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# A Class of Oscillatory Singular Integrals with Rough Kernels and Fewnomials Phases

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## Abstract

This paper is concerned with the oscillatory singular integral operator $T_Q$ defined by

$$T_Q f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y) \frac{\Omega(y)}{|y|^n} e^{iQ(|y|)} dy,$$

where $Q(t) = \sum_{1 \le i \le m} a_i t^{\alpha_i}$ is a real-valued polynomial on $\mathbb{R}$, $\Omega$ is a homogenous function of degree zero on $\mathbb{R}^n$ with mean value zero on the unit sphere $S^{n-1}$. Under the assumption of that $\Omega \in H^1(S^{n-1})$, the authors show that $T_Q$ is bounded on the weighted Lebesgue spaces $L^p(\omega)$ for $1 < p < \infty$ and $\omega \in \tilde{A}_p^I(\mathbb{R}_+)$ with the uniform bound only depending on $m$, the number of monomials in polynomial $Q$, not on the degree of $Q$ as in the previous results. This result is new even in the case $\omega \equiv 1$, which can also be regarded as an improvement and generalization of the result obtained by Guo in [New York J. Math. 23 (2017), 1733-1738].


**Keywords** Oscillatory singular integrals · Rough kernels · $A_p$ weight

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## 1 Introduction

Let $n \geq 2$ and $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Suppose that $\Omega \in L^1(S^{n-1})$ is a homogeneous function of degree zero on $\mathbb{R}^n$ and satisfies the cancellation property

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0. \tag{1.1}$$

We consider the oscillatory singular integral operator defined by

$$T_P f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(y)}{|y|^n} f(x-y)dy,$$

where $P(x, y) = \sum_{|\alpha|+|\beta|\leq d} a_{\alpha,\beta} x^\alpha y^\beta$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$ with a fixed degree $d \in \mathbb{N}$, which is closely related to harmonic analysis on nilpotent groups and singular Radon transforms. It follows from [2, 18, 19] that when $\Omega \in C^1(S^{n-1})$, $T_P$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and weak type $(1, 1)$ with the bounds depending on the degree of $P$ and being independent of the coefficients of $P$. Subsequently, Lu and Zhang [17] improved the condition $\Omega \in C^1(S^{n-1})$ to $\Omega \in L^q(S^{n-1})$ for some $1 < q < \infty$. Afterwards, this result was successively extended the cases of that $\Omega \in L\log^+ L(S^{n-1})$, $B_q^{0,0}(S^{n-1})$ and $H^1(S^{n-1})$ in [1, 13, 16].

It should be pointed out that there are the embedding relations among the functions on $S^{n-1}$:

$$C^1(S^{n-1}) \subsetneq L^q(S^{n-1}) \subsetneq L\log^+ L(S^{n-1}),\ B_q^{0,0}(S^{n-1}) \subsetneq H^1(S^{n-1}) \subsetneq L^1(S^{n-1}).$$

On the other hand, when $P(x, y) = P(y)$, by a sparse domination, Lacey and Spencer [14] showed that if $\Omega \in C^1(S^{n-1})$, then $T_P$ is bounded on $L^p(\omega)$ for $1 < p < \infty$ and $\omega \in A_p$, the Muckenhoupt class, with the bound depending on the degree of $P$, being independent of the coefficients of $P$. Ding and Liu [4] considered the following oscillatory singular integral operator

$$T_\lambda f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iQ_\lambda(|y|)} \frac{\Omega(y)}{|y|^n} f(x-y)dy,$$

which is a generalization of the strongly singular convolution operator studied firstly by Fefferman in [9], where $Q_\lambda(r) = \sum_{2\leq k\leq d} \lambda_k r^k$ is a real-valued polynomial on $\mathbb{R}$ and $\lambda = (\lambda_2, \cdots, \lambda_d) \in \mathbb{R}^{d-1}$, and showed that when $\Omega \in H^1(S^{n-1})$, $T_\lambda$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ with bound depending on the degree of $Q_\lambda$, not on the coefficients $\lambda$. Moreover, for the following oscillatory Hilbert transform

$$H_Q f(x) := \text{p.v.} \int_{\mathbb{R}} e^{iQ(t)} f(x-t) \frac{dt}{t} \tag{1.2}$$

with $Q(t) = \sum_{1\leq k\leq m} a_k t^{\alpha_k}$, where $a_k \in \mathbb{R}$ and $\alpha_k$ is a positive integer for each $1 \leq k \leq m$, Guo [12] proved that for a fixed $m \in \mathbb{N}$,

$$\|H_Q f\|_2 \leq C_m \|f\|_2,$$

with $C_m$ is a constant that depends only on $m$, the number of monomials in $Q$, but not on any $a_k$ or $\alpha_k$.

Inspired by the results above, we will study the following oscillatory singular integral operators

$$T_Q f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iQ(|y|)} \frac{\Omega(y)}{|y|^n} f(x-y)dy, \tag{1.3}$$

with $Q(t) = \sum_{1\leq k\leq m} a_k t^{\alpha_k}$ being as in (1.2).

In order to state our result, we first recall some relevant definitions and notation.

**Definition 1.1** ( [5]) Suppose that $\omega(t) \geq 0$ and $\omega \in L^1_{\text{loc}}(\mathbb{R}_+)$. For $1 < p < \infty$, we say that $\omega \in A_p(\mathbb{R}_+)$ if there is a constant $C > 0$ such that for any interval $I \subset \mathbb{R}_+$,

$$\left(|I|^{-1} \int_I \omega(r)dr\right) \left(|I|^{-1} \int_I \omega(r)^{-1/(p-1)}dr\right)^{p-1} \leq C < \infty.$$

If there is a constant $C > 0$ such that

$$M\omega(r) \leq C\omega(r) \quad \text{for} \quad \text{a.e.} \quad r \in \mathbb{R}_+,$$

where $M\omega$ denotes the standard Hardy-Littlewood maximal function of $\omega$ on $\mathbb{R}_+$, then we say $\omega \in A_1(\mathbb{R}_+)$.

**Definition 1.2** ( [6]) If $\omega(x) = v_1(|x|)v_2(|x|)^{1-p}$, where either $v_i \in A_1(\mathbb{R}_+)$is decreasing or $v_i^2 \in A_1(\mathbb{R}_+)$, $i = 1, 2$, then we say $\omega \in \tilde{A}_p(\mathbb{R}_+)$.

**Definition 1.3** ( [5]) For $1 < p < \infty$, we denote

$$\bar{A}_p(\mathbb{R}_+) = \left\{\omega(x) = \omega(|x|) : \omega(t) > 0, \omega(t) \in L_{\text{loc}}(\mathbb{R}_+)\text{and } \omega^2(t) \in A_p(\mathbb{R}_+)\right\}.$$

Let $A^I_p(\mathbb{R}^n)$ be the weight class defined by using all $n$-dimensional cubes with sides parallel to coordinate axes. In what follows, for $p \in (1, \infty)$, any measurable function $f$ and any weight $\omega$, we define

$$\|f\|_{L^p(\omega)} \equiv \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{1/p}.$$

Thus the weighted $L^p$ spaces associate to the weight $\omega$ is defined by

$$L^p\left(\mathbb{R}^n, \omega(x)dx\right) = \left\{f : \|f\|_{L^p(\omega)} < \infty\right\}.$$

It follows from [5] that $\bar{A}_p(\mathbb{R}_+) \subseteq \tilde{A}_p(\mathbb{R}_+)$. Also, if $\omega(t) \in \tilde{A}_p(\mathbb{R}_+)$, then we know from [6] that the Hardy-Littlewood maximal function $M$ is bounded on $L^p(\mathbb{R}^n, \omega(|x|)dx)$. Thus, if $\omega(t) \in \tilde{A}_p(\mathbb{R}_+)$, then $\omega(|x|) \in A_p(\mathbb{R}^n)$, where $A_p(\mathbb{R}^n)$ is the Muckenhoupt weight (see [10] for the definition). Let $\tilde{A}_p^I = \tilde{A}_p \cap A_p^I$ and $\bar{A}_p^I = \bar{A}_p \cap A_p^I$.

Next, we recall the definition of the Hardy space $H^1(S^{n-1})$.

**Definition 1.4**

$$H^1(S^{n-1}) = \left\{\Omega \in L^1(S^{n-1}) : \|\Omega\|_{H^1(S^{n-1})} < \infty\right\},$$

where

$$\|\Omega\|_{H^1(S^{n-1})} = \left\| \sup_{0<r<1} \left| \int_{S^{n-1}} \Omega(y') P_{r(\cdot)}(y') d\sigma(y') \right| \right\|_{L^1(S^{n-1})}.$$

Here $P_{rx'}(y')$ denotes the Poisson kernel on $S^{n-1}$ defined by

$$P_{rx'}(y') = \frac{1-r^2}{|rx'-y'|^n}, \quad 0 \le r < 1 \text{ and } x', y' \in S^{n-1}.$$

See [3, 7] or [11] for the properties of $H^1(S^{n-1})$.

Now we can formulate our main result as follows.

**Theorem 1.5** *Let $m \in \mathbb{N}$, $T_Q$ be given as in (1.3). Suppose that $\Omega \in H^1(S^{n-1})$ and satisfies (1.1). Then for $1 < p < \infty$ and $\omega \in \tilde{A}_p^I(\mathbb{R}_+)$, $T_Q$ is bounded on $L^p(\omega)$. Moreover, there is a constant $C_{m,\omega} > 0$, which depends only on $m$, the number of monomials in $Q$, and the weight $\omega$, not on the degree of $Q$, such that for all $f \in L^p(\omega)$*

$$\left\| T_Q f \right\|_{L^p(\omega)} \le C_{m,\omega} \|\Omega\|_{H^1(S^{n-1})} \|f\|_{L^p(\omega)}. \tag{1.4}$$

***Remark 1.6*** We remark that the bounds in previous results depended on the degree of the polynomial phases and one of our result depends only on the number of monomials in the polynomial phase, which is more precise and is new even for $\omega \equiv 1$. Moreover, comparing with the result of [12], our result presents three novelties: (i) extend to the higher dimension cases; (ii) relax the range of $p$ from $p = 2$ to $1 < p < \infty$; (iii) give a weighted version.

The rest of this paper is organized as follows. In Sect. 2 we will recall some auxiliary lemmas, which will be used in our arguments. The proof of our main result will be given in Sect. 3. We remark that the main ideas in our arguments are taken from [12, 15].

## 2 Preliminaries

In this section, we recall some auxiliary facts and lemmas, which will be used in our arguments.

**Lemma 2.1** (van der Corput [20]) Suppose that $\phi$ is real-valued and smooth in $(a, b)$, and that $\left|\phi^{(k)}(t)\right| \geq 1$ for all $t \in (a, b)$. Then the inequality

$$\left|\int_a^b e^{-i\lambda\phi(t)}dt\right| \leq C_k|\lambda|^{-\frac{1}{k}}$$

holds when
(i) $k \geq 2$, or
(ii) $k = 1$ and $\phi'$ is monotonic.
The bound $C_k$ is a constant that depends only on $k$, but not on any $a, b, \phi$, and $\lambda$.

We define the singular integral operator $T_\Omega$ by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)dy,$$

and the corresponding maximal operator by

$$T_\Omega^* f(x) = \sup_{\varepsilon>0} \left|T_\Omega^\varepsilon f(x)\right|,$$

where

$$T_\Omega^\varepsilon f(x) = \int_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy, \quad \varepsilon > 0.$$

**Lemma 2.2** ( [8]) *Let $1 < p < \infty$, $\Omega$ satisfies (1.1). Suppose that $\Omega \in H^1\left(S^{n-1}\right)$ and $\omega \in \tilde{A}_p^I(\mathbb{R}_+)$. Then both $T_\Omega$ and $T_\Omega^*$ are bounded on $L^p(\omega)$. Moreover, there is the constant $C_\omega > 0$, which depends only on the weight $\omega$, such that*

$$\|T_\Omega f\|_{L^p(\omega)},\ \|T_\Omega^* f\|_{L^p(\omega)} \leq C_\omega \|\Omega\|_{H^1(S^{n-1})} \|f\|_{L^p(\omega)}.$$

**Lemma 2.3** ( [6]) *For $y' \in S^{n-1}$, define the direction Hardy-Littlewood maximal operator by*

$$M_{y'} f(x) := \sup_{r>0} \frac{1}{r^n} \int_0^r \left|f\left(x - ty'\right)\right| dt.$$

*Then for $1 < p < \infty$ and $\omega \in \tilde{A}_p(\mathbb{R}_+)$,*

$$\|M_{y'} f\|_{L^p(\omega)} \leq C_\omega \|f\|_{L^p(\omega)}, \quad \forall f \in L^p(\omega)$$

*with $C_\omega$ independent of $y'$.*

## 3 Proof of Theorem 1.5

Using polar coordinates, we write the integral in (1.3) as

$$T_Q f(x) = \int_{S^{n-1}} \Omega(y') \int_0^\infty \frac{f(x-ty')}{t} e^{iQ(t)} dt d\sigma(y'). \tag{3.1}$$

Employing the ideas in [12], we will split $\mathbb{R}^+ = (0, \infty)$ into different intervals and show that for all but finitely many of these intervals, there always exists a monomial which "dominates" the polynomial $Q$. Hence we assume that $1 < \alpha_1 < \cdots < \alpha_m$. Denote by $d$ the degree of the polynomial $Q$, that is, $d = \alpha_m$. Let $\lambda = 2^{1/d}$. Define $b_k \in \mathbb{Z}$ such that

$$\lambda^{b_k} \le |a_k| < \lambda^{b_k+1}.$$

We define a few bad scales. For $1 \le k_1 < k_2 \le m$, define

$$\mathcal{L}_{\text{bad}}^{(0)}(k_1, k_2) := \left\{ l \in \mathbb{Z} : 2^{-C_0} \left| a_{k_2} \lambda^{\alpha_{k_2} l} \right| \le \left| a_{k_1} \lambda^{\alpha_{k_1} l} \right| \le 2^{C_0} \left| a_{k_2} \lambda^{\alpha_{k_2} l} \right| \right\}. \tag{3.2}$$

Here $C_0 := 2^{10m!}$. Notice that $l$ satisfies

$$-2 - dC_0 + b_{k_2} - b_{k_1} \le \left(\alpha_{k_1} - \alpha_{k_2}\right) l \le dC_0 + b_{k_2} - b_{k_1} + 2.$$

Hence $\mathcal{L}_{\text{bad}}^{(0)}(k_1, k_2)$ is a connected set whose cardinality is smaller than $4dC_0$. Define

$$\mathcal{L}_{\text{good}}^{(0)} := \left( \bigcup_{k_1 \neq k_2} \mathcal{L}_{\text{bad}}^{(0)}(k_1, k_2) \right)^c.$$

Then, the set $\mathcal{L}_{\text{good}}^{(0)}$ has at most $m^2$ connected components, each of which has a monomial "dominated" $Q$. Similarly, we define

$$\mathcal{L}_{\text{bad}}^{(1)}(k_1, k_2) := \left\{ l \in \mathbb{Z} : 2^{-C_0} \left| \alpha_{k_2} \left(\alpha_{k_2} - 1\right) a_{k_2} \lambda^{\alpha_{k_2} l} \right| \le \left| \alpha_{k_1} \left(\alpha_{k_1} - 1\right) a_{k_1} \lambda^{\alpha_{k_1} l} \right| \right.$$
$$\left. \le 2^{C_0} \left| \alpha_{k_2} \left(\alpha_{k_2} - 1\right) a_{k_2} \lambda^{\alpha_{k_2} l} \right| \right\}. \tag{3.3}$$

Moreover,

$$\mathcal{L}_{\text{bad}}^{(1)} := \bigcup_{k_1 \neq k_2} \mathcal{L}_{\text{bad}}^{(1)}(k_1, k_2), \quad \text{and} \quad \mathcal{L}_{\text{good}} := \mathcal{L}_{\text{good}}^{(0)} \setminus \mathcal{L}_{\text{bad}}^{(1)}.$$

Analogously, $\mathcal{L}_{\text{good}}$ has at most $m^4$ connected components, in each of which both $Q$ and $Q''$ are "dominated" by a monomial.

**Case 1. Bad scales:** For bad scales, suppose that we are working on the collection of bad scales $\mathcal{L}_{\text{bad}}^{(0)}(k_1, k_2)$ for some $k_1$ and $k_2$, since the arguments can be given similarly for $\mathcal{L}_{\text{bad}}^{(1)}(k_1, k_2)$ with the same bound. Let $\psi_0$ be a nonnegative smooth bump function supported on $\left[\lambda^{-1}, \lambda^2\right]$ such that

$$\sum_{l\in\mathbb{Z}} \psi_l(t) := \sum_{l\in\mathbb{Z}} \psi_0\left(\frac{t}{\lambda^l}\right) = 1, \ \forall t \neq 0.$$

Thus, we may define

$$T_Q^l f(x) = \int_{S^{n-1}} \Omega(y') \int_0^\infty \frac{f(x-ty')}{t} e^{iQ(t)} \psi_l(t) dt d\sigma(y').$$

Recall that the cardinality of $\mathcal{L}_{\text{bad}}^{(0)}(k_1, k_2)$ is at most $4dC_0$. Now we divide the set $\mathcal{L}_{\text{bad}}^{(0)}(k_1, k_2)$ into subsets of continuous elements such that each subset contains exactly $d$ elements and there may be an exception which can be treated in the same way. Therefore, we obtain

$$\begin{aligned}
&\left|\sum_{l\in\mathcal{L}_{bad}^{(0)}(k_1,k_2)} T_Q^l f(x)\right| \\
&\leq \sum_{l\in\mathcal{L}_{bad}^{(0)}(k_1,k_2)} \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| \int_0^\infty \left|f\left(x-ty'\right)\right| \psi_0\left(\frac{t}{\lambda^l}\right) \frac{dt}{|t|} d\sigma(y') \\
&\leq 4C_0 \sum_{l=l_0}^{l_0+d} \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| \int_0^\infty \left|f\left(x-ty'\right)\right| \psi_0\left(\frac{t}{\lambda^l}\right) \frac{dt}{|t|} d\sigma(y') \\
&\leq 4C_0 \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| \int_0^\infty \left|f\left(x-ty'\right)\right| \sum_{l=l_0}^{l_0+d} \psi_0\left(\frac{t}{\lambda^l}\right) \frac{dt}{|t|} dy \\
&\leq 4C_0 \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| \int_{\lambda^{l_0-1}}^{\lambda^{l_0+d+2}} \left|f\left(x-ty'\right)\right| \frac{dt}{|t|} d\sigma(y') \\
&\leq 4C_0\, \lambda^{d+3} \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| M_{y'} f(x) d\sigma(y') \\
&\leq 64C_0 \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| M_{y'} f(x) d\sigma(y').
\end{aligned}$$

By Lemma 2.3, we get

$$\begin{aligned}
\left\|\sum_{l\in\mathcal{L}_{bad}^{(0)}(k_1,k_2)} T_Q^l f\right\|_{L^p(\omega)} &\leq 64C_0 \int_{S^{n-1}} \left|\Omega(y')\right| \left\|M_{y'} f\right\|_{L^p(\omega)} d\sigma(y') \\
&\leq C_{m,\omega} \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\omega)},
\end{aligned} \tag{3.4}$$

where $C_{m,\omega}$ only depends on $m$ and $\omega$.

**Case 2. Good scales:** For the good scales, suppose we are working on one connected component of $\mathcal{L}_{\text{good}}$, and for each integer $l$ in such a component, we assume that $a_{k_1}t^{\alpha_{k_1}}$ dominates $Q(t)$ in the sense of (3.2), that is,

$$\left|a_{k_1}\lambda^{\alpha_{k_1}l}\right| \geq 2^{C_0}\left|a_{k_1'}\lambda^{\alpha_{k_1'}l}\right| \text{ for every } k_1' \neq k_1,$$

and $a_{k_2}\alpha_{k_2}\left(\alpha_{k_2}-1\right)t^{\alpha_{k_2}-2}$ dominates $Q''(t)$ in the sense of (3.3), that is,

$$\left|a_{k_2}\alpha_{k_2}\left(\alpha_{k_2}-1\right)\lambda^{\alpha_{k_2}l}\right| \geq 2^{C_0}\left|a_{k_2'}\alpha_{k_2'}\left(\alpha_{k_2'}-1\right)\lambda^{\alpha_{k_2'}l}\right| \text{ for every } k_2' \neq k_2.$$

Let us call such a set $\mathcal{L}_{\text{good}}(k_1,k_2)$. Under this assumption, we have the estimates

$$|Q(t)| \leq 2\left|a_{k_1}t^{\alpha_{k_1}}\right| \text{ and } \left|Q''(t)\right| \geq \left|a_{k_1}t^{\alpha_{k_1}-2}\right| \tag{3.5}$$

for every $t \in \left[\lambda^{l-2}, \lambda^{l+1}\right]$ with $l \in \mathcal{L}_{\text{good}}(k_1,k_2)$. Recall that $\lambda = 2^{1/d}$ is the smallest scale that we will work with. This scale is only visible when $a_d t^d$ dominates. When some other monomial dominates, at such a small scale, our polynomial will not have enough room to see the oscillation. Define $\lambda_{k_1} := 2^{1/\alpha_{k_1}}$. We choose this scale because the monomial $a_{k_1}t^{\alpha_{k_1}}$ dominates. Let

$$\Phi_{k_1,k_2}(t) = \sum_{l \in \mathcal{L}_{\text{good}}(k_1,k_2)} \psi_l(t).$$

Notice that here we join all the small scales from $\mathcal{L}_{\text{good}}(k_1,k_2)$ to form a larger scale. Next we will apply a new partition of unity to the function $\Phi_{k_1,k_2}$.

Now let $\psi_0^{(k_1)}$ be a nonnegative smooth bump function supported on $\left[\lambda_{k_1}^{-1}, \lambda_{k_1}^{2}\right]$ such that

$$\sum_{l' \in \mathbb{Z}} \psi_{l'}^{(k_1)}(t) = 1 \text{ for every } t > 0, \text{ with } \psi_{l'}^{(k_1)}(t) := \psi_0^{(k_1)}\left(\frac{t}{\lambda_{k_1}^{l'}}\right).$$

Take $B_{k_1} \in \mathbb{Z}$ such that $\lambda_{k_1}^{-B_{k_1}} \leq \left|a_{k_1}\right| < \lambda_{k_1}^{-B_{k_1}+1}$, we let $\gamma_{k_1} = B_{k_1}/\alpha_{k_1}$. Define

$$T_{l'}^{(k_1)}f(x) = \int_{S^{n-1}} \Omega(y') \int_0^\infty f(x-ty')e^{iQ(t)}\psi_{l'}^{(k_1)}(t)\Phi_{k_1,k_2}(t)\frac{dt}{t}d\sigma(y').$$

We split the sum in $l'$ into two cases.

$$\sum_{l' \in \mathbb{Z}} T_{l'}^{(k_1)}f = \sum_{l' \leq \gamma_{k_1}} T_{l'}^{(k_1)}f + \sum_{l' > \gamma_{k_1}} T_{l'}^{(k_1)}f. \tag{3.6}$$

By comparing each term in the first summand, we obtain

$$\left|\sum_{l'\leq\gamma_{k_1}} T_{l'}^{(k_1)} f(x)\right| \leq \left|\sum_{l'\leq\gamma_{k_1}} \int_{S^{n-1}} \Omega(y') \int_0^\infty f(x-ty'))\psi_{l'}^{(k_1)}(t)\Phi_{k_1,k_2}(t)\frac{dt}{t}d\sigma(y')\right| + \sum_{l'\leq\gamma_{k_1}} \left|\int_{S^{n-1}} \Omega(y') \int_0^\infty f(x-ty')\left(e^{iQ(t)}-1\right)\psi_{l'}^{(k_1)}(t)\Phi_{k_1,k_2}(t)\frac{dt}{t}\right|. \tag{3.7}$$

It is easy to see that the second term on the right side of (3.7) is controlled by

$$\begin{aligned}
&\sum_{l'\leq\gamma_{j_1}} \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| \int_{\lambda_{k_1}^{l'-1}}^{\lambda_{k_1}^{l'+2}} \left|f\left(x-ty'\right)\right| \left|a_{k_1} t^{\alpha_{k_1}}\right| \frac{dt}{t} d\sigma(y') \\
&\leq \sum_{l\in\mathbb{N}} \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| \int_{\lambda_{k_1}^{\gamma_{k_1}-l-2}}^{\lambda_{k_1}^{\gamma_{k_1}-l+1}} \left|f\left(x-ty'\right)\right| \left|a_{j_1}\right| |t|^{\alpha_{k_1}-1} dt d\sigma(y') \\
&\leq \sum_{l\in\mathbb{N}} \lambda_{k_1}^{(\gamma_{k_1}-l+1)(\alpha_{k_1}-1)} \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| \int_{\lambda_{k_1}^{\gamma_{k_1}-l-2}}^{\lambda_{k_1}^{\gamma_{k_1}-l+1}} \left|f\left(x-ty'\right)\right| \left|a_{k_1}\right| dt d\sigma(y') \\
&\leq 8 \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| M_{y'} f(x) d\sigma(y').
\end{aligned}$$

Denote $z(t) = \sum_{l'\leq\gamma_{k_1}} \psi_{l'}^{(k_1)}(t)\Phi_{k_1,k_2}(t)$. Thus, the first term on the right side of (3.7) can be dominated by

$$\begin{aligned}
&\left|\int_{S^{n-1}} \Omega\left(y'\right) \int_{\{t\in\mathbb{R}^+: z(t)=1\}} f\left(x-ty'\right) \frac{dt}{t} d\sigma(y')\right| \\
&\quad + \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| \int_{\{t\in\mathbb{R}^+: z(t)\neq 1\}} \left|f\left(x-ty'\right)\right| \frac{dt}{t} d\sigma(y') \\
&\leq \left|p.v. \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y) dy\right| + \sup_{\varepsilon>0} \left|\int_{|y|\geq\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y) dy\right| \\
&\quad + C \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| \int_{\lambda_{k_1}^{\gamma_{k_1}-2}\leq t\leq\lambda_{k_1}^{\gamma_{k_1}+1}} \left|f\left(x-ty'\right)\right| \frac{dt}{|t|} d\sigma(y') \\
&\leq |T_\Omega f(x)| + T_\Omega^* f(x) + C \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| M_{y'} f(x) d\sigma(y').
\end{aligned}$$

In fact, $\sum_{l'\leq\gamma_{k_1}} \psi_{l'}^{(k_1)}(t)$ and $\Phi_{k_1,k_2}(t)$ are finite term intersections, the interval endpoint of $z(t) = 1$ is very close to the cutoff point of $\sum_{l'\leq\gamma_{k_1}} \psi_{l'}^{(k_1)}(t)$. The part of $z(t) \neq 1$ only contains a finite number of bump functions, then it can be controlled by the direction Hardy-Littlewood maximal operator $M_{y'}$.

Using the weighted $L^p$ boundedness of $T_\Omega$, $T_\Omega^*$ and $M_{y'}$ (see Lemma 2.2 and Lemma 2.3), we know that, under the conditions of Theorem 1.5,

$$\left\| \sum_{l' \le \gamma_{k_1}} T_{l'}^{(k_1)} f \right\|_{L^p(\omega)} \le C_{m,\omega} \|\Omega\|_{H^1(S^{n-1})} \|f\|_{L^p(\omega)}, \quad 1 < p < \infty,\ \omega \in \tilde{A}_p^I(\mathbb{R}_+). \tag{3.8}$$

The next part we need to bound

$$\left\| \sum_{l' > \gamma_{k_1}} T_{l'}^{(k_1)} f \right\|_{L^p(\omega)} \le \sum_{l=1}^{\infty} \|T_{\gamma_{k_1}+l}^{(k_1)}\|_{L^p(\omega)}.$$

In order to do it, we notice the pointwise bound

$$\left| T_{\gamma_{k_1}+l}^{(k_1)} f(x) \right| \le 8 \int_{S^{n-1}} \left|\Omega\left(y'\right)\right| M_{y'} f(x) d\sigma(y'), \tag{3.9}$$

which, together with Lemma 2.3, leads to that for $1 < p < \infty$, $\omega \in \tilde{A}_p(\mathbb{R}_+)$,

$$\|T_{\gamma_{k_1}+l}^{(k_1)} f\|_{L^p(\omega)} \le C_\omega \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\omega)}. \tag{3.10}$$

Now we need to obtain exponential decay for the $L^2$ bounds of $T_{\gamma_{k_1}+l}^{(k_1)}$. By Plancherel's theorem, we just need to estimate

$$\left| \int_0^\infty e^{iQ(t)+ity'\cdot\xi} \psi_{\gamma_{k_1}+l}^{(k_1)}(t) \Phi_{k_1,k_2}(t) \frac{dt}{t} \right| = \left| \int_0^\infty e^{iQ\left(\lambda_{k_1}^{\gamma_{k_1}+l} t\right) + i\lambda_{k_1}^{\gamma_{k_1}+l} ty'\cdot\xi} \psi_0^{(k_1)}(t) \Phi_{k_1,k_2}(\lambda_{k_1}^{\gamma_{k_1}+l} t) \frac{dt}{t} \right|.$$

We calculate the second order derivative of the phase function:

$$\lambda_{k_1}^{2\gamma_{k_1}+2l} \left| Q''\left(\lambda_{k_1}^{\gamma_{k_1}+l} t\right)\right| \ge \frac{1}{2} \left|a_{k_1}\right| \lambda_{k_1}^{B_{k_1}+\alpha_{k_1} l} \ge 2^{l-2}.$$

Thus, by van der Corput's lemma (Lemma 2.1), we obtain

$$\left| \int_0^\infty e^{iQ\left(\lambda_{k_1}^{\gamma_{k_1}+l} t\right) + i\lambda_{k_1}^{\gamma_{k_1}+l} ty'\cdot\xi} \psi_0^{(k_1)}(t) \Phi_{k_1,k_2}(\lambda_{k_1}^{\gamma_{k_1}+l} t) \frac{dt}{t} \right| \le C 2^{-\frac{l}{2}}.$$

Therefore, we have

$$\|T_{\gamma_{k_1}+l}^{(k_1)}(f)\|_{L^2} \le C \|\Omega\|_{L^1(S^{n-1})} 2^{-\frac{l}{2}} \|f\|_{L^2}. \tag{3.11}$$

Interpolating between (3.10) with $\omega \equiv 1$ and (3.11), we obtain that for some $\theta > 0$,

$$\|T^{(k_1)}_{\gamma_{k_1}+l}(f)\|_{L^p} \leq C\|\Omega\|_{L^1(S^{n-1})}2^{-\theta l}\|f\|_{L^p} \text{ for } 1 < p < \infty. \tag{3.12}$$

Also, note that for $\omega \in \tilde{A}^I_p(\mathbb{R}_+)$, there is an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}^I_p(\mathbb{R}_+)$. Thus by (3.10), we have

$$\|T^{(k_1)}_{\gamma_{k_1}+l}(f)\|_{L^p(\omega^{1+\varepsilon})} \leq C_\omega\|\Omega\|_{L^1(S^{n-1})}\|f\|_{L^p(\omega^{1+\varepsilon})}. \tag{3.13}$$

Applying the Stein-Weiss interpolation theorem with change of measure (see [21]) between (3.12) and (3.13), we obtain that for $1 < p < \infty$, $\omega \in \tilde{A}^I_p(\mathbb{R}_+)$ and some $\sigma > 0$,

$$\|T^{(k_1)}_{\gamma_{k_1}+l}(f)\|_{L^p(\omega)} \leq C_\omega\|\Omega\|_{L^1(S^{n-1})}2^{-\sigma l}\|f\|_{L^p(\omega)}. \tag{3.14}$$

This, together with (3.4), (3.6) and(3.8), leads to (1.4) and completes the proof of Theorem 1.5.

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## References

1. Al-Qassem, H., Cheng, L., Fukui, A., Pan, Y.: Weighted estimates for oscillatory singular integrals. J. Funct. Spaces Appl., Art. ID 543748, 3 pp (2013)
2. Chanillo, S., Christ, M.: Weak (1, 1) bounds for oscillatory singular integrals. Duke Math. J. **55**(1), 141–155 (1987)
3. Colzani, L.: Hardy Spaces on Spheres, Ph. D. Thesis, Washington University, St. Louis (1982)
4. Ding, Y., Liu, H.: $L^p$ boundedness of Carleson type maximal operators with nonsmooth kernels. Tohoku Math. J. **63**, 255–267 (2011)
5. Ding, Y., Lu, S.: Weighted $L^p$-boundedness for higher order commutators of oscillatory singular integrals. Tohoku Math. J. **48**, 437–449 (1996)
6. Duoandikoetxea, J.: Weighted norm inequalities for homogeneous singular integral. Trans. Amer. Math. Soc. **336**, 869–880 (1993)
7. Fan, D., Pan, Y.: Singular integral operators with rough kernels supported by subvarieties. Amer. J. Math. **119**, 799–839 (1997)
8. Fan, D., Pan, Y., Yang, D.: A weighted norm inequality for rough singular integrals. Tohoku Math. J. **51**(2), 141–161 (1999)
9. Fefferman, C.: Inequalities for strongly singular convolution operators. Acta Math. **124**, 9–36 (1970)
10. García-Cuerva, J., Rubio de Francia, J.L.: Weighted Norm Inequalities and Related Topics, Mathematics Studies, 116, North-Holland (1985)
11. Grafakos, L., Stefanov, A.: Convolution Calderón-Zygmund singular integral operators with rough kernels, analysis of divergence: control and management of divergent processes, pp. 119–143. Birkhauser, Boston-Basel-Berlin (1999)
12. Guo, S.: A remark on oscillatory integrals associated with fewnomials. New York J. Math. **23**, 1733–1738 (2017)
13. Jiang, Y., Lu, S.: Oscillatory singular integrals with rough kernel, Harmonic analysis in China, mathematics and its applications, vol. 327, pp. 135–145. Kluwer Academic Publishers, Amsterdam (1995)
14. Lacey, M.T., Spencer, S.: Sparse bounds for oscillatory and random singular integrals. New York J. Math. **23**, 119–131 (2017)
15. Li, X., Xiao, L.: Uniform estimates for bilinear Hilbert transforms and bilinear maximal functions associated to polynomials. Amer. J. Math. **138**(4), 907–962 (2016)

16. Lu, S., Wu, H.: Oscillatory singular integrals and commutators with rough kernels. Ann. Sci. Math. Québec **27**(1), 47–66 (2003)
17. Lu, S., Zhang, Y.: Criterion on $L^p$-boundedness for a class of oscillatory singular integrals with rough kernels. Rev. Mat. Iberoamericana **8**(2), 201–219 (1992)
18. Ricci, F., Stein, E.M.: Oscillatory singular integrals and harmonic analysis on nilpotent groups. Proc. Nat. Acad. Sci. U.S.A. **83**(1), 1–3 (1986)
19. Ricci, F., Stein, E.M.: Harmonic analysis on nilpotent groups and singular integrals, I. Oscillatory integrals. J. Funct. Anal. **73**(1), 179–194 (1987)
20. Stein, E.M.: Harmonic analysis: real-variable mathods, orthogonality and oscillatory integrals. Princeton University Press, Princeton (1993)
21. Stein, E.M., Weiss, G.: Interpolation of operators with change of measures. Trans. Amer. Math. Soc. **87**, 159–172 (1958)