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On Ultradifferentiable Regularity of Perturbations by Lower Order Terms of Globally C^∞ Hypoelliptic Ultradifferentiable Pseudodifferential Operators

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Abstract

We prove $\mathcal{M}$-regularity for a class of pseudodifferential operators in ultradifferentiable classes defined on the torus $\mathbb{T}^{m+n}$ which are globally C^∞ hypoelliptic. The same property is also valid for certain perturbations of these operators by lower order terms.

Keywords Ultradifferentiable pseudodifferential operators · Global hypoellipticity · Global hypoellipticity with loss of derivatives · Lower order perturbations

Mathematics Subject Classification 35H10 · 35S05

1 Introduction and Statement of the Results

The question that motivates this work is the following: let P be a linear partial differential operator with coefficients in $C^\omega(\mathbb{T}^N)$, where $\mathbb{T}^N$ denotes the N-dimensional torus, and suppose that P is globally C^∞ hypoelliptic in $\mathbb{T}^N$. Is P globally Gevrey hypoelliptic in $\mathbb{T}^N$? When $P = P_0(D)$ has constant coefficients the answer is positive thanks to the Greenfield-Wallach conditions. If it is not the case, then we have some partial results as in the work of Himonas and Petronilho [6] which proves the following.

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Theorem 1.1 *For $(t, x) \in \mathbb{T}^{m+n}$ let $P = P(t, D_t, D_x)$ be a linear partial differential operator with coefficients in $C^\omega(\mathbb{T}^m)$ and suppose that P is globally C^∞ hypoelliptic in $\mathbb{T}^{m+n}$. If $u \in D'(\mathbb{T}^{m+n})$, $Pu \in G^s(\mathbb{T}^{m+n})$, and $(t, x, \tau, 0) \notin WF_s(u)$, where $(t, x) \in \mathbb{T}^{m+n}$, $\tau \in \mathbb{R}^m \backslash \{0\}$, and if Ker $P \subset G^s(\mathbb{T}^{m+n})$, then $u \in G^s(\mathbb{T}^{m+n})$.*

Here we are denoting by $G^s\left(\mathbb{T}^N\right)$ the space of the s-Gevrey functions for some fixed $s \geq 1$. We also recall that Petronilho [8] improved the last result by proving that the condition ker $P \subset G^s\left(\mathbb{T}^{n+m}\right)$ is superfluous in the statement of Theorem 1.1. Also, Albanese and Jornet [1] generalized the Petronilho's result to the ultradifferential frame.

Open question 1: Does Theorem 1 hold true in more general classes of operators such as pseudodifferential operators in ultradifferentiable classes defined on the torus?

We also are concerned with a very natural problem: the stability of hypoelliptic operators, i.e., we are going to study the stability of hypoellipticity under lower-order perturbations. Concernig this question let us recall some results in the literature about perturbations by lower order terms. Firstly we mention the work by Dickinson, Gramchev and Yoshino in [3] where they considered perturbations of smooth vector fields on $\mathbb{T}^N$ (constant if $N \geq 3$) by zero order smooth classical pseudodifferential operators.

In [9] Parmeggiani studied the problem of perturbations of C^∞ hypoelliptic operators by lower order terms. He proved that hypoellipticity with a finite loss of derivatives of a linear partial differential operator P, along with its formal adjoint P^*, is stable under perturbations by lower order linear partial differential operators whose order depends on the loss of derivatives, see Theorem 2.3 of [9]. In [11] Parenti and Parmeggiani proved a stability result that streamlines and generalizes that of Parmeggiani described above.

In the analytic setup, Chinni and Cordaro, see [2], introduced a new theory about analytic pseudodifferential operators on the N-dimensional torus $\mathbb{T}^N$. One question analyzed by them is the following: assuming that $P(x, D)$ is a linear partial differential operator defined on $\mathbb{T}^N$ with real-analytic coefficients, that $P(x, D)$ is ϵ-subelliptic for some $\epsilon > 0$ and that $P(x, D)$ is globally analytic hypoelliptic on $\mathbb{T}^N$ they ask when is it true that $P(x, D)$ remains globally analytic hypoelliptic when one adds to it an analytic pseudodifferential operator on $\mathbb{T}^N$ of order less than ϵ.

Ferra and Petronilho [4], inspired by the work of Chinni and Cordaro, introduced a new class of smooth pseudodifferential operators on the torus and used it to show that perturbations by lower order terms do not destroy the global hypoellipticity of certain systems of pseudodifferential operators.

We now mention that recently Ferra, Petronilho and Victor [5] generalized Chinni and Cordaro, [2], for ultradifferentiable pseudodifferential operators defined on the torus, in particular they proved the following result (for more detais about the notation see Sect. 2):

Theorem 1.2 *Let $\mathcal{A} = \{a_j(x, D)\}_{j=1}^m$ be a system of pseudodifferential operators in $\mathfrak{D}^{\mathcal{M}}_{\mathfrak{p}_\sigma}(\mathbb{T}^N)$ that is globally $\mathcal{M}$-hypoelliptic with loss of $r \geq 0$ derivatives. If $\mathcal{B} = \{b_j(x, D)\}_{j=1}^m$ is a system of pseudodifferential operators in $\mathfrak{D}^{\mathcal{M}}_{\mathfrak{p}_\tau}(\mathbb{T}^N)$, with $\tau < \sigma - r$,* *then the system* $\mathcal{C}\doteq\{c_j(x,D)=a_j(x,D)+b_j(x,D)\}_{j=1}^m$ *in* $\mathfrak{D}^{\mathcal{M}}_{\mathfrak{p}_\sigma}(\mathbb{T}^N)$ *is globally* $\mathcal{M}$*-hypoelliptic.*

This result motivates the following

Open question 2: For $j=1$, if we replace the hypothesis of global $\mathcal{M}$-hypoellipticity with loss of derivatives in Theorem 1.2, (see [5]), by global C^∞ hypoellipticity or by global C^∞ hypoellipticity with loss of derivatives what kind of results we will get?

Motivated by the results and by the open questions cited above we work on the problem of $\mathcal{M}$-regularity of solutions $u\in C^\infty(\mathbb{T}^n\times\mathbb{T}^m)$ of perturbations of globally hypoelliptic pseudodifferential operators $a(t,D_t,D_x)$, where $(t,x)\in\mathbb{T}^{m+n}$. Here we need to replace the condition $(t,x,\tau,0)\notin WF_s(u)$, where $(t,x)\in\mathbb{T}^{m+n}$, $\tau\in\mathbb{R}^m\backslash\{0\}$, given in [6] and in [8], by an appropriate one for the ultradifferentiable classes. More precisely we will prove the following (see Sect. 2 for the definition of $\mathfrak{s}_{\mathcal{M}}(u)$).

Theorem 1.3 *Let* $a(t,D_t,D_x)\in\mathfrak{D}^{\mathcal{M}}_{\mathfrak{p}_\sigma}(\mathbb{T}^m\times\mathbb{T}^n)$, $\sigma\in\mathbb{R}$, *be an* $\mathcal{M}$*-ultradifferentiable pseudodifferential operator of order* $\sigma\in\mathbb{R}$ *that is globally* C^∞ *hypoelliptic on* $\mathbb{T}^m\times\mathbb{T}^n$. *Then there exists* $r\in\mathbb{R}$ *depending on* $a(t,D_t,D_x)$ *such that for every* $\mathcal{M}$*-ultradifferentiable pseudodifferential operator* $b(t,x,D_t,D_x)\in\mathfrak{D}^{\mathcal{M}}_{\mathfrak{p}_{\sigma'}}(\mathbb{T}^m\times\mathbb{T}^n)$ *of order* $\sigma'\le\sigma-r-1$ *the following holds: if* $u\in C^\infty(\mathbb{T}^m\times\mathbb{T}^n)$ *is such that*

$$[a(t,D_t,D_x)+b(t,x,D_t,D_x)]u=f\in\mathcal{E}_{\mathcal{M}}\left(\mathbb{T}^m\times\mathbb{T}^n\right) \tag{1.1}$$

and $(\tau,0)\notin\mathfrak{s}_{\mathcal{M}}(u)$ *for every* $\tau\in\mathbb{R}^m\backslash 0$, *then* $u\in\mathcal{E}_{\mathcal{M}}(\mathbb{T}^m\times\mathbb{T}^n)$.

Also we have the following application which allow us to consider solutions $u\in\mathcal{D}'(\mathbb{T}^n\times\mathbb{T}^m)$ in Theorem 1.3 instead of smooth solution if the initial operator $a(t,D_t,D_x)$ is globally hypoelliptic with loss of derivatives.

Theorem 1.4 *Let* $a(t,D_t,D_x)\in\mathfrak{D}^{\mathcal{M}}_{\mathfrak{p}_\sigma}(\mathbb{T}^m\times\mathbb{T}^n)$, $\sigma\in\mathbb{R}$, *be an* $\mathcal{M}$*-ultradifferentiable pseudodifferential operator of order* $\sigma\in\mathbb{R}$ *that is globally* C^∞ *hypoelliptic on* $\mathbb{T}^{n+m}$ *with loss of* R *derivatives and let* $b(x,t,D_x,D_t)\in\mathfrak{D}^{\mathcal{M}}_{\mathfrak{p}_{\sigma'}}(\mathbb{T}^m\times\mathbb{T}^n)$, $\sigma'\in\mathbb{R}$, *be an* $\mathcal{M}$*-ultradifferentiable pseudodifferential operator of order* $\sigma'\le\sigma-R-2$. *Also suppose that* $u\in D'(\mathbb{T}^N)$ *satisfy*

$$[a(t,D_x,D_t)+b(x,t,D_x,D_t)]u=f\in\mathcal{E}_{\mathcal{M}}\left(\mathbb{T}^{m+n}\right). \tag{1.2}$$

Assuming that $(\tau,0)\notin\mathfrak{s}_{\mathcal{M}}(u)$ *for every* $\tau\in\mathbb{R}^m\setminus\{0\}$ *we conclude that* $u\in\mathcal{E}_{\mathcal{M}}\left(\mathbb{T}^{m+n}\right)$.

Before proceeding, we would like to make a few comments about the Theorems 1.3 and 1.4. In Theorem 1.3, assuming that $b(t,x,D_t,D_x)$ is identically equal to zero we can replace the hypothesis $u\in C^\infty(\mathbb{T}^m\times\mathbb{T}^n)$ by $u\in D'(\mathbb{T}^m\times\mathbb{T}^n)$ since $\mathcal{E}_{\mathcal{M}}(\mathbb{T}^m\times\mathbb{T}^n)\subset C^\infty(\mathbb{T}^m\times\mathbb{T}^n)$. Thereby, we answer positively to the open question 1. Concerning the open question 2, in Theorem 1.3, we consider the case that the operator $a(t,D_t,D_x)$ is globally C^∞ hypoelliptic and prove that all solutions $u\in C^\infty(\mathbb{T}^n\times\mathbb{T}^m)$ of certains perturbations of operator $a(t,D_t,D_x)$ are $\mathcal{M}$-regular, i.e., $u \in \mathcal{E}_{\mathcal{M}}(\mathbb{T}^m \times \mathbb{T}^n)$, provided that $(\tau, 0) \notin \mathfrak{s}_{\mathcal{M}}(u)$ for every $\tau \in \mathbb{R}^m \backslash \{0\}$. While in Theorem 1.4 we consider the case that operator $a(t, D_t, D_x)$ is globally C^∞ hypoelliptic with loss of derivatives and as an application of Theorem 1.3 we show that all solutions $u \in D'(\mathbb{T}^n \times \mathbb{T}^m)$ of certain perturbations of operator $a(t, D_t, D_x)$ are $\mathcal{M}$-regular, i.e., $u \in \mathcal{E}_{\mathcal{M}}(\mathbb{T}^m \times \mathbb{T}^n)$, provided that $(\tau, 0) \notin \mathfrak{s}_{\mathcal{M}}(u)$ for every $\tau \in \mathbb{R}^m \setminus \{0\}$.

We also would like to point out that in the case $b = 0$ the Theorem 1.4 is a simple consequence of Theorem 1.3 since global C^∞ hypoellipticity with loss of derivatives implies global C^∞ hypoellipticity. Finally, we call attention to the fact that in Theorem 1.3 the order of the perturbation operator $b(t, x, D_t, D_x)$ is $\sigma' < \sigma + r - 1$, where r is not explicit because it comes from a Functional Analysis result whereas in Theorem 1.4 the order of the operator $b(t, x, D_t, D_x)$ is $\sigma' < \sigma + R - 1$ and R is well determined because it is the loss of derivatives of the hypoelicity of the operator $a(t, D_t, D_x)$.

The paper is structured as follows. In Sect. 2 we recall the basic definitions and results about ultradifferentiable functions and the more important facts on an ultradifferentiable pseudodifferential operators. In Sect. 3 we prove some important inequalities about smooth pseudodifferential operators on the torus. In Sect. 4 we present some technical results about the condition $(\tau, 0) \notin \mathfrak{s}_{\mathcal{M}}(u)$ that will be used in the proof of our main theorems. In Sect. 5 we prove Theorem 1.3 and finally, in Sect. 6, we present the proof of Theorem 1.4.

2 Basic Results

In this section we present the basic definitions and results that will be used throughout this text. We denote by $\mathbb{T}^N$ the N-dimensional torus.

2.1 Ultradifferentiable Functions

We say that a sequence of positive real numbers $\mathcal{M} = \{m_n\}_{n \in \mathbb{Z}_+}$ is a *weight sequence* if it satisfies the following properties:

$$m_0 = m_1 = 1, \tag{2.1}$$

$$m_n^2 \leq m_{n-1} m_{n+1}, \ \forall\, n \in \mathbb{N}, \tag{2.2}$$

$$\sup_{j,k \in \mathbb{N}} \left(\frac{m_{j+k}}{m_j m_k} \right)^{\frac{1}{j+k}} < H, \quad \text{with } H > 1. \tag{2.3}$$

We recall that a weight sequence $\mathcal{M} = \{m_n\}_{n \in \mathbb{Z}_+}$ is called *quasianalytic* if

$$\sum_{k=1}^{\infty} \frac{m_{k-1}}{m_k} = \infty.$$

If the sum if finite, then $\mathcal{M} = \{m_n\}_{n \in \mathbb{Z}_+}$ is called a *non-quasianalytic* weight sequence.

Definition 2.1 Let $\mathcal{M} = \{m_n\}_{n\in\mathbb{Z}_+}$ be a weight sequence. We say that a function $f \in C^\infty\left(\mathbb{T}^N\right)$ is periodic ultradifferentiable of class $\{\mathcal{M}\}$ if there exist constants $C, h > 0$ such that

$$\left|D^\alpha f(x)\right| \leq Ch^{|\alpha|} m_{|\alpha|} |\alpha|!, \quad \forall\, x \in \mathbb{T}^N, \ \forall \alpha \in \mathbb{Z}_+^N .$$

The space of the periodic ultradifferentiable functions of class $\{\mathcal{M}\}$ will be denoted by $\mathcal{E}_\mathcal{M}\left(\mathbb{T}^N\right)$. This class is also known as the space of the periodic ultradifferentiable functions of Roumieu type. The space $D'_\mathcal{M}\left(\mathbb{T}^N\right)$ is the topological dual of $\mathcal{E}_\mathcal{M}\left(\mathbb{T}^N\right)$.

When $m_n = n!^{s-1}$ we recover the Gevrey (periodic) functions, in particular for $s = 1$ we have the space of periodic analytic functions.

Now we state certain results that we need below. For its proofs or more results on weight sequences we refer the reader to Komatsu [7], Pilipović [10], Ferra et al. [5] and to the references in these papers.

Proposition 2.2 *If $\mathcal{M} = \{m_n\}_{n\in\mathbb{N}}$ is a weight sequence then we have the following properties:*
(i) $m_n \geq 1$ *for all* $n \in \mathbb{Z}_+$*;*
(ii) *the sequence* $\{m_n^{1/n}\}_{n\in\mathbb{N}}$ *is increasing;*
(iii) $m_j m_k \leq m_{j+k}$ *for all* $j, k \in \mathbb{Z}_+$*;*
(iv) *for each* $k \in \mathbb{Z}_+$ *we can find a constant* $A_k \geq 1$ *such that*

$$m_{n+k}(n+k)! \leq A_k^{n+1} m_n n!, \ \forall\, n \in \mathbb{Z}_+ . \tag{2.4}$$

Remark 2.3 It follows from item (i) of Proposition 2.2 that $\mathcal{E}_\mathcal{M}\left(\mathbb{T}^N\right)$ contains the space of all periodic analytic functions $C^\omega\left(\mathbb{T}^N\right)$ since $|\alpha|! \leq m_{|\alpha|}|\alpha|!$ for all $\alpha \in \mathbb{Z}_+^N$.

Also, there is a characterization of the space of ultradifferentiable functions in terms of the Fourier transform. Recall that for $u \in D'_\mathcal{M}\left(\mathbb{T}^N\right)$, we define

$$\hat{u}(\xi) = \frac{1}{(2\pi)^N}\left\langle u, e^{-i\langle x,\xi\rangle}\right\rangle, \xi \in \mathbb{Z}^N .$$

Theorem 2.4 *Let $\mathcal{M} = \{m_n\}_{n\in\mathbb{Z}_+}$ be a weight sequence. A function $\varphi \in C^\infty\left(\mathbb{T}^N\right)$ belongs to $\mathcal{E}_\mathcal{M}\left(\mathbb{T}^N\right)$ if and only if there exists $C, h > 0$ such that*

$$\left|\hat{\varphi}(\xi)\right| \leq C \inf_{n\in\mathbb{Z}_+}\left(\frac{h^n m_n n!}{(1+|\xi|)^n}\right), \quad \forall\, \xi \in \mathbb{Z}^N . \tag{2.5}$$

Moreover, if $\{C_\xi\}_{\xi\in\mathbb{Z}^N}$ is a sequence such that (2.5) holds true with C_ξ in place of $\hat{\varphi}(\xi)$, then there exists an unique function $\varphi \in \mathcal{E}_\mathcal{M}\left(\mathbb{T}^N\right)$ such that $\hat{\varphi}(\xi) = C_\xi$ for all $\xi \in \mathbb{Z}^N$.

The latter result motivates us to introduce the following (see also [2], Sect. 3)

Definition 2.5 Given $u \in D'_{\mathcal{M}}\left(\mathbb{T}^N\right)$, we denote by $\mathfrak{s}_{\mathcal{M}}(u)$ the complementary set of every $\xi_0 \in \mathbb{R}^N \setminus 0$ such that there exist $C, h > 0$ and an open cone $\Gamma \subset \mathbb{R}^N \setminus 0$ containing ξ_0 satisfying

$$\left|\hat{u}(\xi)\right| \leq C \inf_{n \in \mathbb{Z}_+} \left(\frac{h^n m_n n!}{(1+|\xi|)^n}\right), \quad \forall \, \xi \in \mathbb{Z}^N \cap \Gamma. \tag{2.6}$$

2.2 Pseudodifferential Operators

Here we present the basic properties of the pseudodifferential operators which we work with in this text. The discrete symbol of a continuous and linear operator $A : C^\infty(\mathbb{T}^N) \longrightarrow C^\infty(\mathbb{T}^N)$ is the function $a : \mathbb{T}^N \times \mathbb{Z}^N \longrightarrow \mathbb{C}$ defined by $a(x, \eta) = e^{-i\langle x, \eta\rangle} A\left(e^{i\langle x, \eta\rangle}\right)$ and we shall use the notation $A = a(x, D)$ and call $a(x, D)$ a pseudodifferential operator. If $\varphi \in C^\infty(\mathbb{T}^N)$ then by linearity and continuity we have

$$a(x, D)\varphi(x) = \sum_{\xi \in \mathbb{Z}^N} a(x, D)\left(e^{i\langle x, \xi\rangle}\right)\widehat{\varphi}(\xi) = \sum_{\xi \in \mathbb{Z}^N} e^{i\langle x, \xi\rangle} a(x, \xi)\widehat{\varphi}(\xi) \in C^\infty(\mathbb{T}^N),$$

from which one can prove that

$$(\widehat{a(x, D)u})(\xi) = \sum_{\eta \in \mathbb{Z}} \hat{a}(\xi - \eta, \eta)\hat{u}(\eta). \tag{2.7}$$

The main object of this work is a specific class of pseudodifferential operators which was introduced in [5]. There the reader can find the proofs of the statements used in this section that we did not include here as well as more properties satisfied by these operators. This class is more appropriate to the ultradifferentiable framework.

Definition 2.6 Let $\mathcal{M}$ be a weight sequence and $\sigma \in \mathbb{R}$. We say that a continuous and linear operator $a(x, D) : C^\infty(\mathbb{T}^N) \to C^\infty(\mathbb{T}^N)$ belongs to $\mathfrak{D}^{\mathcal{M}}_{\mathfrak{p}_\sigma}(\mathbb{T}^N)$ if its discrete symbol $a(x, \xi)$ satisfies one of the following equivalent conditions:

1. There exist positive constants C_1 and h_1 such that

$$|D^\alpha_x a(x, \eta)| \leq C_1 h_1^{|\alpha|} m_{|\alpha|} |\alpha|!(1 + |\eta|)^\sigma, \; \forall \, x \in \mathbb{T}^N, \eta \in \mathbb{Z}^N, \alpha \in \mathbb{Z}^N_+. \tag{2.8}$$

2. There exist positive constants $C_2, h_2 > 0$ such that

$$|\widehat{a}(\xi, \eta)| \leq \frac{C_2 h_2^k m_k k!(1 + |\eta|)^\sigma}{(1 + |\xi|)^k}, \; \forall \, k \in \mathbb{Z}_+, (\xi, \eta) \in \mathbb{Z}^{2N}. \tag{2.9}$$

We also say that $a(x, D)$ is an $\mathcal{M}$-ultradifferentiable pseudodifferential operator of order σ. Also, if $a(x, D) \in \mathfrak{D}^{\mathcal{M}}_{\mathfrak{p}_\sigma}(\mathbb{T}^N)$, then one can prove that $a(x, D)$ defines a continuous and linear operator on $\mathcal{E}_{\mathcal{M}}(\mathbb{T}^N)$.

Remark In [4] the authors introduced a class of *smooth* pseudodifferential operators (of order σ) $\mathfrak{S}\mathfrak{p}_\sigma(\mathbb{T}^N)$, $\sigma \in \mathbb{R}$, which consists of all operators $a(x, D)$ such that given $\alpha \in \mathbb{Z}_+^N$, one can find $C_\alpha > 0$ such that

$$\left|D_x^\alpha a(x, \eta)\right| \leq C_\alpha (1 + |\eta|)^\sigma, \ \forall\ \eta \in \mathbb{Z}^N, x \in \mathbb{T}^N. \tag{2.10}$$

Since (2.8) implies (2.10), it is clear that $\mathfrak{D}_{\mathfrak{p}_\sigma}^{\mathcal{M}}(\mathbb{T}^N) \subset \mathfrak{S}\mathfrak{p}_\sigma(\mathbb{T}^N)$. Also, the operators belonging to this broader class define a continuous linear operator $a(x, D) : D'\left(\mathbb{T}^N\right) \longrightarrow D'\left(\mathbb{T}^N\right)$. More precisely, $a(x, D)$ can be extend to a linear and continuous operator $a(x, D) : H^s\left(\mathbb{T}^N\right) \longrightarrow H^{s-\sigma}\left(\mathbb{T}^N\right)$ for every $s \in \mathbb{R}$. □

3 Globally Hypoelliptic Operators

In this section we are going to prove some inequalities about globally hypoelliptic operators in the class $\mathfrak{S}\mathfrak{p}_\sigma(\mathbb{T}^N)$ that will be useful for our main results. Recall that an operator $a(x, D)$ is *globally C^∞ hypoelliptic in* $\mathbb{T}^N$ when the conditions $u \in D'\left(\mathbb{T}^N\right)$ and $a(x, D)u \in C^\infty\left(\mathbb{T}^N\right)$ imply $u \in C^\infty\left(\mathbb{T}^N\right)$.

In order to deal with perturbations of globally hypoelliptic operators we are going to state and prove some results. They are basically an extension of the results in [6] for the pseudodifferential operators.

Lemma 3.1 *If $a(x, D) \in \mathfrak{S}\mathfrak{p}_\sigma(\mathbb{T}^N)$ is a globally hypoelliptic operator in $\mathbb{T}^N$ then given $\ell, k \in \mathbb{Z}$, there exist $j \in \mathbb{Z}_+$ and $C > 0$ such that*

$$\|\varphi\|_\ell \leq C\left(\|a(x, D)\varphi\|_j + \|\varphi\|_k\right), \quad \forall \varphi \in C^\infty\left(\mathbb{T}^N\right).$$

In particular, there exist $j \in \mathbb{Z}_+$ and $C > 0$ such that

$$\|\varphi\|_1 \leq C\left(\|a(x, D)\varphi\|_j + \|\varphi\|_{-1}\right), \quad \forall \varphi \in C^\infty\left(\mathbb{T}^N\right).$$

Proof We consider in $C^\infty\left(\mathbb{T}^N\right)$ the locally convex, metrizable topology defined by the seminorms

$$\|\varphi\|_{j,k} = \|a(x, D)\varphi\|_j + \|\varphi\|_k, \quad \forall \varphi \in \mathbb{C}^\infty\left(\mathbb{T}^N\right), j \in \mathbb{Z}_+, k \in \mathbb{Z},$$

where $\|\cdot\|_s$ denotes the usual Sobolev norm on $\mathbb{T}^N$. Since $a(x, D)$ is globally hypoelliptic, $C^\infty\left(\mathbb{T}^N\right)$ endowed with this topology becomes a Fréchet. By the open mapping theorem, this topology must coincide with the standard one in $C^\infty\left(\mathbb{T}^N\right)$, whence the result. □

4 About the Condition $(\tau, 0) \notin \mathfrak{s}_{\mathcal{M}}(u)$

In view of our main result Theorem 1.3 we shall denote from now on the N-dimensional torus by $\mathbb{T}^N = \mathbb{T}^m_t \times \mathbb{T}^n_x$ and in the next result we recall that $\|\cdot\|_s$, $s \in \mathbb{R}$, stands for the usual Sobolev norm in the space $H^s\left(\mathbb{T}^N\right)$.

Proposition 4.1 *Suppose that $f \in \mathcal{E}_{\mathcal{M}}(\mathbb{T}^m \times \mathbb{T}^n)$. Given $s \in \mathbb{R}$, there exist constants $C, h > 0$ such that*

$$\left\|(1 + |\Delta_x|^{1/2})^k f\right\|_s \leq Ch^k m_k k!, \;\; \forall\, k \in \mathbb{Z}_+.$$

Proof Since $f \in \mathcal{E}_{\mathcal{M}}(\mathbb{T}^N)$, there exist constants $C_0, h_0 > 0$ such that

$$\left|D_t^\alpha D_x^\beta f(t,x)\right| \leq C_0 h_0^{|\alpha|+|\beta|} m_{|\alpha|+|\beta|}(|\alpha|+|\beta|)!, \;\; \forall\, t \in \mathbb{T}^m, x \in \mathbb{T}^n, \alpha \in \mathbb{Z}^m_+, \beta \in \mathbb{Z}^n_+.$$

We know that given $\zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{Z}^N$ and $q \in \mathbb{Z}_+$ there exist $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{Z}^N_+$ such that $|\gamma| = q$ and $|\zeta|^q \leq c^q|\zeta^\gamma|$, where $c = \sqrt{N}$. By using this fact for $N = n + m$ and putting $\zeta = (\tau_1, \ldots, \tau_m, \xi_1, \ldots, \xi_n) = (\tau, \xi)$ and $\alpha = (\gamma_1, \ldots, \gamma_m)$ and $\beta = (\gamma_{m+1}, \ldots, \gamma_{m+n})$, we obtain the following: given $\tau \in \mathbb{Z}^m$, $\xi \in \mathbb{Z}^n$ and $q \in \mathbb{Z}_+$, there exist $\alpha_q \in \mathbb{Z}^m_+$ and $\beta_q \in \mathbb{Z}^n_+$ such that $|\alpha_q| + |\beta_q| = q$ and

$$|(\tau, \xi)|^q \leq c^q \left|\tau^{\alpha_q}\xi^{\beta_q}\right|,$$

where $c > 0$ does not depend on either (τ, ξ) or q. Now we take $p \in \mathbb{N}$ such that $p \geq |s|$ and we obtain that

$$\begin{aligned}
&\left\|(1 + |\Delta_x|^{1/2})^k f\right\|_s^2 \\
&\leq \left\|(1 + |\Delta_x|^{1/2})^k f\right\|_p^2 = \sum_{\tau\in\mathbb{Z}^m, \xi\in\mathbb{Z}^n} (1 + |(\tau,\xi)|)^{2p}\,(1+|\xi|)^{2k}\left|\hat{f}(\tau,\xi)\right|^2 \\
&\leq \sum_{\tau\in\mathbb{Z}^m, \xi\in\mathbb{Z}^n} (1 + |(\tau,\xi)|)^{-2n-2m}\,(1 + |(\tau,\xi)|)^{2p+2k+2n+2m}\left|\hat{f}(\tau,\xi)\right|^2 \\
&= \sum_{\tau\in\mathbb{Z}^m, \xi\in\mathbb{Z}^n} (1 + |(\tau,\xi)|)^{-2n-2m}\left((1 + |(\tau,\xi)|)^{p+k+n+m}\left|\hat{f}(\tau,\xi)\right|\right)^2 \\
&\leq \sum_{\tau\in\mathbb{Z}^m, \xi\in\mathbb{Z}^n} (1 + |(\tau,\xi)|)^{-2n-2m}\left(\sum_{j=0}^{p+k+m+n}\binom{p+k+m+n}{j}|(\tau,\xi)|^j\left|\hat{f}(\tau,\xi)\right|\right)^2 \\
&\leq c^{p+k+m+n}\sum_{\tau\in\mathbb{Z}^m, \xi\in\mathbb{Z}^n} (1 + |(\tau,\xi)|)^{-2n-2m}\left(\sum_{j=0}^{p+k+m+n}\binom{p+k+m+n}{j}\left|\tau^{\alpha_j}\xi^{\beta_j}\right|\left|\hat{f}(\tau,\xi)\right|\right)^2 \\
&= c^{p+k+m+n}\sum_{\tau\in\mathbb{Z}^m, \xi\in\mathbb{Z}^n} (1 + |(\tau,\xi)|)^{-2n-2m}\left(\sum_{j=0}^{p+k+m+n}\binom{p+k+m+n}{j}\left|\widehat{D_t^{\alpha_j}D_x^{\beta_j}f}(\tau,\xi)\right|\right)^2.
\end{aligned}$$

Since

$$\left|\widehat{D_t^{\alpha_j} D_x^{\beta_j}} f(\tau,\xi)\right| \leq \frac{1}{(2\pi)^{m+n}} \int_{\mathbb{T}^m \times \mathbb{T}^n} \left| D_t^{\alpha_j} D_x^{\beta_j} f(t,x)\right| dt dx$$
$$\leq C_0 h_0^{|\alpha_j|+|\beta_j|} m_{|\alpha_j|+|\beta_j|} (|\alpha_j|+|\beta_j|)!$$
$$= C_0 h_0^j m_j j!,$$

and since $\{m_n\}_{n\in\mathbb{N}}$ is increasing (recall Proposition 2.2 item (iii) and that $m_1 = 1$), if we set $C_1 = c^{p+m+n}(1+h_0)^{p+m+n} C_0^2 \sum_{\tau\in\mathbb{Z}^m,\xi\in\mathbb{Z}^n} (1+|(\tau,\xi)|)^{-2n-2m}$ then it follows that

$$\begin{aligned}
&\left\|(1+|\Delta_x|^{1/2})^k f\right\|_s^2 \\
&\leq c^{p+k+m+n} \sum_{\tau\in\mathbb{Z}^m,\xi\in\mathbb{Z}^n} (1+|(\tau,\xi)|)^{-2n-2m} \left(\sum_{j=0}^{p+k+m+n} \binom{p+k+m+n}{j} C_0 h_0^j m_j j! \right)^2 \\
&\leq c^{p+k+m+n} C_0^2 (1+h_0)^{p+k+m+n} \sum_{\tau\in\mathbb{Z}^m,\xi\in\mathbb{Z}^n} (1+|(\tau,\xi)|)^{-2n-2m} m_{p+k+m+n}(p+k+m+n)! \\
&= c^k C_1 (1+h_0)^k m_{p+k+m+n}(p+k+m+n)!
\end{aligned}$$

and the proof follows since the last inequalities and (2.4) show that

$$\left\|(1+|\Delta_x|^{1/2})^k f\right\|_s \leq C h^k m_k k!,$$

for some $C, h > 0$.

We need one more auxiliary result to prove our main theorem.

Lemma 4.2 *Suppose that $u \in C^\infty(\mathbb{T}^m \times \mathbb{T}^n)$ satisfy the following conditions:*

1. *$(\tau, 0) \notin \mathfrak{s}_{\mathcal{M}}(u)$ for every $\tau \in \mathbb{R}^m \setminus 0$.*
2. *There exist constants $C_0, h_0 > 0$ such that $\left\|(1+|\Delta_x|^{1/2})^k u\right\|_0 \leq C_0 h_0^k m_k k!$ for every $k \in \mathbb{Z}_+$.*

Then $u \in \mathcal{E}_{\mathcal{M}}(\mathbb{T}^N)$.

Proof We first consider

$$A = \left\{(\tau,\xi) \in \mathbb{R}^m \times \mathbb{R}^n : |\tau| = 1, \xi = 0\right\}.$$

By using that $(\tau, 0) \notin \mathfrak{s}_{\mathcal{M}}(u)$ for every $|\tau| = 1$ and the compactness of A, it follows that there exist open cones $\Gamma_1, \ldots, \Gamma_k \subset \mathbb{R}^{m+n}$ such that $A \subset \Gamma_1 \cup \ldots \cup \Gamma_k$ and there exist constants $C, h > 0$ such that

$$\left|\hat{u}(\tau,\xi)\right| \leq C \inf_{p\in\mathbb{Z}_+} \left(\frac{h^p m_p p!}{(1+|(\tau,\xi)|)^p} \right), \quad \forall\, (\tau,\xi) \in \left(\mathbb{Z}^m \times \mathbb{Z}^n\right) \cap (\Gamma_1 \cup \ldots \cup \Gamma_k).$$

Note that if $\mathbb{Z}^m \times \mathbb{Z}^n \subset \Gamma_1 \cup \ldots \cup \Gamma_k$ the result is proved. Otherwise, since A is compact, $\mathbb{R}^{m+n} \setminus (\Gamma_1 \cup \ldots \cup \Gamma_k)$ is closed and they are disjoint, we have that

$$0 < \alpha \doteq d\left(A, \mathbb{R}^{m+n} \setminus (\Gamma_1 \cup \ldots \cup \Gamma_k)\right)$$

and then $(\tau, \xi) \in \Gamma_1 \cup \ldots \cup \Gamma_k$ whenever $d((\tau, \xi), A) < \alpha$. If $c > 0$ satisfy $1 - \frac{c^2}{1+c^2} \leq \frac{\alpha^2}{2}$ then

$$(\tau, \xi) \in S^{m+n-1}, |\tau| > c|\xi| \Longrightarrow d((\tau, \xi), A) < \alpha. \tag{4.1}$$

Indeed, suppose that $(\tau, \xi) \in S^{m+n-1}$ and $|\tau| > c|\xi|$. Since $\xi = 0$ and $|\tau|^2 + |\xi|^2 = 1$ ensure that (τ, ξ) is already an element of A we can suppose that $\xi \neq 0$. So we have that

$$c^2|\xi|^2 < |\tau|^2 = 1 - |\xi|^2,$$

which in turn implies that

$$|\xi|^2 < \frac{1}{1+c^2}.$$

Thus

$$|\tau|^2 = 1 - |\xi|^2 > 1 - \frac{1}{1+c^2} = \frac{c^2}{1+c^2}$$

and then

$$|\xi|^2 = 1 - |\tau|^2 < 1 - \frac{c^2}{1+c^2} \leq \frac{\alpha^2}{2}. \tag{4.2}$$

Since $|\tau| \leq 1$, we obtain from (4.2) that

$$(1 - |\tau|)^2 \leq 1 - |\tau|^2 < \frac{\alpha^2}{2} \tag{4.3}$$

If $\tau_0 = \frac{\tau}{|\tau|}$ then $(\tau_0, 0) \in A$ and from (4.2) and (4.3) we obtain that

$$\begin{aligned} d((\tau, \xi), A)^2 &\leq |(\tau, \xi) - (\tau_0, 0)|^2 = |\tau - \tau_0|^2 + |\xi|^2 = \left|\tau - \frac{\tau}{|\tau|}\right|^2 + |\xi|^2 \\ &< \left|\left(1 - \frac{1}{|\tau|}\right)\tau\right|^2 + \frac{\alpha^2}{2} = (1 - |\tau|)^2 + \frac{\alpha^2}{2} \leq \alpha^2, \end{aligned}$$

which proves (4.1). Now let $(\tau, \xi) \in \mathbb{Z}^m \times \mathbb{Z}^n$, $|\tau| > c|\xi|$. If $\lambda = \frac{1}{|(\tau,\xi)|}$ and $(\tau', \xi') = \lambda(\tau, \xi)$ then $(\tau', \xi') \in S^{m+n-1}$ and $|\tau'| > c|\xi'|$, so $(\tau', \xi') \in A$. Hence there exists

$j \in \{1, \ldots, k\}$ such that $(\tau', \xi') \in \Gamma_j$ and since the latter is a cone, we obtain that $(\tau, \xi) \in \Gamma_j$, from which we conclude that

$$\left|\hat{u}(\tau, \xi)\right| \leq C \inf_{p \in \mathbb{Z}_+} \left(\frac{h^p m_p p!}{(1 + |(\tau, \xi)|)^p} \right), \quad \forall\, (\tau, \xi) \in \left(\mathbb{Z}^m \times \mathbb{Z}^n\right), |\tau| > c|\xi|. \tag{4.4}$$

Our objective now is to extend (4.4) for every $(\tau, \xi) \in \mathbb{Z}^m \times \mathbb{Z}^n$, so we have to prove that

$$\left|\hat{u}(\tau, \xi)\right| \leq C \inf_{p \in \mathbb{Z}_+} \left(\frac{h^p m_p p!}{(1 + |(\tau, \xi)|)^p} \right), \quad \forall\, (\tau, \xi) \in \left(\mathbb{Z}^m \times \mathbb{Z}^n\right), |\tau| \leq c|\xi|. \tag{4.5}$$

If $(\tau, \xi) \in \mathbb{Z}^m \times \mathbb{Z}^n$ is such that $|\tau| \leq c|\xi|$ and $p \in \mathbb{Z}_+$. Then

$$|(\tau, \xi)|^2 = |\tau|^2 + |\xi|^2 \leq (1 + c)|\xi|^2,$$

which yields

$$(1 + |(\tau, \xi)|)^p \leq (1 + (1 + c)^{1/2}|\xi|)^p \leq (1 + c)^{p/2}(1 + |\xi|)^p.$$

From this inequality and our hypothesis we obtain the following:

$$\begin{aligned}
(1 + |(\tau, \xi)|)^p \left|\hat{u}(\tau, \xi)\right| &\leq (1 + c)^{p/2}(1 + |\xi|)^p \left|\hat{u}(\tau, \xi)\right| \\
&\leq (1 + c)^{p/2} \left(\sum_{\tau' \in \mathbb{Z}^m, \xi' \in \mathbb{Z}^n} (1 + |\xi'|)^{2p} \left|\hat{u}(\tau', \xi')\right|^2 \right)^{1/2} \\
&= (1 + c)^{p/2} \left\| (1 + |\Delta_x|^{1/2})^p u \right\|_0 \\
&\leq C_0 \left((1 + c)^{1/2} h_0 \right)^p m_p p!
\end{aligned}$$

and since $p \in \mathbb{Z}_+$ is arbitrary, we conclude that

$$\left|\hat{u}(\tau, \xi)\right| \leq C_0 \inf_{p \in \mathbb{Z}_+} \left(\frac{\left((1 + c)^{1/2} h_0\right)^p m_p p!}{(1 + |(\tau, \xi)|)^p} \right).$$

By increasing C and h if necessary we obtain (4.5). □

Now we are able to prove our main result of this work.

5 Proof of Theorem 1.3

Proof We first use Lemma 3.1: since we are assuming that $a(t, D_t, D_x)$ is globally hypoelliptic in $\mathbb{T}^m \times \mathbb{T}^n$, there exists $j \in \mathbb{Z}_+$ such that

$$\left\|(1+|\Delta_x|^{1/2})^k u\right\|_0 \le C\left(\left\|a(t,D_t,D_x)(1+|\Delta_x|^{1/2})^k u\right\|_j + \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\right), \forall\, k \in \mathbb{Z}_+.$$

In particular, if we take

$$r = j + \sigma, \tag{5.1}$$

then for every $k \in \mathbb{Z}_+$

$$\left\|(1+|\Delta_x|^{1/2})^k u\right\|_0 \le C\left(\left\|a(t,D_t,D_x)(1+|\Delta_x|^{1/2})^k u\right\|_{r-\sigma} + \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\right), \tag{5.2}$$

Now we observe that if $P = a(t, D_t, D_x) + b(t, x, D_t, D_x)$ then

$$\begin{aligned}\left\|a(t,D_t,D_x)(1+|\Delta_x|^{1/2})^k u\right\|_{r-\sigma} &= \left\|(1+|\Delta_x|^{1/2})^k a(t,D_t,D_x)u\right\|_{r-\sigma} \\ &\le \left\|(1+|\Delta_x|^{1/2})^k Pu\right\|_{r-\sigma} + \left\|(1+|\Delta_x|^{1/2})^k b(t,x,D_t,D_x)u\right\|_{r-\sigma},\end{aligned}$$

so from (5.2) we obtain

$$\begin{aligned}\left\|(1+|\Delta_x|^{1/2})^k u\right\|_0 \le C\Big(&\left\|(1+|\Delta_x|^{1/2})^k Pu\right\|_{r-\sigma} + \left\|(1+|\Delta_x|^{1/2})^k b(t,x,D_t,D_x)u\right\|_{r-\sigma} + \\ &+ \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\Big)\end{aligned} \tag{5.3}$$

for every $k \in \mathbb{Z}_+$. Let us estimate now the term $\left\|(1+|\Delta_x|^{1/2})^k b(t,x,D_t,D_x)u\right\|_{r-\sigma}$. It follows from (2.9) that there exist constants $C, h > 0$ such that

$$|\widehat{b}(\tau',\xi',\tau,\xi)| \le \frac{Ch^k m_k k!(1+|(\tau,\xi)|)^{\sigma'}}{(1+|(\tau',\xi')|)^k}, \ \forall\, k \in \mathbb{Z}_+, \tau,\tau' \in \mathbb{Z}^m, \xi,\xi' \in \mathbb{Z}^n. \tag{5.4}$$

If we denote $v = (1+|\Delta_x|^{1/2})^k b(t,x,D_t,D_x)u$, then keeping in mind that $\tau, \tau' \in \mathbb{Z}^m$ and $\xi, \xi' \in \mathbb{Z}^n$ in the all the sums below we have by (2.7) that

$$\begin{aligned}\|v\|^2_{r-\sigma} &= \sum_{\tau,\xi}(1+|(\tau,\xi)|)^{2r-2\sigma}\left|\hat{v}(\tau,\xi)\right|^2 \\ &\le \sum_{\tau,\xi}(1+|(\tau,\xi)|)^{2r-2\sigma}\left|(1+|\xi|)^k(b(t,x,\widehat{D_t},D_x)u)(\tau,\xi)\right|^2 \\ &= \sum_{\tau,\xi}(1+|(\tau,\xi)|)^{2r-2\sigma}\left|(1+|\xi|)^k\sum_{\tau',\xi'}\hat{b}(\tau-\tau',\xi-\xi',\tau',\xi')\hat{u}(\tau',\xi')\right|^2\end{aligned}$$

$$\leq \sum_{\tau,\xi}(1+|(\tau,\xi)|)^{2r-2\sigma}\Big(\sum_{\tau',\xi'}(1+|\xi-\xi'|+|\xi'|)^k|\hat{b}(\tau-\tau',\xi-\xi',\tau',\xi')||\hat{u}(\tau',\xi')|\Big)^2$$

$$\leq \sum_{\tau,\xi}(1+|(\tau,\xi)|)^{2r-2\sigma}\Big(\sum_{\tau',\xi'}\sum_{j=0}^{k}\binom{k}{j}(1+|\xi-\xi'|)^j|\xi'|^{k-j}|\hat{b}(\tau-\tau',\xi-\xi',\tau',\xi')||\hat{u}(\tau',\xi')|\Big)^2,$$

so

$$\|v\|_{r-\sigma} \leq \Bigg(\sum_{\tau,\xi}(1+|(\tau,\xi)|)^{2r-2\sigma}$$
$$\Big(\sum_{j=0}^{k}\sum_{\tau',\xi'}\binom{k}{j}(1+|\xi-\xi'|)^j|\xi'|^{k-j}|\hat{b}(\tau-\tau',\xi-\xi',\tau',\xi')||\hat{u}(\tau',\xi')|\Big)^2\Bigg)^{\frac{1}{2}}.$$

By the Minkowisky inequality for integrals the right side of the last inequality can be majored by

$$\sum_{j=0}^{k}\binom{k}{j}\Bigg(\sum_{\tau,\xi}(1+|(\tau,\xi)|)^{2r-2\sigma}\underbrace{\Big(\sum_{\tau',\xi'}(1+|\xi-\xi'|)^j|\xi'|^{k-j}|\hat{b}(\tau-\tau',\xi-\xi',\tau',\xi')||\hat{u}(\tau',\xi')|\Big)^2}_{\doteq (A)}\Bigg)^{\frac{1}{2}}.$$

Now it follows from the Cauchy-Schwarz inequality that

$$(A) \leq \underbrace{\Big(\sum_{\tau',\xi'}(1+|\xi-\xi'|)^{2j}|\hat{b}(\tau-\tau',\xi-\xi',\tau',\xi')|\Big)}_{\doteq (B)}$$
$$\Big(\sum_{\tau',\xi'}|\xi'|^{2k-2j}|\hat{b}(\tau-\tau',\xi-\xi',\tau',\xi')||\hat{u}(\tau',\xi')|^2\Big). \tag{5.5}$$

We choose $q \in \mathbb{N}$ such that $|\sigma'| \leq q$ and it follows from (5.4), from Peetre's inequality, (2.4), (2.3) and the inequality $(2j)! \leq 2^{2j}j!^2$ that

$$(B) \leq \sum_{\tau',\xi'}(1+|\xi-\xi'|)^{2j}\frac{Ch^{2j+q+2N}m_{2j+q+2N}(2j+q+2N)!(1+|(\tau',\xi')|)^{\sigma'}}{(1+|(\tau-\tau',\xi-\xi')|)^{2j+q+2N}}$$
$$\leq (1+|(\tau,\xi)|)^{\sigma'}\sum_{\tau',\xi'}(1+|\xi-\xi'|)^{2j}\frac{Ch^{2j+q+2N}m_{2j+q+2N}(2j+q+2N)!(1+|(\tau-\tau',\xi-\xi')|)^{|\sigma'|}}{(1+|(\tau-\tau',\xi-\xi')|)^{2j+q+2N}}$$
$$\leq (1+|(\tau,\xi)|)^{\sigma'}\sum_{\tau',\xi'}(1+|\xi-\xi'|)^{2j}\frac{Ch^{2j+q+2N}m_{2j+q+2N}(2j+q+2N)!}{(1+|(\tau-\tau',\xi-\xi')|)^{2j+2N}}$$
$$\leq (1+|(\tau,\xi)|)^{\sigma'}\sum_{\tau',\xi'}(1+|\xi-\xi'|)^{2j}\frac{Ch^{q+2N}(A_{q+2N}h)^{2j}m_{2j}(2j)!}{(1+|(\tau-\tau',\xi-\xi')|)^{2j+2N}}$$
$$\leq (1+|(\tau,\xi)|)^{\sigma'}Ch^{q+2N}(A_{q+2N}h)^{2j}m_{2j}(2j)!\sum_{\tau',\xi'}\frac{1}{(1+|(\tau-\tau',\xi-\xi')|)^{2N}}$$
$$\leq (1+|(\tau,\xi)|)^{\sigma'}Ch^{q+2N}(2HA_{q+2N}h)^{2j}(m_j j!)^2\sum_{\tau',\xi'}\frac{1}{(1+|(\tau-\tau',\xi-\xi')|)^{2N}}.$$

So if we take $C_1^2 = Ch^{q+2N}\sum_{\tau',\xi'}\frac{1}{(1+|(\tau',\xi')|)^{2N}}$ and $h_2 = 2HA_{q+2N}h$, then

$$(B) \le C_1^2 h_2^{2j}(1+|(\tau,\xi)|)^{\sigma'}(m_j j!)^2$$

and from (5.5) we obtain that

$$(A) \le C_1^2 h_2^{2j}(1+|(\tau,\xi)|)^{\sigma'}(m_j j!)^2 \Big(\sum_{\tau',\xi'}|\xi'|^{2k-2j}|\hat{b}(\tau-\tau',\xi-\xi',\tau',\xi')||\hat{u}(\tau',\xi')|^2\Big),$$

which in turn gives that

$$\|v\|_{r-\sigma} \le C_1\sum_{j=0}^{k}\binom{k}{j}h_2^j m_j j!(C), \tag{5.6}$$

where

$$(C) = \Big(\sum_{\tau,\xi}\sum_{\tau',\xi'}(1+|(\tau,\xi)|)^{2r-2\sigma+\sigma'}|\xi'|^{2k-2j}|\hat{b}(\tau-\tau',\xi-\xi',\tau',\xi')||\hat{u}(\tau',\xi')|^2\Big)^{\frac{1}{2}}.$$

Recalling that $q \in \mathbb{N}$ satisfy $|\sigma'| \le q$, we use again (5.4), the Peetre's inequality, (2.4) and (2.3) to obtain the following:

$$\begin{aligned}(C) &\le \Big(\sum_{\tau,\xi}\sum_{\tau',\xi'}(1+|(\tau,\xi)|)^{2r-2\sigma+\sigma'}|\xi'|^{2k-2j}\frac{Ch^{q+2N}m_{q+2N}(q+2N)!(1+|(\tau',\xi')|)^{\sigma'}}{(1+|(\tau-\tau',\xi-\xi')|)^{q+2N}}|\hat{u}(\tau',\xi')|^2\Big)^{\frac{1}{2}}\\ &\le \Big(\sum_{\tau,\xi}\sum_{\tau',\xi'}\big(1+\big|(\tau',\xi')\big|\big)^{2r-2\sigma+2\sigma'}|\xi'|^{2k-2j}\frac{Ch^{q+2N}m_{q+2N}(q+2N)!}{(1+|(\tau-\tau',\xi-\xi')|)^{2N}}|\hat{u}(\tau',\xi')|^2\Big)^{\frac{1}{2}}\\ &= \Big(\sum_{\tau',\xi'}\Big(\sum_{\tau,\xi}\frac{Ch^{q+2N}m_{q+2N}(q+2N)!}{(1+|(\tau-\tau',\xi-\xi')|)^{2N}}\Big)\big(1+\big|(\tau',\xi')\big|\big)^{2r-2\sigma+2\sigma'}|\xi'|^{2k-2j}|\hat{u}(\tau',\xi')|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Taking $C_2^2 = \sum_{\tau,\xi}\frac{Ch^{q+2N}m_{q+2N}(q+2N)!}{(1+|(\tau,\xi)|)^{2N}}$ we conclude that

$$\begin{aligned}(C) &\le C_2\sum_{\tau',\xi'}\big(1+\big|(\tau',\xi')\big|\big)^{2r-2\sigma+2\sigma'}|\xi'|^{2k-2j}|\hat{u}(\tau',\xi')|^2\Big)^{\frac{1}{2}}\\ &\le C_2\sum_{\tau',\xi'}\big(1+\big|(\tau',\xi')\big|\big)^{2r-2\sigma+2\sigma'}\big(1+|\xi'|\big)^{2k-2j}|\hat{u}(\tau',\xi')|^2\Big)^{\frac{1}{2}}\\ &= C_2\sum_{\tau',\xi'}\big(1+\big|(\tau',\xi')\big|\big)^{2r-2\sigma+2\sigma'}|\big(\widehat{(1+|\Delta_x|^{1/2})^{k-j}u}\big)(\tau',\xi')|^2\Big)^{\frac{1}{2}}\end{aligned}$$

$$= C_2 \left\| (1+|\Delta_x|^{1/2})^{k-j} u \right\|_{r-\sigma+\sigma'}.$$

Hence (5.6) and the last inequalities show that

$$\left\| (1+|\Delta_x|^{1/2})^k b(t,x,D_t,D_x)u \right\|_{r-\sigma} \leq C_3 \sum_{j=0}^{k} \binom{k}{j} h_2^j m_j j! \left\| (1+|\Delta_x|^{1/2})^{k-j} u \right\|_{r-\sigma+\sigma'},$$

where $C_3 = C_1 C_2$. Hence we can write (for $k \geq 1$)

$$\begin{aligned} &\left\| (1+|\Delta_x|^{1/2})^k b(t,x,D_t,D_x)u \right\|_{r-\sigma} \\ &\leq C_3 \Big(\left\| (1+|\Delta_x|^{1/2})^k u \right\|_{r-\sigma+\sigma'} + h_2^k m_k k! \, \|u\|_{r-\sigma+\sigma'} + (D) \Big), \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} (D) &= \sum_{j=1}^{k-1} \binom{k}{j} h_2^j m_j j! \left\| (1+|\Delta_x|^{1/2})^{k-j} u \right\|_{r-\sigma+\sigma'} \\ &= \sum_{j=1}^{k-1} \frac{k!}{(k-j)!} h_2^j m_j \left\| (1+|\Delta_x|^{1/2})^{k-j} u \right\|_{r-\sigma+\sigma'}. \end{aligned}$$

In order to estimate (D) we first recall the Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a, b \geq 0, \ \frac{1}{p} + \frac{1}{q} = 1.$$

For $1 \leq j \leq k-1$ we choose $p = \frac{k}{k-j}$ and $q = \frac{k}{j}$. Thus for any $\xi \in \mathbb{Z}^n$ and $\rho > 0$ we have

$$\begin{aligned} (1+|\xi|)^{k-j} &= (1+|\xi|)^{k-j} \rho^{\frac{k-j}{k}} \rho^{\frac{j-k}{k}} \\ &\leq \frac{\left((1+|\xi|)^{k-j} \rho^{\frac{k-j}{k}} \right)^{\frac{k}{k-j}}}{\frac{k}{k-j}} + \frac{\left(\rho^{\frac{j-k}{k}} \right)^{\frac{k}{j}}}{\frac{k}{j}} \\ &= \frac{k-j}{k} (1+|\xi|)^k \rho + \frac{j}{k} \rho^{-\frac{k-j}{j}}. \end{aligned}$$

Now for each $j \in \{1, \ldots, k-1\}$ and $\rho_j > 0$ that will be chosen later we obtain

$$\begin{aligned} &\left\| (1+|\Delta_x|^{1/2})^{k-j} u \right\|_{r-\sigma+\sigma'}^2 \\ &= \sum_{\tau',\xi'} (1+|(\tau,\xi)|)^{2r-2\sigma+2\sigma'} (1+|\xi|)^{2k-2j} \left| \hat{u}(\tau,\xi) \right|^2 \\ &\leq \sum_{\tau',\xi'} (1+|(\tau,\xi)|)^{2r-2\sigma+2\sigma'} \left(\frac{k-j}{k} (1+|\xi|)^k \rho_j + \frac{j}{k} \rho_j^{-\frac{k-j}{j}} \right)^2 \left| \hat{u}(\tau,\xi) \right|^2 \end{aligned}$$

$$\leq 2 \sum_{\tau',\xi'} (1+|(\tau,\xi)|)^{2r-2\sigma+2\sigma'} \left[\left(\frac{k-j}{k}\right)^2 (1+|\xi|)^{2k} \rho_j^2 + \left(\frac{j}{k}\rho_j^{-\frac{k-j}{j}}\right)^2 \right] \left|\hat{u}(\tau,\xi)\right|^2$$
$$= 2\left(\frac{k-j}{k}\right)^2 \rho_j^2 \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{r-\sigma+\sigma'}^2 + 2\left(\frac{j}{k}\rho_j^{-\frac{k-j}{j}}\right)^2 \|u\|_{r-\sigma+\sigma'}^2 .$$

Thus we conclude that

$$\begin{aligned}(D) &\leq \sqrt{2}\sum_{j=1}^{k-1} \frac{k!}{(k-j)!} h_2^j m_j \left(\frac{k-j}{k}\right) \rho_j \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{r-\sigma+\sigma'} \\ &\quad + \sqrt{2}\sum_{j=1}^{k-1} \frac{k!}{(k-j)!} \frac{j}{k} h_2^j m_j \rho_j^{-\frac{k-j}{j}} \|u\|_{r-\sigma+\sigma'} \\ &= \underbrace{\sqrt{2}\sum_{j=1}^{k-1} \frac{(k-1)!}{(k-j-1)!} h_2^j m_j \rho_j \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{r-\sigma+\sigma'}}_{\doteq (D_1)} \\ &\quad + \underbrace{\sqrt{2}\sum_{j=1}^{k-1} \frac{(k-1)!}{(k-j)!} j h_2^j m_j \rho_j^{-\frac{k-j}{j}} \|u\|_{r-\sigma+\sigma'}}_{\doteq (D_2)} . \end{aligned} \tag{5.8}$$

If we take $\rho_j = h_2^{-j} m_j^{-1} \frac{(k-j-1)!}{(k-1)!} \frac{1}{\sqrt{2}2^j}$, then

$$(D_1) \leq \sum_{j=1}^{k-1} \frac{1}{2^j} \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{r-\sigma+\sigma'} \leq \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{r-\sigma+\sigma'} . \tag{5.9}$$

In order to deal with (D_2) first notice that

$$\begin{aligned}(D_2) &= \sqrt{2}\sum_{j=1}^{k-1} \frac{(k-1)!}{(k-j)!} j h_2^j m_j \rho_j^{-\frac{k}{j}} \rho_j \|u\|_{r-\sigma+\sigma'} \\ &\leq \sum_{j=1}^{k-1} \frac{j}{2^j} \frac{(k-j-1)!}{(k-j)!} \rho_j^{-\frac{k}{j}} \|u\|_{r-\sigma+\sigma'} \\ &= \sum_{j=1}^{k-1} \frac{j}{2^j} \frac{(k-j-1)!}{(k-j)!} \left(h_2^{-j} m_j^{-1} \frac{(k-j-1)!}{(k-1)!} \frac{1}{\sqrt{2}2^j}\right)^{-\frac{k}{j}} \|u\|_{r-\sigma+\sigma'} \\ &= \sum_{j=1}^{k-1} \frac{j}{2^j} \frac{(k-j-1)!}{(k-j)!} \left(h_2^j m_j \frac{(k-1)!}{(k-j-1)!} \sqrt{2}2^j\right)^{\frac{k}{j}} \|u\|_{r-\sigma+\sigma'} \end{aligned}$$

$$\leq \sum_{j=1}^{k-1} \frac{j}{2^j} \frac{(k-j-1)!}{(k-j)!} (2\sqrt{2}h_2)^k \left(m_j \frac{(k-1)!}{(k-j-1)!} \right)^{\frac{k}{j}} \|u\|_{r-\sigma+\sigma'} . \tag{5.10}$$

Since

$$j \frac{(k-j-1)!}{(k-j)!} \leq j \leq k \leq 2^k, \quad \forall 1 \leq j \leq k-1$$

and (recall Proposition 2.2 (item ii))

$$\left(m_j \frac{(k-1)!}{(k-j-1)!} \right)^{\frac{k}{j}} \leq \left[m_j (k-1)^j \right]^{\frac{k}{j}} \leq \left(m_j^{\frac{1}{j}} \right)^k k^k \leq \left(m_k^{\frac{1}{k}} \right)^k e^k k! = e^k m_k k!$$

we obtain from (5.10) that

$$(D_2) \leq (4\sqrt{2}h_2 e)^k m_k k! \|u\|_{r-\sigma+\sigma'} \sum_{j=1}^{k-1} \frac{1}{2^j} \leq h_3^k m_k k! \|u\|_{r-\sigma+\sigma'} , \tag{5.11}$$

where $h_3 = 4\sqrt{2}h_2 e$. From (5.8), (5.9) and (5.11) we have that

$$(D) \leq (D_1) + (D_2) \leq \left\| (1 + |\Delta_x|^{1/2})^k u \right\|_{r-\sigma+\sigma'} + h_3^k m_k k! \|u\|_{r-\sigma+\sigma'} .$$

This last inequality and (5.7) yield

$$\begin{aligned} &\left\| (1 + |\Delta_x|^{1/2})^k b(t, x, D_t, D_x) u \right\|_{r-\sigma} \\ &\quad \leq C_3 \Big(2 \left\| (1 + |\Delta_x|^{1/2})^k u \right\|_{r-\sigma+\sigma'} + \left(h_2^k + h_3^k \right) m_k k! \|u\|_{r-\sigma+\sigma'} \Big). \end{aligned} \tag{5.12}$$

If $h_4 = \max\{h_2, h_3\}$ then

$$h_2^k + h_3^k \leq 2h_4^k$$

and from (5.12) we conclude, taking $C_4 = 2C_3$, that

$$\begin{aligned} &\left\| (1 + |\Delta_x|^{1/2})^k b(t, x, D_t, D_x) u \right\|_{r-\sigma} \\ &\quad \leq C_4 \Big(\left\| (1 + |\Delta_x|^{1/2})^k u \right\|_{r-\sigma+\sigma'} + h_4^k m_k k! \|u\|_{r-\sigma+\sigma'} \Big). \end{aligned} \tag{5.13}$$

From (5.3), (5.13), the inequality $r - \sigma + \sigma' \leq -1$ and supposing that $C_4 \geq 1$, we obtain that

$$
\begin{aligned}
&\left\|(1+|\Delta_x|^{1/2})^k u\right\|_0 \\
&\quad \le C\Big(\left\|(1+|\Delta_x|^{1/2})^k Pu\right\|_{r-\sigma} + C_4\Big(\left\|(1+|\Delta_x|^{1/2})^k u\right\|_{r-\sigma+\sigma'} \\
&\qquad + h_4^k m_k k! \,\|u\|_{r-\sigma+\sigma'}\Big) + \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\Big) \\
&\quad \le C_4 C\Big(\left\|(1+|\Delta_x|^{1/2})^k Pu\right\|_{r-\sigma} + h_4^k m_k k! \,\|u\|_{r-\sigma+\sigma'} + 2\left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\Big) \\
&\quad \le C_5\Big(\left\|(1+|\Delta_x|^{1/2})^k Pu\right\|_{r-\sigma} + h_4^k m_k k! \,\|u\|_{r-\sigma+\sigma'} + \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\Big),
\end{aligned}
$$

where $C_5 = 2C_4 C$. Now we can use Proposition 4.1 and we obtain $C_6, h_5 > 0$ such that

$$
\left\|(1+|\Delta_x|^{1/2})^k Pu\right\|_{r-\sigma} \le C_6 h_5^k m_k k!, \ \ \forall\ k \in \mathbb{Z}_+,
$$

so by supposing that $C_6 \ge 1$ and considering $h_6 = \max\{h_4, h_5\}$ we have

$$
\begin{aligned}
\left\|(1+|\Delta_x|^{1/2})^k u\right\|_0 &\le C_5\Big(C_6 h_5^k m_k k! + h_4^k m_k k! \,\|u\|_{r-\sigma+\sigma'} + \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\Big) \\
&\le C_5\Big((C_6 h_5^k + h_4^k)\,\|u\|_{r-\sigma+\sigma'}\, m_k k! + \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\Big) \\
&\le C_5 C_6\Big(2h_6^k\,\|u\|_{r-\sigma+\sigma'}\, m_k k! + \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\Big) \\
&\le C_5 C_6 2\Big(h_6^k\,\|u\|_{r-\sigma+\sigma'}\, m_k k! + \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\Big) \\
&\le C_5 C_6 2(1+\|u\|_{r-\sigma+\sigma'})\Big(h_6^k m_k k! + \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\Big) \\
&= C_7\Big(h_6^k m_k k! + \left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}\Big), \qquad (5.14)
\end{aligned}
$$

where $C_7 = 2C_5 C_6(1+\|u\|_{r-\sigma+\sigma'})$. Since

$$
\begin{aligned}
\left\|(1+|\Delta_x|^{1/2})^k u\right\|_{-1}^2 &= \sum_{\tau,\xi}(1+|(\tau,\xi)|)^{-2}(1+|\xi|)^{2k}|\hat{u}(\tau,\xi)|^2 \\
&\le \sum_{\tau,\xi}(1+|\xi|)^{-2}(1+|\xi|)^{2k}|\hat{u}(\tau,\xi)|^2 \\
&= \left\|(1+|\Delta_x|^{1/2})^{k-1} u\right\|_0^2,
\end{aligned}
$$

from (5.14) we obtain

$$
\left\|(1+|\Delta_x|^{1/2})^k u\right\|_0 \le C_7\Big(h_6^k m_k k! + \left\|(1+|\Delta_x|^{1/2})^{k-1} u\right\|_0\Big), \quad \forall k \in \mathbb{N}. \quad (5.15)
$$

Now we claim that if $C_8 > 0$ and $M > 1$ satisfy $C_8 \ge \|u\|_0$ and $C_7\left(\frac{1}{C_8} + \frac{1}{Mh_6}\right) \le 1$ then

$$
\left\|(1+|\Delta_x|^{1/2})^k u\right\|_0 \le C_8(Mh_6)^k m_k k!, \quad \forall k \in \mathbb{Z}_+. \qquad (5.16)
$$

The proof of (5.16) will be done by induction on $k \in \mathbb{Z}_+$. For $k = 0$, (5.16) is a consequence of our choice of C_8 and for the induction step we first use (5.15) and it follows from the induction hypothesis that

$$\begin{aligned}\left\|(1+|\Delta_x|^{1/2})^k u\right\|_0 &\leq C_7\Big(h_6^k m_k k! + C_8(Mh_6)^{k-1} m_{k-1}(k-1)!\Big)\\ &= C_8(Mh_6)^k m_k k! C_7\left(\frac{1}{C_8 M^k} + \frac{m_{k-1}(k-1)!}{Mh_6 m_k k!}\right).\end{aligned}$$

Hence it suffices to prove that $C_7\left(\frac{1}{C_8 M^k} + \frac{m_{k-1}(k-1)!}{Mh_6 m_k k!}\right) \leq 1$ for every $k \in \mathbb{Z}_+$. By using again that the sequence $m_k k!$ is increasing and our choices of C_8 and M we have

$$\begin{aligned}C_7\left(\frac{1}{C_8 M^k} + \frac{m_{k-1}(k-1)!}{Mh_6 m_k k!}\right) &\leq C_7\left(\frac{1}{C_8 M^k} + \frac{1}{Mh_6}\right)\\ &\leq C_7\left(\frac{1}{C_8} + \frac{1}{Mh_6}\right)\\ &\leq 1.\end{aligned}$$

We now use Lemma 4.2 and the proof is complete. □

6 Proof of Theorem 1.4

Since $\mathfrak{D}^{\mathcal{M}}_{\mathfrak{p}_\sigma}(\mathbb{T}^N) \subset \mathfrak{S}\mathfrak{p}_\sigma(\mathbb{T}^N)$, see 2.2, in this section we shall prove that when the operator $a(t, D_t, D_x)$ is globally hypoelliptic with loss of derivatives, then the hypothesis $u \in C^\infty(\mathbb{T}^m \times \mathbb{T}^n)$ of Theorem 1.3 can be dropped. First we recall the notion of global hypoellipticity with loss of derivatives (see [4]).

Definition 6.1 We say that the an operator $a(x, D) \in \mathfrak{S}\mathfrak{p}_\sigma(\mathbb{T}^N)$ is globally hypoelliptic with loss of R derivatives if for any distribution $u \in D'(\mathbb{T}^N)$ such that $a_j(x, D)u \in H^t(\mathbb{T}^N)$ for some $t \in \mathbb{R}$, we have $u \in H^{t+\sigma-R}(\mathbb{T}^N)$.

The next result improves the information of Lemma 3.1 when the operator is globally hypoelliptic with loss of derivatives.

Lemma 6.2 *Let $a(x, D) \in \mathfrak{S}\mathfrak{p}_\sigma(\mathbb{T}^N)$ be a globally hypoelliptic pseudodifferential operator with loss of $R \geq 0$ derivatives. If $\sigma' < \sigma - R$ then for each $t \in \mathbb{R}$ there exists a constant $C = C_t > 0$ such that*

$$\|\varphi\|_{t+\sigma-R} \leq C\left(\|a(x, D)\varphi\|_t + \|u\|_{t+\sigma'}\right), \quad \forall \varphi \in C^\infty(\mathbb{T}^N). \tag{6.1}$$

Proof For $t \in \mathbb{R}$ fixed we define the set

$$\mathcal{F}_t(\mathbb{T}^N) = \left\{v \in H^{t+\sigma'}(\mathbb{T}^N) : a_j(x, D)v \in H^t(\mathbb{T}^N),\, j = 1, \ldots, m\right\}$$

and we consider the following norm in $\mathcal{F}_t(\mathbb{T}^N)$:

$$|||v||| \doteq \|v\|_{t+\sigma'} + \|a(x, D)v\|_t \,, \ \forall\, v \in \mathcal{F}_t.$$

Since $(\mathcal{F}_t(\mathbb{T}^N), |||v|||)$ is a Banach space then by using the Closed Graph Theorem one can prove that the inclusion $i : \mathcal{F}_t(\mathbb{T}^N) \longrightarrow H^{t+\sigma-R}(\mathbb{T}^N)$ is continuous and the proof is complete. □

***Proof of Theorem* 1.4** Since $a(t, D_x, D_t)$ has order $\sigma \in \mathbb{R}$ and is globally hypoelliptic with loss of $R > 0$ derivatives then $a(t, D_x, D_t)$ is globally hypoelliptic on $\mathbb{T}^N$. By using that $\sigma' \leq \sigma - R - 2 < \sigma - R$, it follows from Theorem 3.4 in [4] that the operator $a(t, D_x, D_t) + b(x, t, D_x, D_t)$ has order σ and loses R derivatives what guarantees that it is globally hypoelliptic on $\mathbb{T}^N$. Thus we conclude that $u \in C^\infty(\mathbb{T}^N)$.

Now if we take $t = R - \sigma + 1$ in (6.1), we obtain $C > 0$ such that

$$\begin{aligned} \|\varphi\|_1 &\leq C\left(\|a(x, D)\varphi\|_{R-\sigma+1} + \|\varphi\|_{R-\sigma+1+\sigma'}\right) \\ &\leq C\left(\|a(x, D)\varphi\|_{R-\sigma+1} + \|\varphi\|_{-1}\right), \ \forall \varphi \in C^\infty\left(\mathbb{T}^m \times \mathbb{T}^n\right). \end{aligned}$$

Let $j \in \mathbb{Z}$ be the smallest integer greater than or equal to $R - \sigma + 1$. Then

$$\|\varphi\|_1 \leq C\left(\|a(x, D)\varphi\|_j + \|u\|_{-1}\right)$$

that is precisely the inequality of Lemma 3.1. Now the proof is an easy consequence of Theorem 1.3, or more specifically, of its proof: from (5.1), if

$$\sigma' \leq \sigma - r - 1 = \sigma - j - \sigma - 1 = -j - 1 \leq \sigma - R - 2,$$

then $u \in \mathcal{E}_{\mathcal{M}}\left(\mathbb{T}^N\right)$. The proof is complete. □

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