Journal of Fourier Analysis and Applications (2024) 30:8
https://doi.org/10.1007/s00041-023-10065-9

A Family of Fractal Fourier Restriction Estimates with Implications on the Kakeya Problem

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Received: 18 January 2023 / Revised: 13 July 2023 / Accepted: 12 December 2023 /
Published online: 10 January 2024

Abstract
In a recent paper, Du and Zhang (Ann Math 189:837–861, 2019) proved a fractal Fourier restriction estimate and used it to establish the sharp L^2 estimate on the Schrödinger maximal function in $\mathbb{R}^n$, $n \geq 2$. In this paper, we show that the Du–Zhang estimate is the endpoint of a family of fractal restriction estimates such that each member of the family (other than the original) implies a sharp Kakeya result in $\mathbb{R}^n$ that is closely related to the polynomial Wolff axioms. We also prove that all the estimates of our family are true in $\mathbb{R}^2$.

Keywords Extension operator · Kakeya conjecture · Weighted restriction estimates

Mathematics Subject Classification 42B10 · 42B20 · 28A75

1 Introduction

Let $Ef = E_{\mathcal{P}} f$ be the extension operator associated with the unit paraboloid $\mathcal{P} = \{\xi \in \mathbb{R}^n : \xi_n = \xi_1^2 + \ldots + \xi_{n-1}^2 \leq 1\}$ in $\mathbb{R}^n$:

$$Ef(x) = \int_{\mathbb{B}^{n-1}} e^{-2\pi i x \cdot (\omega, |\omega|^2)} f(\omega) d\omega,$$

where $\mathbb{B}^{n-1}$ is the unit ball in $\mathbb{R}^{n-1}$.

Communicated by Stephane Jaffard.

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Our starting point is the following fractal restriction theorem of Du and Zhang [4]. (Throughout this paper, we denote a cube in $\mathbb{R}^n$ of center x and side-length r by $\widetilde{B}(x,r)$.)

Theorem 1-A (Du and Zhang [4, Corollary 1.6]) *Suppose $n \geq 2$, $1 \leq \alpha \leq n$, $R \geq 1$, $X = \cup_k \widetilde{B}_k$ is a union of lattice unit cubes in $\widetilde{B}(0,R) \subset \mathbb{R}^n$, and*

$$\gamma = \sup \frac{\#\{\widetilde{B}_k : \widetilde{B}_k \subset \widetilde{B}(x',r)\}}{r^\alpha},$$

where the sup is taken over all pairs $(x',r) \in \mathbb{R}^n \times [1,\infty)$ satisfying $\widetilde{B}(x',r) \subset \widetilde{B}(0,R)$. Then to every $\epsilon > 0$ there is a constant C_ϵ such that

$$\int_X |Ef(x)|^2 dx \leq C_\epsilon R^\epsilon \gamma^{2/n} R^{\alpha/n} \|f\|^2_{L^2(\mathbb{B}^{n-1})} \tag{1}$$

for all $f \in L^2(\mathbb{B}^{n-1})$.

In [4], Theorem 1 was used to derive the sharp L^2 estimate on the Schrödinger maximal function (see [4, Theorem 1.3] and the paragraph following the statement of [4, Corollary 1.6]). The authors of [4] also used Theorem 1 to obtain new results on the Hausdorff dimension of the sets where Schrödinger solutions diverge (see [11]), achieve progress on Falconer's distance set conjecture in geometric measure theory (see [6]), and improve on the decay estimates of spherical means of Fourier transforms of measures (see [16]).

The purpose of this paper is threefold:

- Show that Theorem 1 is a borderline sharp Kakeya result in the sense that (1) is the endpoint of a family of estimates (see (2) in the statement of Conjecture 1.1) such that each member of the family (other than (1)) implies a certain sharp Kakeya result that we will formulate in §3 below.
- Show that the sharp Kakeya result is true in certain cases in $\mathbb{R}^3$; see Theorem 4.1.
- Prove Conjecture 1.1 in $\mathbb{R}^2$ (see Theorem 5.1) in the hope that this will shed some light on whether it would be possible to modify the Du-Zhang argument to also prove it in higher dimensions and consequently obtain the Kakeya result without having to pass through the restriction conjecture.

Conjectute 1.1 (when $\beta = 2/n$ or $n = 2$, this is a theorem) *Suppose n, α, R, X, and γ are as in the statement of Theorem 1.*

Let β be a parameter satisfying $1/n \leq \beta \leq 2/n$, and define the exponent p by

$$p = 2 + \frac{n-\alpha}{n-1}\Big(\frac{2}{n} - \beta\Big).$$

Then to every $\epsilon > 0$ there is a constant C_ϵ such that

$$\int_X |Ef(x)|^p dx \leq C_\epsilon R^\epsilon \gamma^\beta R^{\alpha/n} \|f\|^p_{L^p(\mathbb{B}^{n-1})} \tag{2}$$

for all $f \in L^p(\mathbb{B}^{n-1})$.

We note that when $\beta = 2/n$, (2) becomes (1), so, to prove Conjecture 1.1 we need to perform the following trade: lower the power of γ in (1) from $2/n$ to β in return for raising the Lebesgue space exponent from 2 to p.

We will show below that if (2) holds for any $\beta < 2/n$, then we obtain the sharp Kakeya result of §3.

As noted above, in dimension $n = 2$, (2) is true for all $1/2 \leq \beta \leq 1$ (and hence Conjecture 1.1 is a theorem in the plane). We will prove this in the last three sections of the paper by using weighted bilinear restriction estimates and the broad-narrow strategy of [1].

Before we discuss the implications of Conjecture 1.1 to the Kakeya problem, it will be convenient to write (2) in an equivalent form, which is, perhaps, more user-friendly. This is the purpose of the next section.

2 Writing (2) in an Equivalent Form

Suppose $n \geq 1$ and $0 < \alpha \leq n$. Following [12] (see also [3] and [13]), for Lebesgue measurable functions $H : \mathbb{R}^n \to [0, 1]$, we define

$$A_\alpha(H) = \inf \left\{ C : \int_{B(x_0,R)} H(x)dx \leq C R^\alpha \text{ for all } x_0 \in \mathbb{R}^n \text{ and } R \geq 1 \right\},$$

where $B(x_0, R)$ denotes the ball in $\mathbb{R}^n$ of center x_0 and radius R. We say H is a *weight of fractal dimension* α if $A_\alpha(H) < \infty$. We note that $A_\beta(H) \leq A_\alpha(H)$ if $\beta \geq \alpha$, so we are not really assigning a dimension to the function H; the phrase "H is a weight of dimension α" is merely another way for us to say that $A_\alpha(H) < \infty$.

Proposition 2.1 *Suppose n, α, R, X, γ, β, and p are as in the statement of Conjecture 1.1. Then the estimate (2) holds if and only if to every $\epsilon > 0$ there is a constant C_ϵ such that*

$$\int_{B(0,R)} |Ef(x)|^p H(x)dx \leq C_\epsilon R^\epsilon A_\alpha(H)^\beta R^{\alpha/n} \|f\|^p_{L^p(\mathbb{B}^{n-1})} \tag{3}$$

for all functions $f \in L^p(\mathbb{B}^{n-1})$ and weights H of fractal dimension α.

Proof Let H be the characteristic function of X. By the definition of γ, we have

$$\int_{\widetilde{B}(x_0,r)} H(x)dx \leq \gamma\, (r+2)^\alpha \leq \gamma\, (3r)^\alpha$$

for all $x_0 \in \mathbb{R}^n$ and $r \geq 1$. Thus H is a weight on $\mathbb{R}^n$ of fractal dimension α, and $A_\alpha(H) \leq 3^\alpha \gamma$. This immediately shows that (3) implies (2).

To prove the reverse implication, we follow [4, Proof of Theorem 2.2].

We consider a covering $\{\widetilde{B}\}$ of $B(0, R)$ by unit lattice cubes. Since every unit cube is contained in a ball of radius $\sqrt{n}$, we have $\int_{\widetilde{B}} H(x)dx \leq A_\alpha(H)n^{\alpha/2}$, so, if we define $v(\widetilde{B}) = A_\alpha(H)^{-1} \int_{\widetilde{B}} H(x)dx$ and $V_k = \{\widetilde{B} : 2^{k-1} < n^{-\alpha/2}v(\widetilde{B}) \leq 2^k\}$, then

$$B(0, R) \subset \cup \widetilde{B} \subset \cup_{k=-\infty}^{0} V_k.$$

We note that

$$\begin{aligned} \int_{\widetilde{B}} H(x)dx &\leq \Big(\int_{\widetilde{B}} H(x)^{1/\beta}dx\Big)^\beta \leq \Big(\int_{\widetilde{B}} H(x)dx\Big)^\beta \\ &= \Big(A_\alpha(H)v(\widetilde{B})\Big)^\beta \leq n^{\alpha\beta/2}A_\alpha(H)^\beta 2^{k\beta} \end{aligned} \tag{4}$$

for all $\widetilde{B} \in V_k$, where we have used the assumptions $\beta \leq 2/n \leq 1$ and $\|H\|_{L^\infty} \leq 1$.

The vast majority of the sets V_k are negligible for us. In fact, letting k_1 be the sup of the set $\{k \in \mathbb{Z} : 2^k \leq R^{-1000n/\beta}\}$, we see that

$$\begin{aligned} \int_{\cup_{k=-\infty}^{k_1}\cup_{\widetilde{B}\in V_k}} |Ef(x)|^p H(x)dx &\leq \|f\|^p_{L^1(\mathbb{B}^{n-1})} \sum_{k=-\infty}^{k_1} \sum_{\widetilde{B}\in V_k} \int_{\widetilde{B}} H(x)dx \\ &\leq CA_\alpha(H)^\beta \|f\|^p_{L^1(\mathbb{B}^{n-1})} \sum_{k=-\infty}^{k_1} R^n 2^{k\beta} \\ &\leq CR^{-999n}A_\alpha(H)^\beta \|f\|^p_{L^1(\mathbb{B}^{n-1})}, \end{aligned}$$

where we used (4) on the line before the last, and the fact that $2^{k_1} \leq R^{-1000n/\beta}$ on the last line. Therefore, we only need to estimate

$$\int_{\cup_{k=k_1+1}^{0}\cup_{\widetilde{B}\in V_k}} |Ef(x)|^p H(x)dx = \sum_{k=k_1+1}^{0} \sum_{\widetilde{B}\in V_k} \int_{\widetilde{B}} |Ef(x)|^p H(x)dx.$$

Letting $k_0 \in \{k_1 + 1, k_1 + 2, \ldots, 0\}$ be the integer satisfying

$$\sum_{\widetilde{B}\in V_{k_0}} \int_{\widetilde{B}} |Ef(x)|^p H(x)dx = \max_{k_1+1\leq k\leq 0}\Big[\sum_{\widetilde{B}\in V_k} \int_{\widetilde{B}} |Ef(x)|^p H(x)dx\Big],$$

we see that

$$\begin{aligned} &\int_{B(0,R)} |Ef(x)|^p H(x)dx \\ &\quad \leq (-k_1) \sum_{\widetilde{B}\in V_{k_0}} \int_{\widetilde{B}} |Ef(x)|^p H(x)dx + CR^{-999n}A_\alpha(H)^\beta \|f\|^p_{L^1(\mathbb{B}^{n-1})}. \end{aligned} \tag{5}$$

Since $-k_1 \lesssim \log(2R)$, it follows that we only need to estimate

$$\sum_{\widetilde{B} \in V_{k_0}} \int_{\widetilde{B}} |Ef(x)|^p H(x) dx.$$

We start by using the uncertainty principle in the following form. Let $d\sigma$ be the pushforward of the $(n-1)$-dimensional Lebesgue measure under the map $T : \mathbb{B}^{n-1} \to \mathcal{P}$ given by $T(\omega) = (\omega, |\omega|^2)$. Since the measure $d\sigma$ is compactly supported and $Ef = \widehat{g d\sigma}$, where g is the function on $\mathcal{P}$ defined by the equation $f = g \circ T$, it follows that there is a non-negative rapidly decaying function ψ on $\mathbb{R}^n$ such that

$$\sup_{\widetilde{B}} |Ef|^p \lesssim |Ef|^p * \psi(c(\widetilde{B})),$$

where $c(\widetilde{B})$ is the center of $\widetilde{B}$. Thus

$$\int_{\widetilde{B}} |Ef(x)|^p H(x) dx \lesssim \Big(\int_{\widetilde{B}} H(x) dx \Big) |Ef|^p * \psi(c(\widetilde{B})).$$

From (4) we know that $\int_{\widetilde{B}} H(x)dx \lesssim A_\alpha(H)^\beta 2^{k_0\beta}$ for all $\widetilde{B} \in V_{k_0}$. Also,

$$\begin{aligned} |Ef|^p * \psi(c(\widetilde{B})) &= \int_{B(c(\widetilde{B}), R^\epsilon)} |Ef(x)|^p \psi(c(\widetilde{B}) - x) dx \\ &\quad + \int_{B(c(\widetilde{B}), R^\epsilon)^c} |Ef(x)|^p \psi(c(\widetilde{B}) - x) dx \\ &\lesssim \int_{B(c(\widetilde{B}), R^\epsilon)} |Ef(x)|^p dx + R^{-1000n} \|f\|^p_{L^1(\mathbb{B}^{n-1})} \end{aligned}$$

and

$$\sum_{\widetilde{B} \in V_{k_0}} \chi_{B(c(\widetilde{B}), R^\epsilon)} \lesssim R^{n\epsilon}, \tag{6}$$

so

$$\begin{aligned} &\sum_{\widetilde{B} \in V_{k_0}} \int_{\widetilde{B}} |Ef(x)|^p H(x) dx \\ &\quad \lesssim R^{n\epsilon} A_\alpha(H)^\beta 2^{k_0\beta} \int_V |Ef(x)|^p dx + A_\alpha(H)^\beta R^{-999n} \|f\|^p_{L^1(\mathbb{B}^{n-1})}, \end{aligned} \tag{7}$$

where $V = \cup_{\widetilde{B} \in V_{k_0}} B(c(\widetilde{B}), R^\epsilon)$.

We now let $\{\widetilde{B}^*\}$ be the set of all the unit lattice cubes that intersect V, and $X = \cup \widetilde{B}^*$. We plan to apply (2) on this set X, but we first need to estimate γ.

Let B_r be a ball in $\mathbb{R}^n$ of radius $r \geq R^\epsilon$ (if $1 \leq r \leq R^\epsilon$, then, clearly, $\#\{\widetilde{B}^* : \widetilde{B}^* \subset B_r\} \lesssim R^{n\epsilon}$), and V_r the subset of V_{k_0} that consists of all unit cubes $\widetilde{B}$ such that $B(c(\widetilde{B}), 2R^\epsilon) \cap B_r \neq \emptyset$. If B_r intersects any of the cubes $\widetilde{B}^*$ that make up X, then B_r intersects $B(c(\widetilde{B}), 2R^\epsilon)$ for some $\widetilde{B} \in V_r$. Therefore,

$$\#\{\widetilde{B}^* : \widetilde{B}^* \subset B_r\} \lesssim R^{n\epsilon}\#(V_r).$$

Our assumption $r \geq R^\epsilon$, tells us that

$$\cup_{\widetilde{B}\in V_r} B(c(\widetilde{B}), 2R^\epsilon) \subset B_{5r},$$

so (using (6))

$$\begin{aligned} R^{n\epsilon}\int_{B_{5r}} H(x)dx &\gtrsim \sum_{\widetilde{B}\in V_r}\int_{B(c(\widetilde{B}),2R^\epsilon)} H(x)dx \\ &\geq \sum_{\widetilde{B}\in V_r}\int_{\widetilde{B}} H(x)dx = \sum_{\widetilde{B}\in V_r} v(\widetilde{B})A_\alpha(H) \geq \#(V_r)\, n^{\alpha/2}2^{k_0-1}A_\alpha(H). \end{aligned}$$

On the other hand,

$$\int_{B_{5r}} H(x)dx \leq A_\alpha(H)(5r)^\alpha,$$

so $\#(V_r) \lesssim R^{n\epsilon}2^{-k_0}r^\alpha$, and so

$$\#\{\widetilde{B}^* : \widetilde{B}^* \subset B_r\} \lesssim R^{2n\epsilon}2^{-k_0}r^\alpha.$$

Therefore, $\gamma \lesssim R^{2n\epsilon}2^{-k_0}$.

Applying (2), we now obtain

$$\int_V |Ef(x)|^p dx \leq \int_X |Ef(x)|^p dx \lesssim R^{5\epsilon}2^{-k_0\beta}R^{\alpha/n}\|f\|^p_{L^p(\mathbb{B}^{n-1})},$$

which, combined with (5) and (7), implies that

$$\begin{aligned} \int_{B(0,R)} |Ef(x)|^p H(x)dx &\lesssim R^{(n+6)\epsilon}(2^{k_0})^{\beta-\beta}A_\alpha(H)^\beta R^{\alpha/n}\|f\|^p_{L^p(\mathbb{B}^{n-1})} \\ &= R^{(n+6)\epsilon}A_\alpha(H)^\beta R^{\alpha/n}\|f\|^p_{L^p(\mathbb{B}^{n-1})}, \end{aligned}$$

which is our desired estimate (3).

3 Conjecture 1.1 Implies a Sharp Kakeya Result

Let Ω be a subset of $\mathbb{R}^n$ that obeys the following property: there is a number α between 1 and n such that

$$|\Omega \cap B_R| \leq CR^\alpha \tag{8}$$

for all balls B_R in $\mathbb{R}^n$ of radius $R \geq 1$. (Given $E \subset \mathbb{R}^n$ a Lebesgue measurable set, we let $|E|$ denote its Lebesgue measure.)

For large L, we divide the unit paraboloid $\mathcal{P}$ into finitely overlapping caps θ_j each of radius L^{-1}, and we associate with each θ_j a family $\mathbb{T}_j$ of parallel $1 \times L$ tubes that tile $\mathbb{R}^n$ and point in the direction normal to θ_j at its center. We let N be the cardinality of the set

$$J = \{j : \text{there is a tube of } \mathbb{T}_j \text{ that lies in } \Omega \cap B(0, 5L)\}. \tag{9}$$

It is easy to see that the Kakeya conjecture (in its maximal operator form) implies the following bound on N: to every $\epsilon > 0$ there is a constant C_ϵ such that

$$N \leq C_\epsilon L^\epsilon L^{\alpha-1} \tag{10}$$

for all $L \geq 1$. In fact, [2, Proposition 2.2] presents a proof of the fact that the Kakeya conjecture implies (10) in the case when Ω is a neighborhood of an algebraic variety. This proof easily extends to general sets Ω satisfying (8). (For the connection between neighborhoods of algebraic varieties and the condition (8), we refer the reader to [14].)

We note that (10) implies that if $\Omega \cap B(0, 5L)$ contains at least one tube from each direction (i.e. at least one tube from each of the $\sim L^{n-1}$ families $\mathbb{T}_j$), then $\alpha = n$.

In the special case when Ω is a neighborhood of an algebraic variety, this bound on N was proved by Guth [7] in $\mathbb{R}^3$, conjectured by Guth [8] to be true in $\mathbb{R}^n$ for all $n \geq 3$, and proved by Zahl [17] in $\mathbb{R}^4$; see also [9]. The conjecture of [8] was then settled in all dimensions by Katz and Rogers in [10].

In this section we prove that Conjecture 1.1 about the extension operator implies that all sets $\Omega \subset \mathbb{R}^n$ that satisfy the dimensionality condition (8) will also possess the Kakeya property (10). Here is the precise statement.

Theorem 3.1 *Suppose* (3) *(or equivalently* (2)*) holds for some* $1/n \leq \beta < 2/n$. *Then (10) holds for all Lebesgue measurable sets* $\Omega \subset \mathbb{R}^n$ *that obey* (8).

Proof We first write the set J as $\{j_1, j_2, \ldots, j_N\}$, and for each $1 \leq l \leq N$, we let T_l be a tube from $\mathbb{T}_{j_l}$ that lies in $\Omega \cap B(0, 5L) = \Omega \cap B_{5L}$. Then

$$\begin{aligned} NL = \sum_{l=1}^{N} |T_l| &= \sum_{l=1}^{N} \int_{B_{5L}\cap\Omega} \chi_{T_l}(x)\, dx = \int_{B_{5L}\cap\Omega} \sum_{l=1}^{N} \chi_{T_l}(x)\, dx \\ &= L^{2(n-1)} \int_{B_{5L}\cap\Omega} \sum_{l=1}^{N} \Big(\frac{1}{L^{n-1}}\, \chi_{T_l}(x)\Big)^2 dx. \end{aligned} \tag{11}$$

Recall that $\mathbb{T}_{j_l}$ is a family of parallel $1 \times L$ tubes that tile $\mathbb{R}^n$ and point in the direction normal to the L^{-1}-cap θ_{j_l}. The projection of θ_{j_l} into $\mathbb{B}^{n-1}$ is an L^{-1}-ball.

We denote this ball by B_l and let ω_l be its center and χ_l its characteristic function. Then

$$|E\chi_l(x)| = \Big|\int_{B_l} e^{-2\pi i x\cdot(\omega,|\omega|^2)}d\omega\Big| = \Big|\int_{B_l} e^{-2\pi i\left((x_1+2x_2\omega_l)(\omega-\omega_l)+x_2|\omega-\omega_l|^2\right)}d\omega\Big|$$

for all $x = (x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Since $|\omega - \omega_l| \le L^{-1}$ for all $\omega \in B_l$, it follows that $|E\chi_l(x)| \gtrsim |B_l| \sim L^{-(n-1)}$ on the set $\{x \in \mathbb{R}^n : |x_1 + 2x_2\omega_l| \lesssim L \text{ and } |x_2| \lesssim L^2\}$, and hence $|E\chi_l(Lx)| \gtrsim L^{-(n-1)}$ on the set $\{x \in \mathbb{R}^n : |x_1+2x_2\omega_l| \lesssim 1 \text{ and } |x_2| \lesssim L\}$. Since $|\omega_l| \le 1$, this last set contains a $1 \times L$ tube $\widetilde{T}_l$ that is parallel to the normal vector of the cap θ_{j_l} at its center $(\omega_l, |\omega_l|^2)$. Moreover,

$$|E\chi_l(Lx)| \gtrsim \frac{1}{L^{n-1}}\,\chi_{\widetilde{T}_l}(x)$$

for all $x \in \mathbb{R}^n$.

The tube $\widetilde{T}_l$ is parallel to the tube T_l that we chose at the beginning of the proof and has the same dimensions, so $T_l = v + \widetilde{T}_l$ for some vector $v \in \mathbb{R}^n$, and so

$$|E\chi_l(Lx)| \gtrsim \frac{1}{L^{n-1}}\,\chi_{T_l}(x+v)$$

for all $x \in \mathbb{R}^n$. Defining the function f_l on $\mathbb{R}^{n-1}$ by

$$f_l(\omega) = e^{2\pi i Lv\cdot(\omega,|\omega|^2)}\chi_l(\omega),$$

we see that $Ef_l(x) = E\chi_l(x - Lv)$, so that

$$|Ef_l(Lx)| = |E\chi_l(Lx - Lv)| = |E\chi_l(L(x-v))| \gtrsim \frac{1}{L^{n-1}}\,\chi_{T_l}(x)$$

for all $x \in \mathbb{R}^n$. Returning to (11) and letting $H = \chi_\Omega$, we arrive at

$$NL \lesssim L^{2(n-1)} \int_{B_{5L}} \sum_{l=1}^{N} |Ef_l(Lx)|^2 H(x)dx.$$

Next, we let $\epsilon_l = \pm 1$ be random signs, define the function $f : \mathbb{B}^{n-1} \to \mathbb{C}$ by $f = \sum_{l=1}^N \epsilon_l f_l$, and use Khintchin's inequality to get

$$NL \lesssim L^{2(n-1)}\,\mathbb{E}\Big(\int_{B_{5L}} |Ef(Lx)|^2 H(x)dx\Big),$$

where $\mathbb{E}$ is the expectation sign. Since $p \geq 2$, we can apply Hölder's inequality in the inner integral to get

$$\begin{aligned} NL &\lesssim L^{2(n-1)}\Big(\int_{B_{5L}} H(x)dx\Big)^{1-(2/p)}\,\mathbb{E}\Big(\int_{B_{5L}} |Ef(Lx)|^p H(x)dx\Big)^{2/p} \\ &\lesssim L^{2(n-1)}L^{\alpha(1-(2/p))}\,\mathbb{E}\Big(\int_{B_{5L}} |Ef(Lx)|^p H(x)dx\Big)^{2/p}. \end{aligned}$$

Applying the change of variables $u = Lx$ and defining the weight H^* by $H^*(u) = H(x) = H(u/L)$, this becomes

$$NL \lesssim L^{2(n-1)}L^{\alpha(1-(2/p))}L^{-2n/p}\,\mathbb{E}\Big(\int_{B_{5L^2}} |Ef(u)|^p H^*(u)du\Big)^{2/p},$$

so that

$$NL^{3-n} \lesssim L^{(n+\alpha)(1-(2/p))}\,\mathbb{E}\Big(\int_{B_{5L^2}} |Ef(u)|^p H^*(u)du\Big)^{2/p}. \tag{12}$$

We note that

$$\begin{aligned} \int_{B(u_0,R)} H^*(u)du &= L^n \int_{B(u_0/L,R/L)} H(x)dx \\ &\leq L^n A_\alpha(H)\Big(\frac{R}{L}\Big)^\alpha = L^{n-\alpha}A_\alpha(H)R^\alpha \end{aligned}$$

if $R \geq L$. On the other hand, if $R \leq L$, then

$$\int_{B(u_0,R)} H^*(u)du \lesssim R^n = R^{n-\alpha}R^\alpha \leq L^{n-\alpha}R^\alpha.$$

Therefore,

$$A_\alpha(H^*) \lesssim L^{n-\alpha}.$$

We are now in a good shape to apply (3), which tells us that

$$\begin{aligned} \int_{B_{5L^2}} |Ef(u)|^p H^*(u)du &\lesssim (L^2)^\epsilon A_\alpha(H^*)^\beta (L^2)^{\alpha/n}\|f\|^p_{L^p(\mathbb{B}^{n-1})} \\ &\lesssim L^{2\epsilon}L^{(n-\alpha)\beta}L^{2\alpha/n}\frac{N}{L^{n-1}}. \end{aligned}$$

Inserting this back in (12), we get

$$NL^{3-n} \lesssim L^{2\epsilon}L^{(n+\alpha)(1-(2/p))}\Big(L^{(n-\alpha)\beta}L^{2\alpha/n}L^{1-n}N\Big)^{2/p},$$

so that

$$N^{1-(2/p)}L^{3-n} \lesssim L^{2\epsilon}L^{(n+\alpha)(1-(2/p))}\left(L^{(n-\alpha)(\beta-\frac{2}{n})+2-\frac{2\alpha}{n}}L^{\frac{2\alpha}{n}}\frac{L^{-2}}{L^{n-3}}\right)^{2/p},$$

so that

$$N^{1-(2/p)} \lesssim L^{2\epsilon}L^{(n-3)(1-(2/p))}L^{(n+\alpha)(1-(2/p))}L^{(n-\alpha)(\beta-\frac{2}{n})(\frac{2}{p})}. \tag{13}$$

Therefore,

$$N \lesssim L^{O(\epsilon)}L^{n-3}L^{\alpha}L^{n}L^{\frac{(n-\alpha)(\beta-\frac{2}{n})(\frac{2}{p})}{1-\frac{2}{p}}} = L^{O(\epsilon)}L^{2n-3+\alpha}L^{\frac{(n-\alpha)(\beta-\frac{2}{n})}{\frac{p}{2}-1}}.$$

But

$$\frac{(n-\alpha)(\beta-\frac{2}{n})}{\frac{p}{2}-1} = (n-\alpha)\left(\beta-\frac{2}{n}\right)\frac{2(n-1)}{(n-\alpha)(\frac{2}{n}-\beta)} = -2(n-1) = 2-2n,$$

so

$$N \lesssim L^{O(\epsilon)}L^{2n-3+\alpha+2-2n} = L^{O(\epsilon)}L^{\alpha-1}.$$

At this point, it might be helpful for the reader to observe how the above argument breaks down in the $p=2$ case: recalling that

$$p = 2 + \frac{n-\alpha}{n-1}\Big(\frac{2}{n}-\beta\Big),$$

we see that $\beta = 2/n$ and (13) becomes $1 \lesssim L^{2\epsilon}$, which tells us nothing.

4 Proof of (10) in the Regime $1 \le \alpha \le 2$ in $\mathbb{R}^3$

The fact that the Kakeya conjecture is true in $\mathbb{R}^2$ tells us that (10) is also true there. In this section, we use Wolff's hairbrush argument from [15], as adapted by Guth in [7], to prove the following bound on N.

Theorem 4.1 *In $\mathbb{R}^3$, we have*

$$N \lesssim \begin{cases} (\log L)^2 L^{\alpha-1} & if 1 \le \alpha \le 2, \\ (\log L)^2 L^{2\alpha-3} & if 2 \le \alpha \le 3. \end{cases}$$

Proof Let Ω be a subset of $\mathbb{R}^3$ that obeys (8). As we did in the previous section, for large L, we consider a decomposition $\{\theta_j\}$ of $\mathcal{P}$ into finitely overlapping caps each of radius L^{-1}, and we associate with each θ_j a family $\mathbb{T}_j$ of parallel $1 \times L$ tubes that tile $\mathbb{R}^3$ and point in the direction of the normal vector v_j of $\mathcal{P}$ at the center of θ_j. The quantity N that we need to estimate is the cardinality of the set J as defined in (9).

For each $j \in J$, we let T_j be a member of $\mathbb{T}_j$ that lies in $\Omega \cap B(0, 5L)$, and $S = \{T_j\}$. Of course, $N = \#(S)$.

We tile $\Omega \cap B(0, 5L)$ by unit lattice cubes $\widetilde{B}$. Then (8) tells us that

$$\#(\{\widetilde{B}\}) \lesssim L^{\alpha}. \tag{14}$$

Also, each tube T_j intersects $\sim L$ of the cubes $\widetilde{B}$.

We now define the function $f : \{\widetilde{B}\} \to \mathbb{Z}$ by

$$f(\widetilde{B}) = \#\{T_j \in S : T_j \cap \widetilde{B} \neq \emptyset\}.$$

Then

$$\sum_{\widetilde{B}} f(\widetilde{B}) \sim NL.$$

So, by Cauchy–Schwarz and (14),

$$NL \lesssim \Big(\sum_{\widetilde{B}} f(\widetilde{B})^2\Big)^{1/2} \Big(\#(\{\widetilde{B}\})\Big)^{1/2} \lesssim \Big(\sum_{\widetilde{B}} f(\widetilde{B})^2\Big)^{1/2} L^{\alpha/2},$$

and so

$$\sum_{\widetilde{B}} f(\widetilde{B})^2 \gtrsim N^2 L^{2-\alpha},$$

which means that the set

$$\{(\widetilde{B}, T_i, T_j) : T_i, T_j \in S,\ T_i \cap \widetilde{B} \neq \emptyset,\ \text{and } T_j \cap \widetilde{B} \neq \emptyset\}$$

has cardinality $\gtrsim N^2 L^{2-\alpha}$. Therefore, the set

$$X = \{(\widetilde{B}, T_i, T_j) : T_i, T_j \in S,\ T_i \cap \widetilde{B} \neq \emptyset,\ T_j \cap \widetilde{B} \neq \emptyset \text{ and } i \neq j\}$$

has cardinality

$$\geq C_1 N^2 L^{2-\alpha} - \sum_{\widetilde{B}} f(\widetilde{B}) \geq C_1 N^2 L^{2-\alpha} - C_2 NL.$$

If $C_1 N^2 L^{2-\alpha} \leq 5C_2 NL$, then $N \leq (5C_2/C_1) L^{\alpha-1}$ and the theorem will be proved. So, we may assume that $N \geq C_3 L^{\alpha-1}$ for some large constant C_3. Therefore, $\#(X) \gtrsim N^2 L^{2-\alpha}$.

For $l \in \mathbb{N}$, we define X_l to be the subset of X for which

$$\frac{2^{l-1}}{L} \leq \mathrm{Angle}(v_i, v_j) \leq \frac{2^l}{L}.$$

Since the angle between any two tubes in our set S ranges between L^{-1} and 1, it follows by the pigeonhole principle that $\#(X) \lesssim (\log L)\#(X_{l_0})$ for some $l_0 \in \mathbb{N}$. Denoting $2^{l_0}L^{-1}$ by θ, and X_{l_0} by X', we have $L^{-1} \leq \theta \leq 1$ and $\#(X') \gtrsim N^2L^{2-\alpha}(\log L)^{-1}$.

There are N tubes in S. By the pigeonhole principle, one of the tubes must appear in $\gtrsim N^2L^{2-\alpha}(\log L)^{-1}/N = NL^{2-\alpha}(\log L)^{-1}$ of the elements of X'. We call this tube T, and we define

$$\mathbb{H} = \{T_j \in S : (\widetilde{B}, T, T_j) \in X'\}.$$

Let v be the direction of the tube T. Since the angle between v and v_j is $\sim \theta$, it follows that $|T \cap T_j| \lesssim \theta^{-1}$. So, the set $\{\widetilde{B} : (\widetilde{B}, T, T_j) \in X'\}$ has cardinality $\lesssim \theta^{-1}$, and so

$$\#(\mathbb{H}) \gtrsim \frac{NL^{2-\alpha}(\log L)^{-1}}{\theta^{-1}} = \theta NL^{2-\alpha}(\log L)^{-1}.$$

To finish the proof, we need to also have an upper bound on $\#(\mathbb{H})$. We first observe that

$$\bigcup_{T_j \in \mathbb{H}} T_j \subset \Omega \cap \mathbf{B},$$

where $\mathbf{B}$ is a box in $\mathbb{R}^3$ of dimensions $L \times \theta L \times \theta L$. Since $\mathbf{B}$ can be covered by $\sim L/(\theta L)$ balls of radius θL, and since $\theta L \geq 1$, the dimensionality property (8) tells us that

$$\Big|\bigcup_{T_j \in \mathbb{H}} T_j\Big| \lesssim \theta^{-1}(\theta L)^{\alpha}.$$

Next, we use the (by now) standard fact that the tubes T_j in $\mathbb{H}$ are morally disjoint (see [7, Lemma 4.9] for a very nice explanation of this idea) to see that

$$\Big|\bigcup_{T_j \in \mathbb{H}} T_j\Big| \gtrsim \frac{\#(\mathbb{H})\,|T_j|}{\log L} = \frac{\#(\mathbb{H})\,L}{\log L}.$$

Therefore,

$$\#(\mathbb{H}) \lesssim (\log L)\theta^{-1}L^{-1}(\theta L)^{\alpha} = (\log L)(\theta L)^{\alpha-1}.$$

Comparing the lower and upper bounds we now have on the cardinality of $\mathbb{H}$, we conclude that

$$\theta NL^{2-\alpha}(\log L)^{-1} \lesssim (\log L)(\theta L)^{\alpha-1}.$$

Therefore,

$$N \lesssim (\log L)^2\theta^{\alpha-2}L^{2\alpha-3}.$$

If $\alpha \geq 2$, then the fact that $\theta \leq 1$ tells us that

$$N \leq (\log L)^2 L^{2\alpha-3}.$$

If $1 \leq \alpha < 2$, then the fact that $\theta \geq 1/L$ tells us that

$$N \lesssim (\log L)^2 L^{2-\alpha} L^{2\alpha-3} = (\log L)^2 L^{\alpha-1}.$$

It might be interesting for the reader to observe that the sharp result that we get in the case $1 \leq \alpha < 2$ is due to the fact that we are using 'substantial' information about θ (namely, $\theta \geq 1/L$), whereas in the case $2 \leq \alpha \leq 3$ we only can use the relatively 'unsubstantial' information that $\theta \leq 1$.

We note that if $\Omega \subset \mathbb{R}^3$ obeys (8) and $\Omega \cap B(0, 5L)$ contains at least one tube from each direction (i.e. at least one tube from each of the $\sim L^2$ families $\mathbb{T}_j$), then Theorem 4.1 implies that $\alpha \geq 5/2$ (cf. [15]).

5 Proof of Conjecture 1.1 in the Plane

The rest of the paper is concerned in proving that Conjecture 2.1 is true in $\mathbb{R}^2$. In view of Proposition 2.1, this task will be accomplished as soon as we prove Theorem 5.1 below.

We alert the reader that the extension operator in Theorem 5.1 is the one associated with the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ and is given by

$$Ef(x) = \int_{\mathbb{S}^1} e^{-2\pi i x\cdot\xi} f(\xi) d\sigma(\xi)$$

for $f \in L^1(\sigma)$, where σ is induced Lebesgue measure on $\mathbb{S}^1$. The proof for the extension operator associated with the unit parabola is similar (and a little easier).

Theorem 5.1 *Suppose* $1 \leq \alpha \leq 2$ *and* $R \geq 1$. *Let* β *be a parameter satisfying* $1/2 \leq \beta \leq 1$, *and define the exponent* p *by*

$$p = 2 + (2-\alpha)(1-\beta).$$

Then to every $\epsilon > 0$ *there is a constant* C_ϵ *such that*

$$\int_{B(0,R)} |Ef(x)|^p H(x) dx \leq C_\epsilon R^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \|f\|^p_{L^p(\sigma)} \tag{15}$$

for all functions $f \in L^p(\sigma)$ *and weights* H *of fractal dimension* α.

The proof of Theorem 5.1 will use ideas from [16], [5], [12], and [4]. The overarching idea, however, is the broad-narrow strategy of [1]. Implementing this strategy involves

- proving a bilinear estimate (see (22) in Subsection 7.1 below) that will be used to control Ef on the broad set
- proving a linear estimate (see (28) in Subsection 7.2 below) that will be used to establish (15) when the function f is supported on an arc of small size (i.e. σ-measure), which will provide the base of a recursive process
- carrying out a recursive process on the size of the function's support that will establish (15) for general f.

The main new idea in the proof of Theorem 5.1 is a localization of the weight argument that will help us in deriving the bilinear estimate (22). We use this argument to take advantage of the locally constant property of the Fourier transform, and we will end this section by formulating the intuition that lies behind it in a lemma.

Given a function $f : \mathbb{R}^n \to \mathbb{C}$ and a number $K > 0$, we say that f is essentially constant at scale K if there is a constant C such that

$$\sup_{Q_K} |f| \leq C \inf_{Q_K} |f| \tag{16}$$

for all cubes $Q_K \subset \mathbb{R}^n$ of side-length K.

Lemma 5.1 *Suppose $1 \leq \alpha \leq 2$, $1/2 \leq \beta \leq 1$, $R > K^2 \geq 1$, and Q is a box in $\mathbb{R}^2$ of dimensions[1] $R/K \times R$. Also, suppose that f is a non-negative function on $\mathbb{R}^2$ that is essentially constant at scale K, and H is a weight on $\mathbb{R}^2$ of fractal dimension α. Then*

$$\int_Q f(x)H(x)dx \lesssim K^{-m} A_\alpha(H)^\beta R^{\alpha/2} \|f\|_{L^2(\widetilde{Q})}$$

for some $m \geq 0$ (in fact, $m = \beta - (1/2) + (1-\beta)(\alpha - 1)$), where $\widetilde{Q}$ is a box of dimensions $2R/K \times 2R$ that has the same center as Q, and the implicit constant depends only on α and β and the constant C from (16).

Proof We tile $\mathbb{R}^2$ by cubes $\widetilde{B}_l$ of side-length K. If $\widetilde{B}_l \cap Q \neq \emptyset$, we let c_l be the center of $\widetilde{B}_l$ and write

$$\begin{aligned}\int_Q f(x)H(x)dx &= \sum_l \int_{\widetilde{B}_l \cap Q} f(x)H(x)dx \lesssim \sum_l f(c_l) \int_{\widetilde{B}_l} H(x)dx \\ &= \sum_l K^{-2} \int_{\widetilde{B}_l} f(c_l)H'(y)dy \lesssim K^{-2} \int_{\widetilde{Q}} f(y)H'(y)dy,\end{aligned}$$

where $H' : \mathbb{R}^2 \to [0, \infty)$ is given by

$$H'(y) = \int_{\widetilde{B}_l} H(x)dx \qquad \text{for} \qquad y \in \widetilde{B}_l.$$

[1] Boxes of such dimensions are a common feature in this context; see [4, Subsection 3.2] and Subsection 6.2 below.

For $y \in \widetilde{B}_l$, we have

$$\begin{aligned} H'(y) &= \Big(\int_{\widetilde{B}_l} H(x)dx\Big)^{1-\theta}\Big(\int_{\widetilde{B}_l} H(x)dx\Big)^{\theta} \\ &\leq K^{2(1-\theta)} A_\alpha(H)^\theta (\sqrt{2}K)^{\alpha\theta}, \end{aligned}$$

where $0 \leq \theta \leq 1$ is a parameter that will be determined later in the argument.

Next, we define the function $\mathcal{H} : \mathbb{R}^2 \to [0, 1]$ by

$$\mathcal{H}(y) = 2^{-\alpha\theta/2} A_\alpha(H)^{-\theta} K^{-2(1-\theta)-\alpha\theta} H'(y)$$

and observe that

$$\begin{aligned} \int_{B(x_0,r)} \mathcal{H}(y)dy &\leq K^2 A_\alpha(H)^{-\theta} K^{-2(1-\theta)-\alpha\theta} \int_{B(x_0,3r)} H(y)dy \\ &\leq K^2 A_\alpha(H)^{-\theta} K^{-2(1-\theta)-\alpha\theta} A_\alpha(H)(3r)^\alpha = 3^\alpha A_\alpha(H)^{1-\theta} K^{\theta(2-\alpha)} r^\alpha \end{aligned}$$

for all $x_0 \in \mathbb{R}^2$ and $r \geq K$. On the other hand, when $1 \leq r \leq K$ we use the fact that

$$\begin{aligned} \mathcal{H}(y) &\leq A_\alpha(H)^{-\theta} K^{-2(1-\theta)-\alpha\theta} H'(y) \leq A_\alpha(H)^{-\theta} K^{-2(1-\theta)-\alpha\theta} \sup_l \int_{\widetilde{B}_l} H(x)dx \\ &\lesssim A_\alpha(H)^{-\theta} K^{-2(1-\theta)-\alpha\theta} A_\alpha(H) K^\alpha = A_\alpha(H)^{1-\theta} K^{\theta(2-\alpha)} K^{\alpha-2} \end{aligned}$$

for all $y \in \mathbb{R}^2$ to see that

$$\int_{B(x_0,r)} \mathcal{H}(y)dy \lesssim A_\alpha(H)^{1-\theta} K^{\theta(2-\alpha)} K^{\alpha-2} r^2 \leq A_\alpha(H)^{1-\theta} K^{\theta(2-\alpha)} r^\alpha$$

(because $K^{\alpha-2} \leq r^{\alpha-2}$). Therefore, $\mathcal{H}$ is a weight on $\mathbb{R}^2$ of fractal dimension α with

$$A_\alpha(\mathcal{H}) \lesssim A_\alpha(H)^{1-\theta} K^{\theta(2-\alpha)}.$$

Going back to our integral, we now have

$$\int_Q f(x)H(x)dx \lesssim A_\alpha(H)^\theta K^{\theta(\alpha-2)} \int_{\widetilde{Q}} f(y)\mathcal{H}(y)dy.$$

Bounding the integral on the right-hand side by Cauchy–Schwarz, this becomes

$$\int_Q f(x)H(x)dx \lesssim A_\alpha(H)^\theta K^{\theta(\alpha-2)} \Big(\int_{\widetilde{Q}} \mathcal{H}(y)dy\Big)^{1/2} \|f\|_{L^2(\widetilde{Q})}.$$

But $\widetilde{Q}$ can be covered by $\sim K$ balls of radius R/K, so

$$\begin{aligned}\int_{\widetilde{Q}} \mathcal{H}(y)dy &\lesssim K A_\alpha(\mathcal{H})(K^{-1}R)^\alpha \\ &\lesssim K A_\alpha(H)^{1-\theta} K^{\theta(2-\alpha)}(K^{-1}R)^\alpha,\end{aligned} \tag{17}$$

and so

$$\int_Q f(x)H(x)dx \lesssim K^{1/2} A_\alpha(H)^{(1+\theta)/2} K^{\theta(\alpha-2)/2}(K^{-1}R)^{\alpha/2}\|f\|_{L^2(\widetilde{Q})}.$$

We now determine θ by solving the equation $(1+\theta)/2 = \beta$, which gives $\theta = 2\beta - 1$, and we arrive at

$$\int_Q f(x)H(x)dx \lesssim K^{-m} A_\alpha(H)^\beta R^{\alpha/2}\|f\|_{L^2(\widetilde{Q})}$$

with $m = \beta - (1/2) + (1-\beta)(\alpha - 1)$.

6 Preliminaries for the Proof of Theorem 5.1

This section contains basic facts that we need to prove Theorem 5.1 that we include to make the paper as self-contained as possible.

6.1 The L^1 Norm of a Rapidly Decaying Function over a Box

In the rigorous version of the localization argument that we described in the previous section, instead of integrating over a proper $R/K \times R$ box, we will be integrating against a Schwartz function that is essentially supported on such a box. It is easy to see that (17) continues to be true in this case. Here are the details.

Lemma 6.1 *Suppose $0 < \alpha \le n$, $R \ge K^2 \ge 1$, and Ψ is a non-negative Schwartz function on $\mathbb{R}^n$. Then*

$$\int \Psi\Big(\frac{x_1 - v_1}{RK^{-1}}, \ldots, \frac{x_{n-1} - v_{n-1}}{RK^{-1}} \frac{x_n - v_n}{R}\Big)H(x)dx \lesssim K A_\alpha(H)(K^{-1}R)^\alpha \tag{18}$$

for all weights H on $\mathbb{R}^n$ of fractal dimension α.

Proof Suppose $R_1, \ldots, R_n > 0$ and Ψ is a non-negative Schwartz function. For $l = 0, 1, 2, \ldots$, we let χ_l be the characteristic function of the box in $\mathbb{R}^n$ of center 0 and dimensions $2^{l+1}R_1 \times \ldots \times 2^{l+1}R_n$, and $B_l = B(0, 2^l)$. Then

$$\begin{aligned}
&\Psi\Big(\frac{x_1-\nu_1}{R_1}, \ldots, \frac{x_n-\nu_n}{R_n}\Big) \\
&\le \Big(\sup_{B_0} \Psi\Big)\chi_{B_0}\Big(\frac{x_1-\nu_1}{R_1}, \ldots, \frac{x_n-\nu_n}{R_n}\Big) \\
&\quad + \sum_{l=1}^{\infty}\Big(\sup_{B_l\setminus B_{l-1}} \Psi\Big)\chi_{B_l\setminus B_{l-1}}\Big(\frac{x_1-\nu_1}{R_1}, \ldots, \frac{x_n-\nu_n}{R_n}\Big) \\
&\lesssim \sum_{l=0}^{\infty} 2^{-Nl}\chi_l(x-\nu)
\end{aligned}$$

for all $x, \nu \in \mathbb{R}^n$ and $N \in \mathbb{N}$, so that

$$\int \Psi\Big(\frac{x_1-\nu_1}{R_1}, \ldots, \frac{x_n-\nu_n}{R_n}\Big)H(x)dx \lesssim \sum_{l=0}^{\infty} 2^{-Nl}\int_{P_l} H(x)dx,$$

where P_l is the box in $\mathbb{R}^n$ of center ν and dimensions $2^{l+1}R_1 \times \ldots \times 2^{l+1}R_n$.

In the special case $R_1 = \ldots = R_{n-1} = R/K$ and $R_n = R$ with $R \ge K^2 \ge 1$ (as in (17)), this gives

$$\int \Psi\Big(\frac{x_1-\nu_1}{RK^{-1}}, \ldots, \frac{x_{n-1}-\nu_{n-1}}{RK^{-1}}\frac{x_n-\nu_n}{R}\Big)H(x)dx \lesssim KA_\alpha(H)(K^{-1}R)^\alpha \tag{19}$$

for all weights H on $\mathbb{R}^n$ of fractal dimension α.

6.2 A Property of $R/K \times \cdots \times R/K \times R$ Boxes

Suppose $R \ge K^2 \ge 1$, Q is an $R/K \times \cdots \times R/K \times R$ box in $\mathbb{R}^n$. A box $Q^* \subset \mathbb{R}^n$ of dimensions $(R/K)^{-1} \times \cdots \times (R/K)^{-1} \times R^{-1}$ and with the same axes as Q is called a dual box of Q. This subsection is about the following observation.

Lemma 6.2 *Suppose Q^* is a dual box of Q whose $(R/K)^{-1} \times \cdots \times (R/K)^{-1}$-face is tangent to the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ at some point e. Then Q^* lies in the R^{-1}-neighborhood of $\mathbb{S}^{n-1}$.*

Proof Let $\delta = K^{-1}$. Then Q^* has dimensions $(R\delta)^{-1} \times \ldots \times (R\delta)^{-1} \times R^{-1}$ and its $(R\delta)^{-1} \times \ldots \times (R\delta)^{-1}$-face is tangent to $\mathbb{S}^{n-1}$ at e.

Without any loss of generality, we may assume that $e = (0, \ldots, 0, 1)$.

Suppose $y \in Q^*$. Then

$$\begin{aligned}
|y|^2 &= y_1^2 + \ldots + y_{n-1}^2 + (y_n - 1 + 1)^2 = y_1^2 + \ldots + y_{n-1}^2 + (y_n - 1)^2 \\
&\quad + 2(y_n - 1) + 1
\end{aligned}$$

so that

$$\big||y|^2 - 1\big| \le y_1^2 + \ldots + y_{n-1}^2 + |y_n - 1|^2 + 2|y_n - 1|$$

so that

$$\big||y| - 1\big|\,\big||y| + 1\big| \le y_1^2 + \ldots + y_{n-1}^2 + 3|y_n - 1|$$

so that

$$\big||y| - 1\big| \le y_1^2 + \ldots + y_{n-1}^2 + 3|y_n - 1| \le \frac{n-1}{(R\delta)^2} + \frac{3}{R} \lesssim \frac{1}{R},$$

where we have used the fact that

$$\frac{1}{(R\delta)^2} = \frac{1}{R}\frac{K^2}{R} \le \frac{1}{R}.$$

6.3 The Kakeya Information Underlying the Bilinear Estimate

Suppose $\delta > 0$, $R \ge \delta^{-1}$, and J_1 and J_2 are subsets of the circular arc $\{e^{i\theta} : \pi/4 \le \theta \le 3\pi/4\}$ such that $\mathrm{Dist}(J_1, J_2) \ge 3\delta$.

Let N_1 and N_2 be the R^{-1}-neighborhoods of J_1 and J_2, respectively. In this subsection, we derive the following well-known bound on the Lebesgue measure of the set $(x + N_1) \cap N_2$ for $x \in \mathbb{R}^2$.

Lemma 6.3 *We have*

$$|(x + N_1) \cap N_2| \le \frac{\pi}{2R^2\delta} \tag{20}$$

for a.e. $x \in \mathbb{R}^2$.

Proof Since we are interested in the L^∞-norm of the function

$$x \longmapsto \int \chi_{x+N_1}(y)\chi_{N_2}(y)dy,$$

we let $h \in L^1(\mathbb{R}^2)$ be a non-negative function and consider the integral

$$I = \int\int \chi_{x+N_1}(y)\chi_{N_2}(y)dyh(x)dx.$$

Writing

$$I = \int\int \chi_{N_1}(y - x)\chi_{N_2}(y)h(x)dydx = \int \chi_{N_2}(y)\int \chi_{N_1}(y - x)h(x)dxdy,$$

and applying the change of variables $u = y - x$ in the inner integral, we see that

$$I = \int \chi_{N_2}(y) \int \chi_{N_1}(u) h(y-u) du dy = \int_{N_2} \int_{N_1} h(y-u) du dy.$$

Changing into polar coordinates, this becomes

$$I = \int_{1-R^{-1}}^{1+R^{-1}} \int_{1-R^{-1}}^{1+R^{-1}} \int_{\tilde{J}_1} \int_{\tilde{J}_2} h(re^{i\theta} - se^{i\varphi}) rs d\theta d\varphi dr ds,$$

where $\tilde{J}_1 = N_1 \cap \mathbb{S}^1$ and $\tilde{J}_2 = N_2 \cap \mathbb{S}^1$.

We define

$$T(\theta, \varphi) = re^{i\theta} - se^{i\varphi} = (r\cos\theta - s\cos\varphi, r\sin\theta - s\sin\varphi).$$

The Jacobian of this transformation is

$$J_T(\theta, \varphi) = \begin{vmatrix} -r\sin\theta & s\sin\varphi \\ r\cos\theta & -s\cos\varphi \end{vmatrix} = rs\sin(\theta - \varphi).$$

So

$$\int_{\tilde{J}_1} \int_{\tilde{J}_2} rsh(re^{i\theta} - se^{i\varphi}) d\theta d\varphi = \int_{\tilde{J}_1 \times \tilde{J}_2} \frac{h(T(\theta, \varphi))|J_T|}{|\sin(\theta - \varphi)|} d(\theta, \varphi).$$

But $|\theta - \varphi| \le \pi/2$, so

$$|\sin(\theta - \varphi)| \ge \frac{2}{\pi}|\theta - \varphi| \ge \frac{2}{\pi} \operatorname{Dist}(\tilde{J}_1, \tilde{J}_2) \ge \frac{2\delta}{\pi},$$

and so

$$\begin{aligned} \int_{\tilde{J}_1} \int_{\tilde{J}_2} rsh(re^{i\theta} - se^{i\varphi}) d\theta d\varphi &\le \frac{\pi}{2\delta} \int_{\tilde{J}_1 \times \tilde{J}_2} h \circ T(\theta, \varphi) |J_T(\theta, \varphi)| d(\theta, \varphi) \\ &= \frac{\pi}{2\delta} \int_X h(x, y) d(x, y) \ \le \ \frac{\pi}{2\delta} \|h\|_{L^1}. \end{aligned}$$

Thus

$$I \le \int_{1-R^{-1}}^{1+R^{-1}} \int_{1-R^{-1}}^{1+R^{-1}} \frac{\pi}{2\delta} \|h\|_{L^1} dr ds = \frac{\pi}{2\delta R^2} \|h\|_{L^1},$$

and (20) follows by duality.

7 Proof of Theorem 5.1

As the paragraph following the statement of Theorem 5.1 says, our proof of this theorem relies on ideas from [16], [5], [1], [12], and [4].

7.1 The Bilinear Estimate

Following [1, pp. 1281–1283], we write the ball $B(0, R)$ as a disjoint union of two sets, one *broad*, the other *narrow* (see Subsection 7.3 below for the definition of these two sets). To estimate the $L^p(Hdx)$-norm of Ef on the broad set, we consider a bilinear estimate.

For the rest of the paper, we will use the following notation. If ϕ is a function on $\mathbb{R}^2$ and $\rho > 0$, then ϕ_ρ is the function given by $\phi_\rho(\cdot) = \rho^{-2}\phi(\rho^{-1}\cdot)$.

Lemma 7.1 *Suppose f is supported in an arc I and g is supported in an arc J with $\sigma(I) \sim \sigma(J) \sim \delta$ and $\delta \le$ Dist$(I, J) \le R^\epsilon\delta$. Also, suppose that*

$$(10)R^\epsilon \le \frac{1}{\delta} \le \frac{R\delta}{10}. \tag{21}$$

Then

$$\int_{B(0,R)} |Ef(x)Eg(x)|^{p/2}H(x)dx \le R^\epsilon C_B A_\alpha(H)^\beta R^{\alpha/2}\|f\|^{p/2}_{L^p(\sigma)}\|g\|^{p/2}_{L^p(\sigma)}. \tag{22}$$

Proof Let η be a C_0^∞ function on $\mathbb{R}^2$ satisfying $|\widehat{\eta}| \ge 1$ on $B(0, 1)$. Then

$$\begin{aligned}\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx &= \int_{B(0,R)} |\widehat{fd\sigma}(x)\widehat{gd\sigma}(x)|H(x)dx\\
&\le \int_{B(0,R)} |\widehat{fd\sigma}(x)\widehat{gd\sigma}(x)|\,|\widehat{\eta}(x/R)|^2H(x)dx\\
&= \int_{B(0,R)} |(\eta_{R^{-1}} * fd\sigma)\widehat{\ }(x)(\eta_{R^{-1}} * gd\sigma)\widehat{\ }(x)|H(x)dx\\
&= \int_{B(0,R)} |\widehat{F}(x)\widehat{G}(x)|H(x)dx,\end{aligned}$$

where $F = \eta_{R^{-1}} * fd\sigma$ and $G = \eta_{R^{-1}} * gd\sigma$.

Applying the Cauchy-Schwarz inequality in the convolution integral with respect to the measure $|\eta_{R^{-1}}(\xi - \cdot)|d\sigma$, we see that

$$\begin{aligned}\|F\|^2_{L^2} &\le \int \Big(\int |f(\theta)|^2\,|\eta_{R^{-1}}(\xi-\theta)|d\sigma(\theta)\Big)\Big(\int |\eta_{R^{-1}}(\xi-\theta)|d\sigma(\theta)\Big)d\xi\\
&\lesssim R\int\int |f(\theta)|^2\,|\eta_{R^{-1}}(\xi-\theta)|d\sigma(\theta)d\xi\\
&= R\int |f(\theta)|^2\int |\eta_{R^{-1}}(\xi-\theta)|d\xi d\sigma(\theta) \;=\; R\,\|\eta\|_{L^1}\,\|f\|^2_{L^2(\sigma)},\end{aligned}$$

where in the second inequality we used the fact that

$$\int |\eta_{R^{-1}}(\xi-\theta)|d\sigma(\theta) \lesssim R^2\sigma(B(\xi, R^{-1}) \lesssim R.$$

Therefore,

$$\|F\|_{L^2} \lesssim R^{1/2}\|f\|_{L^2(\sigma)} \qquad \text{and} \qquad \|G\|_{L^2} \lesssim R^{1/2}\|g\|_{L^2(\sigma)}. \tag{23}$$

Since F is supported in the R^{-1}-neighborhood of I and G is supported in the R^{-1}-neighborhood of J, we see (via (21)) that F is supported in a ball of radius $(\delta/2)+(\delta/10)=(3\delta/5)$ and similarly for G. So $F*G$ is supported in a ball of radius $(6\delta/5)$, say $B(\xi_0, (6\delta/5))$. Via the locally constant property of the Fourier transform, this fact tells us that the Fourier transform of $F*G$ is essentially constant at scale $K=\delta^{-1}$, and hence allows us to implement the localization of the weight argument that we described in Section 5 at the intuitive level, and which we now carry out rigorously.

Let ϕ be a Schwartz function which is equal to 1 on $B(0, 6/5)$. Then $\phi_\delta(\xi-\xi_0)=\delta^{-2}$ on $B(\xi_0, \frac{6\delta}{5})$, so that

$$F*G=\delta^2\phi_\delta(\cdot-\xi_0)\big(F*G\big)$$

and

$$\widehat{F}(x)\widehat{G}(x)=\delta^2\Big(\phi_\delta(\cdot-\xi_0)\big(F*G\big)\Big)^{\wedge}(x)=\delta^2\big(\phi_\delta(\cdot-\xi_0)\big)^{\wedge}*\widehat{F*G}(x).$$

Since $\big(\phi_\delta(\cdot-\xi_0)\big)^{\wedge}(x)=e^{-2\pi i x\cdot\xi_0}\widehat{\phi}(\delta x)$, it follows that

$$\begin{aligned}\widehat{F}(x)\widehat{G}(x)&=\delta^2\int \big(\phi_\delta(\cdot-\xi_0)\big)^{\wedge}(x-y)\widehat{F*G}(y)dy\\&=\delta^2\int e^{-2\pi i(x-y)\cdot\xi_0}\widehat{\phi}(\delta(x-y))\widehat{F*G}(y)dy,\end{aligned}$$

so that

$$|\widehat{F}(x)\widehat{G}(x)|\le\delta^2\int|\widehat{\phi}(\delta(x-y))|\,|\widehat{F*G}(y)|dy.$$

Therefore,

$$\int_{B(0,R)}|Ef(x)Eg(x)|H(x)dx\le\delta^2\int|\widehat{F*G}(y)|\int|\widehat{\phi}(\delta(x-y))|H(x)dxdy. \tag{24}$$

For $l = 0, 1, 2, \ldots$, we let $B_l = B(y, 2^l \delta^{-1})$ and write

$$\begin{aligned}
&\int |\widehat{\phi}(\delta(x-y))| H(x)dx \\
&= \int_{B_0} |\widehat{\phi}(\delta(x-y))| H(x)dx + \sum_{l=1}^{\infty} \int_{B_l \setminus B_{l-1}} |\widehat{\phi}(\delta(x-y))| H(x)dx \\
&\leq \int_{B_0} \frac{C_N H(x)}{(1+\delta|x-y|)^N} dx + \sum_{l=1}^{\infty} \int_{B_l \setminus B_{l-1}} \frac{C_N H(x)}{(1+\delta|x-y|)^N} dx \\
&\leq C_N \int_{B_0} H(x)dx + \sum_{l=1}^{\infty} \frac{C_N}{(1+\delta \frac{2^{l-1}}{\delta})^N} \int_{B_l} H(x)dx.
\end{aligned}$$

We now let $0 \leq \theta \leq 1$ be a parameter that will be determined later and write

$$\begin{aligned}
\int_{B_l} H(x)dx &= \Big(\int_{B_l} H(x)dx\Big)^{1-\theta} \Big(\int_{B_l} H(x)dx\Big)^{\theta} \\
&\leq |B_l|^{1-\theta} \Big(A_\alpha(H)\Big(\frac{2^l}{\delta}\Big)^{\alpha}\Big)^{\theta} \\
&\leq C_\theta \Big(\frac{2^l}{\delta}\Big)^{2(1-\theta)+\alpha\theta} A_\alpha(H)^\theta,
\end{aligned}$$

where we have used the fact that $1/\delta \geq 1$, and we obtain

$$\begin{aligned}
&\int |\widehat{\phi}(\delta(x-y))| H(x)dx \\
&\leq C_{N,\theta} \Big(\frac{1}{\delta}\Big)^{n(1-\theta)+\alpha\theta} A_\alpha(H)^\theta + \sum_{l=1}^{\infty} \frac{C_N}{(1+2^{l-1})^N} \Big(\frac{2^l}{\delta}\Big)^{2(1-\theta)+\alpha\theta} A_\alpha(H)^\theta \\
&\leq C_{N,\theta} A_\alpha(H)^\theta \Big(\frac{1}{\delta}\Big)^{2(1-\theta)+\alpha\theta}.
\end{aligned}$$

Also,

$$\int_{B(x_0,r)} \int |\widehat{\phi}(\delta(x-y))| H(x)dxdy = \int\int \chi_{B(x_0,r)}(y) |\widehat{\phi}(\delta(x-y))| dy H(x)dx.$$

Applying the change of variables $z = \delta(x-y)$ in the inner integral, we get

$$\begin{aligned}
\int_{B(x_0,r)} \int |\widehat{\phi}(\delta(x-y))| H(x)dxdy &= \frac{1}{\delta^2} \int\int \chi_{B(x_0,r)}\big(x - \frac{z}{\delta}\big) |\widehat{\phi}(z)| dz H(x)dx \\
&= \frac{1}{\delta^2} \int |\widehat{\phi}(z)| \int \chi_{B(x_0,r)}\big(x - \frac{z}{\delta}\big) H(x)dxdz.
\end{aligned}$$

But

$$\int \chi_{B(x_0,r)}\big(x-\frac{z}{\delta}\big)H(x)dx=\int_{B(x_0+\frac{z}{\delta},r)} H(x)dx \le A_\alpha(H)r^\alpha$$

for all $x_0 \in \mathbb{R}^n$ and $r \ge 1$, so

$$\int_{B(x_0,r)}\int |\widehat{\phi}(\delta(x-y))|H(x)dxdy \le \frac{1}{\delta^2}\,\|\widehat{\phi}\|_{L^1}\,A_\alpha(H)\,r^\alpha$$

for all $x_0 \in \mathbb{R}^2$ and $r \ge 1$.

For $y \in \mathbb{R}^2$, define

$$\mathcal{H}(y)=\frac{\delta^{2(1-\theta)+\alpha\theta}}{C_{N,\theta}A_\alpha(H)^\theta}\int |\widehat{\phi}(\delta(x-y))|H(x)dx.$$

In view of the above discussion, we have

$$\|\mathcal{H}\|_{L^\infty} \le 1 \qquad \text{and} \qquad \int_{B(x_0,r)} \mathcal{H}(y)dy \le C A_\alpha^{1-\theta}\delta^{(\alpha-2)\theta}r^\alpha$$

for all $x_0 \in \mathbb{R}^2$ and $r \ge 1$. Thus $\mathcal{H}$ is a weight on $\mathbb{R}^2$ of fractal dimension α with

$$A_\alpha(\mathcal{H}) \le C A_\alpha(H)^{1-\theta}\delta^{(\alpha-2)\theta}.$$

Going back to (24), we now have

$$\begin{aligned}\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx &\le \delta^2\frac{C_{N,\theta}A_\alpha(H)^\theta}{\delta^{2(1-\theta)+\alpha\theta}}\int |\widehat{F*G}(y)|\mathcal{H}(y)dy\\ &= C_{N,\theta}\,\delta^{(2-\alpha)\theta}A_\alpha(H)^\theta\int |\widehat{F*G}(y)|\mathcal{H}(y)dy.\end{aligned} \tag{25}$$

Next, we let Q^* be the box in frequency space (where the circle is located) of dimensions $(R\delta)^{-1}\times R^{-1}$, centered at the origin, and with the $(R\delta)^{-1}$-side (i.e. the long side) parallel to the line segment that connects the midpoint of I to that of J. We also let $\{Q_l\}$ be a tiling of $\mathbb{R}^2$ by boxes dual to Q^* (i.e. each Q_l is an $R\delta \times R$ box whose $R\delta$-side is parallel to the $(R\delta)^{-1}$-side of Q^*) with centers $\{\nu_l\}$, ψ be a C_0^∞ function on $\mathbb{R}^2$, and we define

$$\psi_l(\xi)=(R\delta)R\;\psi(R\delta\xi_1,R\xi_2)\;e^{2\pi i\nu_l\cdot\xi}.$$

In the definition of ψ_l, we are assuming that the line joining the midpoint of I to that of J is horizontal (i.e. parallel to the ξ_1-axis). This assumption makes the presentation a little smoother and, of course, does not cost us any loss of generality.

We assume further that the Fourier transform of ψ is non-negative and satisfies $\widehat{\psi} \geq 1/2$ on $[-1/2, 1/2] \times [-1/2, 1/2]$. Then

$$\widehat{\psi_l}(x) = \widehat{\psi}\Big(\frac{x_1 - \nu_{l,1}}{R\delta}, \frac{x_2 - \nu_{l,2}}{R}\Big) \geq \frac{1}{2} \qquad \text{if} \qquad x \in Q_l.$$

By the Schwartz decay of $\widehat{\psi}$, we have $\sum_{m\in\mathbb{Z}^2} \widehat{\psi}(\cdot - m)^k \lesssim 1$ for any $k \in \mathbb{N}$. Also, $\{\nu_l\}$ is basically $R\delta\mathbb{Z} \times R\mathbb{Z}$, so

$$\sum_{l=1}^{\infty} \widehat{\psi_l}(R\delta x_1, Rx_2)^k = \sum_{l=1}^{\infty} \widehat{\psi}\Big(\frac{R\delta x_1 - \nu_{l,1}}{R\delta}, \frac{Rx_2 - \nu_{l,2}}{R}\Big)^k = \sum_{m\in\mathbb{Z}^2} \widehat{\psi}(x-m)^k \lesssim 1,$$

and so

$$\sum_{l=1}^{\infty} \widehat{\psi_l}(x)^k \lesssim 1 \tag{26}$$

for all $x \in \mathbb{R}^2$.

Going back to (25), we can now write

$$\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx \lesssim \delta^{(2-\alpha)\theta} A_\alpha(H)^\theta \sum_{l=1}^{\infty} \int |\widehat{F}(x)\widehat{G}(x)|\widehat{\psi_l}(x)^3 \mathcal{H}(x)dx.$$

Letting $F_l = \psi_l * F$ and $G_l = \psi_l * G$, this becomes

$$\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx \lesssim \delta^{(2-\alpha)\theta} A_\alpha(H)^\theta \sum_{l=1}^{\infty} \int |\widehat{F_l}(x)\widehat{G_l}(x)|\widehat{\psi_l}(x)\mathcal{H}(x)dx.$$

By Cauchy–Schwarz,

$$\int |\widehat{F_l}(x)\widehat{G_l}(x)|\widehat{\psi_l}(x)\mathcal{H}(x)dx \leq \|\widehat{F_l}\widehat{G_l}\|_{L^2} \|\widehat{\psi_l}(x)\mathcal{H}\|_{L^2}.$$

Applying (19) from Subsection 6.1 with $n = 2$ and $K = \delta^{-1}$, we have

$$\begin{aligned} \int \widehat{\psi_l}(x)^2 \mathcal{H}(x)^2 dx &\lesssim \int \widehat{\psi_l}(x)\mathcal{H}(x)dx \lesssim A_\alpha(\mathcal{H}) \frac{R}{R\delta}(R\delta)^\alpha \\ &\lesssim A_\alpha(H)^{1-\theta} \delta^{(\alpha-2)\theta} R^\alpha \delta^{\alpha-1} = A_\alpha(H)^{1-\theta} \delta^{(\alpha-2)\theta+\alpha-1} R^\alpha, \end{aligned}$$

so that

$$\|\widehat{\psi_l}(x)\mathcal{H}\|_{L^2} \lesssim A_\alpha(H)^{(1-\theta)/2} \delta^{((\alpha-2)\theta+\alpha-1)/2} R^{\alpha/2}.$$

Therefore,

$$\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx \lesssim A_\alpha(H)^{(1+\theta)/2}\delta^{((2-\alpha)\theta+\alpha-1)/2}R^{\alpha/2}\sum_{l=1}^{\infty}\|\widehat{F_l}\widehat{G_l}\|_{L^2}.$$

Letting $\beta = (1+\theta)/2$ (since $0 \le \theta \le 1$, we have $1/2 \le \beta \le 1$), this becomes

$$\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx \lesssim A_\alpha(H)^{\beta}\delta^{(2-\alpha)\beta+\alpha-(3/2)}R^{\alpha/2}\sum_{l=1}^{\infty}\|\widehat{F_l}\widehat{G_l}\|_{L^2}.$$

We now let A_l be the support of F_l, B_l be the support of G_l, and define the function $\lambda_l : \mathbb{R}^2 \to [0,\infty)$ by $\lambda_l(\xi) = |(\xi - A_l)\cap B_l|$. Applying Plancherel's theorem followed by Cauchy–Schwarz, we see that

$$\|\widehat{F_l}\widehat{G_l}\|_{L^2}^2 = \int |F_l * G_l(\xi)|^2 d\xi \le \|\lambda_l\|_{L^\infty}\int |F_l|^2 * |G_l|^2(\xi)d\xi.$$

By Young's inequality,

$$\int |F_l|^2 * |G_l|^2(\xi)d\xi \le \||F_l|^2\|_{L^1}\||G_l|^2\|_{L^1} = \|F_l\|_{L^2}^2\|G_l\|_{L^2}^2,$$

so the only problem is to estimate $\|\lambda_l\|_{L^\infty}$. We will do this by using the Kakeya bound (20) of Subsection 6.3.

Our assumptions on the arcs I and J imply that the angle between any two points in $I \cup J$ is $\lesssim R^\epsilon\delta$. Also, for each l, the function ψ_l is supported in the $(R\delta)^{-1} \times R^{-1}$ box Q^* of center $(0,0)$ and with the long side parallel to the line joining the midpoints of I and J. So, if $e \in I \cup J$, then the translate $Q^* + e$ of Q^* is contained in an $(R\delta)^{-1} \times R^{\epsilon-1}$ box with the $(R\delta)^{-1}$-side tangent to $\mathbb{S}^1$ at e. Therefore, the property of boxes of this form that was presented in Subsection 6.2 tells us that $Q^* + e$ is contained in the $R^{\epsilon-1}$-neighborhood of $\mathbb{S}^1$. Therefore, the sets A_l and B_l satisfy the requirements needed for us to apply (20) and conclude

$$\|\lambda_l\|_{L^\infty} \lesssim \frac{R^\epsilon}{R^2\delta}.$$

Putting together what we have proved in the previous two paragraphs, we obtain

$$\|\widehat{F_l}\widehat{G_l}\|_{L^2}^2 \lesssim \frac{R^\epsilon}{R^2\delta}\|F_l\|_{L^2}^2\|G_l\|_{L^2}^2,$$

and hence

$$\begin{aligned}&\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx\\&\quad \lesssim R^{\epsilon} A_{\alpha}(H)^{\beta} \delta^{(2-\alpha)\beta+\alpha-(3/2)} \frac{R^{\alpha/2}}{(R^2\delta)^{1/2}} \sum_{l=1}^{\infty} \|F_l\|_{L^2} \|G_l\|_{L^2}\\&\quad = R^{\epsilon} A_{\alpha}(H)^{\beta} \delta^{(2-\alpha)(\beta-1)} \frac{R^{\alpha/2}}{R} \sum_{l=1}^{\infty} \|F_l\|_{L^2} \|G_l\|_{L^2}.\end{aligned}$$

By Cauchy–Schwarz and Plancherel,

$$\sum_{l=1}^{\infty} \|F_l\|_{L^2} \|G_l\|_{L^2} \leq \Big(\sum_{l=1}^{\infty} \|\widehat{F_l}\|_{L^2}^2\Big)^{1/2} \Big(\sum_{l=1}^{\infty} \|\widehat{G_l}\|_{L^2}^2\Big)^{1/2}.$$

Also, by (26),

$$\sum_{l=1}^{\infty} \|\widehat{F_l}\|_{L^2}^2 = \int |\widehat{F}(x)|^2 \sum_{l=1}^{\infty} \widehat{\psi_l}(x)^2 dx \lesssim \|\widehat{F}\|_{L^2}^2 = \|F\|_{L^2}^2$$

and similarly for $\sum_{l=1}^{\infty} \|\widehat{G_l}\|_{L^2}^2$, so

$$\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx \lesssim R^{\epsilon} A_{\alpha}(H)^{\beta} \delta^{(2-\alpha)(\beta-1)} \frac{R^{\alpha/2}}{R} \|F\|_{L^2} \|G\|_{L^2}.$$

Recalling (23), our bilinear estimate becomes

$$\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx \lesssim R^{\epsilon} A_{\alpha}(H)^{\beta} \delta^{(2-\alpha)(\beta-1)} R^{\alpha/2} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\sigma)}.$$

Writing

$$\begin{aligned}&\int_{B(0,R)} |Ef(x)Eg(x)|^{p/2} H(x)dx\\&\quad = \int_{B(0,R)} |Ef(x)Eg(x)|^{(p/2)-1} |Ef(x)Eg(x)|H(x)dx\\&\quad \leq \|f\|_{L^1(S)}^{(p/2)-1} \|g\|_{L^1(S)}^{(p/2)-1} \int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx\\&\quad \leq C_B R^{\epsilon} A_{\alpha}(H)^{\beta} R^{\alpha/2} \delta^{(2-\alpha)(\beta-1)} \|f\|_{L^1(\sigma)}^{(p/2)-1} \|f\|_{L^2(\sigma)} \|g\|_{L^1(\sigma)}^{(p/2)-1} \|g\|_{L^2(\sigma)}\end{aligned}$$

and applying (33) (see the appendix), we arrive at our desired bilinear estimate

$$\int_{B(0,R)} |Ef(x)Eg(x)|^{p/2} H(x)dx \leq R^{\epsilon} C_B A_{\alpha}(H)^{\beta} R^{\alpha/2} \|f\|_{L^p(\sigma)}^{p/2} \|g\|_{L^p(\sigma)}^{p/2}.$$

7.2 The Linear Estimate

In this subsection, we work in $\mathbb{R}^n$ with $n \geq 2$.

Lemma 7.2 *Suppose f is supported in a cap of radius $\delta/2$. Also, suppose that*

$$(10)R^{\epsilon} \leq \frac{1}{\delta} \leq \frac{R}{10}. \tag{27}$$

Then

$$\int_{B(0,R)} |Ef(x)|^p H(x)dx \leq C_L A_\alpha(H)^\beta \delta^{-2\alpha/n}(\delta^2 R)\|f\|^p_{L^p(\sigma)}. \tag{28}$$

Proof Let η be a C_0^∞ function on $\mathbb{R}^n$ satisfying $|\widehat{\eta}| \geq 1$ on $B(0, 1)$, and $F = \eta_{R^{-1}} * f d\sigma$. Then

$$\int_{B(0,R)} |Ef(x)|^2 H(x)dx \leq \int_{B(0,R)} |\widehat{F}(x)|^2 H(x)dx.$$

Also, let ψ be a C_0^∞ function on $\mathbb{R}^n$, and $\{B_l\}$ be a finitely overlapping cover of $\mathbb{R}^n$ by balls dual to $B(0, \delta)$ (i.e. δ^{-1}-balls) with centers $\{\nu_l\}$, and set

$$\psi_l(\xi) = \delta^{-n}\psi(\delta^{-1}\xi)\, e^{2\pi i \nu_l \cdot \xi}.$$

We assume further that $\widehat{\psi}$ is non-negative and $\geq 1/2$ on the unit ball. Then

$$\widehat{\psi_l}(x) = \widehat{\psi}(\delta(x - \tau_l)) \geq \frac{1}{2}$$

if $|\delta(x - \tau_l)| \leq 1$, i.e. if $x \in B_l$. Thus

$$\int_{B(0,R)} |Ef(x)|^2 H(x)dx \lesssim \sum_{l=1}^{\infty} \int |\widehat{F}(x)\widehat{\psi_l}(x)|^2 \widehat{\psi_l}(x) H(x)dx.$$

Since $1/n \leq \beta \leq 2/n$, we can apply Hölder's inequality with the dual exponents $1/(1-\beta)$ and $1/\beta$ to get

$$\int_{B(0,R)} |Ef(x)|^2 H(x)dx \lesssim \sum_{l=1}^{\infty} \|\widehat{F * \psi_l}\|^2_{L^{2/(1-\beta)}} \|\widehat{\psi_l} H\|_{L^{1/\beta}}.$$

Since $\|H\|_{L^\infty} \leq 1$, we have

$$\|\widehat{\psi_l} H\|^{1/\beta}_{L^{1/\beta}} \leq \int \widehat{\psi_l}(x)^{1/\beta} H(x)dx,$$

and hence (by the proof of (19))

$$\|\widehat{\psi_l} H\|_{L^{1/\beta}}^{1/\beta} \lesssim A_\alpha(H)\Big(\frac{1}{\delta}\Big)^\alpha.$$

Also, by Hausdorff–Young,

$$\|\widehat{F * \psi_l}\|_{L^{2/(1-\beta)}} \leq \|F * \psi_l\|_{L^{2/(1+\beta)}}.$$

Therefore,

$$\int_{B(0,R)} |Ef(x)|^2 H(x)dx \lesssim A_\alpha(H)^\beta \delta^{-\alpha\beta} \sum_{l=1}^\infty \|F * \psi_l\|_{L^{2/(1+\beta)}}^2.$$

Since (27) tells us $1/R \leq \delta/10$, it follows that F is supported in a ball of radius $(\delta/2)+(\delta/10) = (3/5)\delta$, say $B(\xi_0, 3\delta/5)$. Moreover, since ψ_l is supported in $B(0,\delta)$, it follows by Hölder's inequality and Plancherel's theorem that

$$\|F * \psi_l\|_{L^{2/(1+\beta)}}^2 \lesssim \delta^{n\beta} \|F * \psi_l\|_{L^2}^2 = \delta^{n\beta} \|\widehat{F}\widehat{\psi_l}\|_{L^2}^2.$$

Thus

$$\begin{aligned}\int_{B(0,R)} |Ef(x)|^2 H(x)dx &\lesssim A_\alpha(H)^\beta \delta^{-\alpha\beta} \delta^{n\beta} \sum_{l=1}^\infty \int |\widehat{F}(\xi)\widehat{\psi_l}(\xi)|^2 d\xi \\ &= A_\alpha(H)^\beta \delta^{(n-\alpha)\beta} \int |\widehat{F}(\xi)|^2 \sum_{l=1}^\infty |\widehat{\psi_l}(\xi)|^2 d\xi \\ &\lesssim A_\alpha(H)^\beta \delta^{(n-\alpha)(\beta-(2/n))} \delta^{2-(2\alpha/n)} \|F\|_{L^2}^2.\end{aligned}$$

But we know from (23) (whose proof shows that it is true in $\mathbb{R}^n$ for all $n \geq 2$) that $\|F\|_{L^2} \lesssim \sqrt{R}\,\|f\|_{L^2(\sigma)}$, so

$$\int_{B(0,R)} |Ef(x)|^2 H(x)dx \lesssim A_\alpha(H)^\beta \delta^{-2\alpha/n} (\delta^2 R) \delta^{(n-\alpha)(\beta-(2/n))} \|f\|_{L^2(\sigma)}^2.$$

Writing

$$|Ef(x)|^p = |Ef(x)|^{p-2} |Ef(x)|^2 \leq \|f\|_{L^1(\sigma)}^{p-2} |Ef(x)|^2$$

and using (32) (see the appendix), we now see that

$$\begin{aligned}&\int_{B(0,R)} |Ef(x)|^p H(x)dx \\ &\quad\lesssim A_\alpha(H)^\beta \delta^{-2\alpha/n} (\delta^2 R) \delta^{(n-\alpha)(\beta-(2/n))} \|f\|_{L^1(\sigma)}^{p-2} \|f\|_{L^2(\sigma)}^2 \\ &\quad\lesssim A_\alpha(H)^\beta \delta^{-2\alpha/n} (\delta^2 R) \|f\|_{L^p(\sigma)}^p,\end{aligned}$$

which proves (28).

7.3 The Recursive Process

We let $0 < \epsilon < 10^{-2}$ and $R \geq 1$ be two numbers satisfying $R \geq (1000)^{1/(1-4\epsilon)}$. We also let δ be as in Lemma 7.1 (so that δ obeys (21)). We're going to prove our estimate by implementing a recursive process over δ.

Base of the recursion: Here $\delta = R^{-1/2}$. Plugging this value of δ into (28) in dimension $n = 2$, we get

$$\int_{B(0,R)} |Ef(x)|^p H(x)dx \leq C_L A_\alpha(H)^\beta R^{\alpha/2} \|f\|^p_{L^p(\sigma)}.$$

The recursive step: We state this in the following lemma.

Lemma 7.3 *Suppose that the estimate*

$$\int_{B(0,R)} |Ef(x)|^p H(x)dx \leq C R^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \|f\|^p_{L^p(\sigma)} \tag{29}$$

holds for every function $f \in L^1(\sigma)$ that is supported in an arc of σ-measure $\leq \delta$, and δ obeys (21). Then the estimate

$$\int_{B(0,R)} |Eg(x)|^p H(x)dx \leq C' R^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \|g\|^p_{L^p(\sigma)} \tag{30}$$

holds for every function $g \in L^1(\sigma)$ that is supported in an arc of σ-measure $\leq R^\epsilon\delta$, where

$$C' = 3^p C + (10)^p R^{(p+2)\epsilon} C_B.$$

Proof Suppose δ satisfies the condition (21):

$$(10)R^\epsilon \leq \frac{1}{\delta} \leq \frac{R\delta}{10},$$

and (29) is true whenever $f \in L^1(\sigma)$, f is supported on an arc $I_\delta \subset \mathbb{S}^1$, and $\sigma(I_\delta) \leq \delta$. We need to show that (30) is true whenever $g \in L^1(\sigma)$, g is supported on an arc $I_{R^\epsilon\delta} \subset \mathbb{S}^1$, and $\sigma(I_{R^\epsilon\delta}) \leq R^\epsilon\delta$, where

$$C' = 3^p C + (10)^p R^{(p+2)\epsilon} C_B.$$

We let $K = R^\epsilon$ and cover the support of g by K arcs τ each of measure δ. We then write $g = \sum_\tau f_\tau$ with each function f_τ supported in the arc τ.

Following [1] and [7], for $x \in \mathbb{R}^2$, we define the significant set of x by

$$S(x) = \{\tau : |Ef_\tau(x)| \geq \frac{1}{10K}|Eg(x)|\}.$$

Then

$$|Eg(x)| \leq \Big| \sum_{\tau \in S(x)} Ef_\tau(x) \Big| + \frac{1}{10}|Eg(x)|,$$

so that

$$|Eg(x)| \leq \frac{10}{9}\Big| \sum_{\tau \in S(x)} Ef_\tau(x) \Big|. \tag{31}$$

The narrow set $\mathcal{N}$ and the broad set $\mathcal{B}$ are now defined as

$$\mathcal{N} = B(0, R) \cap \{x \in \mathbb{R}^2 : \#S(x) \leq 2\} \qquad \text{and} \qquad \mathcal{B} = B(0, R) \setminus \mathcal{N}.$$

We will estimate $\int_{\mathcal{N}} |Eg(x)|^p H(x)dx$ by induction and $\int_{\mathcal{B}} |Eg(x)|^p H(x)dx$ by using the bilinear estimate.

By (29) and (31),

$$\begin{aligned}
\int_{\mathcal{N}} |Eg(x)|^p H(x)dx &\leq 2^{p-1}\Big(\frac{10}{9}\Big)^p \int_N \sum_{\tau \in S(x)} |Ef_\tau(x)|^p H(x)dx \\
&\leq \Big(\frac{20}{9}\Big)^p \int_N \sum_\tau |Ef_\tau(x)|^p H(x)dx \\
&\leq 3^p \sum_\tau C R^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \|f_\tau\|^p_{L^p(\sigma)} \\
&= 3^p C R^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \|g\|^p_{L^p(\sigma)}.
\end{aligned}$$

To every $x \in \mathcal{B}$ there are two caps $\tau_x, \tau'_x \in S(x)$ so that $\mathrm{Dist}(\tau_x, \tau'_x) \geq \delta$. Writing

$$|Eg(x)|^p = |Eg(x)|^{p/2}|Eg(x)|^{p/2} \leq (10K|Ef_{\tau_x}(x)|)^{p/2}(10K|Ef_{\tau'_x}(x)|)^{p/2},$$

we see that

$$|Eg(x)|^p \leq (10K)^p \sum_{\tau,\tau' :\ \mathrm{Dist}(\tau,\tau') \geq \delta} |Ef_\tau(x)|^{p/2}|Ef_{\tau'}(x)|^{p/2}.$$

Using the bilinear estimate (22), it follows that

$$\begin{aligned}
&\int_{\mathcal{B}} |Eg(x)|^p H(x)dx \\
&\leq (10K)^p \sum_{\tau,\tau':\ \mathrm{Dist}(\tau,\tau')\geq\delta} \int_{\mathcal{B}} |Ef_\tau(x)|^{p/2}|Ef_{\tau'}(x)|^{p/2}H(x)dx \\
&\leq (10K)^p C_B R^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \sum_{\tau,\tau':\ \mathrm{Dist}(\tau,\tau')\geq\delta} \|f_\tau\|_{L^p(\sigma)}^{p/2}\|f_{\tau'}\|_{L^p(\sigma)}^{p/2} \\
&\leq (10)^p K^p C_B R^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \sum_{\tau,\tau':\ \mathrm{Dist}(\tau,\tau')\geq\delta} \|g\|_{L^p(\sigma)}^{p/2}\|g\|_{L^p(\sigma)}^{p/2}.
\end{aligned}$$

Therefore,

$$\int_{\mathcal{B}} |Eg(x)|^p H(x)dx \leq (10)^p K^{p+2} C_B A_\alpha(H)^\beta R^{\alpha/2}\|g\|_{L^p(\sigma)}^p.$$

Combining the narrow and broad estimates, we arrive at (30).

The recursion: Starting with the base of the induction, where $\delta = R^{-1/2}$ and $C = C_L$, and applying Lemma 7.3 k times, we arrive at an estimate that holds for every function $f \in L^1(\sigma)$ that is supported on an arc of σ-measure $\leq \delta_k = R^{k\epsilon}\delta = R^{k\epsilon}/\sqrt{R}$, with constant

$$C_k = 3^{kp}C_L + (10)^p R^{(p+2)\epsilon} C_B \sum_{l=0}^{k-1} 3^{lp} = 3^{kp}C_L + (10)^p R^{(p+2)\epsilon} C_B \frac{1-3^{kp}}{1-3^p}.$$

At the step before the last, $k = (1/(2\epsilon)) - 2$ and $\delta_k = R^{[(1/(2\epsilon))-2]\epsilon}/\sqrt{R} = R^{-2\epsilon}$, which is a valid value of δ (i.e. $\delta_k = R^{-2\epsilon}$ obeys (28), because $10R^\epsilon \leq 1/R^{-2\epsilon} \leq R^{1-2\epsilon}/10$). Applying Lemma 7.3 one last time, we get the estimate

$$\int_{B(0,R)} |Ef(x)|^p H(x)dx \leq CR^\epsilon A_\alpha(H)^\beta R^{\alpha/2}\|f\|_{L^p(\sigma)}^p$$

for every function $f \in L^1(\sigma)$ that is supported on an arc of σ-measure $\leq R^{-\epsilon}$, where the constant C satisfies

$$C \leq 3^{p/(2\epsilon)}\Big(C_L + \frac{(10)^p R^{(p+2)\epsilon}}{3^p - 1} C_B\Big).$$

Since the circle $\mathbb{S}^1$ can be covered by $\sim R^\epsilon$ such arcs, (15) follows and Theorem 5.1 is proved.

Appendix: Calculation Giving the Right Exponent for the Restriction estimate

Suppose $0 < \delta \le 1$, $1 \le \alpha \le n$, $1/n \le \beta \le 2/n$, σ is induced Lebesgue measure on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and $f, g \in L^1(\sigma)$ are functions satisfying $\sigma(\text{supp } f), \sigma(\text{supp } g) \le \delta^{n-1}$. We are looking for an exponent $p \ge 2$ so that

$$\|f\|_{L^1(\sigma)}^{p-2}\|f\|_{L^2(\sigma)}^2 \le \delta^{(n-\alpha)((2/n)-\beta)}\|f\|_{L^p(\sigma)}^p \tag{32}$$

and

$$\|f\|_{L^1(\sigma)}^{(p/2)-1}\|f\|_{L^2(\sigma)}\|g\|_{L^1(\sigma)}^{(p/2)-1}\|g\|_{L^2(\sigma)} \le \delta^{(n-\alpha)((2/n)-\beta)}\|f\|_{L^p(\sigma)}^{p/2}\|g\|_{L^2(\sigma)}^{p/2}. \tag{33}$$

We have

$$\|f\|_{L^1(\sigma)} \le \sigma(\text{supp } f)^{1-(1/p)}\|f\|_{L^p(\sigma)} \le \delta^{(n-1)(p-1)/p}\|f\|_{L^p(\sigma)}$$

and

$$\|f\|_{L^2(\sigma)}^2 \le \sigma(\text{supp } f)^{1-(2/p)}\Big(\int |f|^{2(p/2)}d\sigma\Big)^{2/p} \le \delta^{(n-1)(p-2)/p}\|f\|_{L^p(\sigma)}^2,$$

so

$$\begin{aligned}\|f\|_{L^1(\sigma)}^{p-2}\|f\|_{L^2(\sigma)}^2 &\le \delta^{(n-1)(p-2)(p-1)/p}\|f\|_{L^p(\sigma)}^{p-2}\delta^{(n-1)(p-2)/p}\|f\|_{L^p(\sigma)}^2\\ &= \delta^{(n-1)(p-2)}\|f\|_{L^p(\sigma)}^p,\end{aligned}$$

so $(n-1)(p-2) = (n-\alpha)((2/n)-\beta)$, and so

$$p = 2 + \frac{n-\alpha}{n-1}\Big(\frac{2}{n}-\beta\Big).$$

Therefore, (32) holds with the above value of p. Using (32), we now have

$$\begin{aligned}&\|f\|_{L^1(\sigma)}^{(p-2)/2}\|f\|_{L^2(\sigma)}\|g\|_{L^1(\sigma)}^{(p-2)/2}\|g\|_{L^2(\sigma)}\\ &\le \Big(\delta^{(n-\alpha)((2/n)-\beta)}\|f\|_{L^p(\sigma)}^p\Big)^{1/2}\Big(\delta^{(n-\alpha)((2/n)-\beta)}\|g\|_{L^p(\sigma)}^p\Big)^{1/2},\end{aligned}$$

which is the inequality in (33).

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