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Separating Fourier and Schur Multipliers

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Abstract

Let G be a locally compact unimodular group, let $1 \leq p < \infty$, let $\phi \in L^\infty(G)$ and assume that the Fourier multiplier M_ϕ associated with ϕ is bounded on the noncommutative L^p-space $L^p(VN(G))$. Then $M_\phi \colon L^p(VN(G)) \to L^p(VN(G))$ is separating (that is, $\{a^*b = ab^* = 0\} \Rightarrow \{M_\phi(a)^*M_\phi(b) = M_\phi(a)M_\phi(b)^* = 0\}$ for any $a, b \in L^p(VN(G))$) if and only if there exists $c \in \mathbb{C}$ and a continuous character $\psi \colon G \to \mathbb{C}$ such that $\phi = c\psi$ locally almost everywhere. This provides a characterization of isometric Fourier multipliers on $L^p(VN(G))$, when $p \neq 2$. Next, let Ω be a σ-finite measure space, let $\phi \in L^\infty(\Omega^2)$ and assume that the Schur multiplier associated with ϕ is bounded on the Schatten space $S^p(L^2(\Omega))$. We prove that this multiplier is separating if and only if there exist a constant $c \in \mathbb{C}$ and two unitaries $\alpha, \beta \in L^\infty(\Omega)$ such that $\phi(s,t) = c\,\alpha(s)\beta(t)$ a.e. on Ω^2. This provides a characterization of isometric Schur multipliers on $S^p(L^2(\Omega))$, when $p \neq 2$.

Keywords Fourier multipliers · Schur multipliers · Noncommutative L^p-spaces · Isometries

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1 Introduction

Let Γ be a locally compact abelian group, let $1 \leq p \neq 2 < \infty$ and let $T : L^p(\Gamma) \to L^p(\Gamma)$ be a bounded Fourier multiplier. A classical theorem going back to Parrott [27] and Strichartz [33] asserts that T is an isometry if and only there exists $c \in \mathbb{C}$, with $|c| = 1$, and $u \in \Gamma$ such that $T = c\tau_u$. Here $\tau_u : L^p(\Gamma) \to L^p(\Gamma)$ is the translation operator defined by $\tau_u(f) = f(\cdot - u)$.

In the last decade, Fourier multipliers on noncommutative L^p-spaces associated with group von Neumann algebras emerged as a major topic in noncommutative analysis, with applications to approximation properties of operator algebras, to singular integrals and Calderon-Zygmund operators, as well as to noncommutative probability and quantum information. See in particular [2, 9, 16, 17, 20, 25, 26]. It therefore became a natural issue to understand the structure of isometric Fourier multipliers in the noncommutative framework. Indeed, the original motivation for this work was to extend the Parrott-Strichartz theorem to this setting.

To be more specific, let G be a locally compact group, let $VN(G)$ denote its group von Neumann algebra and let $\lambda : L^1(G) \to VN(G)$ be the contractive representation associated with the left regular representation of G. Assume that G is unimodular. This ensures that the Plancherel weight τ_G on $VN(G)$ is actually a normal semifinite faithful trace. For any $1 \leq p < \infty$, let $L^p(VN(G))$ be the noncommutative L^p-space associated with $(VN(G), \tau_G)$. A Fourier multiplier $T : L^p(VN(G)) \to L^p(VN(G))$ is an operator of the form

$$T(\lambda(f)) = \lambda(\phi f),$$

where ϕ is a fixed element of $L^\infty(G)$ and f lies in a suitable dense subspace of $L^1(G)$. We set $T = T_\phi$ in this case. See the beginning of Sect. 3 for more details.

We generalize the Parrott-Strichartz theorem by showing the following result, in which $VN(G)$ plays the role of Γ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$: If $p \neq 2$, a Fourier multiplier $T_\phi : L^p(VN(G)) \to L^p(VN(G))$ is an isometry if and only if there exists $c \in \mathbb{T}$ and a continuous character $\psi : G \to \mathbb{T}$ such that $\phi = c\psi$ locally almost everywhere.

We actually consider the more general class of separating Fourier multipliers. Following [22], we say that an operator $T : L^p(M) \to L^p(M)$ acting on some noncommutative L^p-space $L^p(M)$ is separating if for any disjoint $a, b \in L^p(M)$ (that is, $a^*b = ab^* = 0$), the images $T(a), T(b)$ are disjoint as well. It is well-known that if $p \neq 2$, any isometry on $L^p(M) \to L^p(M)$ is separating. This follows from Yeadon's characterization of isometries on noncommutative L^p spaces [36] (see also [22]). We prove that for any $1 \leq p < \infty$ (including the case $p = 2$), a Fourier multiplier T_ϕ on $L^p(VN(G))$ is separating if and only if there exists $c \in \mathbb{C}$ and a continuous character $\psi : G \to \mathbb{T}$ such that $\phi = c\psi$ locally almost everywhere.

The above two characterizations theorems are established in Sect. 3. Section 4 provides complements on Fourier multipliers.

Section 5 is devoted to Schur multipliers acting on Schatten classes. Let (Ω, μ) be a σ-finite measure space, let $\phi \in L^\infty(\Omega^2)$ and let T_ϕ denote the associated Schur multiplier acting on the Hilbert-Schmidt space $S^2(L^2(\Omega))$ (see below for details). Let

$1 \leq p < \infty$ and assume that T_ϕ is bounded on the Schatten space $S^p(L^2(\Omega))$. We show that $T_\phi \colon S^p(L^2(\Omega)) \to S^p(L^2(\Omega))$ is separating if and only if there exist a constant $c \in \mathbb{C}$ and two unitaries $\alpha, \beta \in L^\infty(\Omega)$ such that $\phi(s,t) = c\,\alpha(s)\beta(t)$ a.e. on Ω^2. In the case when $p \neq 2$, this provides a characterization of isometric Schur multipliers on $S^p(L^2(\Omega))$.

2 Preliminaries on Separating Maps

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. Assume that $\mathcal{M} \subset B(H)$ acts on some Hilbert space H. Let $L^0(\mathcal{M})$ denote the $*$-algebra of all closed, densely defined (possibly unbounded) operators on H, which are τ-measurable. For any $1 \leq p < \infty$, the noncommutative L^p-space $L^p(\mathcal{M})$, associated with $(\mathcal{M}, \tau)$, can be defined as

$$L^p(\mathcal{M}) := \left\{x \in L^0(\mathcal{M}) \,:\, \tau(|x|^p) < \infty\right\}.$$

Let $\|x\|_p := \tau(|x|^p)^{\frac{1}{p}}$ for any $x \in L^p(\mathcal{M})$. Then $L^p(\mathcal{M})$ equipped with $\| \cdot \|_p$ is a Banach space. We let $L^\infty(\mathcal{M}) := \mathcal{M}$ for convenience and we let $\|.\|_\infty$ denote the operator norm. For any $1 \leq p < \infty$, let $p' = \frac{p}{p-1}$ be the conjugate index of p. For any $x \in L^p(\mathcal{M})$ and $y \in L^{p'}(\mathcal{M})$, the product xy belongs to $L^1(\mathcal{M})$ and $|\tau(xy)| \leq \|x\|_p \|y\|_{p'}$. We further have an isometric identification $L^p(\mathcal{M})^* \simeq L^{p'}(\mathcal{M})$ for the duality pairing given by

$$\langle y, x\rangle = \tau(xy), \qquad x \in L^p(\mathcal{M}),\ y \in L^{p'}(\mathcal{M}).$$

We let $L^p(\mathcal{M})^+$ denote the cone of positive elements of $L^p(\mathcal{M})$. The reader is referred to [29] and the references therein for details on the algebraic operations on $L^0(\mathcal{M})$, the construction of $L^p(\mathcal{M})$, and for further properties.

We mention that if $\mathcal{M} = B(H)$ for some Hilbert space H, then the usual trace $\mathrm{tr} \colon B(H)^+ \to [0, \infty]$ is a normal semifinite faithful one and the resulting noncommutative L^p-spaces associated with $(B(H), \mathrm{tr})$ are the Schatten classes $S^p(H)$.

We say that $a, b \in L^0(\mathcal{M})$ are disjoint if $a^*b = ab^* = 0$. We say that a bounded operator $T \colon L^p(\mathcal{M}) \to L^p(\mathcal{M})$, $1 \leq p \leq \infty$, is separating if whenever $a, b \in L^p(\mathcal{M})$ are disjoint then $T(a)$ and $T(b)$ are disjoint as well.

A Jordan $*$-homomorphism on a von Neumann algebra $\mathcal{M}$ is a linear map $J : \mathcal{M} \to \mathcal{M}$ that satisfies $J(a^2) = J(a)^2$ and $J(a^*) = J(a)^*$, for every $a \in \mathcal{M}$. It is clear that a Jordan $*$-homomorphism is positive, i.e. if $a \in \mathcal{M}^+$ then $J(a) \in \mathcal{M}^+$. We warn the reader that Jordan $*$-homomorphisms are not always $*$-homomorphisms. For example, the transposition map on matrices is a Jordan $*$-homomorphism.

However we have the following lemma, in which part (1) follows from the identity $(a+b)^2 = a^2 + b^2 + (ab + ba)$ and part (2) is given by [18, 10.5.22(iii)].

Lemma 2.1 *Let $J : \mathcal{M} \to \mathcal{M}$ be a Jordan $*$-homomorphism.*

(1) *For all $a, b \in \mathcal{M}$, we have $J(ab + ba) = J(a)J(b) + J(b)J(a)$.*

(2) *If* $a, b \in \mathcal{M}$ *satisfy* $ab = ba$, *then we have* $J(ab) = J(a)J(b)$.

We also record the following properties for further use. Here a map $J : \mathcal{M} \to \mathcal{M}$ is called normal if it is weak*-continuous.

Lemma 2.2 *Let* $J : \mathcal{M} \to \mathcal{M}$ *be a normal Jordan* $*$*-homomorphism.*

(1) *The kernel* $\ker(J)$ *is a* w^**-closed ideal of* $\mathcal{M}$.
(2) *If* $(e_i)_i$ *is a bounded net of* $\mathcal{M}$ *such that* $e_i \to 0$ *and* $e_i^* \to 0$ *in the strong operator topology, then* $J(e_i) \to 0$ *in the strong operator topology.*

Proof (1) Let $J : \mathcal{M} \to \mathcal{M}$ be a normal Jordan $*$-homomorphism. A well-known theorem asserts that there exist two von Neumann algebras $\mathcal{M}_1, \mathcal{M}_2$, a von Neumann algebra embedding $\mathcal{M}_1 \overset{\infty}{\oplus} \mathcal{M}_2 \subseteq \mathcal{M}$, a normal $*$-homomorphism $\pi : \mathcal{M} \to \mathcal{M}_1$ and a normal anti $*$-homomorphism $\sigma : \mathcal{M} \to \mathcal{M}_2$, such that $J(a) = \pi(a) \oplus \sigma(a)$ for all $a \in \mathcal{M}$. (See [32, Theorem 3.3] or [11, Corollary 7.4.9.] for this result.) Then $\ker(J) = \ker(\pi) \cap \ker(\sigma)$, hence $\ker(J)$ is a weak*-closed ideal.

(2) Let $(e_i)_i$ be a bounded net of $\mathcal{M}$ such that $e_i \to 0$ and $e_i^* \to 0$ strongly. Writing $(e_i^* e_i \zeta | \eta) = (e_i \zeta | e_i \eta)$, we see that $(e_i^* e_i \zeta | \eta) \to 0$ for all $\zeta, \eta \in \mathcal{H}$. Since $(e_i^* e_i)_i$ is bounded, this implies that $e_i^* e_i \to 0$ in the weak*-topology of $\mathcal{M}$. Consequently, $\pi(e_i^* e_i) \to 0$ weakly. Writing $\|\pi(e_i)\zeta\|^2 = (\pi(e_i^* e_i)\zeta | \zeta)$, we deduce that $\pi(e_i) \to 0$ strongly. Likewise, using $e_i e_i^*$ instead of $e_i^* e_i$, we have that $\sigma(e_i) \to 0$ strongly. Thus, $J(e_i) \to 0$ strongly. □

The next statement plays a fundamental role in the study of separating maps. It was established independently in [22] and [14].

Proposition 2.3 [22, Remark 3.3 and Proposition 3.11] *Let* $1 \leq p < \infty$. *A bounded operator* $T : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ *is separating if and only if there exist a normal Jordan* $*$*-homomorphism* $J : \mathcal{M} \to \mathcal{M}$, *a partial isometry* $w \in \mathcal{M}$, *and a positive operator* B *affiliated with* $\mathcal{M}$, *which verify the following conditions:*

(a) $T(a) = wBJ(a)$, *for all* $a \in \mathcal{M} \cap L^p(\mathcal{M})$;
(b) $w^*w = J(1) = s(B)$;
(c) every spectral projection of B *commutes with* $J(a)$, *for all* $a \in \mathcal{M}$.

Here $s(B)$ *denotes the support of* B. *Furthermore, the triple* (w, B, J) *is unique.*

It was shown by Yeadon [36] that all isometries on $L^p(\mathcal{M})$, $p \neq 2$, are separating and have the above mentioned factorisation. For this reason, for T as above, we refer to (w, B, J) as the Yeadon triple of T.

Lemma 2.4 *Let* $T : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ *be a separating map and let* (w, B, J) *denote its Yeadon triple. If* T *has dense range, then* $J(1) = 1$ *and* w *is a unitary.*

Proof The proof is an easy modification of [24, Remark 3.2]. □

Lemma 2.5 *Let* $1 \leq p, q < \infty$. *Let* $T : L^p(\mathcal{M}) + L^q(\mathcal{M}) \to L^p(\mathcal{M}) + L^q(\mathcal{M})$ *and assume that* $T : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ *and* $T : L^q(\mathcal{M}) \to L^q(\mathcal{M})$ *are bounded. If* $T : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ *is separating, then* $T : L^q(\mathcal{M}) \to L^q(\mathcal{M})$ *is separating as well.*

Proof Let $\mathcal{E} := \{e \in \mathcal{M} : e \text{ is a projection with } \tau(e) < \infty\}$. Suppose that $T: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is separating. Since, $\mathcal{E} \subseteq L^p(\mathcal{M}) \cap L^q(\mathcal{M})$, the operator $T: L^q(\mathcal{M}) \to L^q(\mathcal{M})$ also preserves disjointness on $\mathcal{E}$. By [22, Remark 3.12 (i)], $T: L^q(\mathcal{M}) \to L^q(\mathcal{M})$ is separating. □

3 A Characterization of Separating Fourier Multipliers

Let G be a locally compact group with left Haar measure μ defined on the σ-algebra of Borel sets. We will write ds for $d\mu(s)$. Denote by λ the left regular representation of G defined by

$$\lambda: G \to B(L^2(G)); \quad [\lambda(s)f](t) = f(s^{-1}t).$$

The left regular representation λ determines a representation of $L^1(G)$ also denoted by λ and defined by

$$\lambda: L^1(G) \to B(L^2(G)), \quad \lambda(g)\eta = g * \eta,$$

for all $g \in L^1(G)$ and $\eta \in L^2(G)$. Here, the convolution is $g * \eta(t) = \int_G g(s)\eta(s^{-1}t)ds$. We have that $\lambda(g) = \int_G g(s)\lambda(s)\,ds$, where the operator integral is understood in the strong operator sense.

For any function $g: G \to \mathbb{C}$, we let

$$\check{g}(t) = g(t^{-1}) \qquad \text{and} \qquad g^*(t) = \overline{g(t^{-1})},$$

for all $t \in G$.

We denote by e the unit element of G. Also for any Borel set $A \subseteq G$, we let χ_A denote the indicator function of A.

Let $VN(G) \subseteq B(L^2(G))$ be the von Neumann algebra generated by $\{\lambda(s) : s \in G\}$. This coincides with the von Neumann algebra generated by $\{\lambda(g) : g \in L^1(G)\}$. When G is abelian, we have $VN(G) \simeq L^\infty(\widehat{G})$, where $\widehat{G}$ is the dual group of G.

We let $(\cdot|\cdot)$ denote the inner product on $L^2(G)$. The Fourier algebra of the group G is defined as

$$A(G) = \{(\lambda(\cdot)\zeta|\eta) : \zeta, \eta \in L^2(G)\} \subseteq C_b(G).$$

This is a Banach algebra for the pointwise product and the norm defined, for any $\psi \in A(G)$, by

$$\|\psi\|_{A(G)} = \inf\{\|\zeta\|_2\|\eta\|_2\},$$

where the infimum runs over all $\zeta, \eta \in L^2(G)$ such that $\psi = (\lambda(\cdot)\zeta|\eta)$. We note for further use that equivalently, we can write $A(G) = \{\zeta * \eta : \zeta, \eta \in L^2(G)\}$.

We recall that $A(G)^* \simeq VN(G)$ isometrically for the duality pairing given by

$$\langle \lambda(s), \psi \rangle = \psi(s), \qquad \psi \in A(G),\ s \in G. \tag{3.1}$$

Assume that G is unimodular. We will use the so-called Plancherel trace

$$\tau_G : VN(G)^+ \longrightarrow [0, +\infty],$$

for which we refer to [35, Sect. VII.3] (see also [3]). We note that τ_G is a normal semifinite faithful trace. This allows to consider the noncommutative L^p-spaces $L^p(VN(G))$ associated with τ_G. We recall that if G is discrete then G is unimodular and τ_G is normalised. Also, if G is abelian then G is unimodular and $L^p(VN(G)) = L^p(\widehat{G})$, where $\widehat{G}$ denotes the dual group of G.

It is well-known that for any $g \in L^1(G) \cap L^2(G)$, $\lambda(g) \in L^2(VN(G))$ with $\|\lambda(g)\|_2 = \|g\|_2$ (see e.g. [3, Sect. 6.1]). Consequently, the restriction of λ to $L^1(G) \cap L^2(G)$ extends to an isometry from $L^2(G)$ into $L^2(VN(G))$. It turns out that the latter is onto, which yields a unitary identification

$$L^2(VN(G)) \simeq L^2(G). \tag{3.2}$$

Using the notation $U_\lambda \colon L^2(G) \to L^2(VN(G))$ for the above unitary mapping, we have

$$\tau_G\big(U_\lambda(\zeta)U_\lambda(\eta)\big) = \int_G \zeta(t)\check{\eta}(t)\, dt, \qquad \zeta, \eta \in L^2(G). \tag{3.3}$$

Since $L^1(VN(G))^* \simeq VN(G)$, we have an isometric identification

$$A(G) \simeq L^1(VN(G)).$$

It is not hard to deduce from (3.1) and (3.3) that this identification is given by the mapping $A(G) \to L^1(VN(G))$ taking $\zeta * \eta$ to $U_\lambda(\check{\eta})U_\lambda(\check{\zeta})$, for all $\zeta, \eta \in L^2(G)$. Details are left to the reader.

Let $C_c(G)$ denote the space of continuous and compactly supported functions on G. We let $C_c(G) * C_c(G)$ denote the linear span of $f_1 * f_2$, where $f_1, f_2 \in C_c(G)$. It is well-known that

$$\lambda\big(C_c(G) * C_c(G)\big) \subseteq L^1(VN(G)) \cap VN(G),$$

and that $\lambda\big(C_c(G) * C_c(G)\big)$ is dense in $L^p(VN(G))$, for all $1 \leq p < \infty$. For a proof, we refer to [7, Proposition 3.4] for the case $p = 1$, and to [5, Proposition 4.7] for the other cases. We also note that since $C_c(G) * C_c(G)$ is dense in $L^1(G)$, $\lambda\big(C_c(G) * C_c(G)\big)$ is weak*-dense in $VN(G)$.

Lemma 3.1 *Suppose that $K \subseteq G$ is a compact set. There is a function $\psi \in A(G)$ such that $\psi(s) = \mu(K)$, for all $s \in K$.*

Proof For a proof, we refer to [19, Proposition 2.3.2.]. □

Some of the formulations of the main results in this article are easier when the group G is σ-compact, meaning that G is the countable union of compact subsets. The following well-known lemma relates this property with other countability properties that the group G can have. We provide a proof for the sake of completeness.

Lemma 3.2 *Let G be a locally compact group. Then the following implications hold:*

$$G \text{ is second countable } \implies G \text{ is } \sigma\text{-compact} \iff \text{the Haar measure of } G \text{ is } \sigma\text{-finite}.$$

Moreover, the remaining implication is false. That is, there exists a (σ-)compact group which is not second countable.

Proof If G is second countable, then by definition, its topology admits a countable basis. Since G is locally compact, this basis can be chosen to consist of relatively compact sets $O_k,\ k \geq 1$. Thus,

$$G = \bigcup_{k\in\mathbb{N}} O_k = \bigcup_{k\in\mathbb{N}} \overline{O_k},$$

so G is σ-compact.

Next, we show that for the Haar measure μ, σ-compactness and σ-finiteness are equivalent. Recall [8, Proposition 2.4] that G admits an open, closed, σ-compact subgroup G_0. Thus, $G = \bigcup_{y\in Y} yG_0$ for some subset $Y \subseteq G$ representing the left cosets, where the union is disjoint. We claim that Y can be chosen at most countable if and only if G is σ-compact, if and only if the Haar measure is σ-finite. Indeed, if Y is at most countable, as yG_0 is σ-compact for all $y \in Y$, G is σ-compact. If G is σ-compact, say $G = \bigcup_{n\in\mathbb{N}} K_n$ with compact K_n, then $\mu(K_n) < \infty$ for any $n \in \mathbb{N}$, hence G is σ-finite. Finally, suppose that G is σ-finite, say $G = \bigcup_{n\in\mathbb{N}} H_n$ with $\mu(H_n) < \infty$ for any $n \in \mathbb{N}$. According to [8, Proposition 2.22], the fact that $\mu(H_n)$ is finite implies that $Y_n := \{y \in Y :\ H_n \cap yG_0 \neq \emptyset\}$ is at most countable. Thus, $Y' := \{y \in Y :\ G \cap yG_0 = \bigcup_{n\in\mathbb{N}}(H_n \cap yG_0) \neq \emptyset\} = \bigcup_{n\in\mathbb{N}} Y_n$ is also at most countable. But clearly, $Y' = Y$.

Finally, for the last statement, it suffices to take the compact non first countable group $G = \mathbb{T}^{\mathbb{R}}$. □

In the sequel we use the space $L^\infty(G)$. Its definition requires some care. When G is σ-compact, $L^\infty(G)$ is defined in the usual way. But when G is not σ-compact, by the above Lemma 3.2, the left Haar measure μ is not σ-finite. In this case, if we define $L^\infty(G)$ in the usual way, the duality of $L^1(G)$ and $L^\infty(G)$ may break down. As it is explained in [8, Sect. 2.3], it is possible to salvage this duality by modifying the definition of $L^\infty(G)$ as follows. A set $E \subseteq G$ is called locally Borel if $E \cap F$ is Borel whenever F is Borel and $\mu(F) < \infty$. A locally Borel set is locally null if $\mu(E \cap F) = 0$ whenever F is Borel and $\mu(F) < \infty$. A function $f : G \to \mathbb{C}$ is locally measurable if $f^{-1}(A)$ is locally Borel for every Borel set $A \subseteq \mathbb{C}$. A property is true locally almost everywhere if it is true except on a locally null set.

With these definitions in hand, let $L^\infty(G)$ be the space of all locally measurable functions $\phi\colon G \to \mathbb{C}$ that are bounded except on a locally null set, modulo the functions that are zero locally almost everywhere. Then $L^\infty(G)$ is a Banach space with the norm

$$\|\phi\|_\infty = \inf\left\{c : |\phi(t)| \le c \text{ locally almost everywhere}\right\}.$$

We note that for any $1 \le p < \infty$, any $f \in L^p(G)$ has a σ-finite support hence for all $\phi \in L^\infty(G)$, ϕf is a well-defined element of $L^p(G)$.

One may therefore define $\int_G \phi f$ for all $\phi \in L^\infty(G)$ and all $f \in L^1(G)$ and this duality pairing yields an isometric identification $L^\infty(G) \simeq L^1(G)^*$. When G is σ-compact, $L^\infty(G)$ defined as above coincides with the usual one.

Definition 3.3 For any $\phi \in L^\infty(G)$, let $M_\phi\colon \lambda(C_c(G)*C_c(G)) \to VN(G)$ be defined by

$$M_\phi(\lambda(f)) := \lambda(\phi f), \qquad f \in C_c(G) * C_c(G).$$

For any $1 \le p < \infty$, we say that M_ϕ is a bounded Fourier multiplier on $L^p(VN(G))$ if the above map extends to a bounded operator (still denoted by) $M_\phi\colon L^p(VN(G)) \to L^p(VN(G))$. In the sequel we abbreviate this by saying that "$M_\phi\colon L^p(VN(G)) \to L^p(VN(G))$ is a bounded Fourier multiplier".

Likewise, if $p = \infty$, we say that M_ϕ is a bounded Fourier multiplier on $VN(G)$ if the above map extends to a bounded weak*-continuous operator $M_\phi\colon VN(G) \to VN(G)$.

Let $1 \le p < \infty$. We recall from [3, Sect. 6.1] that if $M_\phi\colon L^p(VN(G)) \to L^p(VN(G))$ is a bounded Fourier multiplier, then $M_\phi\colon L^{p'}(VN(G)) \to L^{p'}(VN(G))$ is a bounded Fourier multiplier as well, where p' is the conjugate index of p. Moreover, $M_{\check{\phi}}\colon L^{p'}(VN(G)) \to L^{p'}(VN(G))$ is a bounded Fourier multiplier, and this operator is actually the adjoint of $M_\phi\colon L^p(VN(G)) \to L^p(VN(G))$.

Thanks to (3.2), we have that for any $\phi \in L^\infty(G)$, $M_\phi\colon L^2(VN(G)) \to L^2(VN(G))$ is a bounded Fourier multiplier, with

$$\|M_\phi\colon L^2(VN(G)) \longrightarrow L^2(VN(G))\| = \|\phi\|_{L^\infty(G)}. \tag{3.4}$$

Indeed, let us denote as before by $U_\lambda\colon L^2(G) \to L^2(VN(G))$ the unitary mapping taking any $f \in L^1(G) \cap L^2(G)$ to $\lambda(f)$, see (3.2). Let $\pi\colon L^\infty(G) \to B(L^2(G))$ be defined by $[\pi(\phi)](f) = \phi f$, for all $\phi \in L^\infty(G)$ and all $f \in L^2(G)$. Then π is an isometry. Since for any $\phi \in L^\infty(G)$, $U_\lambda \pi(\phi) U_\lambda^*$ coincides with M_ϕ on $C_c(G)*C_c(G)$, we obtain that M_ϕ is a bounded Fourier multiplier on $L^2(VN(G))$ and that $M_\phi\colon L^2(VN(G)) \to L^2(VN(G))$ coincides with $U_\lambda \pi(\phi) U_\lambda^*$. Since $U_\lambda \pi(\,\cdot\,) U_\lambda^*$ is an isometry, the equality (3.4) follows. (See also [3, Lemma 6.5].) It follows from above that a Fourier multiplier $M_\phi\colon L^2(VN(G)) \to L^2(VN(G))$ satisfies

$$M_\phi(U_\lambda(f)) = U_\lambda(\phi f), \qquad f \in L^2(G).$$

Remark 3.4 If $\phi_1, \phi_2 \in L^\infty(G)$ are such that M_{ϕ_1} and M_{ϕ_2} coincide on $\lambda(C_c(G) * C_c(G))$, then by (3.4), the operator $M_{\phi_1-\phi_2} = M_{\phi_1} - M_{\phi_2} \colon L^2(VN(G)) \to L^2(VN(G))$ is equal to 0, hence $\phi_1 = \phi_2$ locally almost everywhere.

For the rest of this section, we fix a net $(f_i)_{i\in I}$ in $C_c(G) * C_c(G)$ such that $f_i \geq 0$ and $\int_G f_i(s)\,ds = 1$ for all $i \in I$, the supports of f_i's are contained in some compact neighborhood V_i of e where the net $(V_i)_{i\in I}$, is decreasing, and $\bigcap_{i\in I} V_i = \{e\}$. We set $e_i := \lambda(f_i)$. Then for all $i \in I$, we have

$$e_i \in L^1(VN(G)) \cap VN(G) \qquad \text{and} \qquad \|e_i\|_{VN(G)} \leq 1.$$

Lemma 3.5 *We both have $e_i \to 1$ and $e_i^* \to 1$ in the strong operator topology.*

Proof Let $\zeta \in L^2(G)$. Since each f_i is non-negative and L^1-normalized, we have

$$\|e_i(\zeta) - \zeta\|_{L^2(G)} = \left\| \int_G f_i(t)\,(\lambda(t)\zeta - \zeta)\;dt \right\|_{L^2(G)} \leq \int_G f_i(t)\,\|\lambda(t)\zeta - \zeta\|_{L^2(G)}\,dt.$$

Since $\lambda(t) \to 1$ as $t \to e$ in the strong operator topology, the assumptions on $(f_i)_{i\in I}$ ensure that the right hand-side tends to 0, when $i \to \infty$. Hence $\|e_i(\zeta) - \zeta\|_{L^2(G)} \to 0$.

This shows that $e_i \to 1$ strongly. Since $e_i^* = \lambda(f_i^*)$ and the f_i^* have the same features as the f_i, we also have that $e_i^* \to 1$ strongly. □

We let $C_b(G)$ be the space of bounded and continuous functions on G.

The following lemma shows that the definition of bounded Fourier multipliers $VN(G) \to VN(G)$ considered in this paper coincide with the one in [6].

Lemma 3.6 *Let $\phi \in L^\infty(G)$ and assume that M_ϕ is a bounded Fourier multiplier on $VN(G)$. Then there exists $\psi \in C_b(G)$ such that $\phi = \psi$ locally almost everywhere, and*

$$M_\psi(\lambda(s)) = \psi(s)\lambda(s), \qquad s \in G.$$

Proof Assume that $M_\phi \colon VN(G) \to VN(G)$ is a bounded Fourier multiplier. Let us show that for any $s \in G$,

$$M_\phi(\lambda(s)) \in \mathrm{Span}\{\lambda(s)\}. \tag{3.5}$$

Fix $s \in G$. For all $i \in I$, $\lambda(s)e_i = \int_G f_i(t)\lambda(st)dt$, hence $M_\phi(\lambda(s)e_i) = \int_G \phi(st) f_i(t)\lambda(st)\,dt$, where these integrals are defined in the strong operator topology. Therefore, for all $\varphi \in A(G)$, we have

$$\langle M_\phi(\lambda(s)e_i), \varphi\rangle = \int_G \phi(st) f_i(t)\langle \lambda(st), \varphi\rangle\,dt = \int_G \phi(st) f_i(t)\varphi(st)\,dt.$$

Let $\varphi \in A(G)$ such that $\varphi(s) = 0$. From the above, we have

$$|\langle M_\phi(\lambda(s)e_i), \varphi\rangle| \leq \|\phi\|_\infty \int_G f_i(t)|\varphi(st) - \varphi(s)|\,dt.$$

Since φ is continuous, the right hand-side of the above inequality tends to zero when $i \to \infty$. Moreover by Lemma 3.5, $\lambda(s)e_i \to \lambda(s)$ in the strong operator topology. Since $(e_i)_{i\in I}$ is bounded, this implies that $\lambda(s)e_i \to \lambda(s)$ in the weak*-topology. Therefore, $M_\phi(\lambda(s)e_i) \to M_\phi(\lambda(s))$ in the weak*-topology. Hence, $\langle M_\phi(\lambda(s)e_i), \varphi\rangle \to \langle M_\phi(\lambda(s)), \varphi\rangle$. We obtain that $\langle M_\phi(\lambda(s)), \varphi\rangle = 0$. Hence, $M_\phi(\lambda(s)) \in \{\varphi : \varphi(s) = 0\}^\perp$ in the duality $A(G)^* \simeq VN(G)$. Since $\mathrm{Span}\{\lambda(s)\}_\perp = \{\varphi : \varphi(s) = 0\}$, we deduce (3.5).

Let $\psi : G \to \mathbb{C}$ be the unique function such that for all $s \in G$, $M_\phi(\lambda(s)) = \psi(s)\lambda(s)$. The pre-adjoint $A(G) \to A(G)$ of M_ϕ is the pointwise multiplication by ψ. According to the comment following [6, 1.1. Definition], the function ψ is therefore continuous.

Next, we show that $\phi = \psi$ locally almost everywhere. For any $f \in C_c(G) * C_c(G)$, we have $\lambda(f) = \int_G f(t)\lambda(t)\,dt$. This SOT-integral is absolutely convergent in $VN(G)$, hence

$$\begin{aligned} M_\phi(\lambda(f)) &= \int_G f(t)M_\phi(\lambda(t))\,dt \\ &= \int_G f(t)\psi(t)\lambda(t)\,dt \\ &= M_\psi(\lambda(f)). \end{aligned}$$

This implies that $M_\phi = M_\psi$ on $C_c(G) * C_c(G)$, and the result follows by Remark 3.4.

For the regularity of G needed in the proof of [6, 1.1. Definition], we refer to [19, Theorem 2.3.8 p. 53]. See also the discussion following [3, Definition 6.3]. □

Let $\mathbb{T}$ denote the unit circle of $\mathbb{C}$. A homomorphism $\varphi : G \to \mathbb{T}$ is called a character. We let $Hom(G, \mathbb{T})$ denote the collection of all characters on G. There is a natural isomorphism between $Hom(G, \mathbb{T})$ and $Hom(\frac{G}{[G,G]}, \mathbb{T})$, where $[G, G]$ denotes the commutator subgroup of G [13, (23.8) Theorem p. 358-359]. When G is a perfect group, i.e. $[G, G] = G$, the only character on G is the trivial one, that is, $Hom(G, \mathbb{T}) = \{1\}$. Examples of perfect groups include non-abelian simple groups and the special linear groups $SL_n(\mathbb{K})$, for a fixed field $\mathbb{K}$.

The following is the main result of this section. In view of this theorem and the observation above, we see that there are groups with relatively few separating Fourier multipliers.

Theorem 3.7 *Assume that G is a locally compact unimodular σ-compact group, let $1 \le p < \infty$ and let $\phi \in L^\infty(G)$. The following are equivalent.*

(i) *The mapping M_ϕ is a bounded Fourier multiplier on $L^p(VN(G))$, and the operator $M_\phi : L^p(VN(G)) \to L^p(VN(G))$ is separating.*

(ii) *There exist a constant $c \in \mathbb{C}$ and a continuous character $\psi : G \to \mathbb{T}$ such that $\phi = c\psi$ almost everywhere.*

The proof will be given after a series of intermediate results.

Lemma 3.8 *Let $\mathcal{M}$ be a semifinite von Neumann algebra. Let $(x_j)_j$ be a net in $\mathcal{M} \cap L^2(\mathcal{M})$ with $\sup \|x_j\|_\infty < \infty$. If $\|x_j\|_2 \to 0$, then $x_j \to 0$ in the weak*-topology of $\mathcal{M}$.*

Proof Take $y \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$. We have that

$$|\langle y, x_j \rangle| \leq \|y\|_2 \|x_j\|_2 \to 0,$$

and therefore $\langle y, x_j \rangle \to 0$. Since $L^1(\mathcal{M}) \cap L^2(\mathcal{M})$ is dense in $L^1(\mathcal{M})$ and $(x_j)_j$ is bounded in $\mathcal{M}$, this implies that $\langle y, x_j \rangle \to 0$, for all $y \in L^1(\mathcal{M})$. That is, $x_j \to 0$ in the weak*-topology of $\mathcal{M}$. ◻

In the following lemma, G is not necessarily σ-compact.

Lemma 3.9 *Let $\phi \in L^\infty(G)$. Assume that the Fourier multiplier $M_\phi \colon L^2(VN(G)) \to L^2(VN(G))$ is separating and non-zero.*

(1) *For any compact $K \subseteq G$, the restriction $\phi|_K$ is non-zero almost everywhere on K.*
(2) *The operator $M_\phi \colon L^2(VN(G)) \to L^2(VN(G))$ has dense range.*
(3) *For any compact $K \subseteq G$, there exists a continuous function $\Phi \colon K \to \mathbb{C}$ such that $\phi|_K = \Phi$ almost everywhere on K.*

Proof (1) Let K be a compact subset of G. Set $N_K(\phi) := \{s \in K : \phi(s) = 0\}$. We show that $N_K(\phi)$ has measure zero. Assume on the contrary that $N_K(\phi)$ has positive measure. We show that there exists $a_0 \in K$ such that for any open neighbourhood V of a_0, $\mu(N_K(\phi) \cap V) > 0$. Assume on the contrary that for any $a \in K$ there is an open neighbourhood of a, V_a, such that $\mu(N_K(\phi) \cap V_a) = 0$. Since K is compact and $\{V_a\}_{a \in K}$ covers K, there is a finite subcover $\{V_{a_j}\}_{j=1}^n$ that covers K as well. Now, note that

$$\begin{aligned} \mu(N_K(\phi)) &= \mu(N_K(\phi) \cap (\cup_{j=1}^n V_{a_j})) = \mu(\cup_{j=1}^n (N_K(\phi) \cap V_{a_j})) \\ &\leq \sum_{j=1}^n \mu(N_K(\phi) \cap V_{a_j}) = 0, \end{aligned}$$

which contradicts the fact that $\mu(N_K(\phi)) > 0$.

Let $(U_i)_{i \in I}$ be a net of neighbourhoods of e, the unit element of G, directed by inclusion, with $\cap_{i \in I} U_i = \{e\}$. Then for all $i \in I$, we have $\mu(a_0 U_i \cap N_K(\phi)) > 0$ and we may define

$$h_i := \frac{1}{\mu(a_0 U_i \cap N_K(\phi))} \chi_{U_i} \chi_{N_K(\phi)}(a_0 \cdot).$$

For any $i \in I$, we have that $h_i \geq 0$, $\int h_i = 1$, and $h_i \in L^1(G) \cap L^2(G)$. Moreover, $\mathrm{supp}(h_i) = U_i \cap a_0^{-1} N_K(\phi)$ and therefore, $\bigcap_{i \in I} \mathrm{supp}(h_i) = \{e\}$ and $\phi h_i(a_0^{-1} \cdot) = 0$, for all $i \in I$.

Let (w, B, J) be the Yeadon triple of $M_\phi \colon L^2(VN(G)) \to L^2(VN(G))$. For any $i \in I$, let $\varepsilon_i := \lambda(h_i(a_0^{-1} \cdot))$. We have that $\varepsilon_i \in L^2(VN(G)) \cap VN(G)$ and $\|\varepsilon_i\|_{VN(G)} \leq 1$. By Proposition 2.3(a), $M_\phi(\varepsilon_i) = wBJ(\varepsilon_i)$. Also, $M_\phi(\varepsilon_i) = \lambda(\phi h_i(a_0^{-1} \cdot)) = \lambda(0) = 0$. Therefore, $wBJ(\varepsilon_i) = 0$. We now apply Proposition

2.3(b). Since $w^*w = s(B)$, we have $BJ(\varepsilon_i) = w^*wBJ(\varepsilon_i)$, hence $BJ(\varepsilon_i) = 0$. Further $0 \leq J(\varepsilon_i) \leq J(1) = s(B)$, hence $J(\varepsilon_i)$ is valued in $\ker(B)^\perp$. Hence the equality $BJ(\varepsilon_i) = 0$ implies that $J(\varepsilon_i) = 0$, that is $\varepsilon_i \in \ker(J)$. By Lemma 2.2(1), this implies that for any $g \in L^1(G)$ and any $i \in I$, we have $\lambda(g)\varepsilon_i \in \ker(J)$.

Now, for any $f \in L^1(G) \cap L^2(G)$, let $g = f(\cdot a_0)$. We have

$$\lambda(g)\varepsilon_i = \lambda(g * h_i(a_0^{-1}\,\cdot)) = \lambda(g(\cdot a_0^{-1}) * h_i) = \lambda(f * h_i).$$

Since $\|f * h_i - f\|_2 \to 0$, we have $\|\lambda(f * h_i) - \lambda(f)\|_{L^2(VN(G))} \to 0$, by (3.2). Note that,

$$\|\lambda(f * h_i)\|_\infty \leq \|f * h_i\|_1 \leq \|f\|_1 \|h_i\|_1 = \|f\|_1.$$

Hence by Lemma 3.8, $\lambda(f * h_i) \to \lambda(f)$, in the weak*-topology of $VN(G)$. Since $\ker(J)$ is weak*-closed, by Lemma 2.2(1), we obtain that $\lambda(f)$ belongs to $\ker(J)$. Finally since the space $\{\lambda(f) : f \in L^1(G) \cap L^2(G)\}$ is weak*-dense in $VN(G)$, we deduce that $\ker(J) = VN(G)$. Hence $M_\phi = 0$, which is a contradiction.

(2) Consider any $F \in C_c(G) * C_c(G)$. This function has compact support, say $K \subseteq G$. Set $N_\delta = \{s \in K : |\phi(s)| < \delta\}$, for all $\delta > 0$. By part (1), $\phi \neq 0$ almost everywhere on K, hence we have $\lim_{\delta \to 0} \mu(N_\delta) = 0$. For any $\delta > 0$, set $g_\delta := \phi^{-1}\chi_{N_\delta^c} F$. Since $\phi^{-1}\chi_{N_\delta^c} \in L^\infty(G)$, we have $g_\delta \in L^1(G) \cap L^2(G)$. Hence, $\lambda(g_\delta)$ is well-defined, $\lambda(g_\delta) \in L^2(VN(G)) \cap VN(G)$ and

$$M_\phi(\lambda(g_\delta)) = \int \phi(t) g_\delta(t) \lambda(t) dt = \lambda(\chi_{N_\delta^c} F).$$

Therefore, $\lambda(F) - M_\phi(\lambda(g_\delta)) = \lambda(\chi_{N_\delta} F)$, and we have that

$$\|\lambda(F) - M_\phi(\lambda(g_\delta))\|_2 = \|\lambda(\chi_{N_\delta} F)\|_2 = \|\chi_{N_\delta} F\|_2,$$

by (3.2). Since $\mu(N_\delta) \to 0$, when $\delta \to 0$, this implies that $\lim_{\delta \to 0} \|\lambda(F) - M_\phi(\lambda(g_\delta))\|_2 = 0$. Hence, $\lambda(F) \in \overline{\mathrm{ran}(M_\phi)}$. By density of $\lambda(C_c(G) * C_c(G))$ in $L^2(VN(G))$, this implies that the range of M_ϕ is dense in $L^2(VN(G))$.

(3) Let $w^* M_\phi$ be the operator taking any $a \in L^2(VN(G))$ to $w^* M_\phi(a)$. According to [23, Remark 4.2], $w^* M_\phi$ is positive, that is, it maps $L^2(VN(G))^+$ into itself. We use a modification of the argument in [3, Lemma 6.10]. For all $g \in C_c(G)$, $\lambda(g^* * g) \in L^2(VN(G))^+$. Hence, $w^* M_\phi(\lambda(g^* * g)) \in L^2(VN(G))^+$. Consequently,

$$\big(M_\phi(\lambda(g^* * g))\zeta | w(\zeta)\big) \geq 0,$$

for all $\zeta \in L^2(G)$. The calculation in the proof of [3, Lemma 6.10] shows that

$$\big(M_\phi(\lambda(g^* * g))\zeta | w(\zeta)\big) = \int_G \int_G \overline{g(t)} g(s) \phi(t^{-1}s) \Big(\lambda(t^{-1}s)\zeta | w(\zeta)\Big)\, dsdt.$$

This implies that for all $\zeta \in L^2(G)$, the function $s \mapsto \phi(s)\,(\lambda(s)\zeta|w(\zeta))$ is positive definite in the sense of [35, Definition VII.3.20].

By polarization, this implies that for all $\zeta, \zeta' \in L^2(G)$, $s \mapsto \phi(s)\,\big(\lambda(s)\zeta|w(\zeta')\big)$ is a linear combination of positive definite functions. By part (2) and Lemma 2.4, w is a unitary. Hence the above actually shows that for all $\zeta, \eta \in L^2(G)$, $s \mapsto \phi(s)\,(\lambda(s)\zeta|\eta)$ is a linear combination of positive definite functions. By the definition of $A(G)$, this means that for all $\psi \in A(G)$, $\phi\psi$ is a linear combination of positive definite functions. By [35, Corollary 3.22] and its proof, this implies that for all $\psi \in A(G)$, $\phi\psi$ is locally almost everywhere equal to a continuous function.

Let $K \subseteq G$ be a compact set with $\mu(K) > 0$. By Lemma 3.1, there is $\psi \in A(G)$ such that $\psi|_K > 0$. Now, $(\phi\psi)\,|_K$ is almost everywhere equal to a continuous function. Hence, $\phi|_K$ is almost everywhere equal to a continuous function. □

Remark 3.10 Note that if G is σ-compact, the above lemma implies that if $\phi \in L^\infty(G)$ is such that $M_\phi\colon L^2(VN(G)) \to L^2(VN(G))$ is separating, then ϕ is almost everywhere equal to a continuous function.

In the next statement, we use the net $(e_i)_{i\in I}$ defined before Lemma 3.5.

Lemma 3.11 *Let $\phi \in C_b(G)$ and consider the Fourier multiplier $M_\phi\colon L^2(VN(G)) \to L^2(VN(G))$. We have the following convergences in the strong operator topology of $VN(G)$.*

(1) *For all $s \in G$, $M_\phi(\lambda(s)e_i) \xrightarrow[i\to\infty]{} \phi(s)\lambda(s)$.*
(2) *For all $s \in G$, $M_\phi(\lambda(s)e_i^2) \xrightarrow[i\to\infty]{} \phi(s)\lambda(s)$.*
(3) *For all $s \in G$, $M_\phi(e_i\lambda(s)e_i) \xrightarrow[i\to\infty]{} \phi(s)\lambda(s)$.*

Proof (1) By Lemma 3.5, it is enough to show that for all $s \in G$, $M_\phi(\lambda(s)e_i) - \phi(s)\lambda(s)e_i$ converges to 0 in $VN(G)$. Using the assumptions on $(f_i)_{i\in I}$, and the fact that ϕ is continuous, we have that

$$\left\|M_\phi(\lambda(s)e_i) - \phi(s)\lambda(s)e_i\right\|_\infty = \left\|\int_G (\phi(st)f_i(t)\lambda(st) - \phi(s)f_i(t)\lambda(st))\,dt\right\|_\infty \leq \int_G |\phi(st) - \phi(s)|f_i(t)\,dt \xrightarrow[i\to\infty]{} 0.$$

This proves the result.

(2) For all $i \in I$, we have $e_i^2 = \lambda(f_i * f_i)$, $f_i * f_i \geq 0$, $\int_G (f_i * f_i)(s)\,d(s) = 1$, and $\operatorname{supp}(f_i * f_i) \subseteq \operatorname{supp}(f_i)\cdot\operatorname{supp}(f_i)$. Hence using $f_i * f_i$ instead of f_i, the convergence in (2) can be shown exactly in the same way as in (1).

(3) We argue as in (1). Let $s \in G$. Since $e_i \to 1$ in the strong operator topology and $(e_i)_{i\in I}$ is bounded, $e_i\lambda(s)e_i \to \lambda(s)$ in the strong operator topology. Hence we only need to show that $M_\phi(e_i\lambda(s)e_i) - \phi(s)e_i\lambda(s)e_i$ converges to 0 in $VN(G)$. We have that

$$\left\|M_\phi(e_i\lambda(s)e_i) - \phi(s)e_i\lambda(s)e_i\right\|_\infty$$

$$= \left\| \int_G \int_G \phi(rst) f_i(r) f_i(t) \lambda(rst)\, drdt - \int_G \int_G \phi(s) f_i(r) f_i(t) \lambda(rst)\, drdt \right\|_\infty$$
$$\leq \int_G \int_G |\phi(rst) - \phi(s)| f_i(r) f_i(t) drdt \xrightarrow[i\to\infty]{} 0,$$

again using the assumptions on $(f_i)_{i\in I}$, and the fact that ϕ is continuous. □

Proof $\underline{(ii) \Rightarrow (i)}$: We may assume that $c = 1$ and that $\phi = \psi$, i.e. ϕ is a continuous character. Any character is positive definite and maps e to 1. Hence according to [6, Proposition 4.2] and [28, Proposition 3.6], M_ϕ is a bounded Fourier multiplier on $VN(G)$, with $\|M_\phi \colon VN(G) \to VN(G)\| = \|M_\phi(1)\|_\infty = |\phi(e)| = 1$. According to [3, Lemma 6.4 and Lemma 6.6], M_ϕ is therefore a bounded Fourier multiplier on $L^p(VN(G))$ and

$$\|M_\phi \colon L^p(VN(G)) \longrightarrow L^p(VN(G))\| \leq \|M_\phi \colon VN(G) \longrightarrow VN(G)\|.$$

Thus,

$$\|M_\phi \colon L^p(VN(G)) \longrightarrow L^p(VN(G))\| \leq 1. \tag{3.6}$$

Since ϕ^{-1} is also a continuous character, the above argument shows as well that

$$\|M_{\phi^{-1}} \colon L^p(VN(G)) \longrightarrow L^p(VN(G))\| \leq 1. \tag{3.7}$$

For any $f \in C_c(G) * C_c(G)$, we have

$$M_{\phi^{-1}} M_\phi(\lambda(f)) = M_{\phi^{-1}}(\lambda(\phi f)) = \lambda(\phi^{-1}\phi f) = \lambda(f).$$

Similarly, $M_\phi M_{\phi^{-1}}(\lambda(f)) = \lambda(f)$. Hence M_ϕ and $M_{\phi^{-1}}$ are inverse to each other. It therefore follows from (3.6) and (3.7) that $M_\phi \colon L^p(VN(G)) \to L^p(VN(G))$ is an isometry.

If $p \neq 2$, it follows from [36] (see also [22]) that $M_\phi \colon L^p(VN(G)) \to L^p(VN(G))$ is separating. If $p = 2$, consider any $1 < q \neq 2 < \infty$. The above reasoning shows that $M_\phi \colon L^q(VN(G)) \to L^q(VN(G))$ is separating. Applying Lemma 2.5, we deduce that the operator $M_\phi \colon L^2(VN(G)) \to L^2(VN(G))$ is separating.
$\underline{(i) \Rightarrow (ii)}$: We assume that $M_\phi \colon L^p(VN(G)) \to L^p(VN(G))$ is separating. By Lemma 2.5, $M_\phi \colon L^2(VN(G)) \to L^2(VN(G))$ is separating as well. Let (w, B, J) be its Yeadon triple. We may assume that M_ϕ is non-zero. Then by Lemma 3.9(2), $M_\phi \colon L^2(VN(G)) \to L^2(VN(G))$ has dense range. It then follows from Lemma 2.4 that w is a unitary and $J(1) = 1$. According to Remark 3.10, ϕ is almost everywhere equal to a continuous function. Replacing ϕ by this function, we may now assume that $\phi \in C_b(G)$.

For any $s \in G$ we have, by Lemma 3.11 and Lemma 2.1(1),

$$\phi(s)\,(\lambda(s) + \lambda(s)) = \lim_{j\to\infty} M_\phi\left(e_j\lambda(s)e_j + \lambda(s)e_j^2\right)$$

$$= \lim_{j\to\infty} wBJ\left(e_j\lambda(s)e_j + \lambda(s)e_j^2\right)$$
$$= \lim_{j\to\infty} wB\big(J(e_j)J(\lambda(s)e_j) + J(\lambda(s)e_j)J(e_j)\big)$$
$$= \lim_{j\to\infty} \big(M_\phi(e_j)J(\lambda(s)e_j) + M_\phi(\lambda(s)e_j)J(e_j)\big),$$

where the limit is taken in the strong operator topology.

By Lemma 3.5, $\lambda(s)e_j \to \lambda(s)$ and $(\lambda(s)e_j)^* \to \lambda(s)^*$ strongly. Hence by Lemma 2.2(2), $J(\lambda(s)e_j) \to J(\lambda(s))$ strongly. By Lemma 3.11(1), $M_\phi(e_j) \to \phi(e)$ strongly. Since $(M_\phi(e_j))_{j\in I}$ is bounded, we deduce that $M_\phi(e_j)J(\lambda(s)e_j) \to \phi(e)J(\lambda(s))$ strongly. Similarly, $M_\phi(\lambda(s)e_j)J(e_j) \to \phi(s)\lambda(s)J(1) = \phi(s)\lambda(s)$ strongly. It therefore follows from the previous calculation that $\phi(s)\,(\lambda(s) + \lambda(s)) = \phi(e)J(\lambda(s)) + \phi(s)\lambda(s)$, that is, $\phi(s)\lambda(s) = \phi(e)J(\lambda(s))$.

Since ϕ is non-zero, this implies that $\phi(e) \neq 0$. Set $\psi := \phi(e)^{-1}\phi$. It follows from the above that

$$J(\lambda(s)) = \psi(s)\lambda(s), \qquad s \in G. \tag{3.8}$$

We now show that ψ is a character. Let $s, t \in G$ and recall that $\lambda(st) = \lambda(s)\lambda(t)$. On the one hand, we have that $J(\lambda(st)) = \psi(st)\lambda(st)$. On the other hand, we have that $J(\lambda(s))J(\lambda(t)) = \psi(s)\psi(t)\lambda(st)$.

If $st = ts$, then $\lambda(s)\lambda(t) = \lambda(t)\lambda(s)$, hence by Lemma 2.1(2), we have that $J(\lambda(st)) = J(\lambda(s)\lambda(t)) = J(\lambda(s))J(\lambda(t))$. Hence, $\psi(st) = \psi(s)\psi(t)$. Assume now that $st \neq ts$. By Lemma 2.1(1), we have

$$J(\lambda(s)\lambda(t) + \lambda(t)\lambda(s)) = J(\lambda(s))J(\lambda(t)) + J(\lambda(t))J(\lambda(s)).$$

Therefore,

$$\psi(st)\lambda(st) + \psi(ts)\lambda(ts) = \psi(s)\psi(t)\lambda(st) + \psi(t)\psi(s)\lambda(ts).$$

Since $\lambda(st)$ and $\lambda(ts)$ are linearly independent, the above identity implies that $\psi(st) = \psi(s)\psi(t)$. This proves that ψ is a character and therefore, $\phi = \phi(e)\psi$ is a scalar multiple of a character, as requested. □

Let us now give a variant of Theorem 3.7 in the general case when G is not assumed to be σ-compact (see also Remark 3.16). We need the following lemma.

Lemma 3.12 *Let $h_1, h_2 \colon G \to \mathbb{C}$ be two locally measurable functions. The functions h_1 and h_2 are locally almost everywhere equal if and only if for any compact set $K \subseteq G$, $h_1|_K = h_2|_K$, almost everywhere.*

Proof It is enough to show that if $E \subset G$ is locally Borel, then E is locally null if (and only if) $E \cap K$ has measure 0 for any compact set $K \subseteq G$. Assume this property. By [8, Proposition 2.4], G has an open, closed and σ-compact subgroup, G_0. Let Y be a subset of G that contains exactly one element of each of the left cosets of G_0. Set

$E_y := E \cap yG_0$ for any $y \in Y$. Since G_0 is σ-compact, yG_0 is σ-compact as well. It then follows that $\mu(E_y) = 0$, for all $y \in Y$. Recall from the end of [8, Sect. 2.3] that E is locally null if and only if $\mu(E_y) = 0$, for every $y \in Y$. Hence, E is locally null. ☐

Corollary 3.13 *Let G be locally compact unimodular. Let $1 \leq p < \infty$ and let $\phi \in L^\infty(G)$. The following are equivalent.*

(i) *The mapping M_ϕ is a bounded Fourier multiplier on $L^p(VN(G))$, and the operator $M_\phi\colon L^p(VN(G)) \to L^p(VN(G))$ is separating.*
(ii) *There exist a constant $c_0 \in \mathbb{C}$ and a continuous character $\psi\colon G \to \mathbb{T}$ such that $\phi = c_0\psi$ locally almost everywhere.*

Proof The proof of the implication "$(ii) \Rightarrow (i)$" in Theorem 3.7 applies to the non σ-compact case, so we only need to prove that (i) implies (ii).

Assume that $M_\phi\colon L^p(VN(G)) \to L^p(VN(G))$ is separating. As in the proof of Theorem 3.7, we may assume that $p = 2$ and that M_ϕ is non-zero. We claim that

$$\exists \varphi \in C_b(G) \text{ such that } \varphi = \phi, \text{ locally almost everywhere.} \tag{3.9}$$

To prove this, first note that we may assume that the net $(f_i)_{i \in I}$ defined prior to Lemma 3.5 has the following property: there exists a compact neighbourhood K_0 of the unit e such that for all i, $\operatorname{supp}(f_i) \subseteq K_0$. Let $L \subseteq G$ be compact. Let

$$K = K_0 L K_0 K_0 = \{strq : (s,t,r,q) \in K_0 \times L \times K_0 \times K_0\}.$$

This is a compact set hence by Lemma 3.9(3) there is a continuous function $\Phi : K \to \mathbb{C}$ such that $\Phi = \phi|_K$ almost everywhere. The proof of Lemma 3.11 shows that for all $s \in L$, we have the following convergences in the strong operator topology:

$$M_\phi(\lambda(s)e_i) \to \Phi(s)\lambda(s),\ M_\phi(\lambda(s)e_i^2) \to \Phi(s)\lambda(s), \quad \text{and} \quad M_\phi(e_i\lambda(s)e_i) \to \Phi(s)\lambda(s). \tag{3.10}$$

In particular, (take $L = \{e\}$) we obtain the existence of $c_0 \in \mathbb{C}$ such that

$$M_\phi(e_i) \to c_0,$$

in the strong operator topology.

Let (w, B, J) be the Yeadon triple of M_ϕ. The argument in the proof of Theorem 3.7 and the convergence properties (3.10) show that for any L, K, Φ as above we have

$$c_0 J(\lambda(s)) = \Phi(s)\lambda(s), \quad \text{for all } s \in L. \tag{3.11}$$

By Lemma 3.9(1), this implies that $c_0 \neq 0$.

It follows from the above that for all $s \in G$, $J(\lambda(s))$ is proportional to $\lambda(s)$. We therefore have a necessarily unique

$$F\colon G \to \mathbb{C}; \quad J(\lambda(s)) = F(s)\lambda(s), \quad \text{for all } s \in G.$$

Moreover, for any L, K, Φ as above, we have

$$F|_L = c_0^{-1}\Phi|_L.$$

This implies that F is continuous. To prove this, fix $s_0 \in G$ and apply the above with a compact neighbourhood L of s_0. Then the continuity of $\Phi\colon K \to \mathbb{C}$ implies the continuity of $F|_L$, and hence the continuity of F at s_0.

Set $\varphi := c_0 F$. Again for L, K, Φ as above we obtain that $\varphi|_K = \phi|_K$, almost everywhere. By Lemma 3.9, this implies that $\varphi = \phi$, locally almost everywhere. Hence (3.9) is proved.

Since $M_\varphi = M_\phi$, the argument at the end of the proof of Theorem 3.7 shows that $\psi := c_0^{-1}\varphi$ is a character. □

Remark 3.14 It follows from the proof of the implication "$(ii) \Rightarrow (i)$" in Theorem 3.7 that for any continuous character $\psi\colon G \to \mathbb{T}$ and any $1 \leq p < \infty$, the Fourier multiplier $M_\psi : L^p(VN(G)) \to L^p(VN(G))$ is an onto isometry. It therefore follows from Corollary 3.13 that if $\phi \in L^\infty(G) \setminus \{0\}$ is such that $M_\phi\colon L^p(VN(G)) \to L^p(VN(G))$ is bounded and separating, then $\|M_\phi\|^{-1}M_\phi$ is an onto isometry.

Corollary 3.15 *Let $1 \leq p \neq 2 < \infty$ and let $\phi \in L^\infty(G)$. The following are equivalent.*

(i) *The mapping M_ϕ is a bounded Fourier multiplier on $L^p(VN(G))$, and the operator $M_\phi\colon L^p(VN(G)) \to L^p(VN(G))$ is an isometry.*
(ii) *There exists $\delta \in \mathbb{T}$ such that $\delta\phi$ is locally almost everywhere equal to a continuous character.*

Proof It follows from the proof of the implication "$(ii) \Rightarrow (i)$" in Theorem 3.7 that for any continuous character $\psi\colon G \to \mathbb{T}$, $M_\psi : L^p(VN(G)) \to L^p(VN(G))$ is an isometry. Thus, (ii) implies (i). Conversely, assume (i). Since $p \neq 2$, any isometry on $L^p(VN(G))$ is separating, by [36] (see also [22]). Hence by Corollary 3.13, there exist $c \in \mathbb{C}$ and a continuous character $\psi\colon G \to \mathbb{T}$ such that $\phi = c\psi$ locally almost everywhere. Then $M_\phi = cM_\psi$, hence $\|M_\phi\| = |c|\|M_\psi\|$, hence $|c| = 1$. This yields (ii), with $\delta = c^{-1}$. □

Note that Corollary 3.15 is not true in the case $p = 2$. Indeed, let $\phi \in L^\infty(G)$. It follows from the discussion following (3.2) that $M_\phi\colon L^2(VN(G)) \to L^2(VN(G))$ is an isometry if and only if $|\phi| = 1$ locally almost everywhere. Yet in general, plenty of these isometric Fourier multipliers are not separating. See Sect. 4 for more on this.

Remark 3.16 Corollary 3.13 may be wrong if one replaces "locally almost everywhere" by "almost everywhere" in (ii). Indeed as in [8, Sect. 2.3], take $G = \mathbb{R} \times \mathbb{R}_{\text{disc}}$, where the second factor is equipped with the discrete topology. Consider $Y = \{0\} \times \mathbb{R}_{\text{disc}}$ which is a closed subset of G, hence Borel, and set $\phi = \chi_Y$. For any compact set $K \subseteq G$, we have $\mu(K \cap Y) = 0$, by [8, Proposition 2.22]. Hence $\phi|_K = 0$ almost everywhere. By Lemma 3.12, this implies that $\phi = 0$ locally almost everywhere. Thus ϕ satisfies the properties of Corollary 3.13, with $M_\phi = 0$.

However by [8, Proposition 2.22] again, $\{s \in G : \phi(s) \neq 0\} = Y$ has infinite Haar measure, hence ϕ is not almost everywhere equal to 0. Consequently, ϕ cannot be almost everywhere equal to a constant times a continuous character.

Remark 3.17 In the case when G is discrete, continuity on G is automatic and two locally almost everywhere equal functions are equal. Therefore, in the statement of Corollary 3.13, we can replace part (ii) by the following slightly simpler statement: there exist $c \in \mathbb{C}$ and a character $\psi : G \to \mathbb{C}$ such that $\phi = c\psi$.

Remark 3.18 De Cannière and Haagerup [6] defined Fourier multipliers on $VN(G)$, including the case when G is not unimodular. Let $\phi \in C_b(G)$ and assume that ϕ induces a Fourier multiplier $M_\phi : VN(G) \to VN(G)$ in the sense of [6, Proposition 1.2]. Assume that M_ϕ is separating. If G is unimodular, then $M_\phi : L^2(VN(G)) \to L^2(VN(G))$ is separating by [22, Lemma 3.9]. Hence, by Corollary 3.13, ϕ is a multiple of a character.

However, in the general case of a non-unimodular locally compact group, the description of separating Fourier multipliers on $VN(G)$ is open.

Remark 3.19 Let Γ be a locally compact abelian group. Let $G = \widehat{\Gamma}$ be its dual group and recall that $L^\infty(\Gamma) = VN(G)$. Let $1 \leq p < \infty$. For any $u \in \Gamma$, let $\tau_u : L^p(\Gamma) \to L^p(\Gamma)$ be the translation operator defined by $\tau_u(f) = f(\cdot - u)$, for all $f \in L^p(\Gamma)$. Note that if we regard $u \in \Gamma$ as a character $u : G \to \mathbb{T}$, then the associated Fourier multiplier $M_u : L^p(\Gamma) \to L^p(\Gamma)$ coincides with τ_u.

Let $T : L^p(\Gamma) \to L^p(\Gamma)$ be a bounded operator. Then T commutes with translations, that is, $T \circ \tau_u = \tau_u \circ T$ for all $u \in \Gamma$, if and only if T is a Fourier multiplier (see e.g. [21, Chapter 4]). Hence Corollary 3.15 implies the following:

(*) *If $p \neq 2$, an isometry $T : L^p(\Gamma) \to L^p(\Gamma)$ commutes with translations if and only if there exists $c \in \mathbb{T}$ and $u \in \Gamma$ such that $T = c\tau_u$.*

This statement is a classical result due to Parrott [27] and Strichartz [33] and Corollary 3.15 should be regarded as a generalization of the latter.

We note that the two papers [27, 33] show $(*)$ in the case when Γ is not necessarily abelian. If Γ is non-abelian, the statement $(*)$ is not related to Corollary 3.15.

4 Completely Positive and Completely Isometric Fourier Multipliers

In this section, we complement the characterizations of separating and isometric L^p-Fourier multipliers from Sect. 3 with further information. Throughout this section, we assume that G is a unimodular locally compact group.

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. For any $n \geq 1$, we equip $M_n(\mathcal{M})$ with $\mathrm{tr}_n \otimes \tau$, where tr_n is the usual trace on M_n. For any $1 \leq p \leq \infty$, the resulting noncommutative L^p-space $L^p(M_n(\mathcal{M}))$ can be naturally identified (at the algebraic level) with the space of all $n \times n$ matrices $[x_{ij}]_{1 \leq i,j \leq n}$ with entries x_{ij} belonging to $L^p(\mathcal{M})$.

Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ be a bounded operator. For any $n \geq 1$, let $T_n : L^p(M_n(\mathcal{M})) \to L^p(M_n(\mathcal{M}))$ be defined by $T_n\big([x_{ij}]\big) = [T(x_{ij})]$, for all $[x_{ij}]_{1 \leq i,j \leq n}$ in $L^p(M_n(\mathcal{M}))$. Following usual terminology, we say that T is completely positive if T_n is positive for all $n \geq 1$. Likewise, we say that T is a complete contraction if $\|T_n\| \leq 1$ for all $n \geq 1$ and that T is a complete isometry if T_n is an isometry for all $n \geq 1$.

Let $\psi: G \to \mathbb{T}$ be a continuous character. Then ψ is positive definite hence by [6, Proposition 4.2], the Fourier multiplier $M_\phi: VN(G) \to VN(G)$ is completely positive. The proof of the implication "$(ii) \Rightarrow (i)$" in Theorem 3.7 actually shows that $M_\psi: L^p(VN(G)) \to L^p(VN(G))$ is a complete contraction for all $1 \leq p < \infty$, and then that $M_\psi: L^p(VN(G)) \to L^p(VN(G))$ is a complete isometry for all $1 \leq p < \infty$. As a consequence of Corollary 3.13, we therefore obtain the following complement to Remark 3.14.

Corollary 4.1 *Let $1 \leq p < \infty$ and let $\phi \in L^\infty(G) \setminus \{0\}$. Assume that $M_\phi: L^p(VN(G)) \to L^p(VN(G))$ is bounded and separating. Then $\|M_\phi\|^{-1}M_\phi$ is a complete isometry.*

Remark 4.2 Let $1 \leq p < \infty$ and let $\phi \in L^\infty(G) \setminus \{0\}$ such that $M_\phi: L^p(VN(G)) \to L^p(VN(G))$ is bounded and separating. Let (w, B, J) be the Yeadon triple of the latter operator. According to Corollary 4.1 and [15, Theorem 3.2], J is a $*$-homomorphism.

We can make this statement more precise, as follows. Applying Corollary 3.13, let $c \in \mathbb{C}$ let $\psi: G \to \mathbb{T}$ be the continuous character such that $\phi = c\psi$ locally almost everywhere. Then $J: VN(G) \to VN(G)$ is the L^∞-Fourier multiplier associated with ψ, $c = \|M_\phi\|$, $B = |c| \cdot 1$ and $w = c|c|^{-1} \cdot 1$. The easy verification is left to the reader.

Lemma 4.3 *Let $1 \leq p < \infty$ and let $\phi \in L^\infty(G)$. If M_ϕ is a bounded Fourier multiplier on $L^p(VN(G))$ and $M_\phi: L^p(VN(G)) \to L^p(VN(G))$ is an isometry, then $|\phi| = 1$ locally almost everywhere.*

Proof The case $p = 2$ follows from the paragraph preceding Remark 3.16. Assume that $1 \leq p \neq 2 < \infty$. By Corollary 3.13, there exist a constant $c \in \mathbb{C}$ and a continuous character $\psi: G \to \mathbb{T}$ such that $\phi = c\psi$ locally almost everywhere. We noticed before Corollary 4.1 that M_ψ is a complete isometry. Since $cM_\psi = M_\phi$ is also an isometry, we must have that $|c| = 1$. Hence, $|\phi| = |c\psi| = |\psi| = 1$ locally almost everywhere. □

We have the following partial converse of Lemma 4.3.

Proposition 4.4 *Let $\phi \in L^\infty(G)$ such that $|\phi| = 1$ locally almost everywhere, let $1 \leq p < \infty$ and assume that $M_\phi: L^p(VN(G)) \to L^p(VN(G))$ is a bounded Fourier multiplier. If M_ϕ is completely positive, then ϕ coincides locally almost everywhere with a continuous character $\psi: G \to \mathbb{T}$.*

Proof Since M_ϕ is completely positive, it follows from [3, Proposition 6.11] that ϕ is locally almost everywhere equal to a continuous positive definite function. Hence, we may assume that ϕ is continuous (and positive definite). By [35, Proposition VII.3.21], there exist a unitary representation $\pi: G \to B(H)$ on a Hilbert space H and a vector $\xi \in H$ such that

$$\phi(s) = \langle \pi(s)\xi, \xi \rangle, \qquad s \in G. \tag{4.1}$$

Since $|\phi| = 1$, we have $\phi(e) = 1$. Hence it follows from (4.1) that $1 = \phi(e) = \langle \pi(e)\xi, \xi \rangle = \|\xi\|_H^2$. Given $s \in G$, applying the Cauchy-Schwarz inequality we obtain

$$1 = |\phi(s)| = |\langle \pi(s)\xi, \xi \rangle| \leq \|\pi(s)\xi\|_H \|\xi\|_H = \|\xi\|_H^2 = 1.$$

It follows from the equality condition in the Cauchy-Schwarz inequality that there is $\psi(s) \in \mathbb{C}$ such that $\pi(s)\xi = \psi(s)\xi$.

Now, for any $s, t \in G$, on the one hand

$$\pi(st)\xi = \psi(st)\xi,$$

and on the other hand,

$$\pi(s)\pi(t)\xi = \pi(s)\psi(t)\xi = \psi(s)\psi(t)\xi.$$

Hence, $\psi(st) = \psi(s)\psi(t)$. Finally, $\phi(s) = \langle \pi(s)\xi, \xi \rangle = \psi(s)\|\xi\|_H^2 = \psi(s)$ for all $s \in G$. Therefore, $\phi = \psi$ is a character. □

When $p = 1$ and G is assumed to be amenable we can change the assumption of complete positivity in Proposition 4.4 into mere contractivity.

Proposition 4.5 *Let G be an amenable unimodular locally compact group. Let $\phi \in L^\infty(G)$ and assume that $M_\phi \colon L^1(VN(G)) \to L^1(VN(G))$ is a contractive Fourier multiplier. The following are equivalent.*

(i) *M_ϕ is an isometry.*
(ii) *$|\phi| = 1$ locally almost everywhere.*
(iii) *There exist $c \in \mathbb{T}$ and a continuous character $\psi \colon G \to \mathbb{T}$ such that $\phi = c\psi$ locally almost everywhere.*

Proof The implication "$(i) \Rightarrow (ii)$" is established in Lemma 4.3. The implication "$(iii) \Rightarrow (i)$" was already discussed several times (see, for example, Remark 3.14). We now show that "$(ii) \Rightarrow (iii)$". Since M_ϕ is a bounded Fourier multiplier on $L^1(VN(G))$, we may assume that ϕ is continuous, by Lemma 3.6. Further since G is amenable, symbols of Fourier multipliers on $VN(G)$ coincide with the Fourier-Stieltjes algebra of G. This classical result is mentioned in [6, p. 456], see also [12, Theorem 1]. Hence by [7, Lemma 2.14], there exist a unitary representation $\pi : G \to B(H)$ on a Hilbert space H and vectors ξ, η in H such that

$$\phi(s) = \langle \pi(s)\xi, \eta \rangle, \quad s \in G, \qquad \text{and} \qquad \|\xi\|_H = \|\eta\|_H = 1.$$

Assume (ii). Multiplying ϕ by $\overline{\phi(e)}$, we may assume that $\phi(e) = 1$. This implies that $1 = \langle \pi(e)\xi, \eta \rangle = \langle \xi, \eta \rangle$. Since $\|\xi\|_H = \|\eta\|_H = 1$, we deduce that $\eta = \xi$. Thus, ϕ satisfies (4.1). The proof of Proposition 4.4 therefore shows that ϕ is a character. □

5 A Characterization of Separating Schur Multipliers

Let (Ω, Σ, μ) be a σ-finite measure space. For any $f \in L^2(\Omega^2)$, let $S_f\colon L^2(\Omega) \to L^2(\Omega)$ be the bounded operator defined by

$$[S_f(h)](s) = \int_\Omega f(s,t)h(t)\,dt, \qquad h \in L^2(\Omega).$$

We recall that $S_f \in S^2(L^2(\Omega))$ and that the mapping $f \mapsto S_f$ is a unitary operator from $L^2(\Omega^2)$ onto $S^2(L^2(\Omega))$, see e.g. [30, Theorem VI. 23].

Let $\phi \in L^\infty(\Omega^2)$. According to the above identification $L^2(\Omega^2) \simeq S^2(L^2(\Omega))$, one may define a bounded operator $T_\phi\colon S^2(L^2(\Omega)) \to S^2(L^2(\Omega))$ by

$$T_\phi(S_f) = S_{\phi f}, \qquad f \in L^2(\Omega^2). \tag{5.1}$$

Moreover the norm of this operator is equal to $\|\phi\|_\infty$. The operator T_ϕ is called a Schur multiplier.

Let $1 \leq p < \infty$. We say that T_ϕ is a bounded Schur multiplier on the Schatten space $S^p(L^2(\Omega))$ if the restriction of T_ϕ to $S^p(L^2(\Omega)) \cap S^2(L^2(\Omega))$ extends to a bounded operator from $S^p(L^2(\Omega))$ into itself. Schur multipliers as defined in this section go back at least to Haagerup [10] and Spronk [31].

For any $\alpha \in L^\infty(\Omega)$, we let $\mathrm{Mult}_\alpha \in B(L^2(\Omega))$ be the multiplication operator taking h to αh for all $h \in L^2(\Omega)$. Then we let

$$\mathcal{D}(\Omega) = \left\{\mathrm{Mult}_\alpha \,:\, \alpha \in L^\infty(\Omega)\right\}.$$

This is von Neumann sub-algebra of $B(L^2(\Omega))$, which is isomorphic (as a von Neumann algebra) to $L^\infty(\Omega)$. We will use the classical fact that

$$\mathcal{D}(\Omega)' = \mathcal{D}(\Omega), \tag{5.2}$$

where $\mathcal{D}(\Omega)'$ stands for the commutant of $\mathcal{D}(\Omega)$. In other words, a bounded operator $V\colon L^2(\Omega) \to L^2(\Omega)$ belongs to $\mathcal{D}(\Omega)$ if and only if $V \circ \mathrm{Mult}_\alpha = \mathrm{Mult}_\alpha \circ V$ for all $\alpha \in L^\infty(\Omega)$.

We note that for any $\alpha \in L^\infty(\Omega)$, the mapping $x \mapsto \mathrm{Mult}_\alpha \circ x$ is a Schur multiplier. Indeed it coincides with T_ϕ, where the symbol $\phi \in L^\infty(\Omega^2)$ is given by $\phi(s,t) = \alpha(s)$. Likewise, for any $\beta \in L^\infty(\Omega)$, the mapping $x \mapsto x \circ \mathrm{Mult}_\beta$ is a Schur multiplier, with symbol ϕ given by $\phi(s,t) = \beta(t)$.

Theorem 5.1 *Let $\phi \in L^\infty(\Omega^2)$ and let $1 \leq p < \infty$. The following are equivalent.*

(i) *The mapping T_ϕ is a bounded Schur multiplier on $S^p(L^2(\Omega))$, and the resulting operator $T_\phi\colon S^p(L^2(\Omega)) \to S^p(L^2(\Omega))$ is separating.*

(ii) *There exist a constant $c \in \mathbb{C}$ and two unitaries $\alpha, \beta \in L^\infty(\Omega)$ such that*

$$\phi(s,t) = c\,\alpha(s)\beta(t) \qquad \text{for almost every } (s,t) \in \Omega^2.$$

(iii) *There exist a constant* $c \in \mathbb{C}$ *and two unitaries* $\alpha, \beta \in L^\infty(\Omega)$ *such that*

$$T_\phi(x) = c\,\mathrm{Mult}_\alpha \circ x \circ \mathrm{Mult}_\beta, \qquad x \in S^2(L^2(\Omega)). \tag{5.3}$$

Proof $\underline{(ii) \Rightarrow (iii)}$: Let c, α, β as in (ii) and let $x \in S^2(L^2(\Omega))$. Let $f \in L^2(\Omega^2)$ such that $x = S_f$. Then for all $h \in L^2(\Omega)$, we have

$$[x \circ \mathrm{Mult}_\beta(h)](s) = \int_\Omega f(s,t)\beta(t)h(t)\,d\mu(t),$$

hence

$$\begin{aligned}[c\,\mathrm{Mult}_\alpha \circ x \circ \mathrm{Mult}_\beta(h)](s) &= c\alpha(s)\int_\Omega f(s,t)\beta(t)h(t)\,d\mu(t)\\ &= \int_\Omega \phi(s,t)f(s,t)h(t)\,d\mu(t),\end{aligned}$$

for a.e. $s \in \Omega$. This shows (5.3).
$\underline{(iii) \Rightarrow (i)}$: Assume (5.3) for some unitaries $\alpha, \beta \in L^\infty(\Omega)$. It is plain that T_ϕ extends to a bounded operator on $S^p(L^2(\Omega))$ and that the identity (5.3) holds true on $S^p(L^2(\Omega))$.

Let $x, y \in S^p(L^2(\Omega))$ such that $x^*y = xy^* = 0$. Then

$$\big(\mathrm{Mult}_\alpha \circ x \circ \mathrm{Mult}_\beta\big)^*\big(\mathrm{Mult}_\alpha \circ y \circ \mathrm{Mult}_\beta\big) = \mathrm{Mult}_\beta^* \circ x^* \circ \mathrm{Mult}_\alpha^*\mathrm{Mult}_\alpha \circ y \circ \mathrm{Mult}_\beta.$$

Since α is a unitary of $L^\infty(\Omega)$, the operator Mult_α is a unitary of $B(L^2(\Omega))$, hence the right hand-side of the above equality is equal to $\mathrm{Mult}_\beta^* \circ x^*y \circ \mathrm{Mult}_\beta$, hence to 0. Thus $T_\phi(x)^*T_\phi(y) = 0$. Likewise, $T_\phi(x)T_\phi(y)^* = 0$. This shows that $T_\phi\colon S^p(L^2(\Omega)) \to S^p(L^2(\Omega))$ is separating.
$\underline{(i) \Rightarrow (ii)}$: For convenience we let $H = L^2(\Omega)$ throughout this proof. Owing to Lemma 2.5, we may suppose that $p = 2$. We let (w, B, J) denote the Yeadon triple of the separating map $T_\phi\colon S^2(H) \to S^2(H)$.

We may assume that T_ϕ is non-zero. Since $B(H)$ is a factor, it follows from [24, Lemma 4.3] that T_ϕ is 1-1. Applying the definition of T_ϕ, see (5.1), this implies that $\phi \neq 0$ almost everywhere. Applying this definition again, we obtain that T_ϕ has dense range. By Lemma 2.4, we deduce that w is a unitary and that $J(1) = 1$.

Let w^*T_ϕ denote the operator on $S^2(H)$ taking any $x \in S^2(H)$ to $w^*T_\phi(x)$. According to [32, Theorem 3.3] (see also [11, Corollary 7.4.9.]), there exists a projection $q \in B(H)$ such that $x \mapsto qJ(x)$ is a $*$-homomorphism and $x \mapsto (1-q)J(x)$ is an anti-$*$-homomorphism. As explained in [23, Remark 4.3], this implies that w^*T_ϕ is valued in

$$L^2\big(qB(H)q\big) \overset{2}{\oplus} L^2\big((1-q)B(H)(1-q)\big) \subset S^2(H).$$

Since T_ϕ has dense range and w is a unitary, w^*T_ϕ has dense range as well. This forces q to be equal either to 0 or 1. Thus $J\colon B(H)\to B(H)$ is either a $*$-homomorphism or an anti-$*$-homomorphism.

Assume first that J is a $*$-homomorphism. Recall that J is normal. According to the description of normal $*$-homomorphisms (see e.g. [34, Theorem IV.5.5]), there exist a Hilbert space E and a unitary $u\colon H\to H\overset{2}{\otimes}E$ such that

$$J(a)=u^*(a\otimes I_E)u,\qquad a\in B(H).$$

Here $H\overset{2}{\otimes}E$ stands for the Hilbertian tensor product of H and E and we regard

$$B(H)\otimes B(E)\subset B\big(H\overset{2}{\otimes}E\big)$$

in the usual way. For all $x\in S^2(H)$, we have $T_\phi(x)=wBJ(x)$ hence $w^*T_\phi(x)=BJ(x)$. This implies that

$$u(w^*T_\phi(x))u^*=uBu^*(x\otimes I_E),\qquad x\in S^2(H). \tag{5.4}$$

Since B commutes with the range of J, the operator uBu^* commutes with $x\otimes I_E$ for all $x\in S^2(H)$. Consequently, $uBu^*=I_H\otimes c$ for some positive operator c acting on E. Then it follows from (5.4) that $c\in S^2(E)$ and that

$$w^*T_\phi(x)=u^*(x\otimes c)u,\qquad x\in S^2(H).$$

Now recall that w^*T_ϕ has dense range. The above identity therefore implies that $E=\mathbb{C}$. Thus $c\in\mathbb{C}\setminus\{0\}$, u is a unitary of $B(H)$ and $w^*T_\phi(x)=c\,u^*xu$ for all $x\in S^2(H)$. Let $v=wu^*$. This is a unitary of $B(H)$ and we obtain that

$$T_\phi(x)=c\,vxu,\qquad x\in S^2(H).$$

Let $(\,\cdot\,|\,\cdot\,)$ denote the inner product on H. For any $g,h\in H$, let $g\otimes\overline{h}\in B(H)$ denote the rank one operator taking any $\xi\in H$ to $(\xi|h)\,g$. Then $v(g\otimes\overline{h})u=v(g)\otimes\overline{u^*(h)}$.

Schur multipliers commute with each other, hence for any $\delta\in\mathcal{D}(\Omega)$, we have

$$T_\phi\big(\delta x\big)=\delta T_\phi(x),\qquad x\in S^2(H).$$

Thus $v\delta xu=\delta vxu$ for all $\delta\in\mathcal{D}(\Omega)$ and all $x\in S^2(H)$. Applying this with $x=g\otimes\overline{h}$ and using the identities $\delta(g\otimes\overline{h})=\delta(g)\otimes\overline{h}$ and $v(g\otimes\overline{h})=v(g)\otimes\overline{h}$, we deduce that $v(\delta(g)\otimes\overline{h})u=\delta(v(g)\otimes\overline{h})u$ and hence

$$v\delta(g)\otimes\overline{u^*(h)}=\delta v(g)\otimes\overline{u^*(h)},\qquad g,h\in H,\ \delta\in\mathcal{D}(\Omega).$$

Since $u^*\neq 0$, this implies that $v\delta=\delta v$ for all $\delta\in\mathcal{D}(\Omega)$. Thus v commutes with $\mathcal{D}(\Omega)$. According to (5.2), this implies that $v\in\mathcal{D}(\Omega)$. Thus there exists a unitary

$\alpha \in L^\infty(\Omega)$ such that $v = \mathrm{Mult}_\alpha$. Likewise there exists a unitary $\beta \in L^\infty(\Omega)$ such that $u = \mathrm{Mult}_\beta$. We therefore obtain the identity (5.3), from which (ii) follows at once.

Assume now that J is an anti $*$-homomorphism. For any $f \in L^2(\Omega^2)$, let $\widetilde{f} \in L^2(\Omega^2)$ be defined by $\widetilde{f}(s,t) = f(t,s)$, for a.e. $(s,t) \in \Omega^2$. Next, if $x = S_f$, set ${}^t x = S_{\widetilde{f}}$. It is clear that the mapping $x \mapsto {}^t x$ on $S^2(L^2(\Omega))$ extends to a normal anti $*$-homomorphism

$$\rho \colon B(H) \longrightarrow B(H).$$

This mapping is an analog of the transposition map on matrices. Obviously, the composition map $J \circ \rho \colon B(H) \to B(H)$ is a normal $*$-homomorphism. Now arguing as in the $*$-homomorphism case, we obtain the existence of a constant $c \in \mathbb{C} \setminus \{0\}$ and of two unitaries $\alpha, \beta \in L^\infty(\Omega)$ such that

$$T_\phi(x) = c\,\mathrm{Mult}_\alpha \circ {}^t x \circ \mathrm{Mult}_\beta, \qquad x \in S^2(L^2(\Omega)).$$

Since α, β are unitaries, Mult_α and Mult_β are unitaries as well and we have $\mathrm{Mult}_\alpha^{-1} = \mathrm{Mult}_{\overline{\alpha}}$ and $\mathrm{Mult}_\beta^{-1} = \mathrm{Mult}_{\overline{\beta}}$. Writing ${}^t x = c^{-1}\mathrm{Mult}_{\overline{\alpha}} \circ T_\phi(x) \circ \mathrm{Mult}_{\overline{\beta}}$, we therefore deduce that $x \mapsto {}^t x$ is a Schur multiplier.

Let us show that this is impossible, except if $L^2(\Omega)$ has dimension 1. If $x \mapsto {}^t x$ is a Schur multiplier, then there exists $\phi_0 \in L^\infty(\Omega^2)$ such that

$$\phi_0(s,t)g(s)h(t) = h(s)g(t) \qquad a.e.\text{-}(s,t) \in \Omega^2, \tag{5.5}$$

for all $g, h \in L^2(\Omega)$. If $L^2(\Omega)$ has dimension ≥ 2, then there exist $F_1, F_2 \in \Sigma$ such that $0 < \mu(F_1) < \infty$, $0 < \mu(F_2) < \infty$ and $F_1 \cap F_2 = \emptyset$. The indicator functions $g = \chi_{F_1}$ and $h = \chi_{F_2}$ belong to $L^2(\Omega)$. Applying (5.5) to these functions, we obtain that $h(s)g(t) = 0$ for almost every $(s,t) \in F_2 \times F_1$. Since $h(s)g(t) = 1$ for $(s,t) \in F_2 \times F_1$ and

$$(\mu \otimes \mu)(F_2 \times F_1) = \mu(F_2)\mu(F_1) > 0,$$

we get a contradiction.

Now if we are in the trivial case when $L^2(\Omega)$ has dimension 1, then (ii) holds true. □

Remark 5.2 Let $\phi \in L^\infty(\Omega^2)$, let $1 \leq p \neq 2 < \infty$ and assume that T_ϕ is a bounded Schur multiplier on $S^p(L^2(\Omega))$. It follows from Theorem 5.1 and [36] that if T is an isometry, then there exist two unitaries $\alpha, \beta \in L^\infty(\Omega)$ such that $\phi(s,t) = \alpha(s)\beta(t)$ for a.e. $(s,t) \in \Omega^2$. It is clear that the converse is true. For the discrete case (see the following remark), this has been proved in [1].

Remark 5.3 Let I be an index set and let $(e_i)_{i \in I}$ be the standard basis of ℓ_I^2. Any $x \in B(\ell_I^2)$ can be represented by a matrix $[x_{ij}]_{i,j \in I}$ defined by $x_{ij} = (x(e_j)|e_i)$ for all $i, j \in I$. Of course any finitely supported matrix $[x_{ij}]_{i,j \in I}$ represents an element of $B(H)$ (actually a finite rank one), and we let $\|[x_{ij}]\|_p$ denote the $S^p(\ell_I^2)$-norm of this element.

Let $\mathfrak{m} = \{m_{ij}\}_{(i,j)\in I^2}$ be a bounded family of complex numbers. If we apply the definitions of this section to $\Omega = I$ equipped with the counting measure, the Schur multiplier $T_{\mathfrak{m}}$ is defined on finitely supported matrices by

$$T_{\mathfrak{m}}\big([x_{ij}]\big) = [m_{ij}x_{ij}].$$

It therefore follows from Remark 5.2 that the following are equivalent:

(i) There exists $1 \leq p \neq 2 < \infty$ such that

$$\|[m_{ij}x_{ij}]\|_p = \|[x_{ij}]\|_p$$

for all finitely supported matrices $[x_{ij}]_{i,j\in I}$.

(ii) There exist two families $(\alpha_i)_{i\in I}$ and $(\beta_j)_{j\in I}$ in $\mathbb{T}$ such that

$$m_{ij} = \alpha_i\beta_j, \qquad \text{for all } (i, j) \in I^2.$$

We conclude with a characterisation of a particular class of Schur multipliers, the separating Herz-Schur multipliers. Let G be a locally compact σ-compact group (see Lemma 3.2). Suppose $1 \leq p < \infty$. Let $\varphi \in L^\infty(G)$ and define $\phi \in L^\infty(G^2)$ by $\phi(s,t) = \varphi(s^{-1}t)$. The Schur multiplier $T_\varphi^{HS} := T_\phi$ is called a Herz-Schur multiplier (with symbol φ).

In [4, Proposition 4.5], it is shown that a Herz-Schur multiplier $T_\varphi^{HS} \colon B(L^2(G)) \to B(L^2(G))$ with positive definite φ such that $\varphi(e) = 1$, is a conjugation with a unitary if and only if φ is a character.

Corollary 5.4 *Let $1 \leq p < \infty$. Let G be a locally compact σ-compact group. Let T_φ^{HS} be a bounded Herz-Schur multiplier on $S^p(L^2(G))$. Then T_φ^{HS} is separating if and only if there exists a continuous character $\psi \colon G \to \mathbb{T}$ and $c \in \mathbb{C}$ such that $\varphi = c\psi$ almost everywhere.*

Proof If $\varphi(s) = c\psi(s)$ a.e. $s \in G$ for some continuous character ψ, then the symbol ϕ of the Schur multiplier satisfies $\phi(s,t) = \varphi(s^{-1}t) = c\psi(s^{-1}t) = c\psi^{-1}(s)\psi(t)$ for a.e. $(s,t) \in G^2$. Since ψ^{-1} and ψ are clearly unitaries of $L^\infty(G)$, by Theorem 5.1, $T_\varphi^{HS} = T_\phi$ is separating.

Conversely, assume T_φ^{HS} separating. Then by Theorem 5.1, there are unitaries $\alpha, \beta \in L^\infty(G)$ and some $c \in \mathbb{C}$ such that $\varphi(s^{-1}t) = c\alpha(s)\beta(t)$ for a.e. $(s,t) \in G^2$. Let $r \in G$. Then $c\alpha(s)\beta(t) = \varphi(s^{-1}t) = \varphi((rs)^{-1}(rt)) = c\alpha(rs)\beta(rt)$ for a.e. $(s,t) \in G^2$. Leaving the trivial case $c = 0$ aside, we deduce that

$$\frac{\beta(rt)}{\beta(t)} = \frac{\alpha(s)}{\alpha(rs)} \tag{5.6}$$

for a.e. $(s,t) \in G^2$. Thus there is some $s \in G$ such that (5.6) holds for a.e. $t \in G$. Defining $\psi(r)$ as the right hand side of (5.6), we then obtain $\frac{\beta(rt)}{\beta(t)} = \psi(r)$ for a.e.

$t \in G$. The function $\psi : G \to \mathbb{C}$ with this property is necessarily unique. From $\psi(e) = 1$ and

$$\psi(r_1 r_2) = \frac{\beta(r_1 r_2 t)}{\beta(t)} = \frac{\beta(r_1(r_2 t))}{\beta(r_2 t)} \frac{\beta(r_2 t)}{\beta(t)} = \psi(r_1)\psi(r_2) \quad (\text{a.e. } t \in G)$$

we infer that ψ is a character. Since ψ is measurable, by [13, Corollary 22.19 p. 346], it is automatically continuous. From $\beta(rt) = \beta(t)\psi(r)$ for a.e. $t \in G$, we infer by a Fubini argument that there exists some $t \in G$ such that this equality holds for a.e. $r \in G$. Thus, β coincides a.e. with a continuous function. Choosing this continuous representative for β, we obtain that for every r in G, $\beta(rt) = \beta(t)\psi(r)$ for a.e. $t \in G$. Since β is continuous, this implies $\beta(rt) = \beta(t)\psi(r)$ for all r, t in G. In particular, we have that $\beta(r) = \beta(e)\psi(r)$ for all $r \in G$. Using (5.6), the same argument as above shows that $\alpha(s) = \alpha(e)\psi(s^{-1})$ for all s in G. Hence we deduce that $\varphi(s^{-1}t) = c\alpha(e)\beta(e)\psi(s^{-1}t)$ for a.e. $(s, t) \in G^2$. Therefore, φ coincides a.e. with a multiple of a continuous character. ◻

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