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Fractional Leibniz Rules in the Setting of Quasi-Banach Function Spaces

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Abstract

Fractional Leibniz rules are reminiscent of the product rule learned in calculus classes, offering estimates in the Lebesgue norm for fractional derivatives of a product of functions in terms of the Lebesgue norms of each function and its fractional derivatives. We prove such estimates for Coifman–Meyer multiplier operators in the setting of Triebel–Lizorkin and Besov spaces based on quasi-Banach function spaces. In particular, these include rearrangement invariant quasi-Banach function spaces such as weighted Lebesgue spaces, weighted Lorentz spaces and generalizations, and Orlicz spaces. The method used also yields results in weighted mixed Lebesgue spaces and Morrey spaces, where we present applications to the specific case of power weights, as well as in variable Lebesgue spaces.

Keywords Quasi-Banach function spaces · Besov and Triebel–Lizorkin spaces · Bilinear pseudo-differential operators · Fractional Leibniz rules · Algebra property

Mathematics Subject Classification Primary: 42B35 · 35S05 · Secondary: 42B25 · 42B30 · 46E35

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1 Introduction and Main Results

Fractional Leibniz rules in the setting of Lebesgue spaces state that, for $1 \leq p_1, p_2, \widetilde{p}_1, \widetilde{p}_2 \leq \infty$, $1/2 \leq p \leq \infty$ such that $1/p = 1/p_1 + 1/p_2 = 1/\widetilde{p}_1 + 1/\widetilde{p}_2$, and $s > \max\{0, n(1/p - 1)\}$ or $s \in 2\mathbb{N}$, the inequality

$$\left\| E^s(fg) \right\|_{L^p} \lesssim \left\| E^s f \right\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{\widetilde{p}_1}} \left\| E^s g \right\|_{L^{\widetilde{p}_2}} \tag{1.1}$$

holds true with $E^s = D^s$ or $E^s = J^s$, where D^s and J^s are the homogeneous and inhomogeneous fractional differentiation operators of order s, respectively, defined through the Fourier transform by $\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$ and $\widehat{J^s(f)}(\xi) = (1 + |\xi|^2)^{s/2} \widehat{f}(\xi)$, $\xi \in \mathbb{R}^n$.

The inequality (1.1) and related commutator estimates have emerged as essential tools to study a number of nonlinear PDEs, including Euler and Navier-Stokes equations (see Kato–Ponce [34]) and Korteweg–de Vries equations (see Christ–Weinstein [13] and Kenig–Ponce–Vega [35]), as well as the study of smoothing properties of Schrödinger semigroups (see Gulisashvili–Kon [29]). The range $1 < p < \infty$ is addressed in the mentioned works and the case $\frac{1}{2} < p \leq 1$ is treated in Grafakos–Oh [28] and Muscalu–Schlag [45] (see also Koezuka–Tomita [36] and Naibo–Thomson [48]); for the endpoints $p = \infty$ and $p = \frac{1}{2}$, the reader is referred to Bourgain–Li [9] (see also Grafakos–Maldonado–Naibo [27]) and Oh–Wu [50], respectively.

The estimate (1.1) is a particular instance of inequalities in a variety of function spaces where the product fg is replaced by $T_\sigma(f, g)$; here, T_σ is a bilinear pseudodifferential operator associated to $\sigma = \sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, (called a symbol, or a multiplier if independent of x) and defined by

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta. \tag{1.2}$$

Note that for $\sigma \equiv 1$, we recover the product fg. Estimates using T_σ have the form

$$\left\| E^s(T_\sigma(f, g)) \right\|_Z \lesssim \left\| E^s f \right\|_{Z_1} \|g\|_{Z_2} + \|f\|_{\widetilde{Z}_1} \left\| E^s g \right\|_{\widetilde{Z}_2} \tag{1.3}$$

for function spaces Z, Z_1, Z_2, $\widetilde{Z}_1$, and $\widetilde{Z}_2$. For example, variants of these estimates in weighted Lebesgue spaces associated to Muckenhoupt weights are given in [48] for Coifman–Meyer multiplier operators and in Cruz-Uribe–Naibo [16, 17] for $\sigma \equiv 1$; Hart–Torres–Wu [30] proved estimates for multiplier operators in the context of Lebesgue and mixed Lebesgue spaces using minimal assumptions on the smoothness of the multipliers; Oh–Wu [51] obtained results with $\sigma \equiv 1$ in the setting of Lebesgue and mixed Lebesgue spaces associated to power weights; the works [36] and [48] include estimates in the context of local Hardy spaces and weighted Hardy spaces, respectively. The estimates (1.3) have also been studied in the scale of Besov and Triebel–Lizorkin spaces for operators with symbols belonging to bilinear Hörmander classes; see, for instance, the works Bényi [5] and Naibo–Thomson [47] in the scale of Besov spaces, Bényi–Nahmod–Torres [6] in the context of Sobolev spaces, and Naibo [46] and [36] for Besov and Triebel–Lizorkin spaces. For bilinear pseudo-differential operators with symbols closely related to the Hörmander classes, Brummer–Naibo [10] proved estimates in the setting of function spaces that admit a molecular decomposition and a φ-transform in the sense of Frazier–Jawerth [24, 25], and for Coifman–Meyer multiplier operators, [48] worked in the context of weighted Besov and Triebel–Lizorkin spaces with weights in the Muckenhoupt classes. We refer the reader to the survey in Torres [56] for other considerations.

In this article, we prove fractional Leibniz rules of the type (1.3) for Coifman–Meyer multiplier operators in the setting of Triebel–Lizorkin and Besov spaces based on quasi-Banach function spaces. A Coifman–Meyer multiplier operator of order $m \in \mathbb{R}$ is an operator of the type (1.2) given by a smooth, complex-valued multiplier $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, that verifies

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)\right| \leq C_{\alpha,\beta}(|\xi| + |\eta|)^{m-(|\alpha|+|\beta|)}, \qquad \forall (\xi, \eta) \in \mathbb{R}^{2n} \setminus \{(0,0)\}, \tag{1.4}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ and some constant $C_{\alpha,\beta} > 0$. We will also consider an inhomogeneous analog where σ is such that (1.4) holds with $1 + |\xi| + |\eta|$ instead of $|\xi| + |\eta|$.

Quasi-Banach function spaces include a diverse family of function spaces such as weighted mixed Lebesgue spaces, Morrey spaces (after an adjustment), variable Lebesgue spaces, as well as the large class of rearrangement invariant quasi-Banach function spaces, of which weighted Lebesgue spaces, generalized versions of Lorentz spaces, and Orlicz spaces are specific examples. By proving the identification of quasi-Banach function spaces with spaces in the scale of the associated Triebel–Lizorkin spaces, our main results imply a plethora of fractional Leibniz rules in quasi-Banach function spaces, recovering in a unified way many results in the literature and providing new ones. For instance, we recover the following estimates proved in [48]:

$$\left\| D^s(T_\sigma(f,g)) \right\|_{H^p} \lesssim \left\| D^s f \right\|_{H^{p_1}} \|g\|_{H^{p_2}} + \|f\|_{H^{\widetilde{p}_1}} \left\| D^s g \right\|_{H^{\widetilde{p}_2}}, \tag{1.5}$$

for a Coifman–Meyer multiplier σ of order zero, $0 < p_1, \widetilde{p}_2 < \infty$, $0 < p_2, \widetilde{p}_1 \leq \infty$, $0 < p < \infty$ such that $1/p = 1/p_1 + 1/p_2 = 1/\widetilde{p}_1 + 1/\widetilde{p}_2$, $s > \max\{0, n(1/p - 1)\}$, and where $H^q = H^q(\mathbb{R}^n)$ denotes a Hardy space for $0 < q < \infty$ (recall that $H^q(\mathbb{R}^n) = L^q(\mathbb{R}^n)$ for $1 < q < \infty$) and $H^q(\mathbb{R}^n)$ is replaced by $L^\infty(\mathbb{R}^n)$ if $q = \infty$. Notice that, when $\sigma \equiv 1$, the estimate (1.5) improves (1.1) by allowing all indices to be in the wider range $(0, \infty]$ and by admitting the larger H^p-norm on the left hand side. A weighted version of (1.5) also holds with Hardy spaces associated to weights in the Muckenhoupt class $A_\infty(\mathbb{R}^n)$.

More generally, our main results lead to the following novel version of (1.3) in the setting of Hardy spaces based on weighted rearrangement invariant quasi-Banach function spaces,

$$\begin{aligned} &\left\| D^s(T_\sigma(f,g)) \right\|_{H^{X^p}(w)} \\ &\quad \lesssim \left\| D^s f \right\|_{H^{X_1^{p_1}}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \left\| D^s g \right\|_{H^{X_2^{p_2}}(w_2)}, \end{aligned} \tag{1.6}$$

where for a weight v and $0 < q < \infty$, $H^{X^q}(v)$ denotes the Hardy space based on the weighted rearrangement invariant quasi-Banach function spaces $X^q(v)$, w, w_1, and w_2 are weights in the Muckenhoupt class $A_\infty(\mathbb{R}^n)$, the parameters s, p, p_1, and p_2 satisfy appropriate conditions, and σ is a Coifman–Meyer multiplier of order zero. In turn, (1.6) implies

$$\left\| D^s(T_\sigma(f,g)) \right\|_{X^p(w)} \lesssim \left\| D^s f \right\|_{X_1^{p_1}(w_1)} \|g\|_{X_2^{p_2}(w_2)} + \|f\|_{X_1^{p_1}(w_1)} \left\| D^s g \right\|_{X_2^{p_2}(w_2)}, \tag{1.7}$$

for appropriate indices and weights in the Muckenhoupt classes. We refer the reader to Sect. 4 for more details.

Our main results also provide new estimates in the setting of weighted mixed Lebesgue spaces; for instance, if σ is a Coifman–Meyer multiplier of order zero, we obtain

$$\begin{aligned} \left\| D^s(T_\sigma(f,g)) \right\|_{L^p(L^q(w))} \lesssim & \left\| D^s f \right\|_{L^{p_1}(L^{q_1}(w_1))} \|g\|_{L^{p_2}(L^{q_2}(w_2))} \\ & + \|f\|_{L^{p_1}(L^{q_1}(w_1))} \left\| D^s g \right\|_{L^{p_2}(L^{q_2}(w_2))}, \end{aligned} \tag{1.8}$$

for $1 < p, p_1, p_2, q, q_1, q_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$, $s > 0$, and appropriate weights w, w_1, and w_2 in 'mixed' versions of Muckenhoupt classes. See details in Sect. 5.

Other concrete examples implied by our main results include Leibniz rules in settings associated to weighted Lorentz and Orlicz spaces, as well as weighted Morrey and variable Lebesgue spaces. Details can be found in Sects. 4.3, 6, and 7, respectively.

Some particular cases of (1.3) can be recast as

$$\|T_\sigma(f,g)\|_Y \lesssim \|f\|_Y \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_Y, \tag{1.9}$$

where Y is some function space associated to a smoothness parameter (for instance, a Sobolev space, or more generally a Besov or Triebel–Lizorkin space). Such estimates, in particular when $\sigma \equiv 1$, have played a fundamental role in the study of partial differential equations (see, for instance, [2, 10, 40, 48] and the references therein), and they imply that $Y \cap L^\infty(\mathbb{R}^n)$ is an algebra under pointwise multiplication. Our main results give that (1.9) holds for Besov or Triebel–Lizorkin spaces based on a quasi-Banach function space; as a byproduct, the intersection of such spaces with $L^\infty(\mathbb{R}^n)$ is an algebra under pointwise multiplication.

Multiple approaches (which are based on Coifman–Meyer multiplier operators and the bilinear Calderón-Zygmund theory, square-function estimates, vector-valued multiplier theorems, among others) have been put forward to prove fractional Leibniz rules in the spirit of (1.3). In this article, we employed an alternative unifying approach used in [48], where results in the weighted Lebesgue, Lorentz and Morrey spaces, as well as variable Lebesgue spaces were obtained. This method is based on Nikol'skiĭ representations for function spaces and was pioneered for classical spaces in Bourdaud [8], Meyer [42], Nikol'skiĭ [49], Triebel [57], and Yamazaki [58]. We prove such representations for the general setting of Besov and Triebel–Lizorkin spaces based on quasi-Banach function spaces (Theorem 3.4).

This article is organized as follows. Notation, definitions, and properties of function spaces are given in Sect. 2. The statement of the main result on fractional Leibniz rules in Triebel–Lizorkin and Besov spaces based on quasi-Banach function spaces, as well as its corollaries and proof, are given in Sect. 3. In Sect. 4, we strengthen the main result for the particular case of rearrangement invariant quasi-Banach function spaces, and present specific examples in weighted Lebesgue spaces, weighted Lorentz spaces and Orlicz spaces. Other particular applications of the general theory in the setting of quasi-Banach function spaces that are not rearrangement invariant are given in Sect. 5 for weighted mixed Lebesgue spaces, in Sect. 6 for weighted Morrey spaces, and in Sect. 7 for variable Lebesgue spaces. Finally, Appendix A contains extrapolation theorems, Appendix B proves identifications of function spaces with spaces in the scale of Triebel–Lizorkin spaces, Appendix C shows results pertaining to the boundedness of the Hardy-Littlewood maximal operator on the function spaces used, and Appendix D gives the proof of Nikol'skiĭ representations in the context of quasi-Banach function spaces.

2 Preliminaries

In this section, we give some definitions and notation for quasi-Banach function spaces (QBFS), weights, and QBFS-based Triebel–Lizorkin, Besov, and Hardy spaces.

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz class of smooth, rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^n)$ denote its dual space of tempered distributions. We use $\mathcal{S}_0(\mathbb{R}^n)$ to indicate the subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ with vanishing moments of all orders. That is, for $f \in \mathcal{S}(\mathbb{R}^n)$, we have $f \in \mathcal{S}_0(\mathbb{R}^n)$, if, and only if, for any multi-index $\alpha \in \mathbb{N}_0^n$, $\int_{\mathbb{R}^n} x^\alpha f(x)dx = 0$. Its dual space will be denoted by $\mathcal{S}_0'(\mathbb{R}^n)$, which is the class of tempered distributions modulo polynomials, $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$.

Many of our results will be in quasi-Banach function spaces, which we define following Bennett and Sharpley [4] and Cruz-Uribe et al. [19]. Let $(\mathbb{R}^n, \mu)$ be a totally σ-finite, nonatomic measure space and M denote the collection of measurable functions on $(\mathbb{R}^n, \mu)$. A mapping $\rho : M \to [0, \infty]$ is a *Banach function norm* if it satisfies the following properties for all f and g in M:

P1. $\rho(f) = \rho(|f|)$ and $\rho(f) = 0$ if, and only if, $f = 0$ μ-a.e.;
P2. $\rho(f + g) \leq \rho(f) + \rho(g)$;
P3. $\rho(af) = |a|\, \rho(f)$, for all $a \in \mathbb{R}$;
P4. $|f| \leq |g|$ μ-a.e. implies $\rho(f) \leq \rho(g)$;
P5. If $\{f_j\}_{j\in\mathbb{Z}}$ is a sequence of measurable functions such that $|f_j|$ increases to $|f|$ μ-a.e., then $\rho(f_j)$ increases to $\rho(f)$;
P6. If $E \subset \mathbb{R}^n$ is measurable and $\mu(E) < \infty$, then

   i. $\rho(\chi_E) < \infty$,
   ii. there exists $0 < C_E < \infty$, depending only on E and ρ, such that $\int_E |f|\, d\mu \leq C_E\, \rho(f)$.

Given ρ, we define the function space

$$X = \left\{ f \in M : \|f\|_X < \infty \right\},$$

where $\|f\|_X = \rho(f)$. Using properties P4 and P5, it can be shown that $(X, \|\cdot\|_X)$ is a Banach space (see [4, Chap. 1, Theorem 1.6]). We call X a *Banach function space* (BFS).

The *associate space of* X, denoted X', is defined through the Banach function norm

$$\rho'(f) = \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)| \, d\mu : g \in X, \|g\|_X \le 1 \right\}.$$

Given $0 < p < \infty$, we define

$$X^p = \left\{ f \in M : |f|^p \in X \right\},$$

and set $\|f\|_{X^p} = \||f|^p\|_X^{1/p}$.

In the case that property P2 is replaced by

$$\rho(f + g) \le C_\rho(\rho(f) + \rho(g)),$$

for some constant $0 < C_\rho < \infty$, and property P6ii is omitted, we call X a *quasi-Banach function space* (QBFS). A QBFS is also complete (see Caetano et al. [11, Lemma 3.6]), and the definitions of X^p and X' extend to this setting. We note that if X is a BFS, then X^p for $1 \le p < \infty$ and X' are BFSs, while X^p for $0 < p < 1$ can only be guaranteed to be a QBFS.

For most of our results, it will be required that a QBFS X is such that X^{p_0} is a Banach function space for some $1 \le p_0 < \infty$. That is, defining

$$p(X) = \inf \left\{ p_0 \ge 1 : X^{p_0} \text{ is BFS} \right\},$$

we require that $p(X) < \infty$.

We note that if $p(X) < \infty$ and $0 < p, p_1, p_2 \le \infty$ are such that $1/p = 1/p_1 + 1/p_2$, then the following Hölder's inequality holds:

$$\|fg\|_{X^p} \le \|f\|_{X^{p_1}} \|g\|_{X^{p_2}} . \tag{2.1}$$

We next discuss boundedness properties of the Hardy-Littlewood maximal operator on a QBFS X. We define the *Hardy-Littlewood maximal operator* by

$$\mathcal{M}f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \qquad x \in \mathbb{R}^n,$$

where f is a locally integrable function on $\mathbb{R}^n$, the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^n$ containing x, and $|B|$ denotes the Lebesgue measure of

B. For $h > 0$,

$$\mathcal{M}_h f(x) = \left(\mathcal{M}\left(|f|^h\right)\right)^{\frac{1}{h}}(x) = \sup_{B \ni x}\left(\frac{1}{|B|}\int_B |f(y)|^h dy\right)^{\frac{1}{h}}.$$

Let X be a QBFS over $(\mathbb{R}^n, \mu)$. Assume that given $0 < r \leq \infty$ there exists $h > 0$ such that the Fefferman-Stein inequality holds; that is,

$$\left\|\left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_h(f_j)|^r\right)^{\frac{1}{r}}\right\|_X \lesssim \left\|\left(\sum_{j\in\mathbb{Z}} |f_j|^r\right)^{\frac{1}{r}}\right\|_X, \tag{2.2}$$

for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of locally integrable functions defined on $\mathbb{R}^n$, with the corresponding changes when $r = \infty$. Then for such a QBFS X, we define

$$h_X = \sup\{h > 0 : \mathcal{M}_h \text{ is bounded on } X\} \text{ and } h_{X,r} = \sup\{h > 0 : (2.2) \text{ holds}\}.$$

We also define

$$\tau(X) = n\left(\frac{1}{\min(h_X, 1)} - 1\right) \text{ and } \tau_r(X) = n\left(\frac{1}{\min(h_{X,r}, 1)} - 1\right).$$

We note that $h_X \geq h_{X,r}$ and $\tau_r(X) \geq \tau(X)$.

A weight on $\mathbb{R}^n$ is a nonnegative, locally integrable function defined on $\mathbb{R}^n$. Given a weight w on $\mathbb{R}^n$ and $0 < p \leq \infty$, define the weighted Lebesgue space $L^p(w)$ to be the collection of measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx\right)^{\frac{1}{p}} < \infty,$$

with the usual change when $p = \infty$.

For $1 < p < \infty$, the Muckenhoupt class of weights $A_p(\mathbb{R}^n)$ is the collection of weights w on $\mathbb{R}^n$ such that

$$[w]_{A_p(\mathbb{R}^n)} = \sup_Q \left(⨍_Q w(x)dx\right)\left(⨍_Q w(x)^{1-p'}dx\right)^{p-1} < \infty, \tag{2.3}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $⨍_Q f(x)dx = \frac{1}{|Q|}\int_Q f(x)dx$ for a locally integrable function f. Recall that if $1 < p < \infty$, the Hardy-Littlewood maximal operator is bounded on $L^p(w)$ if, and only if, $w \in A_p(\mathbb{R}^n)$ (see Muckenhoupt [44]). We also define

$$A_\infty(\mathbb{R}^n) = \bigcup_{p>1} A_p(\mathbb{R}^n),$$

and, for $w \in A_\infty(\mathbb{R}^n)$,

$$\tau_w = \inf\{\tau \geq 1 : w \in A_\tau(\mathbb{R}^n)\}.$$

We denote the Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ by $\widehat{f}$ or $\mathscr{F} f$. In particular, for $f \in L^1(\mathbb{R}^n)$, we have

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \qquad \forall \xi \in \mathbb{R}^n.$$

For $j \in \mathbb{Z}$, $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f \in \mathcal{S}'(\mathbb{R}^n)$, and $\xi \in \mathbb{R}^n$, define P_j^ϕ such that $\widehat{P_j^\phi f}(\xi) = \widehat{\phi}(2^{-j}\xi)\widehat{f}(\xi)$. If ϕ is supported in an annulus centered at the origin, we use the notation Δ_j^ϕ instead of P_j^ϕ, and if $\widehat{\phi}$ is supported in a ball centered at the origin with $\widehat{\phi}(0) \neq 0$, we use the notation S_j^ϕ instead of P_j^ϕ. For $a \in \mathbb{R}^n$, we indicate translation by a with τ_a; that is, $\tau_a \phi(x) = \phi(x + a)$ for $x \in \mathbb{R}^n$.

2.1 Function Spaces

We now define the Triebel–Lizorkin and Besov spaces we will be working with.

Let $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}(\widehat{\varphi}) \subset \left\{\xi \in \mathbb{R}^n : |\xi| < 2\right\}, \tag{2.4}$$
$$\operatorname{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : \tfrac{1}{2} < |\xi| < 2\}. \tag{2.5}$$

We define $\dot{\mathcal{A}}(\mathbb{R}^n)$ as the class of $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that ψ satisfies (2.5) and

$$\sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1, \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

We denote by $\mathcal{A}(\mathbb{R}^n)$ the class of pairs (φ, ψ) such that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, φ satisfies (2.4), ψ satisfies (2.5), and

$$\widehat{\varphi}(\xi) + \sum_{j \in \mathbb{N}} \widehat{\psi}(2^{-j}\xi) = 1, \qquad \forall \xi \in \mathbb{R}^n.$$

Let X be a QBFS, $0 < r \leq \infty$, and $s \in \mathbb{R}$. For $\psi \in \dot{\mathcal{A}}(\mathbb{R}^n)$, the *homogeneous Triebel–Lizorkin space* $\dot{F}^s_{X,r}$ is the collection of all $f \in \mathcal{S}'_0(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{X,r}} = \left\| \left(\sum_{j \in \mathbb{Z}} \left(2^{sj} |\Delta_j^\psi f| \right)^r \right)^{\frac{1}{r}} \right\|_X < \infty,$$

and the *homogeneous Besov space* $\dot{B}^s_{X,r}$ is the collection of $f \in \mathcal{S}'_0(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{X,r}} = \left(\sum_{j \in \mathbb{Z}} \left(2^{sj} \left\| \Delta^{\psi}_j f \right\|_X \right)^r \right)^{\frac{1}{r}} < \infty.$$

For $(\varphi, \psi) \in \mathcal{A}(\mathbb{R}^n)$, the *inhomogeneous Triebel–Lizorkin space* $F^s_{X,r}$ is the class of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^s_{X,r}} = \left\| S^{\varphi}_0 f \right\|_X + \left\| \left(\sum_{j \in \mathbb{N}} \left(2^{js} \left| \Delta^{\psi}_j f \right| \right)^r \right)^{\frac{1}{r}} \right\|_X < \infty,$$

and the *inhomogeneous Besov space* $B^s_{X,r}$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{X,r}} = \left\| S^{\varphi}_0 f \right\|_X + \left(\sum_{j \in \mathbb{N}} \left(2^{js} \left\| \Delta^{\psi}_j f \right\|_X \right)^r \right)^{\frac{1}{r}} < \infty.$$

In all four definitions, in the case that $r = \infty$, the summation in j is replaced with the supremum in j.

If $h_{X,r} > 0$ for the Triebel–Lizorkin space and $h_X > 0$ for the Besov space, these definitions are independent of φ and ψ; this follows from an application of Lemma 3.5 (see Sect. 3). Moreover, the following *lifting property* holds:

$$\|f\|_{\dot{F}^s_{X,r}} \sim \left\| D^s f \right\|_{\dot{F}^0_{X,r}} \qquad \text{and} \qquad \|f\|_{F^s_{X,r}} \sim \left\| J^s f \right\|_{F^0_{X,r}}, \tag{2.6}$$

with a corresponding statement in the Besov setting. The proofs of these facts are similar to those for analogous results in the classical Triebel–Lizorkin and Besov spaces based on Lebesgue spaces (see [57, Sects. 2.3.8 and 5.2.3]).

We will also consider the following properties for a QBFS X, with $s \in \mathbb{R}$ and $0 < r \le \infty$:

P7. $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{X,r} \hookrightarrow \mathcal{S}'_0(\mathbb{R}^n)$ and $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{X,r} \hookrightarrow \mathcal{S}'_0(\mathbb{R}^n)$;
P8. $\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_{X,r} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{X,r} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$;
P9. $\dot{B}^s_{X,r}$ and $B^s_{X,r}$ are complete.

Remark 2.1 The completeness of $\dot{F}^s_{X,r}$ and $F^s_{X,r}$ follows from the continuous inclusions $\dot{F}^s_{X,r} \hookrightarrow \mathcal{S}'_0(\mathbb{R}^n)$ and $F^s_{X,r} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, respectively. The same is true for $\dot{B}^s_{X,r}$ and $B^s_{X,r}$ if $X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. All these inclusions hold true if $(1 + |x|)^{-N} \in X'$ for some $N > 0$. Moreover, the inclusions $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{X,r}$ and $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{X,r}$, their inhomogeneous counterparts, and the inclusion $\mathcal{S}(\mathbb{R}^n) \hookrightarrow X$ hold if $(1 + |x|)^{-N} \in X$ for some $N > 0$. These claims can be proved using arguments similar to those used for corresponding results in the classical setting (see [57]; see also Liang et al. [39]).

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x)dx \neq 0$. Given a QBFS X, the *Hardy space* H^X is defined as the class of tempered distributions such that

$$\|f\|_{H^X} = \left\| \sup_{0<t<\infty} \left| t^{-n}\phi(t^{-1}\cdot) * f \right| \right\|_X < \infty,$$

while the *local Hardy space* h^X is given by the collection of tempered distributions such that

$$\|f\|_{h^X} = \left\| \sup_{0<t<1} \left| t^{-n}\phi(t^{-1}\cdot) * f \right| \right\|_X < \infty.$$

Note that we have

$$\|f\|_X \leq \|f\|_{h^X} \leq \|f\|_{H^X}, \qquad \forall f \in \mathcal{S}(\mathbb{R}^n), \tag{2.7}$$

due to property P4 of X and the fact that

$$|f(x)| \leq \sup_{0<t<1} \left| t^{-n}(\phi(t^{-1}\cdot) * f)(x) \right| \leq \sup_{0<t<\infty} \left| t^{-n}(\phi(t^{-1}\cdot) * f)(x) \right|. \tag{2.8}$$

We remark that if X is a BFS over $(\mathbb{R}^n, dx)$ such that the Hardy-Littlewood maximal operator is bounded on X', then for $1 < p < \infty$

$$\dot{F}^0_{X^p,2} = F^0_{X^p,2} = H^{X^p} = h^{X^p} = X^p, \tag{2.9}$$

with equivalent norms (see Appendix B for further details).

3 Fractional Leibniz Rules in Quasi-Banach Function Spaces

We next discuss fractional Leibniz rules in the setting of Triebel–Lizorkin and Besov spaces based on QBFSs.

The main result of this section is the following theorem.

Theorem 3.1 *Let $m \in \mathbb{R}$, $0 < r \leq \infty$, $0 < p, p_1, p_2 < \infty$, and $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman–Meyer multiplier of order m. Suppose X, X_1, and X_2 are QBFSs over $(\mathbb{R}^n, \mu)$, $(\mathbb{R}^n, \mu_1)$, and $(\mathbb{R}^n, \mu_2)$, respectively, such that $p(X), p(X_1), p(X_2) < \infty$, properties P7, P8, and P9 are satisfied by X^p with r as given and s as below, and the following Hölder's inequality holds true:*

$$\|fg\|_{X^p} \lesssim \|f\|_{X_1^{p_1}} \|g\|_{X_2^{p_2}}, \qquad \forall f \in X_1^{p_1},\ g \in X_2^{p_2}. \tag{3.1}$$

i) If $h_{X^p,r}, h_{X_1^{p_1},r}, h_{X_2^{p_2},r} > 0$ and $s > \tau_r(X^p)$, then

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{X^p,r}} \lesssim \|f\|_{\dot{F}^{s+m}_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}} + \|f\|_{H^{X_1^{p_1}}} \|g\|_{\dot{F}^{s+m}_{X_2^{p_2},r}}. \tag{3.2}$$

ii) If $h_{X^p}, h_{X_1^{p_1}}, h_{X_2^{p_2}} > 0$ and $s > \tau(X^p)$, then

$$\|T_\sigma(f,g)\|_{\dot B^s_{X^p,r}} \lesssim \|f\|_{\dot B^{s+m}_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}} + \|f\|_{H^{X_1^{p_1}}} \|g\|_{\dot B^{s+m}_{X_2^{p_2},r}}. \tag{3.3}$$

Moreover, if $h_{X^p,r} > 0$ and $s > \tau_r(X^p)$,

$$\|T_\sigma(f,g)\|_{\dot F^s_{X^p,r}} \lesssim \|f\|_{\dot F^{s+m}_{X^p,r}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot F^{s+m}_{X^p,r}}, \tag{3.4}$$

with an analogous statement for the Besov spaces if $h_X > 0$ and $s > \tau(X^p)$.

We note that applying the lifting property (see (2.6)), (3.2) and (3.3) can be respectively written as

$$\left\|D^s(T_\sigma(f,g))\right\|_{\dot F^0_{X^p,r}} \lesssim \left\|D^s f\right\|_{\dot F^m_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}} + \|f\|_{H^{X_1^{p_1}}} \left\|D^s g\right\|_{\dot F^m_{X_2^{p_2},r}}, \tag{3.5}$$

$$\left\|D^s(T_\sigma(f,g))\right\|_{\dot B^0_{X^p,r}} \lesssim \left\|D^s f\right\|_{\dot B^m_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}} + \|f\|_{H^{X_1^{p_1}}} \left\|D^s g\right\|_{\dot B^m_{X_2^{p_2},r}}. \tag{3.6}$$

Analogous estimates hold for (3.4) and its Besov counterpart.

In view of (2.9), if X, X_1, and X_2 are BFSs over $(\mathbb{R}^n, dx)$ such that the Hardy-Littlewood maximal operator is bounded on X', X_1', and X_2', (3.5) and (3.4) with $r = 2$ can be written for symbols of order zero as

$$\left\|D^s(T_\sigma(f,g))\right\|_{X^p} \lesssim \left\|D^s f\right\|_{X_1^{p_1}} \|g\|_{X_2^{p_2}} + \|f\|_{X_1^{p_1}} \left\|D^s g\right\|_{X_2^{p_2}}, \tag{3.7}$$

$$\left\|D^s(T_\sigma(f,g))\right\|_{X^p} \lesssim \left\|D^s f\right\|_{X^p} \|g\|_{L^\infty} + \|f\|_{L^\infty} \left\|D^s g\right\|_{X^p}, \tag{3.8}$$

for $1 < p, p_1, p_2 < \infty$. In the particular case when $\sigma \equiv 1$, (3.7) and (3.8) give the following fractional Leibniz rules:

$$\left\|D^s(fg)\right\|_{X^p} \lesssim \left\|D^s f\right\|_{X_1^{p_1}} \|g\|_{X_2^{p_2}} + \|f\|_{X_1^{p_1}} \left\|D^s g\right\|_{X_2^{p_2}}, \tag{3.9}$$

$$\left\|D^s(fg)\right\|_{X^p} \lesssim \left\|D^s f\right\|_{X^p} \|g\|_{L^\infty} + \|f\|_{L^\infty} \left\|D^s g\right\|_{X^p}. \tag{3.10}$$

Moreover, a version of Theorem 3.1 along with the corresponding estimates (3.5)–(3.10) hold in the inhomogeneous setting with an inhomogeneous Coifman–Meyer multiplier and J^s instead of D^s.

Remark 3.2 In view of (2.1), if X is a QBFS over $(\mathbb{R}^n, \mu)$ such that $p(X) < \infty$ and properties P7, P8, and P9 are satisfied for X^p, then Theorem 3.1 holds true with $X_1 = X_2 = X$ and $1/p = 1/p_1 + 1/p_2$ if the assumptions in Items i) and ii) are satisfied.

Remark 3.3 We note that the proof of Theorem 3.1 shows that different pairs of p_1 and p_2 and X_1 and X_2 can be used on the right hand side of (3.2) and (3.3) as long as the corresponding Hölder inequality (3.1) holds for both pairs.

3.1 Proof of Theorem 3.1

We now prove Theorem 3.1; the proof of the corresponding result for the inhomogeneous case is similar. The proof follows ideas contained in [48], with modifications to extend the logic to the more general QBFS setting.

We need two supporting results to prove Theorem 3.1. First, we have Nikol'skiĭ representations for the QBFS-based Triebel–Lizorkin and Besov spaces.

Theorem 3.4 (Nikol'skiĭ representations) *For $D > 0$, let $\{u_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$ be such that*

$$supp(\widehat{u}_j) \subset B(0, D2^j), \qquad j \in \mathbb{Z}.$$

Suppose X is a QBFS over $(\mathbb{R}^n, \mu)$ that satisfies properties P7, P8, and P9 for r and s as given below.

i) Let $0 < r \le \infty$. If $h_{X,r} > 0$, $s > \tau_r(X)$, and $\left\|\left(\sum_{j\in\mathbb{Z}} \left|2^{js}u_j\right|^r\right)^{\frac{1}{r}}\right\|_X < \infty$, then the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\mathcal{S}'_0(\mathbb{R}^n)$ to an element in $\dot{F}^s_{X,r}$ and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot{F}^s_{X,r}} \lesssim \left\|\left(\sum_{j\in\mathbb{Z}} \left|2^{js}u_j\right|^r\right)^{\frac{1}{r}}\right\|_X, \tag{3.11}$$

where the implicit constant depends only on n, D, s, r, X, and the function ψ used in the definition of $\dot{F}^s_{X,r}$. An analogous statement with $j \in \mathbb{N}_0$ holds true for $F^s_{X,r}$ (where convergence is in $\mathcal{S}'(\mathbb{R}^n)$).

ii) Let $0 < r \le \infty$. If $h_X > 0$, $s > \tau(X)$, and $\left(\sum_{j\in\mathbb{Z}} \left\|2^{js}u_j\right\|_X^r\right)^{\frac{1}{r}} < \infty$, then the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\dot{B}^s_{X,r}$ (in $\mathcal{S}'_0(\mathbb{R}^n)$ if $r = \infty$) and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot{B}^s_{X,r}} \lesssim \left(\sum_{j\in\mathbb{Z}} \left\|2^{js}u_j\right\|_X^r\right)^{\frac{1}{r}}, \tag{3.12}$$

where the implicit constant depends only on n, D, s, r, X, and the function ψ used in the definition of $\dot{B}^s_{X,r}$. An analogous statement with $j \in \mathbb{N}_0$ holds true for $B^s_{X,q}$ (when $r = \infty$, the convergence is in $\mathcal{S}'(\mathbb{R}^n)$).

We remark that if a dominated convergence theorem holds in X, then the convergence in Item i) is in $\dot{F}^s_{X,r}$ and $F^s_{X,r}$ when $0 < r < \infty$. The proof of Theorem 3.4 follows the same ideas as those for the weighted Lebesgue spaces (see [48, Theorem 3.2]) with some modifications due to the fact that a dominated convergence theorem may not hold in X. For completeness, we include the proof of Theorem 3.4 in Appendix D.

We will also need the following lemma from [48].

Lemma 3.5 (Lemma 3.1 from [48]) *Let $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\phi}_1$ and $\widehat{\phi}_2$ have compact supports and $\widehat{\phi}_1\widehat{\phi}_2 = \widehat{\phi}_1$. If $0 < h \le 1$ and $\varepsilon > 0$, it holds that*

$$\left|P_j^{\tau_a\phi_1} f(x)\right| \lesssim (1+|a|)^{\varepsilon+n/h}\mathcal{M}_h\left(P_j^{\phi_2} f\right)(x), \qquad \forall x, a \in \mathbb{R}^n,\ j \in \mathbb{Z},\ f \in \mathcal{S}'(\mathbb{R}^n).$$

We are now ready to show Theorem 3.1.

Proof of Theorem 3.1 As in [48], we begin with a decomposition of T_σ due to the work of Coifman and Meyer in [14].

Fix $\Psi \in \dot{\mathcal{A}}(\mathbb{R}^n)$ and let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\widehat{\Phi}(0) = 1, \qquad \widehat{\Phi}(\xi) = \sum_{j\le 0}\widehat{\Psi}(2^{-j}\xi), \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given N sufficiently large, we write $T_\sigma = T_\sigma^1 + T_\sigma^2$ where, for $f, g \in \mathcal{S}_0(\mathbb{R}^n)$,

$$T_\sigma^1(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\sum_{j\in\mathbb{Z}} C_j(a,b)(\Delta_j^{\tau_a\Psi} f)(x)(S_j^{\tau_b\Phi} g)(x). \tag{3.13}$$

The coefficients $C_j(a,b)$ are such that

$$|C_j(a,b)| \lesssim 2^{jm}, \qquad \forall a, b \in \mathbb{Z}^n, j \in \mathbb{Z},$$

with implicit constant depending on σ. A formula analogous to (3.13) holds for T_σ^2 with the roles of f and g interchanged.

It suffices to work with T_σ^1 and show that

$$\left\|T_\sigma^1(f,g)(x)\right\|_{\dot{F}^s_{X^p,r}} \lesssim \|f\|_{\dot{F}^{s+m}_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}}$$

and

$$\left\|T_\sigma^1(f,g)(x)\right\|_{\dot{B}^s_{X^p,r}} \lesssim \|f\|_{\dot{B}^{s+m}_{X_1^{p_1},r}} \|g\|_{H^{X_2^{p_2}}},$$

with corresponding estimates for (3.4) and its Besov counterpart. Moreover, since it holds that $\left\|\sum_{j\in\mathbb{Z}} f_j\right\|_{\dot{F}^s_{X^p,r}}^{p/p_0} \le \sum_{j\in\mathbb{Z}}\|f_j\|_{\dot{F}^s_{X^p,q}}^{p/p_0}$ where $p_0 > \max(p(X), p, p/r)$ (similarly for $\dot{B}^s_{X^p,r}$), it suffices to prove that given $\varepsilon > 0$, there exist $0 < h_1, h_2 \le 1$ such that for any $f, g \in \mathcal{S}_0(\mathbb{R}^n)$, it holds that

$$\left\|T^{a,b}(f,g)\right\|_{\dot{F}^s_{X^p,r}} \lesssim (1+|a|)^{\varepsilon+n/h_1}(1+|b|)^{\varepsilon+n/h_2}\|f\|_{\dot{F}^{s+m}_{X_1^{p_1},r}}\|g\|_{H^{X_2^{p_2}}}, \tag{3.14}$$

$$\left\|T^{a,b}(f,g)\right\|_{\dot{B}^s_{X^p,r}} \lesssim (1+|a|)^{\varepsilon+n/h_1}(1+|b|)^{\varepsilon+n/h_2}\|f\|_{\dot{B}^{s+m}_{X_1^{p_1},r}}\|g\|_{H^{X_2^{p_2}}}, \tag{3.15}$$

where

$$T^{a,b}(f,g) = \sum_{j\in\mathbb{Z}} C_j(a,b)\left(\Delta_j^{\tau_a\Psi} f\right)\left(S_j^{\tau_b\Phi} g\right)$$

and the implicit constants are independent of a and b. Corresponding estimates to (3.14) and (3.15) suffice for (3.4) and its Besov counterpart.

Assume that r is finite; the usual changes apply when $r=\infty$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\varphi}$ has compact support and $\widehat{\varphi} \equiv 1$ on $\operatorname{supp}(\widehat{\Phi})$. Let $\Theta \in \dot{\mathcal{A}}(\mathbb{R}^n)$ and define ψ such that $\widehat{\psi} = \sum_{\ell=-1}^{1} \widehat{\Theta}(2^{-\ell}\xi)$; then $\widehat{\psi} \equiv 1$ on $\operatorname{supp}(\widehat{\Psi})$.

Due to the supports of Ψ and Φ, we have that

$$\operatorname{supp}\left(\mathscr{F}\left[C_j(a,b)\left(\Delta_j^{\tau_a\Psi} f\right)\left(S_j^{\tau_b\Phi} g\right)\right]\right) \subset \left\{\xi \in \mathbb{R}^n : |\xi| \lesssim 2^j\right\},$$

for all $j \in \mathbb{Z}$ and $a, b \in \mathbb{Z}^n$.

We start with the proof of (3.14). Applying Theorem 3.4, the bound on the coefficients $C_j(a,b)$, and (3.1), we obtain

$$\begin{aligned}
\left\|T^{a,b}(f,g)\right\|_{\dot{F}^s_{X^p,r}} &\lesssim \left\|\left(\sum_{j\in\mathbb{Z}}\left[2^{sj} C_j(a,b)\left(\Delta_j^{\tau_a\Psi} f\right)\left(S_j^{\tau_b\Phi} g\right)\right]^r\right)^{\frac{1}{r}}\right\|_{X^p} \\
&\lesssim \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr}\left|\left(\Delta_j^{\tau_a\Psi} f\right)\left(S_j^{\tau_b\Phi} g\right)\right|^r\right)^{\frac{1}{r}}\right\|_{X^p} \\
&\lesssim \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr}\left|\Delta_j^{\tau_a\Psi} f\right|^r\right)^{\frac{1}{r}}\right\|_{X_1^{p_1}} \left\|\sup_{j\in\mathbb{Z}}\left|S_j^{\tau_b\Phi} g\right|\right\|_{X_2^{p_2}}.
\end{aligned} \tag{3.16}$$

Let $0 < h_1 < \min(h_{X_1^{p_1},r}, 1)$. By Lemma 3.5 and the Fefferman–Stein inequality, we have that

$$\begin{aligned}
\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr}\left|\Delta_j^{\tau_a\Psi} f\right|^r\right)^{\frac{1}{r}}\right\|_{X_1^{p_1}} &\lesssim (1+|a|)^{\varepsilon+n/h_1}\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr}\left|\mathcal{M}_{h_1}\left(\Delta_j^{\psi} f\right)\right|^r\right)^{\frac{1}{r}}\right\|_{X_1^{p_1}} \\
&\lesssim (1+|a|)^{\varepsilon+n/h_1}\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr}\left|\Delta_j^{\psi} f\right|^r\right)^{\frac{1}{r}}\right\|_{X_1^{p_1}} \\
&\sim (1+|a|)^{\varepsilon+n/h_1}\|f\|_{\dot{F}^{s+m}_{X_1^{p_1},r}},
\end{aligned}$$

where the implicit constants are independent of a and f.

Now let $0 < h_2 < \min(h_{X_2^{p_2}}, 1)$. Applying Lemma 3.5 and the boundedness of $\mathcal{M}_{h_2}$ on $X_2^{p_2}$, we have

$$\begin{aligned}\left\| \sup_{j\in\mathbb{Z}} \left| S_j^{\tau_b\Phi} g \right| \right\|_{X_2^{p_2}} &\lesssim (1+|b|)^{\varepsilon+n/h_2} \left\| \mathcal{M}_{h_2}\left(\sup_{j\in\mathbb{Z}} \left| S_j^{\varphi} g \right| \right) \right\|_{X_2^{p_2}} \\ &\lesssim (1+|b|)^{\varepsilon+n/h_2} \left\| \sup_{j\in\mathbb{Z}} \left| S_j^{\varphi} g \right| \right\|_{X_2^{p_2}} \\ &\sim (1+|b|)^{\varepsilon+n/h_2} \|g\|_{H^{X_2^{p_2}}}, \end{aligned} \tag{3.17}$$

where the constants are independent of b and g. All together, this gives (3.14).

For (3.15), again applying Theorem 3.4, the bound on $|C_j(a,b)|$ and (3.1), we have

$$\begin{aligned}\left\| T^{a,b}(f,g) \right\|_{\dot{B}^s_{X^p,r}} &\lesssim \left(\sum_{j\in\mathbb{Z}} \left\| 2^{sj} C_j(a,b) \left(\Delta_j^{\tau_a\Psi} f\right) \left(S_j^{\tau_b\Phi} g\right) \right\|_{X^p}^r \right)^{\frac{1}{r}} \\ &\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr} \left\| \left(\Delta_j^{\tau_a\Psi} f\right) \left(S_j^{\tau_b\Phi} g\right) \right\|_{X^p}^r \right)^{\frac{1}{r}} \\ &\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr} \left\| \left(\Delta_j^{\tau_a\Psi} f\right) \right\|_{X_1^{p_1}}^r \right)^{\frac{1}{r}} \left\| \sup_{j\in\mathbb{Z}} \left| S_j^{\tau_b\Phi} g \right| \right\|_{X_2^{p_2}}. \end{aligned} \tag{3.18}$$

Setting $0 < h_1 < \min(h_{X_1^{p_1}}, 1)$ and applying Lemma 3.5 and the boundedness of $\mathcal{M}_{h_1}$ on $X_1^{p_1}$, we have

$$\begin{aligned}\left\| \Delta_j^{\tau_a\Psi} f \right\|_{X_1^{p_1}} &\lesssim (1+|a|)^{\varepsilon+n/h_1} \left\| \mathcal{M}_{h_1}\left(\Delta_j^{\psi} f\right) \right\|_{X_1^{p_1}} \\ &\lesssim (1+|a|)^{\varepsilon+n/h_1} \left\| \Delta_j^{\psi} f \right\|_{X_1^{p_1}}, \end{aligned}$$

where the implicit constant is independent of a and f.

Therefore,

$$\begin{aligned}\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr} \left\| \Delta_j^{\tau_a\Psi} f \right\|_{X_1^{p_1}}^r \right)^{\frac{1}{r}} &\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)jr} (1+|a|)^{(\varepsilon+n/h_1)r} \left\| \Delta_j^{\psi} f \right\|_{X_1^{p_1}}^r \right)^{\frac{1}{r}} \\ &\sim (1+|a|)^{\varepsilon+n/h_1} \|f\|_{\dot{B}^{s+m}_{X_1^{p_1},r}}. \end{aligned}$$

The factor $\left\| \sup_{j\in\mathbb{Z}} \left| S_j^{\tau_b\phi} g \right| \right\|_{X_2^{p_2}}$ is treated as in (3.17).

This gives the desired inequality (3.15).

For (3.4) and its Besov counterpart, we proceed as in (3.16) and (3.18) with X^p instead of $X_1^{p_1}$ and $\sup_{j\in\mathbb{Z}} \left\| S_j^{\tau_b\phi} g \right\|_{L^\infty}$ instead of $\left\| \sup_{j\in\mathbb{Z}} \left| S_j^{\tau_b\phi} g \right| \right\|_{X_2^{p_2}}$. □

4 Fractional Leibniz Rules in Rearrangement Invariant Quasi-Banach Function Spaces

We turn our attention to a specific class of QBFSs, those that are rearrangement invariant. Working within rearrangement invariant quasi-Banach function spaces (r.i.QBFS), we invoke extrapolation to deduce necessary tools such as the Fefferman–Stein inequality and equivalences between norms to obtain fractional Leibniz rules within these spaces. We first discuss some definitions and relationships, then state the Leibniz rules in this setting. We also include examples of applications in weighted Lebesgue spaces, Orlicz spaces, classical Lorentz spaces, and general Lorentz spaces.

4.1 Preliminaries

We begin with some background on rearrangement invariant quasi-Banach function spaces; for further details, we refer the reader to [4] and [19].

Let $(\mathbb{R}^n, \mu)$ be a measure space as in Sect. 2 and such that $\mu(\mathbb{R}^n) = \infty$. The *distribution function* μ_f of a measurable function f on $\mathbb{R}^n$ is given by

$$\mu_f(\lambda) = \mu\left(\left\{x \in \mathbb{R}^n : |f(x)| > \lambda\right\}\right).$$

For a measurable function f in $(\mathbb{R}^n, \mu)$ and a measurable function g in $(\mathbb{R}^d, \nu)$, we say that f and g are *equimeasurable* if $\mu_f = \nu_g$. A BFS X over $(\mathbb{R}^n, \mu)$ is said to be *rearrangement invariant* if $\|f\|_X = \|g\|_X$ whenever f and g in X are equimeasurable.

The *decreasing rearrangement of* f is the function f_μ^* on $[0, \infty)$ given by

$$f_\mu^*(t) = \inf\left\{\lambda \geq 0 : \mu_f(\lambda) \leq t\right\}.$$

Note that f_μ^* is equimeasurable with f. If X is a r.i.BFS, this yields a representation of X over the measure space $(\mathbb{R}^+, dt)$. Indeed, by the Luxemburg representation theorem (see [4]), there exists a r.i.BFS $\overline{X}$ over $(\mathbb{R}^+, dt)$ such that $f \in X$ if, and only if, $f_\mu^* \in \overline{X}$, and $\|f\|_X = \left\| f_\mu^* \right\|_{\overline{X}}$.

We use the Luxemburg representation theorem to define the Boyd indices of a r.i.BFS X. For $f \in \overline{X}$, the dilation operator D_t, $0 < t < \infty$, is given by $D_t f(x) = f(x/t)$, and we let

$$a_X(t) = \|D_t\|_{B(\overline{X})},$$

where $\|D_t\|_{B(\overline{X})}$ denotes the norm of the operator D_t. The lower and upper Boyd indices are respectively given by

$$p_X = \lim_{t\to\infty} \frac{\log t}{\log a_X(t)} = \sup_{1<t<\infty} \frac{\log t}{\log a_X(t)},$$
$$q_X = \lim_{t\to 0^+} \frac{\log t}{\log a_X(t)} = \inf_{0<t<1} \frac{\log t}{\log a_X(t)}.$$

It holds that $1 \le p_X \le q_X \le \infty$, $p_{X'} = (q_X)'$, and $q_{X'} = (p_X)'$.

We next introduce weighted versions of a r.i.BFS X over the measure space $(\mathbb{R}^n, dx)$. Given $w \in A_\infty(\mathbb{R}^n)$, define

$$X(w) = \left\{ f \in M : \|f_w^*\|_{\overline{X}} < \infty \right\},$$

with norm $\|f\|_X = \left\|f_w^*\right\|_{\overline{X}}$. We note that $X(w)$ is a r.i.BFS over $(\mathbb{R}^n, w(x)dx)$ and we have $(X(w))' = X'(w)$.

The above definitions can be extended to a r.i.QBFS X with $p(X) < \infty$. In this setting, $0 < p_X \le q_X \le \infty$, and if $0 < r < \infty$, then $p_{X^r} = rp_X$, $q_{X^r} = rq_X$, and $(X(w))^r = X^r(w)$ for $w \in A_\infty(\mathbb{R}^n)$.

For $0 < r \le \infty$, $s \in \mathbb{R}$, and $X(w)$, we denote the corresponding homogeneous and inhomogeneous Triebel–Lizorkin spaces as $\dot{F}^s_{X,r}(w)$ and $F^s_{X,r}(w)$, respectively; an analogous notation applies to the Besov setting. Finally, the weighted Hardy space will be denoted by $H^X(w)$ and the weighted local Hardy space will be denoted by $h^X(w)$.

Given $w \in A_\infty(\mathbb{R}^n)$ and a r.i.QBFS X over $(\mathbb{R}^n, dx)$ with finite Boyd indices and $p(X) < \infty$, we have

$$H^X(w) = \dot{F}^0_{X,2}(w) \quad \text{and} \quad h^X(w) = F^0_{X,2}(w), \tag{4.1}$$

with equivalent quasi-norms. Also, if X is a r.i.BFS with Boyd indices $1 < p_X \le q_X < \infty$ and $w \in A_{p_X}(\mathbb{R}^n)$, then

$$\dot{F}^0_{X,2}(w) = X(w) \quad \text{and} \quad F^0_{X,2}(w) = X(w), \tag{4.2}$$

with equivalent norms. See Appendix B for further details.

Regarding the Fefferman–Stein inequality, if X is r.i.QBFS over $(\mathbb{R}^n, dx)$ with $0 < p_X \le q_X < \infty$ and $p(X) < \infty$, $0 < r \le \infty$, $w \in A_\infty(\mathbb{R}^n)$, and $0 < h < \min(p_X/\tau_w, 1/p(X), r)$, we have

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left|\mathcal{M}_h(f_j)\right|^r \right)^{\frac{1}{r}} \right\|_{X(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \left|f_j\right|^r \right)^{\frac{1}{r}} \right\|_{X(w)}, \tag{4.3}$$

with the summation in j replaced by the supremum in j if $r = \infty$. This also gives that for $0 < h < \min(p_X/\tau_w, 1/p(X))$, we have

$$\|\mathcal{M}_h(f)\|_{X(w)} \lesssim \|f\|_{X(w)}.$$

See Appendix C for further details. Note that the results above imply that

$$h_{X(w)} \geq \min(p_X/\tau_w, 1/p(X)) \tag{4.4}$$

and

$$h_{X(w),r} \geq \min(p_X/\tau_w, 1/p(X), r), \tag{4.5}$$

which also gives

$$\tau(X(w)) \leq n\left(\frac{1}{\min(p_X/\tau_w, 1/p(X))} - 1\right) \tag{4.6}$$

and

$$\tau_r(X(w)) \leq n\left(\frac{1}{\min(p_X/\tau_w, 1/p(X), r)} - 1\right). \tag{4.7}$$

We note that [4, p. 77, Theorem 6.6] gives that if X is a r.i.BFS, then

$$L^1(w) \cap L^\infty \hookrightarrow X(w) \hookrightarrow L^1(w) + L^\infty. \tag{4.8}$$

Therefore, if X is a r.i.QBFS with $p(X) < \infty$ and $p > p(X)$, (4.8) implies that $(1 + |x|)^{-N} \in X^p(w)$ for some $N > 0$; the same holds for $(X^p(w))'$ since $(X^p(w))' = (X^p)'(w)$. As a consequence, in view of Remark 2.1, properties P7, P8, and P9 hold for $X^p(w)$ with $0 < r \leq \infty$ and $s \in \mathbb{R}$.

4.2 Leibniz Rules in Rearrangement Invariant Quasi-Banach Function Spaces

We now present our results for fractional Leibniz rules in the r.i.QBFS setting. While we show only the results in the homogeneous case, corresponding results hold as well in the inhomogeneous setting with an inhomogeneous Coifman–Meyer multiplier and the operator J^s.

Corollary 4.1 *Let $m \in \mathbb{R}$, $0 < r \leq \infty$, $0 < p, p_1, p_2 < \infty$, $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman–Meyer multiplier of order m, and $w, w_1, w_2 \in A_\infty(\mathbb{R}^n)$. Suppose X, X_1, and X_2 are r.i.QBFSs over $(\mathbb{R}^n, dx)$ with finite Boyd indices such that $p(X), p(X_1), p(X_2) < \infty$, properties P7, P8, and P9 are satisfied by $X^p(w)$ with r as given and s as below, and the following Hölder inequality holds:*

$$\|fg\|_{X^p(w)} \lesssim \|f\|_{X_1^{p_1}(w_1)} \|g\|_{X_2^{p_2}(w_2)}, \qquad \forall f \in X_1^{p_1}(w_1),\ g \in X_2^{p_2}(w_2).$$

(i) If $s > n\left(\frac{1}{\min(p p_X/\tau_w, 1/p(X^p), r)} - 1\right)$*, then*

$$\|T_\sigma(f,g)\|_{\dot F^s_{X^p,r}(w)} \lesssim \|f\|_{\dot F^{s+m}_{X_1^{p_1},r}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \|g\|_{\dot F^{s+m}_{X_2^{p_2},r}(w_2)}. \tag{4.9}$$

(ii) If $s > n\left(\frac{1}{\min(p p_X/\tau_w, 1/p(X^p))} - 1\right)$*, then*

$$\|T_\sigma(f,g)\|_{\dot B^s_{X^p,r}(w)} \lesssim \|f\|_{\dot B^{s+m}_{X_1^{p_1},r}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \|g\|_{\dot B^{s+m}_{X_2^{p_2},r}(w_2)}. \tag{4.10}$$

Moreover, if $s > n\left(\frac{1}{\min(p p_X/\tau_w, 1/p(X^p), r)} - 1\right)$*,*

$$\|T_\sigma(f,g)\|_{\dot F^s_{X^p,r}(w)} \lesssim \|f\|_{\dot F^{s+m}_{X^p,r}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot F^s_{X^p,r}(w)}, \tag{4.11}$$

with a corresponding estimate for the Besov spaces if $s > n\left(\frac{1}{\min(p p_X/\tau_w, 1/p(X^p))} - 1\right)$*.*

Proof This follows by applying Theorem 3.1 with the r.i.QBFSs $X(w)$, $X_1(w_1)$, and $X_2(w_2)$. Indeed, since $(X(w))^{p_0} = X^{p_0}(w)$, whenever X^{p_0} is a BFS, $(X(w))^{p_0}$ is as well, giving that $p(X(w)) \le p(X) < \infty$; similarly, $p(X_1(w_1)), p(X_2(w_2)) < \infty$. Moreover, (4.5) applied to X^p, $X_1^{p_1}$, and $X_2^{p_2}$ implies that $h_{X^p(w),r}, h_{X_1^{p_1}(w_1),r}, h_{X_2^{p_2}(w_2),r} > 0$, while (4.7) applied to X^p implies $s > \tau_r(X^p(w))$. The argument for (4.10) is similar. □

Applying the lifting property, we obtain the following versions of (4.9) and (4.10):

$$\left\|D^s(T_\sigma(f,g))\right\|_{\dot F^0_{X^p,r}(w)} \lesssim \left\|D^s f\right\|_{\dot F^m_{X_1^{p_1},r}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \left\|D^s g\right\|_{\dot F^m_{X_2^{p_2},r}(w_2)}, \tag{4.12}$$

$$\left\|D^s(T_\sigma(f,g))\right\|_{\dot B^0_{X^p,r}(w)} \lesssim \left\|D^s f\right\|_{\dot B^m_{X_1^{p_1},r}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \left\|D^s g\right\|_{\dot B^m_{X_2^{p_2},r}(w_2)}. \tag{4.13}$$

By (4.1) and (4.12), we obtain the following estimates for symbols of order zero:

$$\left\|D^s(T_\sigma(f,g))\right\|_{H^{X^p}(w)} \lesssim \left\|D^s f\right\|_{H^{X_1^{p_1}}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \left\|D^s g\right\|_{H^{X_2^{p_2}}(w_2)}. \tag{4.14}$$

In particular, for $\sigma \equiv 1$, we have

$$\left\|D^s(fg)\right\|_{H^{X^p}(w)} \lesssim \left\|D^s f\right\|_{H^{X_1^{p_1}}(w_1)} \|g\|_{H^{X_2^{p_2}}(w_2)} + \|f\|_{H^{X_1^{p_1}}(w_1)} \left\|D^s g\right\|_{H^{X_2^{p_2}}(w_2)}. \tag{4.15}$$

Estimates analogous to (4.12)–(4.15) hold for (4.11) and its Besov counterpart.

Finally, we have Leibniz rules in weighted r.i.QBFS as a consequence of (4.14), (4.1), (4.2), and (2.7).

Corollary 4.2 *Let $\sigma(\xi,\eta)$, $\xi,\eta\in\mathbb{R}^n$, be a Coifman–Meyer multiplier of order zero and $w\in A_\infty(\mathbb{R}^n)$. Suppose X, X_1, and X_2 are r.i.QBFSs over $(\mathbb{R}^n,dx)$ with finite Boyd indices, $p(X)$, $p(X_1)$, $p(X_2)<\infty$, and properties P7, P8, and P9 are satisfied by $X^p(w)$ with $r=2$ and s as given below. Assume $0<p<\infty$, $p(X_1)<p_1<\infty$, $p(X_2)<p_2<\infty$, $w_1\in A_{p_1p_{X_1}}(\mathbb{R}^n)$, $w_2\in A_{p_2p_{X_2}}(\mathbb{R}^n)$, and*

$$\|fg\|_{X^p(w)}\lesssim\|f\|_{X_1^{p_1}(w_1)}\|g\|_{X_2^{p_2}(w_2)},\qquad\forall f\in X_1^{p_1}(w_1),\ g\in X_2^{p_2}(w_2).$$

Then if $s>n\left(\frac{1}{\min(pp_X/\tau_w,1/p(X^p))}-1\right)$,

$$\left\|D^s(T_\sigma(f,g))\right\|_{X^p(w)}\lesssim\left\|D^sf\right\|_{X_1^{p_1}(w_1)}\|g\|_{X_2^{p_2}(w_2)}+\|f\|_{X_1^{p_1}(w_1)}\left\|D^sg\right\|_{X_2^{p_2}(w_2)}.$$

In particular,

$$\left\|D^s(fg)\right\|_{X^p(w)}\lesssim\left\|D^sf\right\|_{X_1^{p_1}(w_1)}\|g\|_{X_2^{p_2}(w_2)}+\|f\|_{X_1^{p_1}(w_1)}\left\|D^sg\right\|_{X_2^{p_2}(w_2)}.$$

Moreover, if $p(X)<p<\infty$, $w\in A_{pp_X}(\mathbb{R}^n)$, and $s>n\left(\frac{1}{\min(pp_X/\tau_w,1/p(X^p))}-1\right)$,

$$\left\|D^s(T_\sigma(f,g))\right\|_{X^p(w)}\lesssim\left\|D^sf\right\|_{X^p(w)}\|g\|_{L^\infty}+\|f\|_{L^\infty}\left\|D^sg\right\|_{X^p(w)},\tag{4.16}$$

and in particular,

$$\left\|D^s(fg)\right\|_{X^p(w)}\lesssim\left\|D^sf\right\|_{X^p(w)}\|g\|_{L^\infty}+\|f\|_{L^\infty}\left\|D^sg\right\|_{X^p(w)}.\tag{4.17}$$

Remark 4.3 As a consequence of Remark 3.2, Corollary 4.1 holds in particular if X is r.i.QBFS over $(\mathbb{R}^n,dx)$ with finite Boyd indices, $p(X)<\infty$, $X_1=X_2=X$, $w=w_1=w_2$, $w\in A_\infty(\mathbb{R}^n)$, $0<p,p_1,p_2<\infty$ are such that $1/p=1/p_1+1/p_2$, and $X^p(w)$ satisfies properties P7, P8, and P9 for $0<r\le\infty$ and s as given in the statement of Corollary 4.1.

Remark 4.4 As a consequence of Remark 3.3, different pairs of X_1, X_2 and p_1, p_2 can be used on the right hand side of (4.9) and (4.10).

4.3 Examples

We now give explicit examples of r.i.QBFSs where the results in Sect. 4.2 may be applied.

4.3.1 Weighted Lebesgue Spaces

Corollary 4.1 gives as a particular case the already known fractional Leibniz rules in Triebel–Lizorkin and Besov spaces based on weighted Lebesgue spaces proved in [48], including (1.5) and its weighted version. In this case, we have $X = X_1 = X_2 = L^1(\mathbb{R}^n)$, $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$, $w_1, w_2 \in A_\infty(\mathbb{R}^n)$, and $w = w_1^{p/p_1} w_2^{p/p_2}$. Therefore, $p(X) = p(X_1) = p(X_2) = 1$, $p_X = q_X = 1$, $X^p(w) = L^p(w)$, $X^{p_1}(w_1) = L^{p_1}(w_1)$, and $X^{p_2}(w_2) = L^{p_2}(w_2)$; the lower bounds for s are $n\left(\frac{1}{\min(p/\tau_w, r, 1)} - 1\right)$ in the Triebel–Lizorkin case and $n\left(\frac{1}{\min(p/\tau_w, 1)} - 1\right)$ in the Besov setting.

Corollary 4.2 then gives the fractional Leibniz rules in the weighted Lebesgue spaces for $1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}(\mathbb{R}^n)$, $w_2 \in A_{p_2}(\mathbb{R}^n)$, and $s > n\left(\frac{1}{\min(p/\tau_w, 1)} - 1\right)$ and versions with L^∞ for $p > 1$, $w \in A_p(\mathbb{R}^n)$, and $s > n\left(\frac{1}{\min(p/\tau_w, 1)} - 1\right)$; in particular, we recover the unweighted version (1.1) presented in the introduction.

4.3.2 Classical Weighted Lorentz Spaces

Given $0 < p, q < \infty$, the *classical Lorentz spaces* $L^{p,q}(\mathbb{R}^n)$ are r.i.QBFSs defined through the quasi-norm given by

$$\|f\|_{L^{p,q}} = \left(\int_0^\infty (f^*(s)s^{\frac{1}{p}})^q \frac{ds}{s}\right)^{\frac{1}{q}}, \tag{4.18}$$

where $f^* = f_w^*$ with $w \equiv 1$, extending the scale of Lebesgue spaces since $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

The Boyd indices for $L^{p,q}(\mathbb{R}^n)$ are $p_X = q_X = p$. We note that if $1 \leq p, q < \infty$, $L^{p,q}(\mathbb{R}^n)$ is a r.i.BFS, and since $(L^{p,q}(\mathbb{R}^n))^{p_0} = L^{pp_0,qp_0}(\mathbb{R}^n)$, we have $p(L^{p,q}(\mathbb{R}^n)) = 1/\min(p, q, 1)$. If $X = L^{p,q}(\mathbb{R}^n)$, then $X(w)$ is given by (4.18) by replacing f^* with f_w^*. Corollary 4.1 gives fractional Leibniz rules for Triebel–Lizorkin and Besov spaces based on weighted Lorentz spaces (see also [48]). In this case, we have $0 < p, p_1, p_2, q, q_1, q_2 < \infty$ satisfying $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$, $X = L^{1,q/p}(\mathbb{R}^n)$, $X_1 = L^{1,q_1/p_1}(\mathbb{R}^n)$, $X_2 = L^{1,q_2/p_2}(\mathbb{R}^n)$, and $w = w_1 = w_2$ with $w \in A_\infty(\mathbb{R}^n)$. Therefore, $p_X = q_X = 1$, $X^p(w) = L^{p,q}(w)$, $X_1^{p_1}(w) = L^{p_1,q_1}(w)$, and $X_2^{p_2}(w) = L^{p_2,q_2}(w)$ (see Hunt [31, Theorem 4.5] for Hölder's inequality between these spaces). The lower bound for s is $n\left(\frac{1}{\min(p/\tau_w, q, r, 1)} - 1\right)$ for the Triebel–Lizorkin case and $n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right)$ for the Besov setting.

Corollary 4.2 then gives the following fractional Leibniz rules for weighted Lorentz spaces:

$$\left\|D^s(T_\sigma(f,g))\right\|_{L^{p,q}(w)} \lesssim \left\|D^s f\right\|_{L^{p_1,q_1}(w)} \|g\|_{L^{p_2,q_2}(w)} + \|f\|_{L^{p_1,q_1}(w)} \left\|D^s g\right\|_{L^{p_2,q_2}(w)},$$

with $w \in A_{\min(p_1,p_2)}(\mathbb{R}^n)$, $1 < p_1, p_2, q_1, q_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$, and $s > n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right)$, with corresponding counterparts for (4.16) and (4.17) if $1 < p, q < \infty$ and $w \in A_p(\mathbb{R}^n)$. See also [16, 17] for the case $\sigma \equiv 1$.

4.3.3 Lorentz Λ-Spaces

The *Lorentz* Λ*-spaces* Λ_v^q are defined to be the collection of measurable functions f defined on $\mathbb{R}^n$ such that

$$\|f\|_{\Lambda_v^q} = \left(\int_0^\infty f^*(s)^q v(s) ds\right)^{\frac{1}{q}} < \infty,$$

where $0 < q < \infty$ and v is a weight on $(0, \infty)$ (see Carro et al. [12]).

The classical Lorentz spaces presented in Sect. 4.3.2 are a specific case of the Lorentz-Λ spaces, since $\Lambda_v^q = L^{p,q}(\mathbb{R}^n)$ for $v(s) = s^{q/p-1}$. Choosing $v(s) = s^{q/p-1}(1+\log^+(1/s))^\alpha$, we obtain the Lorentz-Zygmund spaces $\Lambda_v^q = L^{p,q}(\log L)^\alpha$ (see Bennett–Rudnick [3]). Alternatively, if $v(s) = s^{q/p-1}(1+\log^+(1/s))^\alpha(1+\log^+\log^+(1/s))^\beta$, then $\Lambda_v^q = L^{p,q}(\log L)^\alpha(\log\log L)^\beta$ are the generalized Lorentz-Zygmund spaces (see Evans et al. [23]).

As shown in Curbera et al. [20], $X = \Lambda_v^q$ has upper Boyd index $q_X < \infty$ whenever

$$\frac{1}{t}\int_0^t v(x)dx \lesssim v(t), \qquad t > 0.$$

Moreover, if v satisfies

$$\int_t^\infty v(x)x^{-p_0}dx \lesssim \frac{1}{t^{p_0}}\int_0^t v(x)dx, \qquad t > 0,$$

for large enough p_0, $(\Lambda_v^q)^{p_0}$ is a Banach space, so $p(\Lambda_v^q) < \infty$ (see Sawyer [55] and [12]).

4.3.4 Orlicz Spaces

Let ϕ be a Young function; that is, $\phi : [0,\infty) \to [0,\infty)$ is continuous, convex, strictly increasing, and

$$\lim_{t\to 0^+}\frac{\phi(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t\to\infty}\frac{\phi(t)}{t} = \infty.$$

The *Orlicz space* L^ϕ is the collection of measurable functions f defined on $\mathbb{R}^n$ such that

$$\|f\|_{L^\phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n}\phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\} < \infty. \tag{4.19}$$

It can be shown that Orlicz spaces are r.i.BFSs. For $X = L^{\phi}$ and w a weight in $\mathbb{R}^n$, $X(w)$ is given by replacing dx with $w(x)dx$ in (4.19) (see [4, 19, 20]).

In the case that $\phi(x) = x^p$, $1 < p < \infty$, we obtain the Lebesgue space $L^p(\mathbb{R}^n)$. Also, the Zygmund spaces $L^p(\log L)^{\alpha}$ for $1 < p < \infty$ and $\alpha \in \mathbb{R}$, a particular case of the Lorentz–Zygmund spaces of Sect. 4.3.3, result when $\phi(t) = t^p(1 + \log^+ t)^{\alpha}$. These spaces have Boyd indices $p_X = q_X = p$, and $(L^p(\log L)^{\alpha})^{p_0} = L^{pp_0}(\log L)^{\alpha}$. Other examples of Orlicz spaces include $L^p + L^q$ and $L^p \cap L^q$, which are associated with $\phi(t) \sim \max(t^p, t^q)$ and $\phi(t) = \min(t^p, t^q)$, respectively, and have Boyd indices $p_X = \min(p, q)$ and $q_X = \max(p, q)$.

5 Fractional Leibniz Rules in Weighted Mixed Lebesgue Spaces

There are also many applications of Theorem 3.1 in QBFSs that are not rearrangement invariant. In this section, we obtain fractional Leibniz rules in Triebel–Lizorkin and Besov spaces based on weighted mixed Lebesgue spaces as corollaries of Theorem 3.1 and show that particular cases of these estimates include fractional Leibniz rules in weighted mixed Lebesgue spaces. We then analyze these results for spaces with power weights.

5.1 Preliminaries

Let $n = n_1 + n_2, n_1, n_2 \in \mathbb{N}$, and $x = (x_1, x_2) \in \mathbb{R}^n$ with $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. For $0 < p, q < \infty$ and a weight w on $\mathbb{R}^n$, we define the *weighted mixed Lebesgue space* $L^p(L^q(w))$ to be the collection of all measurable functions f defined on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(L^q(w))} = \left(\int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} |f(x_1, x_2)|^q w(x_1, x_2) dx_2\right)^{\frac{p}{q}} dx_1\right)^{\frac{1}{p}} < \infty.$$

Note that $L^p(L^p(w)) = L^p(w)$.

In this setting, we consider a 'mixed' version of the A_p condition, which we denote $A_p(A_q)$. Following the work of Kurtz in [38], we define

$$[w]_{A_p(A_q)} = \sup_{Q_1, Q_2} \left(\fint_{Q_1} \left(\fint_{Q_2} w(x_1, x_2) dx_2\right)^{\frac{p}{q}} dx_1\right) \left(\fint_{Q_1} \left(\fint_{Q_2} w(x_1, x_2)^{1-q'} dx_2\right)^{\frac{p'}{q'}} dx_1\right)^{p-1},$$

where the supremum is taken over all cubes $Q_1 \subset \mathbb{R}^{n_1}$ and $Q_2 \subset \mathbb{R}^{n_2}$.

The collection of weights $A_p(A_q)$ is given by

$$A_p(A_q) = \{w : w \text{ is a weight on } \mathbb{R}^n \text{ and } [w]_{A_p(A_q)} < \infty\}.$$

We note the following relationship between product weights in $A_p(A_q)$ and the traditional Muckenhoupt classes:

Lemma 5.1 (Lemma 3 from [38]) *The weight* $w(x_1, x_2) = u(x_1)v(x_2)$ *is in* $A_p(A_q)$ *if, and only if,* $u^{p/q} \in A_p(\mathbb{R}^{n_1})$ *and* $v \in A_q(\mathbb{R}^{n_2})$. *Moreover,* $[u^{p/q}]_{A_p(\mathbb{R}^{n_1})} \leq [w]_{A_p(A_q)}$, $[v]_{A_q(\mathbb{R}^{n_2})} \leq [w]^{q/p}_{A_p(A_q)}$, *and* $[w]_{A_p(A_q)} \leq [u^{p/q}]_{A_p(\mathbb{R}^{n_1})}[v]^{p/q}_{A_q(\mathbb{R}^{n_2})}$.

In the case $p = q$, we denote the associated collection of weights by $A_{p,\mathcal{R}}(\mathbb{R}^n)$. This class coincides with that when the supremum in (2.3) is taken over the collection of rectangles $\mathcal{R} = \{Q_1 \times Q_2 : Q_1$ and Q_2 are cubes in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively$\}$. Set $A_{\infty,\mathcal{R}}(\mathbb{R}^n) = \cup_{p>1} A_{p,\mathcal{R}}(\mathbb{R}^n)$.

Let $0 < r \leq \infty$, $s \in \mathbb{R}$, w be a weight on $\mathbb{R}^n$, and $0 < p, q < \infty$. When $X = L^p(L^q(w))$, we denote the weighted homogeneous Triebel–Lizorkin space $\dot{F}^s_{X,r}$ as $\dot{F}^s_{p,q,r}(w)$ and the weighted inhomogeneous Triebel–Lizorkin space $F^s_{X,r}$ as $F^s_{p,q,r}(w)$. Analogous notation applies to the scale of Besov spaces. The weighted Hardy space H^X is denoted by $H^{p,q}(w)$ and the weighted local Hardy space h^X is denoted by $h^{p,q}(w)$.

We observe that since $L^p(L^p(w)) = L^p(w)$, $\dot{F}^s_{p,p,r}(w)$ yields the classical weighted homogeneous Triebel–Lizorkin space, and analogous associations apply for $F^s_{p,p,r}(w)$, $\dot{B}^s_{p,p,r}(w)$, $B^s_{p,p,r}(w)$, $H^{p,p}(w)$, and $h^{p,p}(w)$.

Moreover, for $1 < p, q < \infty$ and $w(x_1, x_2) = u(x_1)v(x_2) \in A_p(A_q)$, we have

$$\dot{F}^0_{p,q,2}(w) = F^0_{p,q,2}(w) = H^{p,q}(w) = h^{p,q}(w) = L^p(L^q(w)), \tag{5.1}$$

with equivalent norms. We refer the reader to Appendix B for further details on these relationships.

5.2 Leibniz Rules in $L^p(L^q(w))$

We first state a corollary of Theorem 3.1 in Triebel–Lizorkin and Besov spaces based on weighted mixed Lebesgue spaces. We then present Leibniz rules in weighted mixed Lebesgue spaces.

In this section, we restrict our attention to product weights: if $0 < p, q < \infty$, we consider $w(x_1, x_2) = u(x_1)v(x_2)$, where $u^{p/q} \in A_\infty(\mathbb{R}^{n_1})$ and $v \in A_\infty(\mathbb{R}^{n_2})$.

In general, the mixed Lebesgue spaces $L^p(L^q(w))$ are not necessarily rearrangement invariant (see Blozinski [7]); however, it easily follows that $L^p(L^q(w))$ is a QBFS over $(\mathbb{R}^n, u^{p/q} \times v)$. In this setting, property P6i is only required for measurable sets $E \subset \mathbb{R}^n$ such that $E \subset I_1 \times I_2$ where I_1 and I_2 are measurable sets in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ with finite measures with respect to $u^{p/q}(x_1)dx_1$ and $v(x_2)dx_2$, respectively (see [7]). In the case that $1 \leq p, q < \infty$, $L^p(L^q(w))$ also fulfills properties P2 and P6ii, where the same change made for P6i is implemented for P6ii. We next note that $(L^p(L^q(w)))^{p_0} = L^{pp_0}(L^{qp_0}(w))$, and therefore $p(L^p(L^q(w))) = 1/\min(p, q, 1)$.

Given $0 < r \leq \infty$ and $0 < h < \min(p/\tau_{u^{p/q}}, q/\tau_v, r)$, the following Fefferman–Stein inequality holds (see Theorem C.1):

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_h(f_j)|^r \right)^{\frac{1}{r}} \right\|_{L^p(L^q(w))} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(L^q(w))},$$

for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^n$. This also gives the boundedness of $\mathcal{M}_h$ on $L^p(L^q(w))$: if $0 < h < \min(p/\tau_{u^{p/q}}, q/\tau_v)$, we have

$$\|\mathcal{M}_h(f)\|_{L^p(L^q(w))} \lesssim \|f\|_{L^p(L^q(w))}\,.$$

We then define

$$\tau_{p,q,r}(w) = n\left(\frac{1}{\min(p/\tau_{u^{p/q}}, q/\tau_v, r, 1)} - 1\right),$$
$$\tau_{p,q}(w) = n\left(\frac{1}{\min(p/\tau_{u^{p/q}}, q/\tau_v, 1)} - 1\right).$$

This implies that for $X = L^p(L^q(w))$,

$$h_{X,r} \geq \min\left(\frac{p}{\tau_{u^{p/q}}}, \frac{q}{\tau_v}, r\right) \quad \text{and} \quad h_X \geq \min\left(\frac{p}{\tau_{u^{p/q}}}, \frac{q}{\tau_v}\right),$$

as well as

$$\tau_r(X) \leq \tau_{p,q,r}(w) \quad \text{and} \quad \tau(X) \leq \tau_{p,q}(w).$$

Therefore, we obtain the following corollary to Theorem 3.1:

Corollary 5.2 *Let $m \in \mathbb{R}$, $\sigma(\xi,\eta), \xi, \eta \in \mathbb{R}^n$, be a Coifman–Meyer multiplier of order m, $0 < r \leq \infty$, and $0 < p, p_1, p_2, q, q_1, q_2 < \infty$ be such that $1/p = 1/p_1 + 1/p_2$ and $1/q = 1/q_1 + 1/q_2$. Suppose $w_1(x_1,x_2) = u_1(x_1)v_1(x_2)$ and $w_2(x_1,x_2) = u_2(x_1)v_2(x_2)$ with $u_1^{p_1/q_1}, u_2^{p_2/q_2} \in A_\infty(\mathbb{R}^{n_1})$ and $v_1, v_2 \in A_\infty(\mathbb{R}^{n_2})$; set $w(x_1,x_2) = (w_1(x_1,x_2))^{q/q_1}(w_2(x_1,x_2))^{q/q_2}$ and assume $L^p(L^q(w))$ satisfies properties P7, P8, and P9 with r as given and s as below.*

(i) *If $s > \tau_{p,q,r}(w)$, then*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q,r}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1,q_1,r}(w_1)}\|g\|_{H^{p_2,q_2}(w_2)} + \|f\|_{H^{p_1,q_1}(w_1)}\|g\|_{\dot{F}^{s+m}_{p_2,q_2,r}(w_2)}. \tag{5.2}$$

(ii) *If $s > \tau_{p,q}(w)$, then*

$$\|T_\sigma(f,g)\|_{\dot{B}^s_{p,q,r}(w)} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1,q_1,r}(w_1)}\|g\|_{H^{p_2,q_2}(w_2)} + \|f\|_{H^{p_1,q_1}(w_1)}\|g\|_{\dot{B}^{s+m}_{p_2,q_2,r}(w_2)}. \tag{5.3}$$

In particular, (5.2) and (5.3) hold for $u = u_1 = u_2$, $v = v_1 = v_2$ with $u^{p_1/q_1}, u^{p_2/q_2} \in A_\infty(\mathbb{R}^{n_1})$ and $v \in A_\infty(\mathbb{R}^{n_2})$, in which case $w = w_1 = w_2$. Moreover, if $s > \tau_{p,q,r}(w)$, then

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q,r}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p,q,r}(w)}\,\|g\|_{L^\infty} + \|f\|_{L^\infty}\,\|g\|_{\dot{F}^{s+m}_{p,q,r}(w)}\,, \tag{5.4}$$

with analogous estimates for the Besov spaces if $s > \tau_{p,q}(w)$.

Proof We first note that $w(x_1, x_2) = u(x_1)v(x_2)$ where $u(x_1) = (u_1(x_1))^{q/q_1}(u_2(x_1))^{q/q_2}$ and $v(x_2) = (v_1(x_2))^{q/q_1}(v_2(x_2))^{q/q_2}$. We then have that $u^{p/q} = (u_1^{p_1/q_1})^{p/p_1}(u_2^{p_2/q_2})^{p/p_2}$ belongs to $A_\infty(\mathbb{R}^{n_1})$ since $u_1^{p_1/q_1}, u_2^{p_2/q_2} \in A_\infty(\mathbb{R}^{n_1})$ and $p/p_1 + p/p_2 = 1$; similarly, $v \in A_\infty(\mathbb{R}^{n_2})$ since $v_1, v_2 \in A_\infty(\mathbb{R}^{n_2})$ and $q/q_1 + q/q_2 = 1$. Moreover, a simple computation shows that

$$\|fg\|_{L^p(L^q(w))} \leq \|f\|_{L^{p_1}(L^{q_1}(w_1))} \|g\|_{L^{p_2}(L^{q_2}(w_2))} .$$

We next apply Theorem 3.1 with the spaces $X = L^1(L^{q/p}(w))$, $X_1 = L^1(L^{q_1/p_1}(w_1))$, and $X_2 = L^1(L^{q_2/p_2}(w_2))$, which verify all assumptions required. Therefore, (5.2), (5.3), and (5.4) with its Besov counterpart follow. □

Remark 5.3 Corollary 5.2 requires $L^p(L^q(w))$ to satisfy P7, P8, and P9 for r and s as stated. We first note that if $0 < p, q < \infty$, $w(x_1, x_2) = u(x_1)v(x_2)$ with $u^{p/q} \in A_\infty(\mathbb{R}^{n_1})$ and $v \in A_\infty(\mathbb{R}^{n_2})$, $0 < r \leq \infty$, and $s \in \mathbb{R}$, then the inclusions $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{p,q,r}(w)$, $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{p,q,r}(w)$, and their inhomogeneous counterparts hold since it can be proved that there exists $N > 0$ such that $(1 + |x_1| + |x_2|)^{-N} \in L^p(L^q(w))$ (see Remark 2.1). Moreover, under the same assumptions on the weights and indices, the inclusions $\dot{F}^s_{p,q,r}(w) \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$, $\dot{B}^s_{p,q,r}(w) \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$, and their inhomogeneous counterparts, as well as the completeness of the spaces hold in the following cases:

(1) If $1 \leq p, q < \infty$, it can be proved that $(1 + |x_1| + |x_2|)^{-N} \in (L^p(L^q(w)))'$. By Remark 2.1, the desired inclusions and completeness follow.

(2) Suppose $0 < p, q < \infty$ and u and v satisfy

$$\int_{|x_1 - y_1| \leq t} u^{p/q}(y_1)dy_1 \geq t^{d_1} \quad \text{and} \quad \int_{|x_2 - y_2| \leq t} v(y_2)dy_2 \geq t^{d_2}, \tag{5.5}$$

for all $t > 0$, $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, and some $d_1, d_2 > 0$. In this case, it can be proved that if $f \in L^p(L^q(w)) \cap \mathcal{S}'(\mathbb{R}^n)$ is such that $\operatorname{supp}(\widehat{f}) \subset [-a, a]^n$ for some $a > 0$, then

$$\|f\|_{L^\infty} \lesssim a^{\frac{d_1}{p} + \frac{d_2}{q}} \|f\|_{L^p(L^q(w))} , \tag{5.6}$$

where the implicit constant is independent of f and a. With the estimate (5.6), the proofs of the desired inclusions and completeness follow similar ideas as in the ones for the classical settings (see [57, Sect. 2.3.3]). A proof of (5.6) can be obtained using analogous steps to those in Qui [52, Lemma 2.5]; the unweighted case of (5.6) was treated in Johnsen [33]. For later use, we note that if $u(x_1) = |x_1|^a$ and $v(x_2) = |x_2|^b$ with $a \geq 0$ and $b \geq 0$, then u and v satisfy (5.5) with $d_1 = n_1 + ap/q$ and $d_2 = n_2 + b$ (see Grafakos [26, pp. 505–506]).

Applying the lifting property (see (2.6)), the estimates (5.2) and (5.3) can be recast as

$$\begin{aligned}\|D^s(T_\sigma(f,g))\|_{\dot F^0_{p,q,r}(w)} \lesssim &\|D^s f\|_{\dot F^m_{p_1,q_1,r}(w_1)}\|g\|_{H^{p_2,q_2}(w_2)}\\ &+\|f\|_{H^{p_1,q_1}(w_1)}\|D^s g\|_{\dot F^m_{p_2,q_2,r}(w_2)},\end{aligned} \tag{5.7}$$

$$\begin{aligned}\|D^s(T_\sigma(f,g))\|_{\dot B^0_{p,q,r}(w)} \lesssim &\|D^s f\|_{\dot B^m_{p_1,q_1,r}(w_1)}\|g\|_{H^{p_2,q_2}(w_2)}\\ &+\|f\|_{H^{p_1,q_1}(w_1)}\|D^s g\|_{\dot B^m_{p_2,q_2,r}(w_2)}.\end{aligned} \tag{5.8}$$

Using (5.1) and Remark 5.3 and noting that $\tau_{p,q,2}(w)=0$ for $1<p,q<\infty$ and $w(x_1,x_2)=u(x_1)v(x_2)\in A_p(A_q)$, we obtain the following Leibniz rules in weighted mixed Lebesgue spaces.

Corollary 5.4 *Let $\sigma(\xi,\eta), \xi,\eta\in\mathbb{R}^n$, be a Coifman–Meyer multiplier of order zero and $1<p,p_1,p_2,q,q_1,q_2<\infty$ be such that $1/p=1/p_1+1/p_2$ and $1/q=1/q_1+1/q_2$. Suppose $w_1(x_1,x_2)=u_1(x_1)v_1(x_2)\in A_{p_1}(A_{q_1})$, $w_2(x_1,x_2)=u_2(x_1)v_2(x_2)\in A_{p_2}(A_{q_2})$, and $w(x_1,x_2)=(w_1(x_1,x_2))^{q/q_1}(w_2(x_1,x_2))^{q/q_2}\in A_p(A_q)$. If $s>0$, then*

$$\begin{aligned}\|D^s(T_\sigma(f,g))\|_{L^p(L^q(w))} \lesssim &\|D^s f\|_{L^{p_1}(L^{q_1}(w_1))}\|g\|_{L^{p_2}(L^{q_2}(w_2))}\\ &+\|f\|_{L^{p_1}(L^{q_1}(w_1))}\|D^s g\|_{L^{p_2}(L^{q_2}(w_2))}.\end{aligned} \tag{5.9}$$

Versions of Corollaries 5.2 and 5.4 and the corresponding estimates for (5.7) and (5.8) also hold in the inhomogeneous setting with an inhomogeneous Coifman–Meyer multiplier and the operator J^s.

5.3 Example: Power Weights

Of particular interest are power weights, or weights of the form $|x_1|^a|x_2|^b$ in the homogeneous setting and $\langle x_1\rangle^a\langle x_2\rangle^b$ in the inhomogeneous setting, where $\langle x\rangle^a=(1+|x|^2)^{a/2}$. In this section, we present examples of fractional Leibniz rules for weighted mixed Lebesgue spaces associated to power weights.

Recall that for $1<\tau<\infty$, a power weight $|x|^a$, $x\in\mathbb{R}^n$, is in $A_\tau(\mathbb{R}^n)$ if, and only if, $-n<a<n(\tau-1)$. Therefore, for $u_j(x_1)=|x_1|^{a_j}$ and $v_j(x_2)=|x_2|^{b_j}$, $j=1,2$, to meet the conditions in Corollary 5.2 that $u_j^{p_j/q_j}\in A_\infty(\mathbb{R}^{n_1})$ and $v_j\in A_\infty(\mathbb{R}^{n_2})$, we require that

$$-n_1\frac{q_j}{p_j}<a_j<\infty \quad \text{and} \quad -n_2<b_j<\infty.$$

With these conditions on a_j and b_j, $j=1,2$, Corollary 5.2 holds with $w_1(x_1,x_2)=|x_1|^{a_1}|x_2|^{b_1}$, $w_2(x_1,x_2)=|x_1|^{a_2}|x_2|^{b_2}$, and $w(x_1,x_2)=|x_1|^a|x_2|^b$ where

$$\frac{a}{q}=\frac{a_1}{q_1}+\frac{a_2}{q_2} \quad \text{and} \quad \frac{b}{q}=\frac{b_1}{q_1}+\frac{b_2}{q_2}, \tag{5.10}$$

if $L^p(L^q(w))$ satisfies P7, P8, and P9 for r and s as needed, in particular, if $1 \le p, q < \infty$ or, if $0 < p < 1$ or $0 < q < 1$ and $a, b \ge 0$ (see Remark 5.3).

To obtain Leibniz rules in mixed Lebesgue spaces with power weights we use Corollary 5.4, which requires $w_1 \in A_{p_1}(A_{q_1})$, $w_2 \in A_{p_2}(A_{q_2})$, and $w \in A_p(A_q)$. Therefore, we impose further conditions on the exponents a_1, a_2, b_1, and b_2. Using Lemma 5.1, we require, for $j = 1, 2$,

$$\begin{aligned} &-n_1\frac{q_j}{p_j} < a_j < \frac{q_j n_1}{p_j'} \quad \text{and} \quad -n_2 < b_j < n_2(q_j - 1), \\ &-\frac{n_1}{p} < \frac{a_1}{q_1} + \frac{a_2}{q_2} < \frac{n_1}{p'} \quad \text{and} \quad -\frac{n_2}{q} < \frac{b_1}{q_1} + \frac{b_2}{q_2} < \frac{n_2}{q'}. \end{aligned} \tag{5.11}$$

In particular, in the case $\sigma \equiv 1$ and for those values of a, a_1, a_2, b, b_1, and b_2 as in (5.10) and (5.11), Corollary 5.4 gives

$$\begin{aligned} \|D^s(fg)\|_{L^p(L^q(|x_1|^a|x_2|^b))} \lesssim\ & \|D^s f\|_{L^{p_1}(L^{q_1}(|x_1|^{a_1}|x_2|^{b_1}))}\|g\|_{L^{p_2}(L^{q_2}(|x_1|^{a_2}|x_2|^{b_2}))} \\ &+ \|f\|_{L^{p_1}(L^{q_1}(|x_1|^{a_1}|x_2|^{b_1}))}\|D^s g\|_{L^{p_2}(L^{q_2}(|x_1|^{a_2}|x_2|^{b_2}))}. \end{aligned} \tag{5.12}$$

An analogous result also holds in the inhomogeneous settings.

We note that when $a_1 = a_2$ and $b_1 = b_2$ (therefore, $a = a_1 = a_2$ and $b = b_1 = b_2$), the conditions (5.11) translate to

$$-n_1 \min\left(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q}{p}\right) < a < n_1 \min\left(\frac{q_1}{p_1'}, \frac{q_2}{p_2'}, \frac{q}{p'}\right) \quad \text{and} \quad -n_2 < b < n_2(q-1).$$

Using different methods of proof, fractional Leibniz rules in weighted mixed Lebesgue spaces with power weights were also proved in [51, Theorem 1.6]. In this work, using our notation, they let $1/2 \le p, q \le \infty$, $1 \le p_1, p_2, q_1, q_2 \le \infty$, and $0 \le a, a_1, a_2, b, b_1, b_2 < \infty$ be such that $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$, and satisfy (5.10). For $s > \max\left(n\left(\frac{1}{\min(p,q,1)} - 1\right), 0\right)$ or s a positive even integer, they obtain

$$\begin{aligned} \|J^s(fg)\|_{L^p(L^q(\langle x_1\rangle^a\langle x_2\rangle^b))} \lesssim\ & \|J^s f\|_{L^{p_1}(L^{q_1}(\langle x_1\rangle^{a_1}\langle x_2\rangle^{b_1}))}\|g\|_{L^{p_2}(L^{q_2}(\langle x_1\rangle^{a_2}\langle x_2\rangle^{b_2}))} \\ &+ \|f\|_{L^{p_1}(L^{q_1}(\langle x_1\rangle^{a_1}\langle x_2\rangle^{b_1}))}\|J^s g\|_{L^{p_2}(L^{q_2}(\langle x_1\rangle^{a_2}\langle x_2\rangle^{b_2}))}. \end{aligned} \tag{5.13}$$

6 Fractional Leibniz Rules in Weighted Morrey Spaces

In this section, we present Leibniz rules in weighted Morrey spaces. For $0 < p \le t < \infty$ and $w \in A_\infty$, the *weighted Morrey space* $M_p^t(w)$ consists of measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{M_p^t(w)} = \sup_{B\subset\mathbb{R}^n} w(B)^{\frac{1}{t}-\frac{1}{p}} \left(\int_B |f(x)|^p\, w(x)dx\right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all balls B contained in $\mathbb{R}^n$. It is easy to see that for $t = p$, we recover the traditional weighted Lebesgue space, $L^p(w)$. In this setting, we denote the homogeneous Triebel–Lizorkin and Besov spaces as $\dot{F}^s_{[p,t],r}(w)$ and $\dot{B}^s_{[p,t],r}(w)$, respectively, and the Hardy space as $H^{[p,t]}(w)$. We refer the reader to Rosenthal-Schmeisser [53] for more details about weighted Morrey spaces and to the works of Kozono-Yamazaki [37], Mazzucato [41], and Izuki et al. [32] regarding Morrey-based Triebel–Lizorkin and Besov spaces.

Morrey spaces fail to be QBFSs as they lack property P6ii (see Sawano–Tanaka [54]). However, applying the same argument as that in Theorem 3.1, we obtain the following result.

Theorem 6.1 (Theorem 6.2 from [48]) *Let $m \in \mathbb{R}$ and suppose $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, is a Coifman–Meyer multiplier of order m.*

(i) *If $w \in A_\infty(\mathbb{R}^n)$, $0 < p \le t < \infty$, $0 < p_1 \le t_1 < \infty$, $0 < p_2 \le t_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $1/t = 1/t_1 + 1/t_2$, $0 < r \le \infty$, and $s > n\left(\frac{1}{\min(p/\tau_w, r, 1)} - 1\right)$, then*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{[p,t],r}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{[p_1,t_1],r}(w)} \|g\|_{H^{[p_2,t_2]}(w)} + \|f\|_{H^{[p_1,t_1]}(w)} \|g\|_{\dot{F}^{s+m}_{[p_2,t_2],r}(w)}, \tag{6.1}$$

where different pairs of p_1, p_2 and t_1, t_2 can be used on the right hand side of the inequality above. Moreover,

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{[p,t],r}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{[p,t],r}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{[p,t],r}(w)}. \tag{6.2}$$

(ii) *If $w_1, w_2 \in A_\infty(\mathbb{R}^n)$, $w = w_1^{p/p_1} w_2^{p/p_2}$, $0 < p \le t < \infty$, $0 < p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $s > n\left(\frac{1}{\min(p/\tau_w, r, 1)} - 1\right)$, then*

$$\begin{aligned} \|T_\sigma(f,g)\|_{\dot{F}^s_{[p,t],r}(w)} &\lesssim \|f\|_{\dot{F}^{s+m}_{[p_1,p_1t/p],r}(w_1)} \|g\|_{H^{[p_2,p_2t/p]}(w_2)} \\ &\quad + \|f\|_{H^{[p_1,p_1t/p]}(w_1)} \|g\|_{\dot{F}^{s+m}_{[p_2,p_2t/p],r}(w_2)}. \end{aligned} \tag{6.3}$$

Estimates analogous to (6.1)–(6.3) hold in the Besov setting when $s > n\left(\frac{1}{\min(p/\tau_w, 1)} - 1\right)$.

From Theorem 6.1, we deduce Leibniz rules in weighted Morrey spaces and Hardy spaces based on weighted Morrey spaces. Through an extrapolation theorem in Morrey spaces given in Duoandikoetxea–Rosenthal [22, Corollary 4.3], for $0 < p \le t < \infty$ and $w \in A_\infty(\mathbb{R}^n)$, we obtain

$$H^{[p,t]}(w) = \dot{F}^0_{[p,t],2}(w) \quad \text{and} \quad h^{[p,t]}(w) = F^0_{[p,t],2}(w), \tag{6.4}$$

with equivalent quasi-norms.

These equivalences and (6.1) combined with the lifting property, which holds for Triebel–Lizorkin and Besov spaces based on Morrey spaces, give that under the hypotheses of Theorem 6.1 with $m = 0$,

$$\begin{aligned} \|D^s(T_\sigma(f,g))\|_{H^{[p,t]}(w)} &\lesssim \|D^s f\|_{H^{[p_1,t_1]}(w)}\|g\|_{H^{[p_2,t_2]}(w)} \\ &\quad + \|f\|_{H^{[p_1,t_1]}(w)}\|D^s g\|_{H^{[p_2,t_2]}(w)}. \end{aligned} \tag{6.5}$$

From (6.2) and (6.3) we also have

$$\|D^s(T_\sigma(f,g))\|_{H^{[p,t]}(w)} \lesssim \|D^s f\|_{H^{[p,t]}(w)}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|D^s g\|_{H^{[p,t]}(w)} \tag{6.6}$$

and

$$\begin{aligned} \left\|D^s(T_\sigma(f,g))\right\|_{H^{[p,t]}(w)} &\lesssim \left\|D^s f\right\|_{H^{[p_1,p_1t/p]}(w_1)}\|g\|_{H^{[p_2,p_2t/p]}(w_2)} \\ &\quad + \|f\|_{H^{[p_1,p_1t/p]}(w_1)}\left\|D^s g\right\|_{H^{[p_2,p_2t/p]}(w_2)}. \end{aligned} \tag{6.7}$$

Similarly, for $1 < \overline{p} \le \bar{t} < \infty$ and $w \in A_{\overline{p}}(\mathbb{R}^n)$, we have that, through extrapolation [22, Theorem 4.1],

$$\dot{F}^0_{[\overline{p},\bar{t}],2}(w) = M^{\bar{t}}_{\overline{p}}(w) \quad \text{and} \quad F^0_{[\overline{p},\bar{t}],2}(w) = M^{\bar{t}}_{\overline{p}}(w).$$

Using this, (6.4), (6.5), and the fact that $\|\cdot\|_{M^t_p(w)} \le \|\cdot\|_{H^{[p,t]}(w)}$ for $0 < p \le t < \infty$, under the hypotheses of Theorem 6.1 with $1 < p_1, p_2 < \infty$, $w \in A_{\min(p_1,p_2)}(\mathbb{R}^n)$, and $m = 0$, we have

$$\|D^s(T_\sigma(f,g))\|_{M^t_p(w)} \lesssim \|D^s f\|_{M^{t_1}_{p_1}(w)}\|g\|_{M^{t_2}_{p_2}(w)} + \|f\|_{M^{t_1}_{p_1}(w)}\|D^s g\|_{M^{t_2}_{p_2}(w)}.$$

as well as an analog to (6.6):

$$\|D^s(T_\sigma(f,g))\|_{M^t_p(w)} \lesssim \|D^s f\|_{M^t_p(w)}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|D^s g\|_{M^t_p(w)}.$$

Moreover, if $0 < p \le t < \infty$ and $1 < p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}(\mathbb{R}^n)$, $w_2 \in A_{p_2}(\mathbb{R}^n)$, and $w = w_1^{p/p_1} w_2^{p/p_2}$, then

$$\begin{aligned} \|D^s(T_\sigma(f,g))\|_{M^t_p(w)} &\lesssim \|D^s f\|_{M^{p_1t/p}_{p_1}(w_1)}\|g\|_{M^{p_2t/p}_{p_2}(w_2)} \\ &\quad + \|f\|_{M^{p_1t/p}_{p_1}(w_1)}\|D^s g\|_{M^{p_2t/p}_{p_2}(w_2)}. \end{aligned}$$

We can apply these results with power weights as a specific example. For $1 < p_1, p_2 < \infty$ and $w(x) = |x|^a$, we require that $w \in A_{\min(p_1,p_2)}(\mathbb{R}^n)$, and therefore

$$-n < a < n(\min(p_1, p_2) - 1).$$

Thus for $\frac{1}{2} < p \le t < \infty$, $1 < p_1 \le t_1 < \infty$, and $1 < p_2 \le t_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$, $1/t = 1/t_1 + 1/t_2$, and $w(x) = |x|^a$, with a as above,

$$\|D^s(T_\sigma(f,g))\|_{M^t_p(|x|^a)} \lesssim \|D^s f\|_{M^{t_1}_{p_1}(|x|^a)} \|g\|_{M^{t_2}_{p_2}(|x|^a)} + \|f\|_{M^{t_1}_{p_1}(|x|^a)} \|D^s g\|_{M^{t_2}_{p_2}(|x|^a)},$$

where $s > n\left(\frac{1}{\min(p,1)} - 1\right)$ if $a \le 0$ and $s > n\left(\frac{1}{\min\left(\frac{p}{a/n+1},1\right)} - 1\right)$ if $a > 0$. Similarly, we have

$$\|D^s(T_\sigma(f,g))\|_{M^t_p(|x|^a)} \lesssim \|D^s f\|_{M^t_p(|x|^a)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{M^t_p(|x|^a)}.$$

Further, suppose $0 < p \le t < \infty$ and $1 < p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$; also let $w_1(w) = |x|^{a_1}$ and $w_2(x) = |x|^{a_2}$ with

$$-n < a_j < n(p_j - 1), \qquad j = 1, 2,$$

and $w = w_1^{p/p_1} w_2^{p/p_2} = |x|^b$, where $b = p(a_1/p_1 + a_2/p_2)$. Then we have

$$\begin{aligned} \left\|D^s(T_\sigma(f,g))\right\|_{M^t_p(|x|^b)} &\lesssim \left\|D^s f\right\|_{M^{p_1 t/p}_{p_1}(|x|^{a_1})} \|g\|_{M^{p_2 t/p}_{p_2}(|x|^{a_2})} \\ &\quad + \|f\|_{M^{p_1 t/p}_{p_1}(|x|^{a_1})} \left\|D^s g\right\|_{M^{p_2 t/p}_{p_2}(|x|^{a_2})}, \end{aligned} \tag{6.8}$$

with $s > n\left(\frac{1}{\min(p,1)} - 1\right)$ if $b \le 0$ and $s > n\left(\frac{1}{\min(\frac{p}{b/n+1},1)} - 1\right)$ if $b > 0$.

Moreover, corresponding versions of Corollary 6.1 and (6.5)–(6.8) also hold in the inhomogeneous setting with an inhomogeneous Coifman–Meyer multiplier and the operator J^s.

7 Fractional Leibniz Rules in Variable Lebesgue Spaces

We now discuss applications of Theorem 3.1 in the setting of variable Lebesgue spaces. We begin with some definitions and notation followed by results for variable Lebesgue spaces.

Let $\mathcal{P}_0$ be the collection of measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 0 \quad \text{and} \quad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty.$$

For $p(\cdot) \in \mathcal{P}_0$, the *variable Lebesgue space* $L^{p(\cdot)}$ is the class of all measurable functions such that

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\} < \infty.$$

With this quasi-norm, $L^{p(\cdot)}$ is a QBFS (BFS when $p_- \geq 1$). Note that if $p(x) = p_0$, $0 < p_0 < \infty$, then $L^{p(\cdot)}$ coincides with $L^{p_0}(\mathbb{R}^n)$ with equality of quasi-norms. Similar to the traditional Lebesgue spaces,

$$\left\| |f|^r \right\|_{L^{p(\cdot)}} = \|f\|^r_{L^{rp(\cdot)}}, \tag{7.1}$$

and, if $p_- \geq 1$, $(L^{p(\cdot)})' = L^{p'(\cdot)}$, where $p'(\cdot)$ is defined to be the conjugate exponent of $p(\cdot)$; that is,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \qquad \forall x \in \mathbb{R}^n.$$

Let $\mathcal{D}$ be the collection of $p(\cdot) \in \mathcal{P}_0$ such that the Hardy-Littlewood operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}$. A necessary condition for $p(\cdot) \in \mathcal{D}$ is $p_- > 1$, while log-Hölder continuity conditions are sufficient. Moreover, it can be proved that the following conditions are equivalent for $p(\cdot) \in \mathcal{P}_0$ such that $p_- > 1$:

a) $p(\cdot) \in \mathcal{D}$;
b) $p'(\cdot) \in \mathcal{D}$;
c) $p(\cdot)/q \in \mathcal{D}$ for some $1 < q < p_-$;
d) $(p(\cdot)/q)' \in \mathcal{D}$ for some $1 < q < p_-$.

See Cruz-Uribe et al. [15] and references therein.

A version of Hölder's inequality holds for variable Lebesgue spaces: if $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot) \in \mathcal{P}_0$ such that $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, then

$$\|fg\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}}, \qquad \forall f \in L^{p_1(\cdot)},\ g \in L^{p_2(\cdot)}.$$

The case for exponents in $\mathcal{P}_0$ such that $p_- \geq 1$ is given in [18]; the general case follows from the latter case and (7.1).

Jensen's inequality combined with (7.1) give that if $p(\cdot) \in \mathcal{P}_0$ and $0 < \tau_0 < \infty$ is such that $p(\cdot)/\tau_0 \in \mathcal{D}$, then $p(\cdot)/\tau \in \mathcal{D}$ for $0 < \tau < \tau_0$. Therefore, we define $\mathcal{P}_0^*$ to be the collection of $p(\cdot) \in \mathcal{P}_0$ such that $p(\cdot)/\tau_0 \in \mathcal{D}$ for some $\tau_0 > 0$ and, for $p(\cdot) \in \mathcal{P}_0^*$, we set

$$\tau_{p(\cdot)} = \sup\{\tau > 0 : p(\cdot)/\tau \in \mathcal{D}\}.$$

We observe that $\tau_{p(\cdot)} \leq p_-$. The following version of the Fefferman–Stein inequality follows using [18, Sect. 5.6.8] and (7.1). For $p(\cdot) \in \mathcal{P}_0^*$, $0 < r \leq \infty$, and $0 < h <$

$\min(\tau_{p(\cdot)}, r)$,

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left|\mathcal{M}_h(f_j)\right|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \left|f_j\right|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}} .$$

In particular, for $0 < h < \tau_{p(\cdot)}$, $\mathcal{M}_h$ is bounded on $L^{p(\cdot)}$, i.e.,

$$\|\mathcal{M}_h(f)\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}} .$$

As a consequence, if $X = L^{p(\cdot)}$ and $p(\cdot) \in \mathcal{P}_0^*$, we have $h_{X,r} \geq \min(\tau_{p(\cdot)}, r)$ and $h_X \geq \tau_{p(\cdot)}$, as well as

$$\tau_r(X) \leq n\left(\frac{1}{\min(\tau_{p_{(\cdot)}}, r, 1)} - 1\right) \quad \text{and} \quad \tau(X) \leq n\left(\frac{1}{\min(\tau_{p_{(\cdot)}}, 1)} - 1\right).$$

For $s \in \mathbb{R}$, $0 < r \leq \infty$, and $p(\cdot) \in \mathcal{P}_0$, we denote the homogeneous Triebel–Lizorkin and Besov spaces in this setting as $\dot{F}^s_{p(\cdot),r}$ and $\dot{B}^s_{p(\cdot),r}$, respectively. More general variable exponent Triebel–Lizorkin and Besov spaces, where r and s are replaced with functions are considered in Diening et al. [21] and Almeida–Hästö [1]. The Hardy space with variable exponent $p(\cdot) \in \mathcal{P}_0$ will be denoted $H^{p(\cdot)}$. The corresponding inhomogeneous spaces are denoted analogously.

We then obtain the following fractional Leibniz rules in variable exponent Triebel–Lizorkin and Besov spaces as a corollary to Theorem 3.1. This result was also proven directly in [48, Theorem 6.4] using methods similar to those for Theorem 3.1.

Corollary 7.1 *Let $m \in \mathbb{R}$, $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman–Meyer multiplier of order m, $0 < r \leq \infty$, $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}_0^*$ be such that $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, and assume $L^{p(\cdot)}$ satisfies properties P7, P8, and P9.*

(i) *If $s > n\left(\frac{1}{\min(\tau_{p(\cdot)}, r, 1)} - 1\right)$, it holds that*

$$\|T_\sigma(f, g)\|_{\dot{F}^s_{p(\cdot),r}} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1(\cdot),r}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \|g\|_{\dot{F}^{s+m}_{p_2(\cdot),r}} ; \tag{7.2}$$

(ii) *if $s > n\left(\frac{1}{\min(\tau_{p(\cdot)}, 1)} - 1\right)$, it holds that*

$$\|T_\sigma(f, g)\|_{\dot{B}^s_{p(\cdot),r}} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1(\cdot),r}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \|g\|_{\dot{B}^{s+m}_{p_2(\cdot),r}} , \tag{7.3}$$

where different pairs of $p_1(\cdot)$ and $p_2(\cdot)$ can be used on the right hand sides of (7.2) and (7.3).

Moreover, if $s > n\left(\frac{1}{\min(\tau_{p(\cdot)}, r, 1)} - 1\right)$, then

$$\|T_\sigma(f, g)\|_{\dot{F}^s_{p(\cdot),r}} \lesssim \|f\|_{\dot{F}^s_{p(\cdot),r}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^s_{p(\cdot),r}} , \tag{7.4}$$

with a corresponding estimate holding in the Besov setting if $s > n\left(\frac{1}{\min(\tau_{p(\cdot)},1)} - 1\right)$.

Proof We apply Theorem 3.1 with $X = L^{p(\cdot)}$, $X_1 = L^{p_1(\cdot)}$, $X_2 = L^{p_2(\cdot)}$, and $p = p_1 = p_2 = 1$. Then $X^p = L^{p(\cdot)}$, $X_1^{p_1} = L^{p_1(\cdot)}$, and $X_2^{p_2} = L^{p_2(\cdot)}$ fulfill all conditions of Theorem 3.1 and (7.2), (7.3), and (7.4) with its Besov space counterpart follow. Finally, Remark 3.3 implies that different pairs of $p_1(\cdot)$ and $p_2(\cdot)$ can be used on the right hand sides of (7.2) and (7.3), provided that both pairs satisfy the Hölder relationship with $p(\cdot)$. □

As in Sects. 5 and 6, we can apply the lifting property (2.6) (see also [21, Lemma 4.4]) and write the estimates (7.2) and (7.3) as

$$\left\|D^s(T_\sigma(f,g))\right\|_{\dot F^0_{p(\cdot),r}} \lesssim \left\|D^s f\right\|_{\dot F^m_{p_1(\cdot),r}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \left\|D^s g\right\|_{\dot F^m_{p_2(\cdot),r}}, \quad (7.5)$$

$$\left\|D^s(T_\sigma(f,g))\right\|_{\dot B^0_{p(\cdot),r}} \lesssim \left\|D^s f\right\|_{\dot B^m_{p_1(\cdot),r}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \left\|D^s g\right\|_{\dot B^m_{p_2(\cdot),r}}; \quad (7.6)$$

(7.4) and its Besov counterpart can be also be rewritten in a similar manner.

Now, by using [15, Theorem 1.3], an extrapolation theorem that allows to deduce inequalities in variable Lebesgue spaces from weighted inequalities in Lebesgue spaces, it follows that if $p(\cdot) \in \mathcal{P}_0^*$, then

$$\dot F^0_{p(\cdot),2} = H^{p(\cdot)} \quad \text{and} \quad F^0_{p(\cdot),2} = h^{p(\cdot)}. \quad (7.7)$$

With this in mind, using (7.5) and (7.7), when σ is a Coifman–Meyer multiplier of order zero, Corollary 7.1 gives

$$\left\|D^s(T_\sigma(f,g))\right\|_{H^{p(\cdot)}} \lesssim \left\|D^s f\right\|_{H^{p_1(\cdot)}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \left\|D^s g\right\|_{H^{p_2(\cdot)}}; \quad (7.8)$$

in particular,

$$\left\|D^s(fg)\right\|_{H^{p(\cdot)}} \lesssim \left\|D^s f\right\|_{H^{p_1(\cdot)}} \|g\|_{H^{p_2(\cdot)}} + \|f\|_{H^{p_1(\cdot)}} \left\|D^s g\right\|_{H^{p_2(\cdot)}}. \quad (7.9)$$

Moreover, for $p(\cdot) \in \mathcal{D}$, by (2.9) applied with power q to $X = L^{p(\cdot)/q}$, where q is as in Item c), we have

$$\dot F^0_{p(\cdot),2} = F^0_{p(\cdot),2} = H^{p(\cdot)} = h^{p(\cdot)} = L^{p(\cdot)}, \quad (7.10)$$

with equivalence in norm (see also [21, Theorem 4.2]).

Thus, when $p_1(\cdot), p_2(\cdot) \in \mathcal{D}$, using (7.8), (7.10), and (2.7), we obtain

$$\left\|D^s(T_\sigma(f,g))\right\|_{L^{p(\cdot)}} \lesssim \left\|D^s f\right\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}} + \|f\|_{L^{p_1(\cdot)}} \left\|D^s g\right\|_{L^{p_2(\cdot)}}, \quad (7.11)$$

and, in particular,

$$\left\|D^s(fg)\right\|_{L^{p(\cdot)}} \lesssim \left\|D^s f\right\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}} + \|f\|_{L^{p_1(\cdot)}} \left\|D^s g\right\|_{L^{p_2(\cdot)}}.$$

Corresponding estimates for (7.4) also hold.

We note that (7.11) was proved in [17, Theorem 3.1] using bilinear extrapolation techniques.

Versions of Corollary 7.1, (7.5), (7.6), (7.8), (7.9), and (7.11) hold in the inhomogeneous setting with an inhomogeneous Coifman–Meyer multiplier and the operator J^s.

Appendix A. Extrapolation Theorems

In this appendix, we present extrapolation theorems that will be used in Appendix B and Appendix C to obtain relationships between the spaces X, $F^s_{X,r}$, and H^X, as well as Fefferman–Stein inequalities and the boundedness of the Hardy-Littlewood maximal operator in X.

The extrapolation results presented in this section are given in terms of pairs of functions (f, g). We will use $\mathcal{F}$ to denote a family of pairs of measurable functions that are not identically zero. If for some $0 < \overline{p} < \infty$ and $\overline{w} \in A_{\overline{q}}(\mathbb{R}^n)$ (or $\overline{w} \in A_{\overline{q},\mathcal{R}}(\mathbb{R}^n)$), $1 \leq \overline{q} \leq \infty$, we say that

$$\int_{\mathbb{R}^n} |f(x)|^{\overline{p}}\, \overline{w}(x)dx \lesssim \int_{\mathbb{R}^n} |g(x)|^{\overline{p}}\, \overline{w}(x)dx, \qquad \forall (f, g) \in \mathcal{F}, \tag{A.1}$$

we mean that (A.1) holds for all pairs of functions $(f, g) \in \mathcal{F}$ such that the left-hand side is finite, and the implicit constant depends only on $\overline{p}$ and $[\overline{w}]_{A_{\overline{q}}(\mathbb{R}^n)}$ (or $[\overline{w}]_{A_{\overline{q},\mathcal{R}}(\mathbb{R}^n)}$). In the case that the $L^{\overline{p}}(\overline{w})$ norm in (A.1) is replaced with another function space norm, the inequality should be interpreted the same way.

We first present an extrapolation theorem for weighted mixed Lebesgue spaces, similar to [38, Theorem 2], but for pairs of functions. Its proof is the same as that of [38, Theorem 2].

Theorem A.1 *Suppose that for some $1 < \overline{p} < \infty$ and for all $\overline{w} \in A_{\overline{p},\mathcal{R}}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} |f(x)|^{\overline{p}}\overline{w}(x)dx \lesssim \int_{\mathbb{R}^n} |g(x)|^{\overline{p}}\overline{w}(x)dx, \qquad \forall (f, g) \in \mathcal{F}.$$

If $1 < p, q < \infty$, then for every $w \in A_p(A_q)$ such that $w(x_1, x_2) = u(x_1)v(x_2)$, we have

$$\|f\|_{L^p(L^q(w))} \lesssim \|g\|_{L^p(L^q(w))}, \qquad \forall (f, g) \in \mathcal{F}.$$

For the BFS setting, we will use the following result from [19]:

Theorem A.2 (Corollary 4.8 from [19]) *Suppose that for some $1 \leq \overline{p} < \infty$ and every $\overline{w} \in A_{\overline{p}}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)|^{\overline{p}}\overline{w}(x)dx \lesssim \int_{\mathbb{R}^n} |g(x)|^{\overline{p}}\overline{w}(x)dx, \qquad \forall (f, g) \in \mathcal{F}.$$

If X is a BFS over $(\mathbb{R}^n, dx)$ such that the Hardy-Littlewood maximal function is bounded on X', then for all $1 < p < \infty$,

$$\|f\|_{X^p} \lesssim \|g\|_{X^p}, \qquad \forall (f, g) \in \mathcal{F}.$$

In the r.i.QBFS setting, we need the following extrapolation theorems from [20] and [19]:

Theorem A.3 (Theorem 2.1 from [20]) *Suppose that for some $0 < \overline{p} < \infty$ and all $\overline{w} \in A_\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)|^{\overline{p}} \overline{w}(x) dx \lesssim \int_{\mathbb{R}^n} |g(x)|^{\overline{p}} \overline{w}(x) dx, \qquad \forall (f, g) \in \mathcal{F}.$$

If X is a r.i.QBFS over $(\mathbb{R}^n, dx)$ with Boyd indices $0 < p_X \leq q_X < \infty$ and $p(X) < \infty$, then for all $w \in A_\infty(\mathbb{R}^n)$, we have

$$\|f\|_{X(w)} \lesssim \|g\|_{X(w)}, \qquad \forall (f, g) \in \mathcal{F}.$$

Theorem A.4 (Theorem 4.10 from [19]) *Suppose that for some $1 \leq \overline{p} < \infty$ and every $\overline{w} \in A_{\overline{p}}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)|^{\overline{p}} \overline{w}(x) dx \lesssim \int_{\mathbb{R}^n} |g(x)|^{\overline{p}} \overline{w}(x) dx, \qquad \forall (f, g) \in \mathcal{F}.$$

If X is a r.i.BFS over $(\mathbb{R}^n, dx)$ such that $1 < p_X \leq q_X < \infty$, then for all $w \in A_{p_X}(\mathbb{R}^n)$, we have

$$\|f\|_{X(w)} \lesssim \|g\|_{X(w)}, \qquad \forall (f, g) \in \mathcal{F}.$$

Appendix B. Equivalent Spaces

With the extrapolation theorems stated in Appendix A, we are now ready to demonstrate the relationships between the Triebel–Lizorkin spaces, Hardy spaces, and the spaces they are based on.

We first present some equivalences between Triebel–Lizorkin spaces, Hardy spaces, and the weighted mixed Lebesgue spaces.

Theorem B.1 *We have*

i) for $0 < p < \infty$ and $w \in A_{\infty,\mathcal{R}}(\mathbb{R}^n)$,

$$\dot{F}^0_{p,p,2}(w) = H^{p,p}(w) \quad \text{and} \quad F^0_{p,p,2}(w) = h^{p,p}(w),$$

with equivalent quasi-norms;

ii) for $1 < p < \infty$ *and* $w \in A_{p,\mathcal{R}}(\mathbb{R}^n)$*,*

$$\dot{F}^0_{p,p,2}(w) = F^0_{p,p,2}(w) = H^{p,p}(w) = h^{p,p}(w) = L^p(w),$$

with equivalent norms;

iii) for $1 < p, q < \infty$ *and* $w(x_1, x_2) = u(x_1)v(x_2) \in A_p(A_q)$*,*

$$\dot{F}^0_{p,q,2}(w) = F^0_{p,q,2}(w) = H^{p,q}(w) = h^{p,q}(w) = L^p(L^q(w)),$$

with equivalent norms.

Proof Part i) is a direct consequence of [52, Theorem 1.4(vi)] since $A_{\infty,\mathcal{R}}(\mathbb{R}^n) \subset A_\infty(\mathbb{R}^n)$.

Part ii) follows immediately from Part i) and [52, Remark 4.5], since $A_{p,\mathcal{R}}(\mathbb{R}^n) \subset A_p(\mathbb{R}^n)$.

Part iii) follows from Part ii) and Theorem A.1. □

We have similar results in the BFS setting. The proof uses extrapolation based on Theorem A.2.

Theorem B.2 *Let* X *be a BFS over* $(\mathbb{R}^n, dx)$ *such that the Hardy-Littlewood maximal operator is bounded on* X'*. Then for* $1 < p < \infty$*,*

$$\dot{F}^0_{X^p,2} = F^0_{X^p,2} = H^{X^p} = h^{X^p} = X^p,$$

with equivalent norms.

Finally, in the r.i.QBFS setting we can use Theorems A.3 and A.4 to get the following result:

Theorem B.3 *Let* X *be a r.i.QBFS over* $(\mathbb{R}^n, dx)$ *such that* $p(X) < \infty$*.*

(i) *If* X *has Boyd indices* $0 < p_X \leq q_X < \infty$ *and* $w \in A_\infty(\mathbb{R}^n)$*, then*

$$\dot{F}^0_{X,2}(w) = H^X(w) \quad \text{and} \quad F^0_{X,2}(w) = h^X(w),$$

with equivalent quasi-norms.

(ii) *If* X *is a r.i.BFS with Boyd indices* $1 < p_X \leq q_X < \infty$ *and* $w \in A_{p_X}(\mathbb{R}^n)$*, then*

$$\dot{F}^0_{X,2}(w) = F^0_{X,2}(w) = X(w),$$

with equivalent norms.

Appendix C. Fefferman–Stein Inequalities and Boundedness of $\mathcal{M}$ on Quasi-Banach Function Spaces

In this appendix, we discuss results regarding the Fefferman–Stein inequality and the boundedness of the Hardy-Littlewood maximal operator in weighted mixed Lebesgue spaces and r.i.QBFSs.

Recall that the classical Fefferman–Stein inequality on weighted Lebesgue spaces states that if $0 < p < \infty$, $0 < r \le \infty$, $0 < h < \min(p, r)$, and $w \in A_{p/h}(\mathbb{R}^n)$ (i.e., $0 < h < \min(p/\tau_w, r)$), then

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left|\mathcal{M}_h(f_j)\right|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \left|f_j\right|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)}, \tag{C.1}$$

where the summation in j should be replaced by the supremum in j if $r = \infty$.

We next consider $\mathcal{M}_h$ in the weighted mixed Lebesgue space setting.

Theorem C.1 *Let $0 < p, q < \infty$, $0 < r \le \infty$, $w(x_1, x_2) = u(x_1)v(x_2)$ with $u^{p/q} \in A_\infty(\mathbb{R}^{n_1})$ and $v \in A_\infty(\mathbb{R}^{n_2})$, and $0 < h < \min(p/\tau_{u^{p/q}}, q/\tau_v, r)$. Then for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of locally integrable functions defined on $\mathbb{R}^n$, we have*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_h(f_j)|^r \right)^{\frac{1}{r}} \right\|_{L^p(L^q(w))} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(L^q(w))}, \tag{C.2}$$

with the sum in j replaced by the supremum in j when $r = \infty$.

Proof We show here the case when r is finite; if $r = \infty$, the argument remains the same, exchanging the sum in j for the supremum in j.

We first assume that $h = 1$, and therefore $\tau_{u^{p/q}} < p < \infty$, $\tau_v < q < \infty$, and $1 < r < \infty$. By definition of $\tau_{u^{p/q}}$ and τ_v and Lemma 5.1, this implies $w \in A_p(A_q)$. If $1 < \overline{p} < \infty$ and $\overline{w} \in A_{\overline{p},\mathcal{R}}(\mathbb{R}^n) \subset A_{\overline{p}}(\mathbb{R}^n)$, (C.1) gives

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}(f_j)|^r \right)^{\frac{1}{r}} \right\|_{L^{\overline{p}}(\overline{w})} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^{\overline{p}}(\overline{w})}.$$

Therefore, Theorem A.1 gives that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}(f_j)|^r \right)^{\frac{1}{r}} \right\|_{L^p(L^q(w))} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(L^q(w))}.$$

We now let $0 < p, q < \infty$, $0 < r \le \infty$, and $0 < h < \min(p/\tau_{u^{p/q}}, q/\tau_v, r)$, or, equivalently, $1 < \min\left(\frac{p}{h\tau_{u^{p/q}}}, \frac{q}{h\tau_v}, \frac{r}{h}\right)$. Therefore,

$$\begin{aligned}
\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_h f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(L^q(w))} &= \left\| \left(\sum_{j\in\mathbb{Z}} (\mathcal{M}|f_j|^h)^{\frac{r}{h}} \right)^{\frac{h}{r}} \right\|_{L^{\frac{p}{h}}(L^{\frac{q}{h}}(w))}^{\frac{1}{h}} \\
&\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} (|f_j|^h)^{\frac{r}{h}} \right)^{\frac{h}{r}} \right\|_{L^{\frac{p}{h}}(L^{\frac{q}{h}}(w))}^{\frac{1}{h}} \\
&= \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(L^q(w))},
\end{aligned}$$

where the second equality follows from the previous case. □

Remark C.2 We note that for $p = q$, $0 < h < \min(p, r)$, and $w \in A_{p/h,\mathcal{R}}(\mathbb{R}^n)$, (C.1) implies (C.2), since $A_{p/h,\mathcal{R}}(\mathbb{R}^n) \subset A_{p/h}(\mathbb{R}^n)$.

We immediately have the following corollary regarding the boundedness of $\mathcal{M}_h$ on $L^p(L^q(w))$.

Theorem C.3 *If* $0 < p, q < \infty$, $w(x_1, x_2) = u(x_1)v(x_2)$ *with* $u^{p/q} \in A_\infty(\mathbb{R}^{n_1})$ *and* $v \in A_\infty(\mathbb{R}^{n_2})$, *and* $0 < h < \min(p/\tau_{u^{p/q}}, q/\tau_v)$, *then*

$$\|\mathcal{M}_h(f)\|_{L^p(L^q(w))} \lesssim \|f\|_{L^p(L^q(w))}.$$

We have similar results for a r.i.QBFS X over $(\mathbb{R}^n, dx)$, which we next present. The boundedness of $\mathcal{M}$ was given in Montgomery-Smith [43].

Theorem C.4 *Let* X *be a r.i.QBFS over* $(\mathbb{R}^n, dx)$. *Then* $\mathcal{M}$ *is bounded on* X *if, and only if,* $p_X > 1$.

This leads to the following result for the boundedness of the operator $\mathcal{M}_h$ on X.

Theorem C.5 *Let* X *be a r.i.QBFS over* $(\mathbb{R}^n, dx)$. *Then* $\mathcal{M}_h$ *is bounded on* X *if, and only if,* $0 < h < p_X$.

Recall that for $1 < p < \infty$, $\mathcal{M}$ is bounded on $L^p(w)$ if, and only if, $w \in A_p(\mathbb{R}^n)$. By Theorem A.4, we then have that $\mathcal{M}$ is bounded on $X(w)$ if X is a r.i.BFS with $1 < p_X \le q_X < \infty$ and $w \in A_{p_X}(\mathbb{R}^n)$. As a consequence, we have the following:

Theorem C.6 *Let* X *be a r.i.QBFS over* $(\mathbb{R}^n, dx)$ *such that* $0 < p_X \le q_X < \infty$ *and* $p(X) < \infty$. *If* $w \in A_\infty(\mathbb{R}^n)$ *and* $0 < h < \min(p_X/\tau_w, 1/p(X))$, $\mathcal{M}_h$ *is bounded on* $X(w)$.

Similarly, by (C.1) and Theorem A.4, we have a Fefferman–Stein inequality for a weighted r.i.QBFS $X(w)$, with $w \in A_\infty(\mathbb{R}^n)$.

Theorem C.7 *Suppose X is a r.i.QBFS over $(\mathbb{R}^n, dx)$ such that $0 < p_X \le q_X < \infty$ and $p(X) < \infty$. If $w \in A_\infty(\mathbb{R}^n)$, $0 < r \le \infty$, and $0 < h < \min(p_X/\tau_w, 1/p(X), r)$, then for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of locally integrable functions defined on $\mathbb{R}^n$, we have*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_h f_j|^r \right)^{\frac{1}{r}} \right\|_{X(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{X(w)},$$

with the sum in j replaced by the supremum in j when $r = \infty$.

Appendix D. Nikol'skiĭ Representations

In this appendix, we prove Theorem 3.4. The proof is similar to that in the setting of Triebel–Lizorkin and Besov spaces based on weighted Lebesgue spaces in [48], with modifications due to the fact that a dominated convergence theorem may not hold in X.

First, we introduce some notation. For a QBFS X, $0 < r \le \infty$, and a sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$, we denote

$$\left\| \{f_j\}_{j\in\mathbb{Z}} \right\|_{X(\ell^r)} = \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_X \quad \text{and} \quad \left\| \{f_j\}_{j\in\mathbb{Z}} \right\|_{\ell^r(X)} = \left(\sum_{j\in\mathbb{Z}} \|f_j\|_X^r \right)^{\frac{1}{r}}.$$

We also use the following lemmas from [48]

Lemma D.1 (Lemma A.1 from [48]) *Suppose $0 < h \le 1$, $A > 0$, $R \ge 1$, and $d > n/h$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and f is such that supp$(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \le AR\}$, it holds that*

$$|\phi * f(x)| \lesssim R^{n\left(\frac{1}{h}-1\right)} A^{-n} \left\| (1 + |A\cdot|)^d \phi \right\|_{L^\infty} \mathcal{M}_h f(x),$$

where the implicit constant is independent of A, R, ϕ, and f.

Lemma D.2 (similar to Lemma A.2 in [48]) *Suppose X is a QBFS such that $h_X > 0$. Let $A > 0$, $R \ge 1$, and $d > b > n/\min(h_X, 1)$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and f is such that supp$(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \le AR\}$, it holds that*

$$\|\phi * f\|_X \lesssim R^{b-n} A^{-n} \left\| (1 + |A\cdot|)^d \phi \right\|_{L^\infty} \|f\|_X,$$

where the implicit constant is independent of A, R, ϕ, and f.

Lemma D.3 (Lemma A.3 from [48]) *Let $\tau < 0$, $\lambda \in \mathbb{R}$, $0 < r \leq \infty$, and $k_0 \in \mathbb{Z}$. Then for any sequence $\{d_j\}_{j\in\mathbb{Z}} \subset [0, \infty)$, it holds that*

$$\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^r} \lesssim \left\| \left\{ 2^{j\lambda} d_j \right\}_{j\in\mathbb{Z}} \right\|_{\ell^r},$$

where the implicit constant only depends on k_0, τ, λ, and r.

Proof of Theorem 3.4 We begin by proving the result for finite families of functions. Here, we show the homogeneous case, but the logic for the inhomogeneous case is similar.

Let $\{u_j\}_{j\in\mathbb{Z}}$ be such that $u_j \equiv 0$, except for finitely many j. Suppose D, X, r, and s are as in the hypotheses of the theorem. Let $\psi \in \dot{\mathcal{A}}(\mathbb{R}^n)$.

We first prove (3.11). Fix $0 < h < \min\left(h_{X,r}, 1\right)$ such that $s > n(1/h - 1)$, and let $k_0 \in \mathbb{Z}$ be such that $2^{k_0-1} < D \leq 2^{k_0}$. Then for any $\ell \in \mathbb{Z}$, we have

$$\operatorname{supp}(\widehat{u_\ell}) \subset B(0, 2^\ell D) \subset B(0, 2^{\ell+k_0}).$$

Defining $u = \sum_{\ell\in\mathbb{Z}} u_\ell$, we note that

$$\operatorname{supp}\left(\widehat{\psi}(2^{-j}\cdot)\widehat{u_\ell}\right) \subset B(0, 2^{\ell+k_0}) \cap \left\{2^{j-1} < |\xi| < 2^{j+1}\right\}.$$

This intersection is empty for any $\ell < j - k_0$; therefore, we have the following identity:

$$\Delta_j^\psi u = \sum_{\ell\in\mathbb{Z}} \Delta_j^\psi u_\ell = \sum_{\ell=j-k_0}^{\infty} \Delta_j^\psi u_\ell = \sum_{k=-k_0}^{\infty} \Delta_j^\psi u_{j+k}. \tag{D.1}$$

Applying Lemma D.1 with $\phi(x) = 2^{jn}\psi(2^j x)$, $f = u_{j+k}$, $A = 2^j$, $R = 2^{k+k_0}$, $k, j \in \mathbb{Z}$, $k \geq -k_0$, and $d > n/h$, we obtain

$$\begin{aligned}\left|\Delta_j^\psi u_{j+k}(x)\right| &\lesssim \left(2^{k+k_0}\right)^{n\left(\frac{1}{h}-1\right)} \left(2^j\right)^{-n} \left\| \left(1 + |2^j\cdot|\right)^d 2^{jn}\psi(2^j\cdot)\right\|_{L^\infty} \mathcal{M}_h u_{j+k}(x)\\ &\lesssim 2^{kn\left(\frac{1}{h}-1\right)} \mathcal{M}_h u_{j+k}(x),\end{aligned}$$

where the implicit constant depends only on the parameters stated and ψ. This yields

$$2^{js}\left|\Delta_j^\psi u_{j+k}(x)\right| \lesssim 2^{kn\left(\frac{1}{h}-1\right)} 2^{js}\mathcal{M}_h u_{j+k}(x) = 2^{kn\left(\frac{1}{h}-1-\frac{s}{n}\right)} 2^{s(j+k)} \mathcal{M}_h u_{j+k}(x).$$

Therefore, by (D.1), we have

$$2^{js}\left|\Delta_j^\psi u(x)\right| \lesssim \sum_{k=-k_0}^{\infty} 2^{kn\left(\frac{1}{h}-1-\frac{s}{n}\right)} 2^{s(j+k)} \mathcal{M}_h u_{j+k}(x).$$

We now apply Lemma D.3 with $\tau = n(1/h - 1 - s/n), \lambda = s$, and $d_{j+k} = \mathcal{M}_h u_{j+k}(x)$. Note that $\tau < 0$ by definition. This gives

$$\begin{aligned}\left\|\left\{2^{js}\left|\Delta_j^\psi u(x)\right|\right\}_{j\in\mathbb{Z}}\right\|_{\ell^r} &\lesssim \left\|\left\{\sum_{k=-k_0}^{\infty} 2^{kn\left(\frac{1}{h}-1-\frac{s}{n}\right)} 2^{s(j+k)} \mathcal{M}_h u_{j+k}(x)\right\}_{j\in\mathbb{Z}}\right\|_{\ell^r} \\ &\lesssim \left\|\left\{2^{js}\mathcal{M}_h u_j(x)\right\}_{j\in\mathbb{Z}}\right\|_{\ell^r}.\end{aligned}$$

The desired inequality follows from the monotonicity of the quasi-norm associated to X and the Fefferman–Stein inequality.

We now prove (3.12) for finite families. We apply similar logic, but working instead with the norm inequality from Lemma D.2. Using $\phi(x) = 2^{jn}\psi(2^j x)$, $f = u_{j+k}$, $A = 2^j$, $R = 2^{k+k_0}$, $k, j \in \mathbb{Z}, k \geq -k_0, d > b$, and $n/\min(h_X, 1) < b < n + s$, we have

$$\begin{aligned}\left\|\Delta_j^\psi u_{j+k}\right\|_X &\lesssim \left(2^{k+k_0}\right)^{b-n}\left(2^j\right)^{-n}\left\|\left(1+\left|2^j\cdot\right|\right)^d 2^{jn}\psi\left(2^j\cdot\right)\right\|_{L^\infty}\left\|u_{j+k}\right\|_X \\ &\sim 2^{k(b-n)}\left\|u_{j+k}\right\|_X,\end{aligned}$$

where the implicit constants depend only on the parameters stated and ψ.

Setting p^* such that $|||\cdot||| \sim \|\cdot\|_X$ and $|||f+g|||^{p^*} \leq |||f|||^{p^*} + |||g|||^{p^*}$ (Aoki-Rolewicz Theorem), we obtain

$$\begin{aligned}2^{jsp^*}\left\|\Delta_j^\psi u\right\|_X^{p^*} &\lesssim 2^{jsp^*}\sum_{k=-k_0}^{\infty}\left\|\Delta_j^\psi u_{j+k}\right\|_X^{p^*} \\ &\lesssim \sum_{k=-k_0}^{\infty} 2^{jsp^*}2^{k(b-n)p^*}\left\|u_{j+k}\right\|_X^{p^*} \\ &= \sum_{k=-k_0}^{\infty} 2^{sp^*(j+k)}2^{k(b-n-s)p^*}\left\|u_{j+k}\right\|_X^{p^*}.\end{aligned}$$

Taking ℓ^{r/p^*} norms (quasi-norms when $r/p^* < 1$) and applying Lemma D.3 with $\tau = (b-n-s)p^*, \lambda = sp^*$, and $d_{j+k} = \left\|u_{j+k}\right\|_X^{p^*}$, we have the desired result.

We now show that the result holds for infinite families of functions. We first show the homogeneous Besov space case for $0 < r < \infty$. Let $\left\{u_j\right\}_{j\in\mathbb{Z}}$, X, r, and s be as in the hypotheses.

Let $U_N = \sum_{j=-N}^{N} u_j$. For $M < N$, $\{u_j\}_{M+1\le|j|\le N}$ fulfills the conditions of the theorem, and since the theorem holds for finite families of functions, we have

$$\|U_N - U_M\|_{\dot{B}^s_{X,r}} \lesssim \left\| \left\{2^{js} u_j\right\}_{M+1\le|j|\le N} \right\|_{\ell^r(X)},$$

where the implicit constant is independent of M, N, and the family $\{u_j\}_{j\in\mathbb{Z}}$.

By the assumption that $\left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{\ell^r(X)} < \infty$, the value of $\left\|\{2^{js}u_j\}_{M+1\le|j|\le N}\right\|_{\ell^r(X)}$ must tend to zero as M approaches ∞. Therefore, $\{U_N\}_{N\in\mathbb{Z}}$ is a Cauchy sequence in $\dot{B}^s_{X,r}$, and by the completeness of $\dot{B}^s_{X,r}$, the sum $\sum_{j\in\mathbb{Z}} u_j$ converges in $\dot{B}^s_{X,r}$.

Similarly, we see that

$$\|U_N\|_{\dot{B}^s_{X,r}} \lesssim \left\| \left\{2^{js} u_j\right\}_{-N\le j\le N} \right\|_{\ell^r(X)},$$

where the implicit constant is independent of N and the family $\{u_j\}_{j\in\mathbb{Z}}$. Therefore,

$$\left\| \sum_{j\in\mathbb{Z}} u_j \right\|_{\dot{B}^s_{X,r}} \lesssim \left\| \left\{2^{js} u_j\right\}_{j\in\mathbb{Z}} \right\|_{\ell^r(X)},$$

with the implicit constant independent of the family $\{u_j\}_{j\in\mathbb{Z}}$.

Now we consider the case of infinite families for $\dot{F}^s_{X,r}$ with $0 < r \le \infty$ as well as $\dot{B}^s_{X,\infty}$. Note that $\{2^{(s-\varepsilon)j}u_j\}_{j\ge0}$ and $\{2^{(s+\varepsilon)j}u_j\}_{j<0}$ belong to $\ell^1(X)$ for any $\varepsilon > 0$. Indeed, we have

$$\begin{aligned}\left\| \left\{2^{(s-\varepsilon)j} u_j\right\}_{j\ge0} \right\|_{\ell^1(X)} &= \sum_{j=0}^{\infty} 2^{js}2^{-\varepsilon j} \|u_j\|_X \\ &\lesssim \left\|\{2^{js}u_j\}_{j\ge0}\right\|_{\ell^\infty(X)} \lesssim \left\|\{2^{js}u_j\}_{j\ge0}\right\|_{\ell^r(X)}.\end{aligned}$$

The final expression is finite by assumption. Similar logic shows that $\left\|\{2^{(s+\varepsilon)j}u_j\}_{j<0}\right\|_{\ell^1(X)}$ is also finite.

Choosing $\varepsilon > 0$ such that $s - \varepsilon > \tau(X)$, by the case when $0 < r < \infty$, it follows that $\sum_{j=0}^{N} u_j$ and $\sum_{j=-N}^{-1} u_j$ converge in $\dot{B}^{s-\varepsilon}_{X,1}$ and $\dot{B}^{s+\varepsilon}_{X,1}$, respectively. Therefore, $\{U_N\}_{N\in\mathbb{Z}}$ converges in $\mathcal{S}_0'(\mathbb{R}^n)$. Applying the case for finite sequences for the space $\dot{F}^s_{X,r}$, we have $U_N \in \dot{F}^s_{X,r}$ and

$$\|U_N\|_{\dot{F}^s_{X,r}} \lesssim \left\| \left\{2^{js} u_j\right\}_{-N\le j\le N} \right\|_{X(\ell^r)} \le \left\| \left\{2^{js} u_j\right\}_{j\in\mathbb{Z}} \right\|_{X(\ell^r)}.$$

Since $\dot{F}^s_{X,r}$ has the Fatou Property, we have

$$\lim_{N\to\infty} U_N = \sum_{j\in\mathbb{Z}} u_j \in \dot{F}^s_{X,r}$$

and

$$\left\| \sum_{j\in\mathbb{Z}} u_j \right\|_{\dot{F}^s_{X,r}} \lesssim \left\| \left\{ 2^{js} u_j \right\}_{j\in\mathbb{Z}} \right\|_{X(\ell^r)} .$$

Similar reasoning works for $\dot{B}^s_{X,\infty}$. □

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