mathrm{d}\sigma_1(\xi) = \hat{\sigma}_1(0) < \infty.$$

We have also, by Proposition 12, that

$$\int_{S^{d-1}} Y_{m,k}(\xi)\, e^{2\pi i t x\cdot\xi} \,\mathrm{d}\sigma_1(\xi) = \Psi_{m,k}(t\,x), \quad x \in \mathbb{R}^d,$$

and

$$\begin{aligned} c_{m,k}(t) &= \int_{S^{d-1}} \hat{F}(t\xi)\, Y_{m,k}(\xi)\,\mathrm{d}\sigma_1(\xi) \\ &= \int_{S^{d-1}} \left(\int_{\mathbb{R}^d} F(x)\, e^{-2\pi i t\xi\cdot x}\, dx \right) Y_{m,k}(\xi)\,\mathrm{d}\sigma_1(\xi) \\ &= \int_{S^{d-1}} \left(\int_0^\infty \int_{S^{d-1}} F(r\tau)\, e^{-2\pi i t\xi\cdot r\tau}\,\mathrm{d}\sigma_1(\tau)\, r^{d-1}\, dr \right) Y_{m,k}(\xi)\,\mathrm{d}\sigma_1(\xi) \\ &= \int_0^\infty \int_{S^{d-1}} F(r\tau) \left(\int_{S^{d-1}} e^{-2\pi i t r\tau\cdot\xi}\, Y_{m,k}(\xi)\,\mathrm{d}\sigma_1(\xi) \right) \mathrm{d}\sigma_1(\tau)\, r^{d-1}\, dr \\ &= \int_0^\infty \int_{S^{d-1}} F(r\tau)\, \overline{\Psi_{m,k}(t\,r\,\tau)}\, r^{d-1}\, dr = \int_{\mathbb{R}^d} F(x)\, \overline{\Psi_{m,k}(t\,x)}\, dx, \end{aligned}$$

for $m \geq 0$ and $1 \leq k \leq a(m)$, which completes the proof. □

The next step is now to find conditions equivalent for the identity (b) in Theorem 10 to hold for the measure μ defined in (24).

Proposition 14 *Let μ be the measure in (24). Then, μ satisfies (b) in Theorem 10, i.e. the identity*

$$\int_B |h(x)|^2\, dx = \int_{\mathbb{R}^d} |\hat{h}(\xi)|^2 \,\mathrm{d}\mu(\xi), \quad h \in L^2(B),$$

if and only if, for any measurable function p on $(0, 1)$ satisfying $\int_0^1 |p(r)|^2\, r^{d-1}\, dr < \infty$, the following identities hold:

$$\int_0^1 |p(r)|^2\, r^{d-1}\, dr = d\, \left|\int_0^1 p(r)\, r^{d-1}\, dr\right|^2 + \sum_{n\geq 1} \frac{2}{|J_{d/2-1}(\gamma_n)|^2} \left|\int_0^1 r^{-d/2+1}\, J_{d/2-1}(\gamma_n r)\, p(r)\, r^{d-1} dr\right|^2, \tag{27}$$

and, for every $m \geq 1$,

$$\int_0^1 |p(r)|^2\, r^{d-1}\, dr = \sum_{n\geq 1} \frac{2}{|J_{d/2-1}(\gamma_n)|^2} \left|\int_0^1 r^{-d/2+1}\, J_{d/2+m-1}(\gamma_n r)\, p(r)\, r^{d-1}\, dr\right|^2. \tag{28}$$

Proof For every integer $N \geq 1$, defined the measure

$$\mu_N = \frac{1}{|B|}\, \delta_0 + \sum_{n=1}^{N} \frac{1}{\pi\, \gamma_n\, |J_{d/2-1}(\gamma_n)|^2}\, \sigma_{\gamma_n/2\pi}$$

and let $T_N = \mathcal{F}^{-1}(\mu_N)$. If $h \in L^2(B)$, we have, using Proposition 13, that

$$\begin{aligned} T_N * h = \mathcal{F}^{-1}\left(\hat{h}\, \mathrm{d}\mu_N\right) &= \frac{\hat{h}(0)}{|B|} + \sum_{n=1}^{N} \frac{1}{\pi\, \gamma_n\, |J_{d/2-1}(\gamma_n)|^2}\, \mathcal{F}^{-1}\left(\hat{h}\, \sigma_{\gamma_n/2\pi}\right) \\ &= \frac{1}{|B|} \int_B h(x)\, dx \\ &\quad + \sum_{n=1}^{N} \frac{1}{\pi\, \gamma_n\, |J_{d/2-1}(\gamma_n)|^2} \left(\frac{\gamma_n}{2\pi}\right)^{d-1} \\ &\quad \times \sum_{\substack{m\geq 0 \\ 1\leq k\leq a(m)}} c_{m,k}(\gamma_n/2\pi)\, \Psi_{m,k}(\gamma_n x/2\pi) \\ &= \frac{1}{|B|} \int_B h(x)\, dx \\ &\quad + \sum_{n=1}^{N} \frac{2\, \gamma_n^{d-2}}{(2\pi)^d\, |J_{d/2-1}(\gamma_n)|^2} \\ &\quad \times \sum_{\substack{m\geq 0 \\ 1\leq k\leq a(m)}} c_{m,k}(\gamma_n/2\pi)\, \Psi_{m,k}(\gamma_n x/2\pi), \end{aligned}$$

where

$$\begin{aligned} c_{m,k}(\gamma_n/2\pi) &= \int_{\mathbb{R}^d} h(x)\, \overline{\Psi_{m,k}(\gamma_n\, x/2\pi)}\, dx \\ &= \int_0^1 \int_{S^{d-1}} h(r\tau)\, \overline{\Psi_{m,k}(\gamma_n\, r\, \tau/2\pi)}\, \mathrm{d}\sigma_1(\tau)\, r^{d-1}\, dr. \end{aligned}$$

Using the fact that μ_N, being a bounded measure, satisfies the Bessel condition (8), we obtain, using part (c) of Proposition 9, that

$$\begin{aligned}\int_{\mathbb{R}^d} |\hat{h}(\xi)|^2 \, \mathrm{d}\mu_N(\xi) &= (T_N * h,\ h) \\ &= \frac{1}{|B|} \left| \int_B h(x)\, dx \right|^2 + \sum_{n=1}^{N} \frac{2\,\gamma_n^{d-2}}{(2\pi)^d \, |J_{d/2-1}(\gamma_n)|^2} \\ &\quad \times \sum_{\substack{m \ge 0 \\ 1 \le k \le a(m)}} \left| c_{m,k}(\gamma_n/2\pi) \right|^2 .\end{aligned}$$

Define, for $m \ge 0$ and $1 \le k \le a(m)$, the function

$$h_{m,k}(r) = \int_{S^{d-1}} h(r\tau)\, Y_{m,k}(\tau)\, \mathrm{d}\sigma_1(\tau), \quad 0 \le r \le 1,$$

and note that

$$\begin{aligned}\int_B |h(x)|^2 \, dx &= \int_0^1 \int_{S^{d-1}} |h(r\tau)|^2 \, \mathrm{d}\sigma_1(\tau)\, r^{d-1}\, dr \\ &\sum_{\substack{m \ge 0 \\ 1 \le k \le a(m)}} \int_0^1 |h_{m,k}(r)|^2 \, r^{d-1}\, dr.\end{aligned} \tag{29}$$

Also, since $\dim(\mathcal{H}_0) = 1$, we can take $Y_{0,1}(\tau) = \hat{\sigma}_1(0)^{-1/2}$. Hence,

$$\begin{aligned}\int_B h(x)\, dx &= \int_0^1 \int_{S^{d-1}} h(r\tau)\, \mathrm{d}\sigma_1(\tau)\, r^{d-1}\, dr \\ &= \hat{\sigma}_1(0)^{1/2} \int_0^1 \int_{S^{d-1}} h(r\tau)\, Y_{0,1}(\tau)\, \mathrm{d}\sigma_1(\tau)\, r^{d-1}\, dr \\ &= \hat{\sigma}_1(0)^{1/2} \int_0^1 h_{0,1}(r)\, r^{d-1}\, dr.\end{aligned}$$

and

$$\frac{1}{|B|} \left| \int_B h(x)\, dx \right|^2 = \frac{\hat{\sigma}_1(0)}{|B|} \left| \int_0^1 h_{0,1}(r)\, r^{d-1}\, dr \right|^2 = d \left| \int_0^1 h_{0,1}(r)\, r^{d-1}\, dr \right|^2 .$$

Furthermore, using the fact that, for $r \ge 0$ and $\tau \in S^{d-1}$,

$$\Psi_{m,k}(\gamma_n\, r\, \tau/2\pi) = i^m\, (2\pi)^{d/2}\, \gamma_n^{-d/2+1}\, r^{-d/2+1}\, J_{d/2+m-1}(\gamma_n\, r)\, Y_{m,k}(\tau),$$

we have

$$
\begin{aligned}
&\left|\int_0^1 \int_{S^{d-1}} h(r\tau)\,\overline{\Psi_{m,k}(\gamma_n\, r\, \tau/2\pi)}\,\mathrm{d}\sigma_1(\tau)\, r^{d-1}\, dr\right|^2 \\
&\quad = (2\,\pi)^d \gamma_n^{-d+2} \\
&\quad \left|\int_0^1 r^{-d/2+1}\; J_{d/2+m-1}(\gamma_n r)\left(\int_{S^{d-1}} h(r\tau)\; Y_{m,k}(\tau)\,\mathrm{d}\sigma_1(\tau)\right) r^{d-1}\, dr\right|^2 \\
&\quad = (2\,\pi)^d \gamma_n^{-d+2}\left|\int_0^1 r^{-d/2+1}\; J_{d/2+m-1}(\gamma_n r)\, h_{m,k}(r)\, r\, dr\right|^2.
\end{aligned}
$$

It follows thus that

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\hat{h}(\xi)|^2\,\mathrm{d}\mu_N(\xi) &= d\left|\int_0^1 h_{0,1}(r)\, r^{d-1}\, dr\right|^2 \\
&\quad + \sum_{n=1}^{N} \frac{2}{|J_{d/2-1}(\gamma_n)|^2} \sum_{\substack{m\ge 0\\ 1\le k\le a(m)}} \left|\int_0^1 r^{-d/2+1}\; J_{d/2+m-1}(\gamma_n r)\, h_{m,k}(r)\, r^{d-1}\, dr\right|^2.
\end{aligned}
$$

and, using the Lebesgue monotone convergence theorem, we deduce that

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\hat{h}(\xi)|^2\,\mathrm{d}\mu(\xi) &= d\left|\int_0^1 h_{0,1}(r)\, r^{d-1}\, dr\right|^2 \\
&\quad + \sum_{n\ge 1} \frac{2}{|J_{d/2-1}(\gamma_n)|^2} \sum_{\substack{m\ge 0\\ 1\le k\le a(m)}} \left|\int_0^1 r^{-d/2+1}\; J_{d/2+m-1}(\gamma_n r)\, h_{m,k}(r)\, r^{d-1}\, dr\right|^2.
\end{aligned}
$$

Using (29), the identity

$$
\int_B |h(x)|^2\, dx = \int_{\mathbb{R}^d} |\hat{h}(\xi)|^2\,\mathrm{d}\mu(\xi), \quad h \in L^2(B)
$$

is thus equivalent to having

$$
\begin{aligned}
\sum_{\substack{m\ge 0\\ 1\le k\le a(m)}} \int_0^1 |h_{m,k}(r)|^2\, r^{d-1}\, dr &= d\left|\int_0^1 h_{0,1}(r)\, r^{d-1}\, dr\right|^2 \\
&\quad + \sum_{n\ge 1} \frac{2}{|J_{d/2-1}(\gamma_n)|^2} \sum_{\substack{m\ge 0\\ 1\le k\le a(m)}} \left|\int_0^1 r^{-d/2+1}\; J_{d/2+m-1}(\gamma_n r)\, h_{m,k}(r)\, r^{d-1}\, dr\right|^2,
\end{aligned}
\tag{30}
$$

for every $h \in L^2(B)$. Choosing $h \in L^2(B)$ of the form

$$h(r\tau) = p(r)\, Y_{m,k}(\tau), \quad 0 \le r \le 1,\ \tau \in S^{d-1},$$

for $m \ge 0$ and $1 \le k \le a(m)$, we deduce from this last identity the infinite number of Parseval type identities given in (27) and (28) for the space $L^2((0,1), r^{d-1}\,dr)$ consisting of the measurable functions p on $(0,1)$ satisfying $\int_0^1 |p(r)|^2\, r^{d-1}\,dr < \infty$. Conversely, it is easy to see that the identities (27) and (28) also imply (30), proving our claim. □

As was mentioned earlier, the identity (27) is a well-known fact about Fourier-Bessel series. It remains to prove the the identities (28), which will be our task in the next section.

5 Parseval Frames Involving Bessel Functions

We will prove the identities (28) by induction on m using an approximation argument. The following lemma will play a key role.

Lemma 15 *For any real $\gamma \ge 0$ and any integer $d \ge 2$, the subspace*

$$M_\gamma := \{r^\gamma \frac{d}{dr}\left\{r^{-\gamma}\,\varphi(r)\right\} :\ \varphi \in C_0^\infty\,((0,1))\}$$

is dense in $L^2((0,1), r^{d-1})$.

Proof Let $g \in L^2((0,1), r^{d-1}\,dr)$ satisfy

$$\int_0^1 r^\gamma \frac{d}{dr}\left\{r^{-\gamma}\,\varphi(r)\right\} \overline{g(r)}\, r^{d-1}\,dr = 0, \quad \text{for all } \varphi \in C_0^\infty\,((0,1))\,.$$

This is equivalent to the identity (in the sense of distributions)

$$r^{-\gamma}\frac{d}{dr}\left\{\overline{g(r)}\, r^{d+\gamma-1}\right\} = 0 \quad \text{on } (0,1)$$

which implies that $g(r) = C\, r^{-d-\gamma+1}$, for some constant C. If $C \ne 0$, this implies that

$$\int_0^1 r^{-2d-2\gamma+2}\, r^{d-1}\,dr = \int_0^1 r^{-d-2\gamma+1}\,dr < \infty,$$

or $d + 2\,\gamma < 2$ and leads to a contradiction. Thus $g = 0$ and the conclusion follows from the Hahn–Banach theorem. □

Theorem 16 *If $d \geq 2$ is an integer and $p \in L^2((0,1), r^{d-1}\,dr)$, the identities*

$$\int_0^1 |p(r)|^2\, r^{d-1}\, dr = \sum_{n=1}^{\infty} \frac{2}{|J_{d/2-1}(\gamma_n)|^2} \left| \int_0^1 p(r)\, r^{-d/2+1}\, J_{d/2+m-1}(\gamma_n r)\, r^{d-1}\, dr \right|^2 \tag{31}$$

hold for any $m \geq 1$, i.e. the collection $\left\{ \frac{\sqrt{2}}{|J_{d/2-1}(\gamma_n)|}\, r^{-d/2+1}\, J_{d/2+m-1}(\gamma_n r) \right\}_{n \geq 1}$ is a Parseval frame for $L^2((0,1), r^{d-1}\,dr)$, and this for any $m \geq 1$.

Proof We will use induction on m to prove our claim. Our starting point is the known identity (27) which will help us to prove the case $m = 1$ of (31). We will use the fact that, if $\nu \geq 0$, the function $y(r) = J_\nu(r)$ is solution of the differential equation

$$r^2\, y''(r) + r y'(r) + (r^2 - \nu^2)\, y(r) = 0, \quad r > 0, \tag{32}$$

and also that it satisfies

$$\frac{d}{dr}\left\{ r^{-\nu}\, J_\nu(r) \right\} = -r^{-\nu}\, J_{\nu+1}(r), \quad r > 0.$$

(see [26]). In particular, if $\gamma > 0$, we have

$$\frac{d}{dr}\left\{ r^{-\nu}\, J_\nu(\gamma\, r) \right\} = -\gamma\, r^{-\nu}\, J_{\nu+1}(\gamma\, r), \quad r > 0. \tag{33}$$

Furthermore, the function $z(r) = r^{-\nu}\, J_\nu(r)$ is solution of the differential equation

$$r\, z''(r) + (2\,\nu + 1)\, z'(r) + r\, z(r) = 0, \quad r > 0. \tag{34}$$

Let $\nu = d/2 - 1$ to simplify the notations and note that the identity (27) is equivalent to the validity of the expansion, for any $p \in L^2((0,1), r^{d-1}\,dr)$,

$$p = d \int_0^1 p(r)\, r^{d-1}\, dr + \sum_{n=1}^{\infty} \frac{2}{|J_\nu(\gamma_n)|^2} \left(\int_0^1 p(r)\, r^{-\nu}\, J_\nu(\gamma_n r)\, r^{d-1}\, dr \right) r^{-\nu}\, J_\nu(\gamma_n r),$$

where the series converges in $L^2((0,1), r^{d-1}\,dr)$ and thus a also in $\mathcal{D}'\,((0,1))$. Differentiating both sides of the previous expression and using (33) yields the expansion

$$p' = \sum_{n=1}^{\infty} \frac{2}{|J_\nu(\gamma_n)|^2} \left(\int_0^1 p(r)\, r^{-\nu}\, J_\nu(\gamma_n r)\, r^{d-1}\, dr \right) \frac{d}{dr}\left\{ r^{-\nu}\, J_\nu(\gamma_n r) \right\}$$

$$= -\sum_{n=1}^{\infty} \frac{2}{|J_\nu(\gamma_n)|^2} \left(\gamma_n \int_0^1 p(r)\, r^{-\nu}\, J_\nu(\gamma_n r)\, r^{d-1}\, dr \right) r^{-\nu}\, J_{\nu+1}(\gamma_n r)$$

which holds in $\mathcal{D}'((0, 1))$. Letting $p = \varphi \in C_0^\infty((0, 1))$ and $z(r) = r^{-\nu} J_\nu(r)$, we have, using (34),

$$\begin{aligned}
\gamma_n \int_0^1 \varphi(r)\, r^{-\nu}\, J_\nu(\gamma_n r)\, r^{d-1}\, dr &= \int_0^1 \varphi(r)\, (\gamma_n\, r)\, z(\gamma_n r)\, \gamma_n^\nu\, r^{d-2}\, dr \\
&= -\gamma_n^\nu \int_0^1 \varphi(r) \left[(\gamma_n\, r)\, z''(\gamma_n r) + (d-1)\, z'(\gamma_n r)\right] r^{d-2}\, dr \\
&= -\gamma_n^\nu \int_0^1 \varphi(r) \left[\gamma_n\, r^{d-1}\, z''(\gamma_n r) + (d-1)\, r^{d-2} z'(\gamma_n r)\right] dr \\
&= -\gamma_n^\nu \int_0^1 \varphi(r)\, \frac{d}{dr} \left\{ r^{d-1}\, z'(\gamma_n r) \right\} dr = \gamma_n^\nu \int_0^1 \varphi'(r)\, r^{d-1}\, z'(\gamma_n r)\, dr \\
&= -\gamma_n^\nu \int_0^1 \varphi'(r)\, (\gamma_n r)^{-\nu}\, J_{\nu+1}(\gamma_n r)\, r^{d-1}\, dr \\
&= -\int_0^1 \varphi'(r)\, r^{-\nu}\, J_{\nu+1}(\gamma_n r)\, r^{d-1}\, dr.
\end{aligned}$$

It follows thus that

$$\varphi' = \sum_{n=1}^{\infty} \frac{2}{|J_\nu(\gamma_n)|^2} \left(\int_0^1 \varphi'(r)\, r^{-\nu}\, J_{\nu+1}(\gamma_n r)\, r^{d-1}\, dr \right) r^{-\nu}\, J_{\nu+1}(\gamma_n r)$$

where the series converges in the distributional sense on the interval $(0, 1)$. Applying this identity to the test function $\overline{\varphi'(r)}\, r^{d-1}$, we obtain thus

$$\begin{aligned}
\int_0^1 |\varphi'(r)|^2\, r^{d-1}\, dr &= \langle \varphi'(r), \overline{\varphi'(r)}\, r^{d-1} \rangle \\
&= \sum_{n=1}^{\infty} \frac{2}{|J_\nu(\gamma_n)|^2} \left(\int_0^1 \varphi'(r)\, r^{-\nu}\, J_{\nu+1}(\gamma_n r)\, r^{d-1}\, dr \right) \\
&\quad \times \left\langle r^{-\nu}\, J_{\nu+1}(\gamma_n r), \overline{\varphi'(r)}\, r^{d-1} \right\rangle
\end{aligned}$$

or

$$\int_0^1 |\varphi'(r)|^2\, r^{d-1}\, dr = \sum_{n=1}^{\infty} \frac{2}{|J_{d/2-1}(\gamma_n)|^2} \left| \int_0^1 \varphi'(r)\, r^{-d/2+1}\, J_{d/2}(\gamma_n r)\, r^{d-1}\, dr \right|^2.$$

Using Lemma 15 with $\gamma = 0$, the case $m = 1$ of the identity (31) follows easily from the previous identity by an approximation argument. We now prove the general case by induction.

Let us assume that (31) holds for some $m \geq 1$. This is to equivalent to having the expansion

$$p = \sum_{n=1}^{\infty} \frac{2}{[J_\nu(\gamma_n)]^2} \left(\int_0^1 p(r)\, r^{-\nu}\, J_{\nu+m}(\gamma_n r)\, r^{d-1}\, dr \right) r^{-\nu}\, J_{\nu+m}(\gamma_n r)$$

where the series converges in $L^2((0,1), r^{d-1}\, dr)$ and thus also in $\mathcal{D}'((0,1))$. It follows that the identity

$$r^m \frac{d}{dr} \left\{ r^{-m}\, p \right\} = \sum_{n=1}^{\infty} \frac{2}{[J_\nu(\gamma_n)]^2} \left(\int_0^1 p(r)\, r^{-\nu} J_{\nu+m}(\gamma_n r)\, r^{d-1}\, dr \right) \times r^m \frac{d}{dr} \left\{ r^{-\nu-m}\, J_{\nu+m}(\gamma_n r) \right\}$$

holds in $\mathcal{D}'((0,1))$ which is equivalent, using (33) to the identity

$$r^m \frac{d}{dr} \left\{ r^{-m}\, p \right\} = -\sum_{n=1}^{\infty} \frac{2}{[J_\nu(\gamma_n)]^2} \left(\int_0^1 p(r)\, r^{-\nu}\, \gamma_n\, J_{\nu+m}(\gamma_n r)\, r^{d-1}\, dr \right) \times r^{-\nu}\, J_{\nu+m+1}(\gamma_n r).$$

Now, letting $p = \varphi \in C_0^\infty((0,1))$, we will show that

$$\int_0^1 \varphi(r)\, r^{-\nu}\, \gamma_n\, J_{\nu+m}(\gamma_n r)\, r^{d-1}\, dr = -\int_0^1 r^m \left(r^{-m}\, \varphi(r) \right)' r^{-\nu}\, J_{\nu+m+1}(\gamma_n r)\, r^{d-1}\, dr. \tag{35}$$

Starting with the right-hand side of (35), we have

$$\begin{aligned} &-\int_0^1 r^m \left(r^{-m}\, \varphi(r) \right)' r^{-\nu}\, J_{\nu+m+1}(\gamma_n r)\, r^{d-1}\, dr \\ &= \int_0^1 r^{-m}\, \varphi(r) \frac{d}{dr} \left\{ J_{\nu+m+1}(\gamma_n r)\, r^{d+m-\nu-1} \right\} dr \\ &= \int_0^1 \varphi(r) \left\{ \gamma_n\, J'_{\nu+m+1}(\gamma_n r)\, r^{d-\nu-1} + (d+m-\nu-1)\, J_{\nu+m+1}(\gamma_n r)\, r^{d-\nu-2} \right\} dr. \end{aligned}$$

It follows that the identity (35) is equivalent to

$$0 = r^{-\nu}\, \gamma_n\, J_{\nu+m}(\gamma_n r)\, r^{d-1} - \gamma_n\, J'_{\nu+m+1}(\gamma_n r)\, r^{d-\nu-1} - (d+m-\nu-1)\, J_{\nu+m+1}(\gamma_n r)\, r^{d-\nu-2}$$

or to

$$r\, \gamma_n\, J_{\nu+m}(\gamma_n r) - r\, \gamma_n\, J'_{\nu+m+1}(\gamma_n r) - (d+m-\nu-1)\, J_{\nu+m+1}(\gamma_n r) = 0,$$

for $0 < r < 1$. Since $d + m - \nu - 1 = \nu + m + 1$, this will follow if we can prove the identity

$$r\, J_{\nu+m}(r) - r\, J'_{\nu+m+1}(r) \; - (\nu + m + 1)\, J_{\nu+m+1}(r) \; = 0, \quad r > 0.$$

Since

$$J_{\nu+m+1}(r) = -r^{\nu+m} \left[r^{-\nu-m}\, J_{\nu+m}(r) \right]' = (\nu + m)\, r^{-1}\, J_{\nu+m}(r) - \; J'_{\nu+m}(r),$$

this last identity can be written as

$$r\, J_{\nu+m}(r) - r \left[-(\nu + m)\, r^{-2}\, J_{\nu+m}(r) + (\nu + m)\, r^{-1}\, J'_{\nu+m}(r) - J''_{\nu+m}(r) \right] - (\nu + m + 1) \left[(\nu + m)\, r^{-1}\, J_{\nu+m}(r) - \; J'_{\nu+m}(r) \right] = 0,$$

or, after multiplying by r,

$$r^2\, J''_{\nu+m}(r) + r\, J'_{\nu+m}(r) + [r^2 - \; (\nu + m)^2]\, J_{\nu+m}(r) = 0$$

which is exactly the Bessel equation (32) satisfied by $J_{\nu+m}$. As before, we can deduce that the identity (31) holds, if m is replaced by $m + 1$, for functions of the form $p(r) = r^m \frac{d}{dr} \left\{ r^{-m}\, \varphi(r) \right\}$, where $\varphi \in C_0^\infty((0, 1))$, a collection which is dense in $L^2\left((0, 1),\; r^{d-1}\, dr\right)$ by Lemma 15 applied to $\gamma = m$. Again, an approximation argument shows that (31) holds for all functions in $L^2\left((0, 1),\; r^{d-1}\, dr\right)$, which concludes the proof. □

The previous theorem has, of course, direct counterparts for the space $L^2_{\text{rad}}(B)$ and the unweighted space $L^2((0, 1))$, which we state next.

Corollary 17 *Let $m \geq 1$ be an integer. Then,*

(a) The collection $\left\{ \frac{\sqrt{2}}{|J_{d/2-1}(\gamma_n)|}\, |x|^{-d/2+1}\, J_{d/2+m-1}(\gamma_n |x|) \right\}_{n\geq 1}$ is a Parseval frame for $L^2_{rad}(B)$.

(b) The collection $\left\{ \frac{\sqrt{2}}{|J_{d/2-1}(\gamma_n)|}\, \sqrt{r}\, J_{d/2+m-1}(\gamma_n r) \right\}_{n\geq 1}$ is a Parseval frame for $L^2((0, 1))$.

Although we did not use this fact in the proof, the case $m = 1$ in Theorem 16 and in Corollary 17 is also well-known and, in fact, the Parseval frames are orthonormal bases in this particular case. Indeed, the functions forming the Parseval frame in part (a) of Corollary 17 are known to the radial eigenfunctions $\{\Psi_n\}$ of the Laplace operator on B that satisfy the Dirichlet boundary condition $\Psi_n = 0$ on the boundary of B, and these form an orthonormal basis for $L^2_{\text{rad}}(B)$ (see [5]). This can also be easily deduced from the fact that the collection in part (b) of Corollary 17 for an orthonormal basis for $L^2((0, 1))$, as proved in [14].

We now have all the ingredients to prove the main result of this paper.

Proof of Theorem 4 Since, by construction, the measure μ in Theorem 4 satisfies $\hat{f}\,\mu = \delta_0$ or $f * T = 1$ on $\mathbb{R}^d$, where $f = |B|^{-1}\,\chi_B * \tilde{\chi}_B$ and $T = \mathcal{F}^{-1}(\mu)$, it suffices to show that $T \in \tilde{\mathcal{A}}(B(0,2)$ by Corollary 3. This last property follows immediately from Theorem 10, Proposition 14, Theorem 16 as well as the known identity (27) for Bessel functions. □

6 Additional Properties of the Turán Maximizer and Its Dual

In dimension $d \geq 2$, although we have an explicit formula for the Fourier transform of the dual Turán maximizer T we constructed, i.e. the measure μ defined in (24), we don't have such an explicit expression for T itself. All we known is that $T = \delta_0$ on the ball $B(0, 2)$ and T is thus a measure on that open set. However, the following shows T has to be "singular" near the sphere of radius 2 centered at 0 in the sense that it can not be equal to a measure, i.e. a distribution of order 0, on any ball centered at 0 with radius $2 + \epsilon$, $\epsilon > 0$.

Proposition 18 *Suppose that $d \geq 2$ and let $T = \mathcal{F}^{-1}(\mu)$, where μ is the measure defined in (24). Then, if $\epsilon > 0$, the restriction of T to the ball $B(0, 2+\epsilon)$ is not a (complex) measure.*

Proof We will argue by contradiction. Suppose that $T = \delta_0 + \rho$ where ρ is a distribution supported on the set $\{x \in \mathbb{R}^d,\ |x| \geq 2\}$ whose restriction to the open set $\{x \in \mathbb{R}^d,\ |x| < 2+\epsilon\}$ is a complex measure also denoted by ρ. Let B be the open ball $\{x \in \mathbb{R}^d,\ |x| < 1\}$, let $E = \{y \in \mathbb{R}^d,\ |y| \geq 2\}$ and define $g_n(y) = \chi_B(z_n - y)\,\chi_E(y)$. Note that the convolution $\rho * \chi_B$ is given a. e. by the function

$$f(w) = \int_{|y|\geq 2} \chi_B(w-y)\,\mathrm{d}\rho(y) = \int_{\mathbb{R}} \chi_B(w-y)\,\chi_E(y)\,\mathrm{d}\rho(y) = \int_{\mathbb{R}} g_n(y)\,\mathrm{d}\rho(y)$$

on the open set $\{w \in \mathbb{R}^d,\ |w| < 1+\epsilon/2\}$. Since $T * \chi_B = 1$, we have $\rho * \chi_B = 1 - \chi_B$ and, in particular, $\rho * \chi_B = 1$ a.e. on the set $\{x \in \mathbb{R}^d,\ |x| > 1\}$. Now let $z \in \mathbb{R}^d$ with $|z| = 1$ and consider a sequence (z_n) in $\mathbb{R}^d$ with $1 < |z_n| < 1+\epsilon/2$ such that $z_n \to z$.

Note that, if $|z-y| < 1$, we have $|z_n - y| < 1$ and thus $\chi_E(y) = 0 = g_n(y)$ for n large enough. Also, if $|z-y| > 1$, we have $|z_n - y| > 1$ and thus $\chi_B(z_n - y) = 0 = g_n(y)$ for n large enough. Finally, if $|z-y| = 1$, we have $|y| < 2$ unless $y = 2\,z$. Indeed, if $|y| \geq 2$, we would have

$$2 \leq |y| \leq |z| + |y-z| = 2$$

which implies that $|y| = 2$ and that the vectors z and $y - z$ are multiple of each other yielding $y = 2\,z$. We have thus $\chi_E(y) = 0 = g_n(y)$ if $|z-y| = 1$ and $y \neq 2\,z$. Since a locally bounded radial measure cannot give mass to any non-zero point, we have $|\rho|(\{2\,z\}) = 0$ and we obtain that $g_n(y) \to 0$ ρ-almost everywhere as $n \to \infty$. The Lebesgue dominated convergence theorem then shows that $f(z_n) \to 0$ as $n \to \infty$, contradicting the fact that $f = 1$ a. e. on the set $\{x \in \mathbb{R}^d,\ |x| > 1\}$. □

It is straightforward to extend our main result to balls centered at the origin with arbitrary radius or, more generally, to ellipsoidal regions, i.e. the images of the ball $B(0,2)$ under an invertible linear transformation of $\mathbb{R}^d$. If D is a linear transformation of $\mathbb{R}^d$, we denote by D^t its transpose, by $\det(D)$ its determinant and, if $E \subset \mathbb{R}^d$, by $D(E)$ the set $\{Dx,\ x \in E\}$.

Theorem 19 *Let $d \geq 2$ and suppose that $D : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear transformation. Let $U = D(B(0,2))$, let $V = D(B)$ and let μ be the measure defined in (6). Then,*

(a) U is a Turán domain, i.e. the function $f_1 = |V|^{-1}\,\chi_V * \tilde{\chi}_V$ is a Turán maximizer for U.
(b) The distribution $T_1 = |\det(D)|\,\mathcal{F}^{-1}\left(\mu(D^t\cdot)\right)$ is a dual Turán maximizer for U, where

$$\langle \mu(D^t\cdot), \varphi\rangle = \int_{\mathbb{R}^d} \varphi((D^t)^{-1}\xi)\,\mathrm{d}\mu(\xi), \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Proof Note that

$$U = D(B(0,2)) = D(B - B) = D(B) - D(B) = V - V.$$

Letting $f = |B|^{-1}\,\chi_B * \tilde{\chi}_B$, where $B = B(0,1)$, we have $f_1 = f(D^{-1}\cdot)$. Morover, if (φ_n) is a sequence in $\mathcal{A}(B(0,2))$ converging to f in $\mathcal{S}'(\mathbb{R}^d)$, then the sequence $(\varphi_n(D^{-1}\cdot))$ is a sequence in $\mathcal{A}(U)$ converging to f_1 in $\mathcal{S}'(\mathbb{R}^d)$. It is easily checked that $T_1 \in \tilde{\mathcal{A}}(U)$ and that $\hat{f}_1\,\hat{T}_1 = \delta_0$ on $\mathbb{R}^d$. It follows that $f_1 * T_1 = 1$ on $\mathbb{R}^d$ and the results follows then Corollary 2. □

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