



Multilinear Pseudo-differential Operators with $S_{0,0}$ Class Symbols of Limited Smoothness

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Abstract

We consider the boundedness of the multilinear pseudo-differential operators with symbols in the multilinear Hörmander class $S_{0,0}$. The aim of this paper is to discuss smoothness conditions for symbols to assure the boundedness between local Hardy spaces.

Keywords Multilinear pseudo-differential operators · Multilinear Hörmander symbol classes · Local Hardy spaces · Wiener amalgam spaces

Mathematics Subject Classification 35S05 · 42B15 · 42B35

1 Introduction

First of all, the letter N which is mentioned in this article is understood to be a positive integer unless the contrary is explicitly stated.

For a bounded measurable function $\sigma = \sigma(x, \xi_1, \dots, \xi_N)$ on $(\mathbb{R}^n)^{N+1}$, the (N -fold) multilinear pseudo-differential operator T_σ is defined by

$$T_\sigma(f_1, \dots, f_N)(x) = \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix \cdot (\xi_1 + \dots + \xi_N)} \sigma(x, \xi_1, \dots, \xi_N) \prod_{j=1}^N \widehat{f}_j(\xi_j) d\xi_1 \dots d\xi_N$$

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for $x \in \mathbb{R}^n$ and $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$. The function σ is called the symbol of the operator T_σ .

The subject of the present paper is to investigate the boundedness of the multilinear pseudo-differential operators on several function spaces. In stating this, we use the following terminology with a slight abuse. Let X_1, \dots, X_N , and Y be function spaces on \mathbb{R}^n equipped with quasi-norms $\|\cdot\|_{X_j}$ and $\|\cdot\|_Y$, respectively. If there exists a constant C such that

$$\|T_\sigma(f_1, \dots, f_N)\|_Y \leq C \prod_{j=1}^N \|f_j\|_{X_j}, \quad f_j \in \mathcal{S} \cap X_j, \quad j = 1, \dots, N, \quad (1.1)$$

then we say that T_σ is bounded from $X_1 \times \dots \times X_N$ to Y . The smallest constant C of (1.1) is denoted by $\|T_\sigma\|_{X_1 \times \dots \times X_N \rightarrow Y}$. If \mathcal{A} is a class of symbols, we denote by $\text{Op}(\mathcal{A})$ the class of all operators T_σ corresponding to $\sigma \in \mathcal{A}$. If T_σ is bounded from $X_1 \times \dots \times X_N$ to Y for all $\sigma \in \mathcal{A}$, then we write $\text{Op}(\mathcal{A}) \subset B(X_1 \times \dots \times X_N \rightarrow Y)$. For the spaces X_j and Y , we consider the Lebesgue space L^p , the Hardy space H^p , the local Hardy space h^p , and the spaces BMO and bmo . The definitions of these spaces will be collected in Sect. 2.2.

Notice that, if T_σ is bounded from $X_1 \times \dots \times X_N$ to Y in the sense given above, then, in many cases, we can extend the definition of T_σ defined for $f_j \in \mathcal{S}(\mathbb{R}^n)$ to that for general $f_j \in X_j$ and can prove that (1.1) holds for all $f_j \in X_j$ using some limiting argument.

In this article, we focus on the Hörmander symbol class of $S_{0,0}$ -type. We recall that the class $S_{0,0}^m(\mathbb{R}^n, N)$, $m \in \mathbb{R}$, consists of all smooth functions σ on $(\mathbb{R}^n)^{N+1}$ such that

$$|\partial_x^{\alpha_0} \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \sigma(x, \xi_1, \dots, \xi_N)| \leq C_{\alpha_0, \alpha_1, \dots, \alpha_N} (1 + |\xi_1| + \dots + |\xi_N|)^m$$

holds for all multi-indices $\alpha_0, \alpha_1, \dots, \alpha_N \in (\mathbb{N}_0)^n = (\{0, 1, 2, \dots\})^n$. The linear case, $N = 1$, is the widely known Hörmander class and the following is a classical boundedness result:

Theorem A *Let $0 < p \leq \infty$ and $m \in \mathbb{R}$. Then, the boundedness*

$$\text{Op}(S_{0,0}^m(\mathbb{R}^n, 1)) \subset B(h^p \rightarrow h^p)$$

holds if and only if

$$m \leq \min \left\{ \frac{n}{p}, \frac{n}{2} \right\} - \max \left\{ \frac{n}{p}, \frac{n}{2} \right\},$$

where, if $p = \infty$, h^p should be replaced by bmo .

The “if” part of this result for $p = 2$ was proved by Calderón and Vaillancourt in [4], and then it was generalized to the case $1 < p < \infty$ by Fefferman in [9] and Coifman and Meyer in [5]. Finally, the boundedness for the full range $0 < p \leq \infty$

was obtained by Miyachi in [31] and Päivärinta and Somersalo in [35]. For the “only if” part, see, for instance, [31, Sect. 5] and [28, Theorem 1.5].

The study of the multilinear case, $N \geq 2$, originated with the paper [1] by Bényi and Torres, where they showed that, for $N = 2$ and for $1 \leq p, p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$, x -independent symbols in $S_{0,0}^m(\mathbb{R}^n, 2)$ do not always give rise to bounded operators from $L^{p_1} \times L^{p_2}$ to L^p . Then, the condition of $m \in \mathbb{R}$ for which the multilinear pseudo-differential operators with symbols in the class $S_{0,0}^m(\mathbb{R}^n, N)$ can be bounded among local Hardy spaces was investigated. More precisely, the following holds:

Theorem B *Let $N \geq 2$, $0 < p, p_1, \dots, p_N \leq \infty$, $1/p = 1/p_1 + \dots + 1/p_N$, and $m \in \mathbb{R}$. Then, the boundedness*

$$\text{Op}(S_{0,0}^m(\mathbb{R}^n, N)) \subset B(h^{p_1} \times \dots \times h^{p_N} \rightarrow h^p)$$

holds if and only if

$$m \leq \min \left\{ \frac{n}{p}, \frac{n}{2} \right\} - \sum_{j=1}^N \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\},$$

where, if $p_j = \infty$ for some $j \in \{1, \dots, N\}$, the corresponding h^{p_j} can be replaced by bmo .

The case $N = 2$ was proved by Miyachi and Tomita [32] and the case $N \geq 3$ by Miyachi, Tomita, and the author [28]. For the preceding results considering the subcritical case, see the papers by Michalowski, Rule, and Staubach [30] and by Bényi, Bernicot, Maldonado, Naibo, and Torres [2]. Quite recently, a generalization of Theorem B for $N = 2$ considering boundedness on Sobolev spaces was shown by Shida [37].

Remark that, in Theorems A and B, much smoothness is implicitly assumed for symbols. In the rest of this section, we shall consider smoothness conditions for symbols to assure the boundedness, which is our interest of the present paper. We first recall the linear case. In Miyachi [31], it was shown that the smoothness condition of symbols assumed in Theorem A can be relaxed to, roughly speaking, the smoothness up to $\min\{n/p, n/2\}$ for the space variable x and $\max\{n/p, n/2\}$ for the frequency variable ξ_1 . Moreover, it might be worth mentioning that these values are partially sharp (see [31, Sect. 5]). Some results on this direction can be also found in, for instance, Boulkhemair [3], Coifman and Meyer [5], Cordes [7], Hwang [24], Muramatu [34], and Sugimoto [38] for $p = 2$, and Sugimoto [39] and Tomita [40] for $0 < p < \infty$. For the multilinear case, in [26, 27], it was shown that, for the case $2/N \leq p \leq 2$ and $2 \leq p_1, \dots, p_N \leq \infty$, the assumptions of the smoothness up to $n/2$ for each space and frequency variables are sufficient to have the boundedness in Theorem B. See also Herbert and Naibo [21, 22] for the preceding results.

The purpose of this paper is to extend the partial result on the multilinear case stated above to the whole range of the exponents $0 < p, p_1, \dots, p_N \leq \infty$. We shall determine the smoothness conditions of symbols for the boundedness in Theorem B as

weak as possible. Before stating our main theorem, we introduce a Besov type class to measure the smoothness of symbols. In order to define this class, we use a partition of unity as follows. We take $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying that $\text{supp } \psi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$, $\text{supp } \psi \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$, and $\psi_0 + \sum_{k \in \mathbb{N}} \psi(2^{-k}\cdot) = 1$, and denote $\psi_k := \psi(2^{-k}\cdot)$ for $k \in \mathbb{N}$. We call this $\{\psi_k\}_{k \in \mathbb{N}_0}$ a Littlewood–Paley partition of unity on \mathbb{R}^n . Moreover, we write as $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, $\xi \in \mathbb{R}^d$, to shorten the notations.

Definition 1.1 Let $N \geq 2$, $m \in \mathbb{R}$, and $t \in (0, \infty]$ and let $\{\psi_k\}_{k \in \mathbb{N}_0}$ be a Littlewood–Paley partition of unity on \mathbb{R}^n . For $k = (k_0, k_1, \dots, k_N) \in (\mathbb{N}_0)^{N+1}$, $s = (s_0, s_1, \dots, s_N) \in [0, \infty)^{N+1}$, and $\sigma = \sigma(x, \xi) \in L^\infty((\mathbb{R}^n)^{N+1})$, we write $s \cdot k = \sum_{j=0}^N s_j k_j$ and

$$\Delta_k \sigma(x, \xi) = \psi_{k_0}(D_x) \psi_{k_1}(D_{\xi_1}) \dots \psi_{k_N}(D_{\xi_N}) \sigma(x, \xi).$$

We denote by $S_{0,0}^m(s, t; \mathbb{R}^n, N)$ the set of all $\sigma \in L^\infty((\mathbb{R}^n)^{N+1})$ such that the quasi-norm

$$\|\sigma\|_{S_{0,0}^m(s,t;\mathbb{R}^n,N)} = \left\{ \sum_{k \in (\mathbb{N}_0)^{N+1}} (2^{s \cdot k})^t \|\langle \xi \rangle^{-m} \Delta_k \sigma(x, \xi)\|_{L_{ul,\xi}^2((\mathbb{R}^n)^N)} \|^t_{L^\infty_x(\mathbb{R}^n)} \right\}^{1/t}$$

is finite, with a usual modification when $t = \infty$.

Here, the space L_{ul}^2 is the uniformly local L^2 space, which includes L^∞ (see Sect. 2.2). Using the class in Definition 1.1, the main theorem of the present paper reads as follows.

Theorem 1.2 Let $N \geq 2$, $0 < p$, $p_1, \dots, p_N \leq \infty$, and $1/p = 1/p_1 + \dots + 1/p_N$. If

$$m = \min \left\{ \frac{n}{p}, \frac{n}{2} \right\} - \sum_{j=1}^N \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\}$$

and

$$s_0 = \min \left\{ \frac{n}{p}, \frac{n}{2} \right\}, \quad s_j = \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\}, \quad j = 1, \dots, N,$$

then

$$\text{Op} \left(S_{0,0}^m(s, \min\{1, p\}; \mathbb{R}^n, N) \right) \subset B(h^{p_1} \times \dots \times h^{p_N} \rightarrow h^p),$$

where, if $p_j = \infty$ for some $j \in \{1, \dots, N\}$, the corresponding h^{p_j} can be replaced by h^{∞} .

We end this section with noting the organization of this paper. In Sect. 2, we collect some notations which will be used throughout this paper and give the definitions and properties of some function spaces. In Sect. 3, we first display the key statements, Theorem 3.2 and Proposition 3.4, which contain the essential part of Theorem 1.2. Then, we prove Theorem 3.2 and also consider the boundedness for symbols with classical derivatives. After preparing several lemmas for the proof of Proposition 3.4 in Sect. 4, we actually give its proof in Sect. 5. In Sect. 6, we consider the sharpness of the order m and the smoothness s_0, s_1, \dots, s_N stated in Theorem 1.2.

2 Preliminaries

2.1 Notations

We denote by Q the n -dimensional unit cube $[-1/2, 1/2]^n$ and we write $\ell Q = [-\ell/2, \ell/2]^n$, $\ell > 0$. Then, the cubes $\ell\tau + \ell Q$, $\tau \in \mathbb{Z}^n$, are mutually disjoint and constitute a partition of the Euclidean space \mathbb{R}^n . This implies that integral of a function on \mathbb{R}^n is written as

$$\int_{\mathbb{R}^n} f(x) dx = \sum_{v \in \mathbb{Z}^n} \int_{\ell Q} f(x + \ell v) dx \tag{2.1}$$

for $\ell > 0$. We denote by B_R the closed ball in \mathbb{R}^n of radius $R > 0$ centered at the origin. We denote by $\mathbf{1}_\Omega$ the characteristic function of a set Ω . For $1 \leq p \leq \infty$, p' is the conjugate number of p defined by $1/p + 1/p' = 1$.

For two nonnegative functions $A(x)$ and $B(x)$ defined on a set X , we write $A(x) \lesssim B(x)$ for $x \in X$ to mean that there exists a positive constant C such that $A(x) \leq CB(x)$ for all $x \in X$. We often omit to mention the set X when it is obviously recognized. Also $A(x) \approx B(x)$ means that $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

The symbols $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ denote the Schwartz class of rapidly decreasing smooth functions and the space of tempered distributions on \mathbb{R}^d , respectively. The Fourier transform and the inverse Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ are defined by

$$\begin{aligned} \mathcal{F}f(\xi) &= \widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx, \\ \mathcal{F}^{-1}f(x) &= \check{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) d\xi, \end{aligned}$$

respectively. For a Schwartz function $f(x, \xi_1, \dots, \xi_N)$, $x, \xi_1, \dots, \xi_N \in \mathbb{R}^n$, we denote the partial Fourier transform with respect to the x and ξ_j variables by \mathcal{F}_0 and \mathcal{F}_j , $j = 1, \dots, N$, respectively. We also write the Fourier transform on $(\mathbb{R}^n)^N$ for the ξ_1, \dots, ξ_N variables as $\mathcal{F}_{1, \dots, N} = \mathcal{F}_1 \dots \mathcal{F}_N$. For $m \in \mathcal{S}'(\mathbb{R}^n)$, we defined $m(D)f = \mathcal{F}^{-1}[m\widehat{f}]$ and use the notation $m(D)f(x) = m(D_x)f(x)$ when we indicate which variable is considered. For $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, we denote the short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to g by

$$V_g f(x, \xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot t} \overline{g(t-x)} f(t) dt. \tag{2.2}$$

2.2 Function Spaces

For a measurable subset $E \subset \mathbb{R}^d$, the Lebesgue space $L^p(E)$, $0 < p \leq \infty$, is the set of all those measurable functions f on E such that $\|f\|_{L^p(E)} = (\int_E |f(x)|^p dx)^{1/p} < \infty$ if $0 < p < \infty$ or $\|f\|_{L^\infty(E)} = \text{ess sup}_{x \in E} |f(x)| < \infty$ if $p = \infty$. If $E = \mathbb{R}^n$, we usually write L^p for $L^p(\mathbb{R}^n)$. The uniformly local L^2 space, denoted by $L^2_{ul}(\mathbb{R}^d)$, consists of all those measurable functions f on \mathbb{R}^d such that

$$\|f\|_{L^2_{ul}(\mathbb{R}^d)} = \sup_{v \in \mathbb{Z}^d} \|f(x+v)\|_{L^2_x([-1/2, 1/2]^d)} < \infty.$$

(This notion can be found in [25, Definition 2.3].)

For a countable set K , the sequence space $\ell^q(K)$, $0 < q \leq \infty$, is the set of all those complex sequences $a = \{a_k\}_{k \in K}$ such that $\|a\|_{\ell^q(K)} = (\sum_{k \in K} |a_k|^q)^{1/q} < \infty$ if $0 < q < \infty$ or $\|a\|_{\ell^\infty(K)} = \sup_{k \in K} |a_k| < \infty$ if $q = \infty$. If $K = \mathbb{Z}^n$, we usually write ℓ^q for $\ell^q(\mathbb{Z}^n)$.

Let X, Y, Z be function spaces. We use the notation $\|f\|_X = \|f(x)\|_{X_x}$ when we indicate which variable is measured. We denote the mixed norm by

$$\|f(x, y, z)\|_{X_x Y_y Z_z} = \left\| \|f(x, y, z)\|_{X_x} \right\|_{Y_y Z_z}.$$

(Pay special attention to the order of taking norms.) For X, Y, Z , we consider L^p or ℓ^p .

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x) dx \neq 0$ and let $\phi_t(x) = t^{-n} \phi(x/t)$ for $t > 0$. The space $H^p = H^p(\mathbb{R}^n)$, $0 < p \leq \infty$, consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{H^p} = \|\sup_{0 < t < \infty} |\phi_t * f|\|_{L^p} < \infty$. The space $h^p = h^p(\mathbb{R}^n)$, $0 < p \leq \infty$, consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{h^p} = \|\sup_{0 < t < 1} |\phi_t * f|\|_{L^p} < \infty$. It is known that H^p and h^p do not depend on the choice of the function ϕ up to the equivalence of quasi-norm. Obviously $H^p \subset h^p$. If $1 < p \leq \infty$, then $H^p = h^p = L^p$ with equivalent norms. In more details, see, for instance, Goldberg [16].

The space $BMO = BMO(\mathbb{R}^n)$ consists of all locally integrable functions f on \mathbb{R}^n such that

$$\|f\|_{BMO} = \sup_R \frac{1}{|R|} \int_R |f(x) - f_R| dx < \infty,$$

where $f_R = |R|^{-1} \int_R f(x) dx$ and R ranges over all the cubes in \mathbb{R}^n . The space $bmo = bmo(\mathbb{R}^n)$ consists of all locally integrable functions f on \mathbb{R}^n such that

$$\|f\|_{bmo} = \sup_{|R| \leq 1} \frac{1}{|R|} \int_R |f(x) - f_R| dx + \sup_{|R| \geq 1} \frac{1}{|R|} \int_R |f(x)| dx < \infty,$$

where R denotes cubes in \mathbb{R}^n . Obviously, $L^\infty \subset bmo \subset BMO$ holds. Also, BMO satisfies that $\|\lambda f\|_{BMO} = |\lambda| \|f\|_{BMO}$, $\lambda \in \mathbb{R}$, and $\|f(\lambda \cdot)\|_{BMO} = \|f\|_{BMO}$, $\lambda > 0$ (see, e.g., [17, Sect. 3.1.1]).

Let $\kappa \in \mathcal{S}(\mathbb{R}^n)$ be a function such that the support of κ is compact and

$$\sum_{k \in \mathbb{Z}^n} \kappa(\xi - k) = 1, \quad \xi \in \mathbb{R}^n.$$

For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, the Wiener amalgam space $W_s^{p,q}$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the quasi-norm

$$\|f\|_{W_s^{p,q}} = \left\| \| \langle k \rangle^s \kappa(D - k) f(x) \|_{\ell_k^q(\mathbb{Z}^n)} \right\|_{L_x^p(\mathbb{R}^n)}$$

is finite. If $s = 0$, we write $W^{p,q} = W_0^{p,q}$. The space $W_s^{p,q}$ does not depend on the choice of the function κ up to the equivalence of quasi-norm. The space $W_s^{p,q}$ is a quasi-Banach space (Banach space if $1 \leq p, q \leq \infty$) and $\mathcal{S} \subset W_s^{p,q} \subset \mathcal{S}'$. If $0 < p, q < \infty$, then \mathcal{S} is dense in $W_s^{p,q}$. It is known that $W^{p,2}$ is equivalent to the (standard) amalgams $W(L^2, \ell^p)$, where the space $W(L^q, \ell^p)$ is equipped with the quasi-norm

$$\|f\|_{W(L^q, \ell^p)} = \left\| \|f(x + v)\|_{L_x^q(Q)} \right\|_{\ell_v^p(\mathbb{Z}^n)}.$$

Also, in particular, $W^{\infty,2}$ is equivalent to L_{ul}^2 , since $L_{ul}^2 = W(L^2, \ell^\infty)$. See Feichtinger [10, 11] and Triebel [41] for more details.

Some of the relations between $W_s^{p,q}$ and the spaces L^p , h^p , and bmo will be given below.

Lemma 2.1 *Let $s \in \mathbb{R}$ and $0 < p, p_1, p_2, q_1, q_2 \leq \infty$. Then,*

$$W_s^{p_1, q_1} \hookrightarrow W_s^{p_2, q_2} \quad \text{if } p_1 \leq p_2, \quad q_1 \leq q_2; \tag{2.3}$$

$$h^p \hookrightarrow W_{\alpha(p)}^{p, 2}, \quad \text{where } \alpha(p) = n/2 - \max\{n/2, n/p\}; \tag{2.4}$$

$$bmo \hookrightarrow W^{\infty, 2}. \tag{2.5}$$

Proof The embedding (2.4) is given in [8, Theorem 1.1] for $1 < p \leq \infty$ and in [19, Theorem 1.2] for $0 < p \leq 1$. The explicit proofs of (2.3) and (2.5) can be found in [28, Lemma 2.2]. □

Remark 2.2 The idea of constructing the amalgam space $W(L^q, \ell^p)$, which, in contrast to the L^p space, treats local and global behavior of functions simultaneously, goes back to N. Wiener [45, 46], where special cases were considered. In [23], Holland gave systematic study including some basic properties of the amalgams $W(L^q, \ell^p)$. (For more details, see also, e.g., [14] and references therein.) Then, in [10, 11], Feichtinger introduced a vastly generalized amalgams which enables us to deal with a wide range of function spaces to be used as local or global components. He denoted them by $W(B, C)$ and, according to the suggestion of J. Benedetto, named the spaces $W(B, C)$

as Wiener amalgam spaces in recognition of Wiener’s works. Here, the components B and C measure local and global behavior of functions respectively. (For the detailed definition and basic properties, see also [6, 20].) Feichtinger’s framework allows us, of course, to treat the amalgams $W(L^q, \ell^p)$ and to understand that the space $W(\mathcal{F}L_s^q, L^p)$ is equivalent to the space $W_s^{p,q}$ defined above, where $\mathcal{F}L_s^q$ is the so-called Fourier Lebesgue space. In fact, although we omit the precise definition of $W(B, C)$, the quasi-norm of the space $W(\mathcal{F}L_s^q, L^p)$ is expressed by

$$\begin{aligned} \|f\|_{W(\mathcal{F}L_s^q, L^p)} &= \left\| \|g(\cdot - x)f\|_{\mathcal{F}L_s^q(\mathbb{R}^n)} \right\|_{L_x^p(\mathbb{R}^n)} \\ &= \left\| \|\langle \xi \rangle^s \mathcal{F}[g(\cdot - x)f](\xi)\|_{L_\xi^q(\mathbb{R}^n)} \right\|_{L_x^p(\mathbb{R}^n)} \end{aligned} \tag{2.6}$$

for a window function $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Choosing a window g with compact Fourier supports suitably, we see that the quasi-norm of (2.6) can be equivalent to

$$\left\| \|\langle \xi \rangle^s \check{g}(D - \xi)f(x)\|_{L_\xi^q(\mathbb{R}^n)} \right\|_{L_x^p(\mathbb{R}^n)} \approx \left\| \|\langle k \rangle^s \kappa(D - k)f(x)\|_{\ell_k^q(\mathbb{Z}^n)} \right\|_{L_x^p(\mathbb{R}^n)}.$$

This is more precisely discussed in, e.g., [10, 41]. So, from the historical point of view, it may be a terminology with a slight abuse that we refer to only the space $W_s^{p,q}$ as the Wiener amalgam space as above, because the above-mentioned spaces $W(L^q, \ell^p)$ and L_{ul}^2 are also contained in Wiener amalgam spaces. Moreover, the space $W_s^{p,q}$ probably should be denoted by $W(\mathcal{F}L_s^q, L^p)$, but, in this paper, we use the notation $W_s^{p,q}$ just for the purpose of simplification. For clarification with our choice of this notation, we refer the reader to [8, 19, 36].

We also remark that the quasi-norm of (2.6) is additionally expressed as a constant times

$$\left\| \|\langle \xi \rangle^s \mathcal{F}[\check{g}(\cdot - \xi)\widehat{f}](x)\|_{L_\xi^q(\mathbb{R}^n)} \right\|_{L_x^p(\mathbb{R}^n)},$$

or, by choosing a window g with compact supports suitably, it is equivalent to

$$\left\| \|\langle \xi \rangle^s \tilde{g}(D - x)\widehat{f}(\xi)\|_{L_\xi^q(\mathbb{R}^n)} \right\|_{L_x^p(\mathbb{R}^n)} \approx \left\| \|\langle \xi \rangle^s \kappa(D - k)\widehat{f}(\xi)\|_{L_\xi^q(\mathbb{R}^n)} \right\|_{\ell_k^p(\mathbb{Z}^n)},$$

where $\tilde{g} = g(\cdot - \cdot)$. One can regard the expressions above as the quasi-norm of \widehat{f} in a weighted modulation space. Thus, the Wiener amalgam spaces $W(\mathcal{F}L_s^q, L^p) = W_s^{p,q}$ are sometimes understood to be the Fourier images of modulation spaces. See, e.g., [6, 12] for more precise discussion. More detailed properties of modulation spaces can be also found in, e.g., [13, 15, 18, 29, 41, 44].

3 Main Result

3.1 Refined Version of the Main Theorem

In this subsection, we give a slight extension of Theorem 1.2. To do this, we shall define the following symbol class which can be wider than the class stated in Definition 1.1.

Definition 3.1 Let $N \geq 2$. For $\mathbf{m} = (m_1, \dots, m_N) \in \mathbb{R}^N$, $\mathbf{s} \in [0, \infty)^{N+1}$, and $t \in (0, \infty]$, we denote by $S_{0,0}^{\mathbf{m}}(\mathbf{s}, t; \mathbb{R}^n, N)$ the set of all $\sigma \in L^\infty((\mathbb{R}^n)^{N+1})$ such that the quasi-norm

$$\|\sigma\|_{S_{0,0}^{\mathbf{m}}(\mathbf{s}, t; \mathbb{R}^n, N)} = \left\{ \sum_{\mathbf{k} \in (\mathbb{N}_0)^{N+1}} (2^{\mathbf{s} \cdot \mathbf{k}})^t \left\| \prod_{j=1}^N (\xi_j)^{-m_j} \Delta_{\mathbf{k}} \sigma(x, \boldsymbol{\xi}) \right\|_{L_{ul, \boldsymbol{\xi}}^2((\mathbb{R}^n)^N) L_x^\infty(\mathbb{R}^n)}^t \right\}^{1/t}$$

is finite, with a usual modification when $t = \infty$.

The definition above does not depend on the choice of the Littlewood–Paley partition up to the equivalence of quasi-norms. Also, the same applies to the class given in Definition 1.1.

Notice that if $m_1, \dots, m_N \leq 0$ and if $m_1 + \dots + m_N = m$, then $S_{0,0}^{\mathbf{m}}(\mathbf{s}, t; \mathbb{R}^n, N) \subset S_{0,0}^m(\mathbf{s}, t; \mathbb{R}^n, N)$. Then, we see that the theorem below induce the statement of Theorem 1.2.

Theorem 3.2 Let $N \geq 2$, $0 < p, p_1, \dots, p_N \leq \infty$, and $1/p \leq 1/p_1 + \dots + 1/p_N$.

(1) Let $m_1, \dots, m_N \in \mathbb{R}$ satisfy

$$- \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\} < m_j < \frac{n}{2} - \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\}, \quad j = 1, \dots, N, \quad (3.1)$$

and

$$m_1 + \dots + m_N = \min \left\{ \frac{n}{p}, \frac{n}{2} \right\} - \sum_{j=1}^N \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\}. \quad (3.2)$$

(a) If $0 < p < 2$, $s_0 = n/2$, and $s_j = \max\{n/p_j, n/2\}$, $j = 1, \dots, N$, then

$$\text{Op} \left(S_{0,0}^{\mathbf{m}}(\mathbf{s}, \min\{1, p\}; \mathbb{R}^n, N) \right) \subset B(h^{p_1} \times \dots \times h^{p_N} \rightarrow h^p).$$

(b) If $2 \leq p < \infty$, $s_0 = n/p$, and $s_j = n/2$, $j = 1, \dots, N$, then

$$\text{Op} \left(S_{0,0}^{\mathbf{m}}(\mathbf{s}, 1; \mathbb{R}^n, N) \right) \subset B(h^{p_1} \times \dots \times h^{p_N} \rightarrow L^p).$$

(2) Let $m = - \sum_{j=1}^N \max\{n/p_j, n/2\}$. If $s_0 = 0$ and $s_j = n/2$, $j = 1, \dots, N$, then

$$\text{Op} \left(S_{0,0}^{\mathbf{m}}(\mathbf{s}, 1; \mathbb{R}^n, N) \right) \subset B(h^{p_1} \times \dots \times h^{p_N} \rightarrow L^\infty).$$

In the above assertions, if some of the p_j 's, $j \in \{1, \dots, N\}$, are equal to ∞ , then the conclusions hold with the corresponding h^{p_j} replaced by *bmo*.

Remark 3.3 The boundedness stated in Theorem 3.2 holds still true even if the norm

$$\|f(x, \boldsymbol{\xi})\|_{L_{ul, \boldsymbol{\xi}}^2((\mathbb{R}^n)^N) L_x^\infty(\mathbb{R}^n)}$$

of the classes $S_{0,0}^m(s, t; \mathbb{R}^n, N)$ and $S_{0,0}^m(s, t; \mathbb{R}^n, N)$ is replaced by the better one

$$\sup_{\nu_0, \nu_1, \dots, \nu_N \in \mathbb{Z}^n} \|f(x + \nu_0, \xi_1 + \nu_1, \dots, \xi_N + \nu_N)\|_{L_{(\xi_1, \dots, \xi_N)}^2(Q^N) L_x^{\max\{p, 2\}}(Q)}.$$

This can be seen by a careful following of the proof given in Sect. 5, but since its proof becomes much more complicated, we leave it to the interested readers.

3.2 Key Proposition

Proposition 3.4 below plays a crucial role in our argument and contains the essential part of Theorem 3.2. The proof will be given in the succeeding sections.

Proposition 3.4 *Let $N \geq 2$, $0 < p, p_1, \dots, p_N \leq \infty$, and $1/p \leq 1/p_1 + \dots + 1/p_N$. Suppose $\sigma \in L^\infty((\mathbb{R}^n)^{N+1})$ satisfies $\text{supp } \mathcal{F}\sigma \subset B_{R_0} \times B_{R_1} \times \dots \times B_{R_N}$ for $R_0, R_1, \dots, R_N \in [1, \infty)$.*

(1) *Let $m_1, \dots, m_N \in \mathbb{R}$ satisfy (3.1) and (3.2).*

(a) *If $0 < p < 2$, then*

$$\begin{aligned} & \|T_\sigma\|_{h^{p_1} \times \dots \times h^{p_N} \rightarrow h^p} \\ & \lesssim R_0^{n/2} \prod_{j=1}^N R_j^{\max\{n/p_j, n/2\}} \left\| \prod_{j=1}^N \langle \xi_j \rangle^{-m_j} \sigma(x, \xi) \right\|_{L_{ul, \xi}^2((\mathbb{R}^n)^N) L_x^\infty(\mathbb{R}^n)}. \end{aligned}$$

(b) *If $2 \leq p < \infty$, then*

$$\|T_\sigma\|_{h^{p_1} \times \dots \times h^{p_N} \rightarrow L^p} \lesssim R_0^{n/p} \prod_{j=1}^N R_j^{n/2} \left\| \prod_{j=1}^N \langle \xi_j \rangle^{-m_j} \sigma(x, \xi) \right\|_{L_{ul, \xi}^2((\mathbb{R}^n)^N) L_x^\infty(\mathbb{R}^n)}.$$

(2) *Let $m = -\sum_{j=1}^N \max\{n/p_j, n/2\}$. Then,*

$$\|T_\sigma\|_{h^{p_1} \times \dots \times h^{p_N} \rightarrow L^\infty} \lesssim \prod_{j=1}^N R_j^{n/2} \|\langle \xi \rangle^{-m} \sigma(x, \xi)\|_{L_{ul, \xi}^2((\mathbb{R}^n)^N) L_x^\infty(\mathbb{R}^n)}.$$

*In the above assertions, if some of the p_j 's, $j \in \{1, \dots, N\}$, are equal to ∞ , then the conclusions hold with the corresponding h^{p_j} replaced by *bmo*.*

3.3 Proof of Theorem 3.2

Boulkhemair [3] first pointed out that, in order to investigate the smoothness condition to assure the L^2 -boundedness of the linear pseudo-differential operators, it suffices to consider the boundedness for symbols whose Fourier supports are compact. Our strategy relies heavily on his idea. We shall proceed to the proof of Theorem 3.2. We

decompose the symbol σ into the sum of $\Delta_k \sigma$ over $k \in (\mathbb{N}_0)^{N+1}$. Since the support of $\mathcal{F}(\Delta_k \sigma)$ is included in $B_{R_0} \times B_{R_1} \times \dots \times B_{R_N}$ with $R_j = 2^{k_j+1}$, $j = 0, 1, \dots, N$, we see that Theorem 3.2 follows by applying Proposition 3.4 to the decomposed operators $T_{\Delta_k \sigma}$.

3.4 Symbols with Classical Derivatives

The following proposition shows that symbols that have classical derivatives up to certain order satisfy the conditions of Theorem 3.2.

Proposition 3.5 *Let $N \geq 2$, $m, m_1, \dots, m_N \in \mathbb{R}$, $s_0, s_1, \dots, s_N \in [0, \infty)$, and $t \in (0, \infty]$. If a bounded measurable function σ on $(\mathbb{R}^n)^{N+1}$ satisfies*

$$|\partial_x^{\alpha_0} \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \sigma(x, \xi_1, \dots, \xi_N)| \leq (1 + |\xi_1| + \dots + |\xi_N|)^m$$

or

$$|\partial_x^{\alpha_0} \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \sigma(x, \xi_1, \dots, \xi_N)| \leq \prod_{j=1}^N (1 + |\xi_j|)^{m_j}$$

for $\alpha_j \in (\mathbb{N}_0)^n$ with $|\alpha_j| \leq [s_j] + 1$, where $[s_j]$ is the integer part of s_j , then $\sigma \in S_{0,0}^m(s, t; \mathbb{R}^n, N)$ or $\sigma \in S_{0,0}^{m_j}(s, t; \mathbb{R}^n, N)$, respectively.

To be precise, the above assumptions should be understood that the derivatives of σ taken in the sense of distribution are locally integrable functions on $(\mathbb{R}^n)^{N+1}$ and they satisfy the inequality stated above almost everywhere.

Since statements quite similar to Proposition 3.5 are already proved in [26, Proposition 4.7] and [27, Proposition 5.4], we omit the proof here.

4 Lemmas for the Proof of Proposition 3.4

In this section, we collect some lemmas to prove Proposition 3.4. The following will be used to decompose symbols, which was essentially proved in [38, Lemma 2.2.1]. The explicit proof can be found in [27, Lemma 4.4].

Lemma 4.1 *There exist functions $\kappa \in \mathcal{S}(\mathbb{R}^n)$ and $\chi \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp } \kappa \subset [-1, 1]^n$, $\text{supp } \widehat{\chi} \subset B(0, 1)$, $|\chi| \geq c > 0$ on $[-1, 1]^n$ and*

$$\sum_{v \in \mathbb{Z}^n} \kappa(\xi - v) \chi(\xi - v) = 1, \quad \xi \in \mathbb{R}^n.$$

The two lemmas below play important roles to obtain the boundedness for the multilinear Hörmander class with the critical order (Theorem B) in [28]. We will again use them in the present paper. See [28, Lemmas 2.4 and 2.5] for these proofs.

Lemma 4.2 Let $N \geq 2$, $1 < r < \infty$, and let $a_1, \dots, a_N \in \mathbb{R}$ satisfy $-n/2 < a_j < 0$ and $\sum_{j=1}^N a_j = n/r - Nn/2$. Then the following holds for all nonnegative functions A_1, \dots, A_N on \mathbb{Z}^n :

$$\sum_{v_1, \dots, v_N \in \mathbb{Z}^n} A_0(v_1 + \dots + v_N) \prod_{j=1}^N (1 + |v_j|)^{a_j} A_j(v_j) \lesssim \|A_0\|_{\ell^r(\mathbb{Z}^n)} \prod_{j=1}^N \|A_j\|_{\ell^2(\mathbb{Z}^n)}.$$

Lemma 4.3 Let $N \geq 2$. Then the following holds for all nonnegative functions A_1, \dots, A_N on \mathbb{Z}^n :

$$\sum_{v_1, \dots, v_N \in \mathbb{Z}^n} (1 + |v_1| + \dots + |v_N|)^{-Nn/2} \prod_{j=1}^N A_j(v_j) \lesssim \prod_{j=1}^N \|A_j\|_{\ell^2(\mathbb{Z}^n)}.$$

For $0 < r < \infty$, we denote by S_r the operator

$$S_r(f)(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x - z)|^r}{\langle z \rangle^{n+1}} dz \right)^{1/r} \tag{4.1}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Obviously, S_r is bounded on L^p for $p \geq r$. The lemma below was proved in [27, Lemma 4.1] for the case $r = 2$. We extend it to the general case $0 < r < \infty$.

Lemma 4.4 Let $0 < r < \infty$ and $\kappa \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\text{supp } \kappa \subset [-1, 1]^n$. Then

$$|\kappa(D - \nu)f(x)| \lesssim S_r(\kappa(D - \nu)f)(y) \tag{4.2}$$

holds for any $f \in \mathcal{S}(\mathbb{R}^n)$, $\nu \in \mathbb{Z}^n$, and $x, y \in \mathbb{R}^n$ satisfying $|x - y| \lesssim 1$.

Proof Taking $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\varphi = 1$ on $[-1, 1]^n$ and $\text{supp } \varphi \subset [-2, 2]^n$, we have

$$\kappa(D - \nu)f(x) = \varphi(D - \nu)\kappa(D - \nu)f(x).$$

If $0 < r \leq 1$, the Nikol'skij inequality (see, e.g., [42, Sect. 1.3.2, Remark 1]) gives

$$\begin{aligned} |\kappa(D - \nu)f(x)| &\leq \left\| (\mathcal{F}^{-1}\varphi)(z)\kappa(D - \nu)f(x - z) \right\|_{L^1_{\langle z \rangle}(\mathbb{R}^n)} \\ &\lesssim \left\| (\mathcal{F}^{-1}\varphi)(z)\kappa(D - \nu)f(x - z) \right\|_{L^r_{\langle z \rangle}(\mathbb{R}^n)}. \end{aligned}$$

Here, we remark that the implicit constant above is independent of x and ν . Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $S_r(f)(x) \approx S_r(f)(y)$ for $|x - y| \lesssim 1$, we have

$$\left\| (\mathcal{F}^{-1}\varphi)(z)\kappa(D - \nu)f(x - z) \right\|_{L^r_{\langle z \rangle}(\mathbb{R}^n)} \lesssim S_r(\kappa(D - \nu)f)(x)$$

$$\approx S_r(\kappa(D - \nu)f)(y).$$

Hence, we obtain (4.2) for $0 < r \leq 1$. If $1 < r < \infty$, the Hölder’s inequality gives

$$\begin{aligned} |\kappa(D - \nu)f(x)| &\leq \int_{\mathbb{R}^n} |(\mathcal{F}^{-1}\varphi)(z)|^{1/r'+1/r} |\kappa(D - \nu)f(x - z)| dz \\ &\lesssim \left(\int_{\mathbb{R}^n} |(\mathcal{F}^{-1}\varphi)(z)| |\kappa(D - \nu)f(x - z)|^r dz \right)^{1/r} \\ &\lesssim S_r(\kappa(D - \nu)f)(x). \end{aligned}$$

Hence, using again that $S_r(f)(x) \approx S_r(f)(y)$ for $|x - y| \lesssim 1$, we have (4.2) for $1 < r < \infty$. □

The lemma below may be well-known (see [27, Sect. 2.3]). Also, the readers familiar with Wiener amalgam spaces may realize that the following inequality is immediately deduced from the embedding $W^{p,2} \hookrightarrow h^p$, $0 < p \leq 2$, proved in [19, Theorem 1.2].

Lemma 4.5 *Let $0 < p \leq 2$. If $g \in \mathcal{S}(\mathbb{R}^n)$ satisfies that $|g| \geq c > 0$ on $[-1, 1]^n$ with some positive constant c , then*

$$\|f\|_{h^p(\mathbb{R}^n)} \lesssim \|V_g f(x, \xi)\|_{L^2_{\xi}(\mathbb{R}^n)L^p_x(\mathbb{R}^n)}.$$

Proof Since $W(L^2, \ell^p) \hookrightarrow h^p$ for $0 < p \leq 2$ (see [27, Sect. 2.3]), it suffices to prove

$$\|f\|_{W(L^2, \ell^p)} \lesssim \|V_g f(x, \xi)\|_{L^2_{\xi}(\mathbb{R}^n)L^p_x(\mathbb{R}^n)}. \tag{4.3}$$

Since $|g(x - y)| \geq c$ for $x, y \in Q$, it holds that

$$\|f\|_{W(L^2, \ell^p)} = \|f(x + \nu)\|_{L^2_x(Q)\ell^p_y(\mathbb{Z}^n)} \lesssim \overline{\|g(x - y)f(x + \nu)\|_{L^2_x(Q)\ell^p_y(\mathbb{Z}^n)}}$$

for any $y \in Q$, which implies from the embedding $L^2(\mathbb{R}^n) \hookrightarrow L^2(Q)$ that

$$\begin{aligned} \|f\|_{W(L^2, \ell^p)} &\lesssim \overline{\|g(x - y)f(x + \nu)\|_{L^2_x(Q)\ell^p_y(\mathbb{Z}^n)L^p_y(Q)}} \\ &\leq \overline{\|g(x - y)f(x + \nu)\|_{L^2_x(\mathbb{R}^n)\ell^p_y(\mathbb{Z}^n)L^p_y(Q)}}. \end{aligned}$$

By recalling the definition of V_g stated in (2.2), the last quantity is identical with

$$\overline{\|g(x - y)f(x)\|_{L^2_x(\mathbb{R}^n)L^p_y(\mathbb{R}^n)}} = \|V_g f(y, \xi)\|_{L^2_{\xi}(\mathbb{R}^n)L^p_y(\mathbb{R}^n)}.$$

This completes the proof of (4.3). Here, note that the opposite inequality of (4.3) holds. However, since the equivalence is unnecessary here, we omit the detail. □

The following lemma was already given in [33, Lemma 3.2] for the case $p = 2$ and $R = 1$. We extend it to a bit more general form. Moreover, we remark that the inequality below implies the embedding $L^p \hookrightarrow W^{p,p'}$, $1 \leq p \leq 2$, proved in [8, Theorem 1.1].

Lemma 4.6 *Let $2 \leq p \leq \infty$, $R \geq 1$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\left\| \varphi\left(\frac{D - \nu}{R}\right) f(x) \right\|_{\ell_v^p(\mathbb{Z}^n) L_x^{p'}(\mathbb{R}^n)} \lesssim R^{n/p} \|f\|_{L^{p'}(\mathbb{R}^n)}.$$

Proof With the notation $\Phi = \mathcal{F}^{-1}\varphi$, the expression (2.1) with $\ell = 2\pi$ yields that

$$\begin{aligned} \varphi\left(\frac{D - \nu}{R}\right) f(x) &= R^n \int_{\mathbb{R}^n} e^{iy \cdot \nu} \Phi(Ry) f(x - y) dy \\ &= R^n \int_{2\pi Q} e^{iy \cdot \nu} \left\{ \sum_{v' \in \mathbb{Z}^n} \Phi(R(y + 2\pi v')) f(x - y - 2\pi v') \right\} dy. \end{aligned}$$

We realize that the function $\sum_{v' \in \mathbb{Z}^n} \Phi(R(y + 2\pi v')) f(x - y - 2\pi v')$ is $2\pi\mathbb{Z}^n$ -periodic with respect to the y -variable. Hence, we have by Hausdorff–Young’s inequality

$$\left\| \varphi\left(\frac{D - \nu}{R}\right) f(x) \right\|_{\ell_v^p}^{p'} \lesssim R^{np'} \int_{2\pi Q} \left| \sum_{v' \in \mathbb{Z}^n} \Phi(R(y + 2\pi v')) f(x - y - 2\pi v') \right|^{p'} dy.$$

Since $\sum_{v' \in \mathbb{Z}^n} |\Phi(R(y + 2\pi v'))| \lesssim 1$ for any $y \in \mathbb{R}^n$ and $R \geq 1$, by applying Hölder’s inequality to the sum over v' , the integral of the right hand side is bounded by

$$\int_{2\pi Q} \sum_{v' \in \mathbb{Z}^n} |\Phi(R(y + 2\pi v'))| |f(x - y - 2\pi v')|^{p'} dy = \|\Phi(Ry)\| \|f(x - y)\|^{p'}_{L^1_y},$$

where we again used (2.1) in the identity above. Therefore, we obtain

$$\left\| \varphi\left(\frac{D - \nu}{R}\right) f(x) \right\|_{\ell_v^p}^{p'} \lesssim R^{np'} \|\Phi(Ry)\| \|f(x - y)\|^{p'}_{L^1_y}.$$

Integrating over x , we have

$$\begin{aligned} \left\| \varphi\left(\frac{D - \nu}{R}\right) f(x) \right\|_{\ell_v^p L_x^{p'}}^{p'} &\lesssim R^{np'} \|\Phi(Ry)\| \|f(x - y)\|^{p'}_{L^1_y L^1_x} \\ &\approx R^{n(p'-1)} \|f\|_{L^{p'}}^{p'}, \end{aligned}$$

which completes the proof. □

5 Proof of Proposition 3.4

In this section, we will use the following notation: $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$, $\nu = (\nu_1, \dots, \nu_N) \in (\mathbb{Z}^n)^N$, and $d\xi = d\xi_1 \dots d\xi_N$. Also, we remark that, for any $p, p_j \in (0, \infty]$ satisfying $1/p \leq 1/p_1 + \dots + 1/p_N$, we can choose $\tilde{p}_j \in (0, \infty]$, $j =$

1, . . . , N, such that

$$\frac{1}{p} = \frac{1}{\tilde{p}_1} + \dots + \frac{1}{\tilde{p}_N} \quad \text{and} \quad p_j \leq \tilde{p}_j. \tag{5.1}$$

In fact, for instance, if $p = \infty$, then we can take $\tilde{p}_j = \infty$, and if $p < \infty$, then we can take

$$\frac{1}{\tilde{p}_j} = \frac{1}{p} \frac{1}{p_j} \left(\frac{1}{p_1} + \dots + \frac{1}{p_N} \right)^{-1}.$$

Now, we shall give a proof of Proposition 3.4. First, we decompose T_σ as follows. By Lemma 4.1, the symbol σ can be written as

$$\sigma(x, \xi) = \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \sigma(x, \xi) \prod_{j=1}^N \kappa(\xi_j - v_j) \chi(\xi_j - v_j) = \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \sigma_{\mathbf{v}}(x, \xi) \prod_{j=1}^N \kappa(\xi_j - v_j)$$

with

$$\sigma_{\mathbf{v}}(x, \xi) = \sigma(x, \xi) \prod_{j=1}^N \chi(\xi_j - v_j).$$

Then, by denoting the operators $\kappa(D - v_j)$ by \square_{v_j} , $j = 1, \dots, N$, we can write as

$$\begin{aligned} T_\sigma(f_1, \dots, f_N)(x) &= \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} T_{\sigma_{\mathbf{v}}}(\kappa(D - v_1)f_1, \dots, \kappa(D - v_N)f_N)(x) \\ &= \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} T_{\sigma_{\mathbf{v}}}(\square_{v_1}f_1, \dots, \square_{v_N}f_N)(x). \end{aligned} \tag{5.2}$$

Here we remark that, for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, it holds that $\|\langle v \rangle^s \square_v f\|_{\ell^q L^p} \lesssim \|f\|_{W_s^{p,q}}$, since $\text{supp } \kappa$ is compact. Now, this $T_{\sigma_{\mathbf{v}}}(\square_{v_1}f_1, \dots, \square_{v_N}f_N)$ satisfies the following inequality.

Lemma 5.1 *Let $N \geq 2$. For $j = 1, \dots, N$, let $m, m_j \in (-\infty, 0]$, $r_j \in (0, \infty)$, and $R_0, R_j \in [1, \infty)$. Suppose σ is a bounded continuous function on $(\mathbb{R}^n)^{N+1}$ satisfying $\text{supp } \mathcal{F}\sigma \subset B_{R_0} \times B_{R_1} \times \dots \times B_{R_N}$ and write $W(\xi) = \langle \xi \rangle^m$ or $\prod_{j=1}^N \langle \xi_j \rangle^{m_j}$. Then,*

$$\begin{aligned} &|T_{\sigma_{\mathbf{v}}}(\square_{v_1}f_1, \dots, \square_{v_N}f_N)(x)| \\ &\lesssim W(\mathbf{v}) \|W(\xi)^{-1} \sigma(x, \xi)\|_{L_{ul,\xi}^2((\mathbb{R}^n)^N) L_x^\infty(\mathbb{R}^n)} \prod_{j=1}^N \|S_{r_j}(\square_{v_j}f_j)(y + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})} \end{aligned}$$

holds for any $x, y \in \mathbb{R}^n$ satisfying $|x - y| \lesssim 1$, where $\Lambda_{R_j} = \mathbb{Z}^n \cap [-2R_j - 1, 2R_j + 1]^n$.

Proof Since the support of $(\mathcal{F}_{1,\dots,N}\sigma_{\mathbf{v}})(x, \cdot)$ is included in $B_{2R_1} \times \dots \times B_{2R_N}$ for any $x \in \mathbb{R}^n$,

$$\begin{aligned} T &:= T_{\sigma_{\mathbf{v}}}(\square_{v_1} f_1, \dots, \square_{v_N} f_N)(x) \\ &= \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} (\mathcal{F}_{1,\dots,N}\sigma_{\mathbf{v}})(x, \mathbf{z}) \prod_{j=1}^N \square_{v_j} f_j(x + z_j) \, d\mathbf{z} \\ &= \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} (\mathcal{F}_{1,\dots,N}\sigma_{\mathbf{v}})(x, \mathbf{z}) \prod_{j=1}^N \mathbf{1}_{B_{2R_j}}(z_j) \square_{v_j} f_j(x + z_j) \, d\mathbf{z}. \end{aligned}$$

Since the ball B_{2R_j} is covered by a disjoint union of the unit cubes $\tau + Q$, $\tau \in \Lambda_{R_j}$, the characteristic function $\mathbf{1}_{B_{2R_j}}$ is bounded by the sum of $\mathbf{1}_Q(\cdot - \tau)$ over $\tau \in \Lambda_{R_j}$. This yields

$$\begin{aligned} |T| &\leq \int_{(\mathbb{R}^n)^N} |(\mathcal{F}_{1,\dots,N}\sigma_{\mathbf{v}})(x, \mathbf{z})| \prod_{j=1}^N \mathbf{1}_{B_{2R_j}}(z_j) |\square_{v_j} f_j(x + z_j)| \, d\mathbf{z} \\ &\leq \sum_{\tau_1 \in \Lambda_{R_1}} \dots \sum_{\tau_N \in \Lambda_{R_N}} \int_{(\mathbb{R}^n)^N} |(\mathcal{F}_{1,\dots,N}\sigma_{\mathbf{v}})(x, \mathbf{z})| \prod_{j=1}^N \mathbf{1}_Q(z_j - \tau_j) |\square_{v_j} f_j(x + z_j)| \, d\mathbf{z}. \end{aligned} \tag{5.3}$$

Note that $|(x + z_j) - (y + \tau_j)| \lesssim 1$ if $|x - y| \lesssim 1$ and $z_j - \tau_j \in Q = [-1/2, 1/2]^n$. Then, by Lemma 4.4 and the Cauchy–Schwarz inequality, the integral above is estimated by

$$\begin{aligned} &\left(\prod_{j=1}^N S_{r_j}(\square_{v_j} f_j)(y + \tau_j) \right) \int_{(\mathbb{R}^n)^N} |(\mathcal{F}_{1,\dots,N}\sigma_{\mathbf{v}})(x, \mathbf{z})| \prod_{j=1}^N \mathbf{1}_Q(z_j - \tau_j) \, d\mathbf{z} \\ &= \left(\prod_{j=1}^N S_{r_j}(\square_{v_j} f_j)(y + \tau_j) \right) \int_{Q^N} |(\mathcal{F}_{1,\dots,N}\sigma_{\mathbf{v}})(x, \mathbf{z} + \boldsymbol{\tau})| \, d\mathbf{z} \\ &\leq \left(\prod_{j=1}^N S_{r_j}(\square_{v_j} f_j)(y + \tau_j) \right) \|(\mathcal{F}_{1,\dots,N}\sigma_{\mathbf{v}})(x, \mathbf{z} + \boldsymbol{\tau})\|_{L^2(Q^N)}, \end{aligned} \tag{5.4}$$

where we wrote $\boldsymbol{\tau} = (\tau_1, \dots, \tau_N)$. Combining with (5.3) and (5.4) and then using the Cauchy–Schwarz inequalities to the sums with respect to the τ_j 's, $j = 1, \dots, N$, we have

$$\begin{aligned}
 |T| &\lesssim \sum_{\tau_1 \in \Lambda_{R_1}} \cdots \sum_{\tau_N \in \Lambda_{R_N}} \left(\prod_{j=1}^N S_{r_j}(\square_{v_j} f_j)(y + \tau_j) \right) \|(\mathcal{F}_{1,\dots,N} \sigma_{\mathbf{v}})(x, \mathbf{z} + \boldsymbol{\tau})\|_{L^2_{\mathbf{z}}(Q^N)} \\
 &\leq \left(\prod_{j=1}^N \|S_{r_j}(\square_{v_j} f_j)(y + \tau_j)\|_{\ell^2_{\tau_j}(\Lambda_{R_j})} \right) \|(\mathcal{F}_{1,\dots,N} \sigma_{\mathbf{v}})(x, \mathbf{z} + \boldsymbol{\tau})\|_{L^2_{\mathbf{z}}(Q^N) \ell^2_{\boldsymbol{\tau}}((\mathbb{Z}^n)^N)}.
 \end{aligned}
 \tag{5.5}$$

Here, we apply (2.1) to the $L^2_{\mathbf{z}} \ell^2_{\boldsymbol{\tau}}$ -norm and then use Plancherel’s theorem to have

$$\begin{aligned}
 &\|(\mathcal{F}_{1,\dots,N} \sigma_{\mathbf{v}})(x, \mathbf{z} + \boldsymbol{\tau})\|_{L^2_{\mathbf{z}}(Q^N) \ell^2_{\boldsymbol{\tau}}((\mathbb{Z}^n)^N)}^2 \\
 &= \sum_{\boldsymbol{\tau} \in (\mathbb{Z}^n)^N} \int_{Q^N} |(\mathcal{F}_{1,\dots,N} \sigma_{\mathbf{v}})(x, \mathbf{z} + \boldsymbol{\tau})|^2 d\mathbf{z} \\
 &= \|(\mathcal{F}_{1,\dots,N} \sigma_{\mathbf{v}})(x, \mathbf{z})\|_{L^2_{\boldsymbol{\xi}}((\mathbb{R}^n)^N)}^2 = \|\sigma_{\mathbf{v}}(x, \boldsymbol{\xi})\|_{L^2_{\boldsymbol{\xi}}((\mathbb{R}^n)^N)}^2.
 \end{aligned}
 \tag{5.6}$$

Substituting the identity (5.6) into (5.5), we have

$$|T| \lesssim \|\sigma_{\mathbf{v}}(x, \boldsymbol{\xi})\|_{L^2_{\boldsymbol{\xi}}((\mathbb{R}^n)^N)} \prod_{j=1}^N \|S_{r_j}(\square_{v_j} f_j)(y + \tau_j)\|_{\ell^2_{\tau_j}(\Lambda_{R_j})}.
 \tag{5.7}$$

In what follows, we shall prove that

$$\sup_{x \in \mathbb{R}^n} \|\sigma_{\mathbf{v}}(x, \boldsymbol{\xi})\|_{L^2_{\boldsymbol{\xi}}((\mathbb{R}^n)^N)} \lesssim W(\mathbf{v}) \|W(\boldsymbol{\xi})^{-1} \sigma(x, \boldsymbol{\xi})\|_{L^2_{ul,\boldsymbol{\xi}}((\mathbb{R}^n)^N) L^\infty_x(\mathbb{R}^n)}
 \tag{5.8}$$

for $\mathbf{v} \in (\mathbb{Z}^n)^N$. By using (2.1) as in (5.6), we have

$$\begin{aligned}
 \|\sigma_{\mathbf{v}}(\cdot, \boldsymbol{\xi})\|_{L^2_{\boldsymbol{\xi}}} &= W(\mathbf{v}) \left\| W(\mathbf{v})^{-1} \sigma(\cdot, \boldsymbol{\xi} + \boldsymbol{\mu}) \right. \\
 &\quad \times \left. \prod_{j=1}^N \chi(\xi_j + \mu_j - \nu_j) \right\|_{L^2_{\boldsymbol{\xi}}(Q^N) \ell^2_{\boldsymbol{\mu}}((\mathbb{Z}^n)^N)} \\
 &\lesssim W(\mathbf{v}) \left\| W(\boldsymbol{\xi} + \boldsymbol{\mu})^{-1} W(\boldsymbol{\xi} + \boldsymbol{\mu} - \mathbf{v})^{-1} \right. \\
 &\quad \times \left. \sigma(\cdot, \boldsymbol{\xi} + \boldsymbol{\mu}) \prod_{j=1}^N \chi(\xi_j + \mu_j - \nu_j) \right\|_{L^2_{\boldsymbol{\xi}}(Q^N) \ell^2_{\boldsymbol{\mu}}((\mathbb{Z}^n)^N)}.
 \end{aligned}$$

Here the inequality above holds true since $m, m_j \in (-\infty, 0]$ are assumed. Since $\chi \in \mathcal{S}(\mathbb{R}^n)$, for some sufficiently large number $L > 0$, the $L^2 \ell^2$ -norm above is bounded by

$$\begin{aligned} & \left\| \left\{ W(\xi + \mu)^{-1} \sigma(\cdot, \xi + \mu) \right\} W(\mu - \nu)^{-1} \prod_{j=1}^N \langle \mu_j - \nu_j \rangle^{-L} \right\|_{L^2_{\xi}(\mathbb{Q}^N) \ell^2_{\mu}} \\ & \lesssim \left\| W(\xi)^{-1} \sigma(\cdot, \xi) \right\|_{L^2_{ul,\xi}}, \end{aligned}$$

which gives (5.8). Therefore, by (5.7) and (5.8), we complete the proof. □

Now, we shall proceed to the estimates of the operators considered in Proposition 3.4. In order to simplify the notations appearing in Lemma 5.1, let us denote

$$|\sigma|_m = \| \langle \xi \rangle^{-m} \sigma(x, \xi) \|_{L^2_{ul,\xi}(\mathbb{R}^n) L^{\infty}_x(\mathbb{R}^n)}, \tag{5.9}$$

$$|\sigma|_m = \left\| \prod_{j=1}^N \langle \xi_j \rangle^{-m_j} \sigma(x, \xi) \right\|_{L^2_{ul,\xi}(\mathbb{R}^n) L^{\infty}_x(\mathbb{R}^n)}, \tag{5.10}$$

and further for $0 < p_j \leq \infty$ and $0 < r_j < \infty$

$$F_{\nu_j}^{p_j, r_j}(x) = S_{r_j}(\langle \nu_j \rangle^{\alpha(p_j)} \square_{\nu_j} f_j)(x), \quad \alpha(p_j) = n/2 - \max\{n/2, n/p_j\}. \tag{5.11}$$

5.1 Proof of Proposition 3.4 (1)-(a)

Take a real valued function $g \in \mathcal{S}(\mathbb{R}^n)$ satisfying $|g| \geq c > 0$ on $[-1, 1]^n$ and $\text{supp } \widehat{g} \subset B_1$. We have by Lemma 4.5 and duality

$$\begin{aligned} & \|T_{\sigma}(f_1, \dots, f_N)\|_{h^p} \lesssim \|V_g[T_{\sigma}(f_1, \dots, f_N)](x, \zeta)\|_{L^2_{\zeta}(\mathbb{R}^n) L^p_x(\mathbb{R}^n)} \\ & = \left\| \sup_{h \in L^2(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} V_g[T_{\sigma}(f_1, \dots, f_N)](x, \zeta) h(\zeta) d\zeta \right| \right\|_{L^p_x(\mathbb{R}^n)}. \end{aligned} \tag{5.12}$$

Hence, in what follows, we consider

$$I := \int_{\mathbb{R}^n} V_g[T_{\sigma}(f_1, \dots, f_N)](x, \zeta) h(\zeta) d\zeta \tag{5.13}$$

for $x \in \mathbb{R}^n$ and $h \in L^2(\mathbb{R}^n)$, which is decomposed by (5.2) as

$$I = \sum_{\nu \in (\mathbb{Z}^n)^N} \int_{\mathbb{R}^n} V_g[T_{\sigma_{\nu}}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)](x, \zeta) h(\zeta) d\zeta.$$

Here, we shall observe that

$$\text{supp } V_g[T_{\sigma_{\nu}}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)](x, \cdot) \subset \{ \zeta \in \mathbb{R}^n : |\zeta - (\nu_1 + \dots + \nu_N)| \lesssim R_0 \}. \tag{5.14}$$

In fact, since $\text{supp } \mathcal{F}_0 \sigma_{\mathbf{v}}(\cdot, \boldsymbol{\xi}) \subset B_{R_0}$ and $\text{supp } \kappa(\cdot - \nu_j) \subset \nu_j + [-1, 1]^n$, the identity

$$\begin{aligned} & \mathcal{F}[T_{\sigma_{\mathbf{v}}}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)](\zeta) \\ &= \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} (\mathcal{F}_0 \sigma_{\mathbf{v}})(\zeta - (\xi_1 + \dots + \xi_N), \boldsymbol{\xi}) \prod_{j=1}^N \kappa(\xi_j - \nu_j) \widehat{f}_j(\xi_j) d\xi \end{aligned}$$

implies that

$$\text{supp } \mathcal{F}[T_{\sigma_{\mathbf{v}}}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)] \subset \{\zeta \in \mathbb{R}^n : |\zeta - (\nu_1 + \dots + \nu_N)| \lesssim R_0\}. \tag{5.15}$$

Hence, regarding the short-time Fourier transform given in (2.2) as

$$V_g[T_{\sigma_{\mathbf{v}}}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)](x, \zeta) = \mathcal{F}[g(\cdot - x)T_{\sigma_{\mathbf{v}}}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)](\zeta),$$

we see that (5.14) holds. Now, we take a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\varphi = 1$ on $\{\zeta \in \mathbb{R}^n : |\zeta| \lesssim 1\}$. Then, the expression I considered in (5.13) can be written as

$$\begin{aligned} I &= \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \int_{\mathbb{R}^n} V_g[T_{\sigma_{\mathbf{v}}}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)](x, \zeta) \\ &\quad \times \varphi\left(\frac{\zeta - (\nu_1 + \dots + \nu_N)}{R_0}\right) h(\zeta) d\zeta \\ &= \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \int_{\mathbb{R}^n} g(t) T_{\sigma_{\mathbf{v}}}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)(x + t) \\ &\quad \times \mathcal{F}\left[\varphi\left(\frac{\cdot - (\nu_1 + \dots + \nu_N)}{R_0}\right) h\right](x + t) dt. \end{aligned}$$

By (2.1), we can further rewrite the above as

$$\begin{aligned} I &= \sum_{\mu \in \mathbb{Z}^n} \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \int_Q g(\mu + t) T_{\sigma_{\mathbf{v}}}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)(x + \mu + t) \\ &\quad \times \mathcal{F}\left[\varphi\left(\frac{\cdot - (\nu_1 + \dots + \nu_N)}{R_0}\right) h\right](x + \mu + t) dt. \end{aligned} \tag{5.16}$$

Now, we shall actually estimate the expression in (5.16). Using the fact that, for sufficiently large $L > 0$, $|g(\mu + t)| \lesssim \langle \mu \rangle^{-L}$ holds for $t \in Q$, we have

$$\begin{aligned} |I| &\leq \sum_{\mu \in \mathbb{Z}^n} \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \int_Q |g(\mu + t)| |T_{\sigma_{\mathbf{v}}}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)(x + \mu + t)| \\ &\quad \times \left| \mathcal{F}\left[\varphi\left(\frac{\cdot - (\nu_1 + \dots + \nu_N)}{R_0}\right) h\right](x + \mu + t) \right| dt \end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{\mu \in \mathbb{Z}^n} \langle \mu \rangle^{-L} \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \int_Q |T_{\sigma_{\mathbf{v}}}(\square_{v_1} f_1, \dots, \square_{v_N} f_N)(x + \mu + t)| \\ &\quad \times \left| \mathcal{F} \left[\varphi \left(\frac{\cdot - (v_1 + \dots + v_N)}{R_0} \right) h \right] (x + \mu + t) \right| dt. \end{aligned}$$

Since $|(x + \mu + t) - (x + \mu)| \lesssim 1$ for $t \in Q$, Lemma 5.1 with $W(\xi) = \prod_{j=1}^N \langle \xi_j \rangle^{m_j}$ yields that

$$\begin{aligned} &|T_{\sigma_{\mathbf{v}}}(\square_{v_1} f_1, \dots, \square_{v_N} f_N)(x + \mu + t)| \\ &\lesssim |\sigma|_m \prod_{j=1}^N \langle v_j \rangle^{m_j} \|S_{r_j}(\square_{v_j} f_j)(x + \mu + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})} \end{aligned}$$

holds for any $t \in Q$, where we used the notation (5.10), and thus,

$$\begin{aligned} |I| &\lesssim |\sigma|_m \sum_{\mu \in \mathbb{Z}^n} \langle \mu \rangle^{-L} \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \prod_{j=1}^N \langle v_j \rangle^{m_j} \|S_{r_j}(\square_{v_j} f_j)(x + \mu + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})} \\ &\quad \times \int_Q \left| \mathcal{F} \left[\varphi \left(\frac{\cdot - (v_1 + \dots + v_N)}{R_0} \right) h \right] (x + \mu + t) \right| dt. \end{aligned} \tag{5.17}$$

Here, we observe from the notation (5.11) that

$$S_{r_j}(\square_{v_j} f_j) = \langle v_j \rangle^{-\alpha(p_j)} S_{r_j}(\langle v_j \rangle^{\alpha(p_j)} \square_{v_j} f_j) = \langle v_j \rangle^{-\alpha(p_j)} F_{v_j}^{p_j, r_j}. \tag{5.18}$$

Moreover, recalling $Q = [-1/2, 1/2]^n$, we have by the Cauchy–Schwarz inequality and the Plancherel theorem

$$\begin{aligned} &\int_Q \left| \mathcal{F} \left[\varphi \left(\frac{\cdot - (v_1 + \dots + v_N)}{R_0} \right) h \right] (x + \mu + t) \right| dt \\ &\leq \left\| \mathcal{F} \left[\varphi \left(\frac{\cdot - (v_1 + \dots + v_N)}{R_0} \right) h \right] (x + \mu + t) \right\|_{L_t^2(\mathbb{R}^n)} \\ &= \left\| \varphi \left(\frac{\cdot - (v_1 + \dots + v_N)}{R_0} \right) h \right\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{5.19}$$

Gathering together (5.17), (5.18), and (5.19), we obtain

$$\begin{aligned} |I| &\lesssim |\sigma|_m \sum_{\mu \in \mathbb{Z}^n} \langle \mu \rangle^{-L} \\ &\quad \times \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \prod_{j=1}^N \langle v_j \rangle^{m_j - \alpha(p_j)} \|F_{v_j}^{p_j, r_j}(x + \mu + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})} \left\| \varphi \left(\frac{\cdot - (v_1 + \dots + v_N)}{R_0} \right) h \right\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Since the assumptions (3.1) and (3.2) imply respectively that $-n/2 < m_j - \alpha(p_j) < 0$ and $\sum_{j=1}^N (m_j - \alpha(p_j)) = n/2 - Nn/2$, we have by Lemma 4.2 with $r = 2$

$$|I| \lesssim |\sigma|_m \sum_{\mu \in \mathbb{Z}^n} \langle \mu \rangle^{-L} \left\| \varphi\left(\frac{\cdot - \nu}{R_0}\right) h \right\|_{L^2 \ell_v^2} \prod_{j=1}^N \|F_{v_j}^{p_j, r_j}(x + \mu + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{v_j}^2},$$

and further have by using that $\|\varphi((x - \nu)/R_0)\|_{\ell_v^2} \lesssim R_0^{n/2}$ for any $x \in \mathbb{R}^n$

$$|I| \lesssim R_0^{n/2} \|h\|_{L^2} |\sigma|_m \sum_{\mu \in \mathbb{Z}^n} \langle \mu \rangle^{-L} \prod_{j=1}^N \|F_{v_j}^{p_j, r_j}(x + \mu + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{v_j}^2}. \tag{5.20}$$

Collecting (5.12), (5.13), and (5.20), we obtain

$$\|T_\sigma(f_1, \dots, f_N)\|_{h^p} \lesssim R_0^{n/2} |\sigma|_m \left\| \sum_{\mu \in \mathbb{Z}^n} \langle \mu \rangle^{-L} \prod_{j=1}^N \|F_{v_j}^{p_j, r_j}(x + \mu + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{v_j}^2} \right\|_{L_x^p}.$$

Apply the embedding $\ell^{\min\{1, p\}} \hookrightarrow \ell^1$ to the sum over μ and choose $L > n/\min\{1, p\}$. Then, by Minkowski’s inequality, the L_x^p quasi-norm above is bounded by

$$\begin{aligned} & \left\| \langle \mu \rangle^{-L} \left\| \prod_{j=1}^N \|F_{v_j}^{p_j, r_j}(x + \mu + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{v_j}^2} \right\|_{L_x^p} \right\|_{\ell_\mu^{\min\{1, p\}}} \\ & \approx \left\| \prod_{j=1}^N \|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{v_j}^2} \right\|_{L_x^p}. \end{aligned}$$

We take $\tilde{p}_j \in (0, \infty]$, $j = 1, \dots, N$, satisfying (5.1) and use Hölder’s inequality to have

$$\|T_\sigma(f_1, \dots, f_N)\|_{h^p} \lesssim R_0^{n/2} |\sigma|_m \prod_{j=1}^N \|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{v_j}^2 L_x^{\tilde{p}_j}}. \tag{5.21}$$

Using that $\ell_{\tau_j}^{\min\{2, p_j\}} \hookrightarrow \ell_{\tau_j}^2$, since $\min\{2, p_j\} \leq \tilde{p}_j$, we have by Minkowski’s inequality

$$\begin{aligned} & \|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{v_j}^2 L_x^{\tilde{p}_j}} \leq \|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^{\min\{2, p_j\}}(\Lambda_{R_j}) \ell_{v_j}^2 L_x^{\tilde{p}_j}} \\ & \leq \|F_{v_j}^{p_j, r_j}(x)\|_{\ell_{v_j}^2 L_x^{\tilde{p}_j} \ell_{\tau_j}^{\min\{2, p_j\}}(\Lambda_{R_j})} \approx R_j^{\max\{n/2, n/p_j\}} \|F_{v_j}^{p_j, r_j}\|_{\ell_{v_j}^2 L^{\tilde{p}_j}}. \end{aligned} \tag{5.22}$$

Here, recall the definition of the operator S_r from (4.1) and the fact which $r_j \in (0, \infty)$ can be chosen arbitrarily. Then, by taking $r_j < \min\{1, \tilde{p}_j\}$, Minkowski’s inequality for integrals implies

$$\begin{aligned}
 \|F_{v_j}^{p_j, r_j}\|_{\ell_{v_j}^2 L^{\tilde{p}_j}} &= \|S_{r_j}(\langle v_j \rangle^{\alpha(p_j)} \square_{v_j} f_j)\|_{\ell_{v_j}^2 L^{\tilde{p}_j}} \\
 &= \left\| \int_{\mathbb{R}^n} \langle y \rangle^{-(n+1)} |\langle v_j \rangle^{\alpha(p_j)} \square_{v_j} f_j(x-y)|^{r_j} dy \right\|_{\ell_{v_j}^{2/r_j} L_x^{\tilde{p}_j/r_j}}^{1/r_j} \\
 &\leq \left(\int_{\mathbb{R}^n} \langle y \rangle^{-(n+1)} \|\langle v_j \rangle^{\alpha(p_j)} \square_{v_j} f_j(x)\|_{\ell_{v_j}^2 L_x^{\tilde{p}_j}}^{r_j} dy \right)^{1/r_j} \\
 &\approx \|\langle v_j \rangle^{\alpha(p_j)} \square_{v_j} f_j\|_{\ell_{v_j}^2 L^{\tilde{p}_j}} \lesssim \|f_j\|_{W_{\alpha(p_j)}^{\tilde{p}_j, 2}}.
 \end{aligned} \tag{5.23}$$

Use (2.3) with $p_j \leq \tilde{p}_j$, and then use (2.4) if $p_j < \infty$ and (2.5) if $p_j = \infty$. Then we have

$$\|f_j\|_{W_{\alpha(p_j)}^{\tilde{p}_j, 2}} \lesssim \|f_j\|_{W_{\alpha(p_j)}^{p_j, 2}} \lesssim \|f_j\|_{h^{p_j}}, \tag{5.24}$$

where h^{p_j} can be replaced by bmo when $p_j = \infty$. Hence, gathering (5.22), (5.23), and (5.24), we obtain

$$\|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{v_j}^2 L_x^{\tilde{p}_j}} \lesssim R_j^{\max\{n/2, n/p_j\}} \|f_j\|_{h^{p_j}}. \tag{5.25}$$

Lastly, substituting (5.25) into (5.21), we obtain

$$\|T_\sigma(f_1, \dots, f_N)\|_{h^p} \lesssim |\sigma|_m R_0^{n/2} \prod_{j=1}^N R_j^{\max\{n/2, n/p_j\}} \|f_j\|_{h^{p_j}},$$

which completes the proof of Proposition 3.4 (1)-(a).

5.2 Proof of Proposition 3.4 (1)-(b)

We take a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\varphi = 1$ on $\{\zeta \in \mathbb{R}^n : |\zeta| \lesssim 1\}$. By (5.2) and (5.15), we have

$$\begin{aligned}
 &\|T_\sigma(f_1, \dots, f_N)\|_{L^p} \\
 &= \sup_{h \in L^{p'}} \left| \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \int_{\mathbb{R}^n} T_{\sigma_{\mathbf{v}}}(\square_{v_1} f_1, \dots, \square_{v_N} f_N)(x) h(x) dx \right| \\
 &= \sup_{h \in L^{p'}} \left| \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} \int_{\mathbb{R}^n} T_{\sigma_{\mathbf{v}}}(\square_{v_1} f_1, \dots, \square_{v_N} f_N)(x) \varphi\left(\frac{D + v_1 + \dots + v_N}{R_0}\right) h(x) dx \right|.
 \end{aligned} \tag{5.26}$$

In what follows, we consider

$$II := \sum_{\nu \in (\mathbb{Z}^n)^N} \int_{\mathbb{R}^n} T_{\sigma_\nu}(\square_{\nu_1} f_1, \dots, \square_{\nu_N} f_N)(x) \varphi\left(\frac{D + \nu_1 + \dots + \nu_N}{R_0}\right) h(x) dx. \tag{5.27}$$

Here, observe that Lemma 5.1 holds for $y = x$. Then, by Lemma 5.1 with $W(\xi) = \prod_{j=1}^N \langle \xi_j \rangle^{m_j}$ and the notations (5.10) and (5.11), we have

$$\begin{aligned} |II| &\lesssim |\sigma| m \sum_{\nu \in (\mathbb{Z}^n)^N} \int_{\mathbb{R}^n} \prod_{j=1}^N \langle \nu_j \rangle^{m_j} \|S_{r_j}(\square_{\nu_j} f_j)(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})} \\ &\quad \times \left| \varphi\left(\frac{D + \nu_1 + \dots + \nu_N}{R_0}\right) h(x) \right| dx \\ &= |\sigma| m \int_{\mathbb{R}^n} \sum_{\nu \in (\mathbb{Z}^n)^N} \prod_{j=1}^N \langle \nu_j \rangle^{m_j - \alpha(p_j)} \|F_{\nu_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})} \\ &\quad \times \left| \varphi\left(\frac{D + \nu_1 + \dots + \nu_N}{R_0}\right) h(x) \right| dx. \end{aligned}$$

Note here that (3.1) and (3.2) imply respectively that $-n/2 < m_j - \alpha(p_j) < 0$ and $\sum_{j=1}^N (m_j - \alpha(p_j)) = n/p - Nn/2$. Then, using Lemma 4.2 with $r = p$ and Hölder’s inequality with $\tilde{p}_j \in (0, \infty]$ satisfying (5.1), the integral above is estimated by

$$\begin{aligned} &\int_{\mathbb{R}^n} \left\| \varphi\left(\frac{D + \nu}{R_0}\right) h(x) \right\|_{\ell_v^p} \prod_{j=1}^N \|F_{\nu_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{\nu_j}^2} dx \\ &\leq \left\| \varphi\left(\frac{D + \nu}{R_0}\right) h \right\|_{\ell_v^p L^{p'}} \prod_{j=1}^N \|F_{\nu_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{\nu_j}^2 L_x^{\tilde{p}_j}}, \end{aligned}$$

which implies, from Lemma 4.6, that

$$|II| \lesssim R_0^{n/p} \|h\|_{L^{p'}} |\sigma| m \prod_{j=1}^N \|F_{\nu_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{\nu_j}^2 L_x^{\tilde{p}_j}}.$$

Since $2 \leq \tilde{p}_j \leq \infty$ for the case $2 \leq p < \infty$ of this subsection, Minkowski’s inequality gives

$$\|F_{\nu_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j}) \ell_{\nu_j}^2 L_x^{\tilde{p}_j}} \leq \|F_{\nu_j}^{p_j, r_j}(x)\|_{\ell_{\nu_j}^2 L_x^{\tilde{p}_j} \ell_{\tau_j}^2(\Lambda_{R_j})} \approx R_j^{n/2} \|F_{\nu_j}^{p_j, r_j}\|_{\ell_{\nu_j}^2 L^{\tilde{p}_j}}.$$

Moreover, we have by (5.23) and (5.24)

$$\|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})\ell_{v_j}^2 L_x^{\tilde{p}_j}} \lesssim R_j^{n/2} \|f_j\|_{W_{\alpha(p_j)}^{\tilde{p}_j, 2}} \lesssim R_j^{n/2} \|f_j\|_{h^{p_j}},$$

where h^{p_j} can be replaced by bmo when $p_j = \infty$. Gathering the above inequalities, we obtain

$$|II| \lesssim |\sigma|_m R_0^{n/p} \prod_{j=1}^N R_j^{n/2} \|f_j\|_{h^{p_j}} \|h\|_{L^{p'}}$$

where h^{p_j} can be replaced by bmo when $p_j = \infty$. Combining the above inequality with (5.26) and (5.27), we complete the proof of Proposition 3.4 (1)-(b).

5.3 Proof of Proposition 3.4 (2)

By using (5.2) and Lemma 5.1 with $W(\xi) = \langle \xi \rangle^m$,

$$\begin{aligned} |T_\sigma(f_1, \dots, f_N)(x)| &\leq \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} |T_{\sigma_{\mathbf{v}}}(\square_{v_1} f_1, \dots, \square_{v_N} f_N)(x)| \\ &\lesssim |\sigma|_m \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} (1 + |v_1| + \dots + |v_N|)^m \prod_{j=1}^N \|S_{r_j}(\square_{v_j} f_j)(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})}, \end{aligned}$$

where $|\sigma|_m$ is as in (5.9). Since $m = -Nn/2 + \sum_{j=1}^N \alpha(p_j)$, where $\alpha(p_j) \leq 0$ (see (5.11)), the sum over \mathbf{v} is bounded by

$$\begin{aligned} &\sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} (1 + |v_1| + \dots + |v_N|)^{-Nn/2} \prod_{j=1}^N \langle v_j \rangle^{\alpha(p_j)} \|S_{r_j}(\square_{v_j} f_j)(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})} \\ &= \sum_{\mathbf{v} \in (\mathbb{Z}^n)^N} (1 + |v_1| + \dots + |v_N|)^{-Nn/2} \prod_{j=1}^N \|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})} \end{aligned}$$

(for the notation $F_{v_j}^{p_j, r_j}$, see (5.11)). By Lemma 4.3, the sum above is further estimated by

$$\prod_{j=1}^N \|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})\ell_{v_j}^2},$$

which yields that

$$|T_\sigma(f_1, \dots, f_N)(x)| \lesssim |\sigma|_m \prod_{j=1}^N \|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})\ell_{v_j}^2}. \tag{5.28}$$

As in (5.25), we shall show that

$$\sup_{x \in \mathbb{R}^n} \|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})\ell_{v_j}^2} \lesssim R_j^{n/2} \|f_j\|_{h^{p_j}}, \tag{5.29}$$

where h^{p_j} can be replaced by *bmo* when $p_j = \infty$. Since (5.23) holds for $\tilde{p}_j = \infty$, we have

$$\|F_{v_j}^{p_j, r_j}(x + \tau_j)\|_{\ell_{\tau_j}^2(\Lambda_{R_j})\ell_{v_j}^2 L_x^\infty} \lesssim R_j^{n/2} \|F_{v_j}^{p_j, r_j}\|_{\ell_{v_j}^2 L^\infty} \lesssim R_j^{n/2} \|f_j\|_{W_{\alpha(p_j)}^{\infty, 2}}.$$

Here, since $p_j \leq \infty, j = 1, \dots, N$, the embeddings (2.3)–(2.5) of Lemma 2.1 yield that

$$\|f_j\|_{W_{\alpha(p_j)}^{\infty, 2}} \lesssim \|f_j\|_{W_{\alpha(p_j)}^{p_j, 2}} \lesssim \|f_j\|_{h^{p_j}},$$

where h^{p_j} can be replaced by *bmo* when $p_j = \infty$. This concludes (5.29). Therefore, substituting (5.29) into (5.28), we obtain

$$\|T_\sigma(f_1, \dots, f_N)\|_{L^\infty} \lesssim |\sigma|_m \prod_{j=1}^N R_j^{n/2} \|f_j\|_{h^{p_j}}$$

with h^{p_j} replaced by *bmo* when $p_j = \infty$, which completes the proof of Proposition 3.4 (2).

6 Sharpness

In this section, we consider the sharpness of the conditions of the order $m \in \mathbb{R}$ and the smoothness $s = (s_0, s_1, \dots, s_N) \in [0, \infty)^{N+1}$ stated in Theorem 1.2.

6.1 Sharpness of m of Theorem 1.2

In this subsection, we show the following.

Proposition 6.1 *Let $N \geq 2, p, p_1, \dots, p_N \in (0, \infty], 1/p \leq 1/p_1 + \dots + 1/p_N, m \in \mathbb{R}, s_0, s_1, \dots, s_N \in [0, \infty)$, and $t \in (0, \infty]$. If*

$$\text{Op}(S_{0,0}^m(s, t; \mathbb{R}^n, N)) \subset B(H^{p_1} \times \dots \times H^{p_N} \rightarrow L^p),$$

with L^p replaced by BMO when $p = \infty$, then

$$m \leq \min \left\{ \frac{n}{p}, \frac{n}{2} \right\} - \sum_{j=1}^N \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\}. \tag{6.1}$$

This is immediately obtained by the inclusion $S_{0,0}^m(\mathbb{R}^n, N) \subset S_{0,0}^m(s, t; \mathbb{R}^n, N)$ stated in Proposition 3.5 and the following theorem proved in [28, Theorem 1.5].

Theorem 6.2 *Let $N \geq 2$, $0 < p, p_1, \dots, p_N \leq \infty$, $1/p \leq 1/p_1 + \dots + 1/p_N$, and $m \in \mathbb{R}$. If*

$$\text{Op}(S_{0,0}^m(\mathbb{R}^n, N)) \subset B(H^{p_1} \times \dots \times H^{p_N} \rightarrow L^p),$$

with L^p replaced by BMO when $p = \infty$, then (6.1) holds.

6.2 Sharpness of s_0 of Theorem 1.2

In this subsection, we show the following.

Proposition 6.3 *Let $N \geq 2$, $p, p_1, \dots, p_N \in (0, \infty]$, $s_0, s_1, \dots, s_N \in [0, \infty)$, $t \in (0, \infty]$, and*

$$m = \min \left\{ \frac{n}{p}, \frac{n}{2} \right\} - \sum_{j=1}^N \max \left\{ \frac{n}{p_j}, \frac{n}{2} \right\}. \tag{6.2}$$

Suppose that the estimate

$$\|T_\sigma\|_{H^{p_1} \times \dots \times H^{p_N} \rightarrow L^p} \lesssim \|2^{k \cdot s} \|\langle \xi \rangle^{-m} \Delta_k \sigma(x, \xi)\|_{L_{x,\xi}^\infty((\mathbb{R}^n)^{N+1})} \| \ell_k^t \|_{\ell_k^t((\mathbb{N}_0)^{N+1})} \tag{6.3}$$

holds for all smooth functions σ with the right hand side finite, where L^p is replaced by BMO for $p = \infty$. Then $s_0 \geq \min\{n/p, n/2\}$.

To show this, we will use the following lemma which was given by Wainger [43, Theorem 10] and by Miyachi and Tomita [32, Lemma 6.1].

Lemma 6.4 *Let $0 < a < 1$, $0 < b < n$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. For $\epsilon > 0$, set*

$$f_{a,b,\epsilon}(x) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} e^{-\epsilon|k|} |k|^{-b} e^{i|k|^a} e^{ik \cdot x} \varphi(x).$$

If $1 \leq p \leq \infty$ and $b > n - an/2 - n/p + an/p$, then $\sup_{\epsilon > 0} \|f_{a,b,\epsilon}\|_{L^p(\mathbb{R}^n)} < \infty$.

Now, let us begin with the proof of Proposition 6.3. See also [27, Proposition 7.3].

Proof In this proof, for $p_j \in (0, \infty]$, we define the sets J and J^c by

$$J = \{j \in \{1, \dots, N\} : 2 \leq p_j \leq \infty\}, \quad J^c = \{j \in \{1, \dots, N\} : 0 < p_j < 2\}.$$

It is sufficient to show that the condition $s_0 \geq \min\{n/p, n/2\}$ is deduced under the assumption (6.3) with $t = \infty$. In fact, once this is proved, then replacing s_j by $s_j + \epsilon, \epsilon > 0, j = 0, 1, \dots, N$, we see that (6.3) with $t \in (0, \infty)$ implies $s_0 + \epsilon \geq \min\{n/p, n/2\}$. Thus since $\epsilon > 0$ is arbitrary, we must have $s_0 \geq \min\{n/p, n/2\}$.

Suppose (6.3) holds with $t = \infty$. For $0 < \delta_1, \delta_2 < 1$, we take real-valued radial functions $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{aligned} \text{supp } \varphi &\subset \{\xi \in \mathbb{R}^n : |\xi| \leq \delta_1\}, & \int \varphi \neq 0, & \int \varphi^2 \neq 0, \\ \text{supp } \psi &\subset \{\xi \in \mathbb{R}^n : 2^{-1/2-\delta_2} \leq |\xi| \leq 2^{1/2+\delta_2}\}, \\ \psi &= 1 \quad \text{on} \quad \{\xi \in \mathbb{R}^n : 2^{-1/2+\delta_2} \leq |\xi| \leq 2^{1/2-\delta_2}\}. \end{aligned} \tag{6.4}$$

(We note that $\mathcal{F}^{-1}\psi$ has integral zero.) For $\delta_3 > 0$ and $A \in \mathbb{N}$, we set

$$D_A = \{\ell \in \mathbb{Z}^n : 2^{A-\delta_3} \leq |\ell| \leq 2^{A+\delta_3}\}.$$

Here, notice that there exist $\delta_1, \delta_2, \delta_3 > 0$ such that for any $A \in \mathbb{N}$

$$\psi(2^{-A}\cdot) = 1 \quad \text{on} \quad \text{supp } \varphi(\cdot - \ell) \quad \text{with} \quad \ell \in D_A \tag{6.5}$$

(for instance, take $\delta_1 = 2^{-10}, \delta_2 = 2^{-2}$, and $\delta_3 = 2^{-3}$). For $A \in \mathbb{N}$ and $\epsilon > 0$ we set

$$\begin{aligned} \sigma_A(x, \xi) &= \varphi(x) e^{-ix \cdot (\xi_1 + \dots + \xi_N)} \sum_{\ell_1, \dots, \ell_N \in D_A} \langle \ell \rangle^{m-s_0} \left(\prod_{j \in J} e^{-i|\ell_j|^{a_j}} \right) \left(\prod_{j=1}^N \varphi(\xi_j - \ell_j) \right), \\ f_{a_j, b_j, \epsilon}(x) &= \sum_{\ell_j \in \mathbb{Z}^n \setminus \{0\}} e^{-\epsilon|\ell_j|} |\ell_j|^{-b_j} e^{i|\ell_j|^{a_j}} e^{i\ell_j \cdot x} \mathcal{F}^{-1}\varphi(x), \quad j \in J, \\ f_{j,A}(x) &= 2^{An/p_j} (\mathcal{F}^{-1}\psi)(2^A x), \quad j \in J^c, \end{aligned}$$

where $\ell = (\ell_1, \dots, \ell_N) \in (\mathbb{Z}^n)^N, 0 < a_j < 1$, and $b_j = n - a_j n/2 - n/p_j + a_j n/p_j + \epsilon_j$ with $\epsilon_j > 0$. Here, we choose sufficiently small $\epsilon_j > 0$ satisfying $0 < b_j < n$.

Firstly, we show that

$$\|2^{k \cdot s} \langle \xi \rangle^{-m} \Delta_k \sigma_A(x, \xi)\|_{L_{x, \xi}^\infty \ell_k^\infty} \lesssim 1, \tag{6.6}$$

$$\|f_{a_j, b_j, \epsilon}\|_{H^{p_j}} \lesssim 1, \quad j \in J, \tag{6.7}$$

$$\|f_{j,A}\|_{H^{p_j}} \lesssim 1, \quad j \in J^c, \tag{6.8}$$

with the implicit constants independent of $A \in \mathbb{N}$ and $\epsilon > 0$. Since $H^p = L^p$ for $2 \leq p \leq \infty$, (6.7) follows from Lemma 6.4. Since $\|f_{j,A}\|_{H^{p_j}} = \|\mathcal{F}^{-1}\psi\|_{H^{p_j}}$,

(6.8) holds. In what follows, we shall consider (6.6). Let L_j be a nonnegative integer satisfying $L_j \geq s_j, j = 0, 1, \dots, N$. Observing that the supports of $\varphi(\cdot - \ell_j), \ell_j \in D_A$, are mutually disjoint, we see that

$$|\partial_x^{\alpha_0} \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \sigma_A(x, \xi)| \leq C_{\alpha_0, \alpha_1, \dots, \alpha_N} \langle \xi \rangle^{m-s_0+|\alpha_0|}. \tag{6.9}$$

Then, we obtain

$$|\Delta_k \sigma_A(x, \xi)| \lesssim \begin{cases} \langle \xi \rangle^{m-s_0} 2^{-k_1 L_1 - \dots - k_N L_N}, \\ \langle \xi \rangle^{m-s_0+L_0} 2^{-k_0 L_0 - k_1 L_1 - \dots - k_N L_N} \end{cases} \tag{6.10}$$

for any $k \in (\mathbb{N}_0)^{N+1}$ (see Remark 6.5 below or [26, Sect. 5.3]). By taking $0 \leq \theta_0 \leq 1$ satisfying $s_0 = L_0 \theta_0$, we have

$$\begin{aligned} |\Delta_k \sigma_A(x, \xi)| &= |\Delta_k \sigma_A(x, \xi)|^{1-\theta_0} |\Delta_k \sigma_A(x, \xi)|^{\theta_0} \\ &\lesssim \left(\langle \xi \rangle^{m-s_0} 2^{-k_1 L_1 - \dots - k_N L_N} \right)^{1-\theta_0} \left(\langle \xi \rangle^{m-s_0+L_0} 2^{-k_0 L_0 - k_1 L_1 - \dots - k_N L_N} \right)^{\theta_0} \\ &= \langle \xi \rangle^{m-2^{-k_0 s_0 - k_1 L_1 - \dots - k_N L_N}} \end{aligned} \tag{6.11}$$

for any $k \in (\mathbb{N}_0)^{N+1}$. Thus, we obtain (6.6) with the implicit constant independent of $A \in \mathbb{N}$.

Choosing $\delta_1, \delta_2, \delta_3 > 0$ such that (6.5), we have by the conditions in (6.4)

$$\begin{aligned} T_{\sigma_A}(f_{\dots}, \dots, f_{\dots})(x) &= (2\pi)^{-Nn} \varphi(x) \sum_{\ell_1, \dots, \ell_N \in D_A} \langle \ell \rangle^{m-s_0} \\ &\quad \times \prod_{j \in J} e^{-\epsilon |\ell_j| |\ell_j|^{-b_j}} \int_{\mathbb{R}^n} \varphi(\xi_j - \ell_j)^2 d\xi_j \\ &\quad \times \prod_{j \in J^c} 2^{An(1/p_j-1)} \int_{\mathbb{R}^n} \psi(2^{-A} \xi_j) \varphi(\xi_j - \ell_j) d\xi_j \\ &= C \varphi(x) \sum_{\ell_1, \dots, \ell_N \in D_A} \langle \ell \rangle^{m-s_0} \prod_{j \in J} e^{-\epsilon |\ell_j| |\ell_j|^{-b_j}} \prod_{j \in J^c} 2^{An(1/p_j-1)}. \end{aligned}$$

Hence, collecting (6.6), (6.7), (6.8), and the assumption (6.3) with $t = \infty$, we see that

$$\sum_{\ell_1, \dots, \ell_N \in D_A} \langle \ell \rangle^{m-s_0} \prod_{j \in J} e^{-\epsilon |\ell_j| |\ell_j|^{-b_j}} \prod_{j \in J^c} 2^{An(1/p_j-1)} \lesssim 1$$

with the implicit constant independent of $\epsilon > 0$, where we used that $\|\lambda f\|_{BMO} = |\lambda| \|f\|_{BMO}, \lambda \in \mathbb{R}$, when $p = \infty$. Then, a limiting argument gives that

$$\sum_{\ell_1, \dots, \ell_N \in D_A} \langle \ell \rangle^{m-s_0} \prod_{j \in J} |\ell_j|^{-b_j} \prod_{j \in J^c} 2^{An(1/p_j-1)} \lesssim 1,$$

and thus,

$$2^{ANn} 2^{A(m-s_0)} \prod_{j \in J} 2^{-Ab_j} \prod_{j \in J^c} 2^{An(1/p_j-1)} \lesssim 1.$$

Since this holds for arbitrarily large $A \in \mathbb{N}$, we have

$$Nn + (m - s_0) - \sum_{j \in J} b_j + \sum_{j \in J^c} \left(\frac{n}{p_j} - n \right) \leq 0.$$

Since $b_j \rightarrow n/2$ by taking the limits as $a_j \rightarrow 1$ and $\varepsilon_j \rightarrow 0$, we have

$$s_0 \geq m + \sum_{j \in J} \frac{n}{2} + \sum_{j \in J^c} \frac{n}{p_j},$$

which implies from (6.2) that $s_0 \geq \min\{n/p, n/2\}$. This completes the proof. □

Remark 6.5 In this remark, we shall show (6.10) in detail. Also, for the sake of simplicity, we write $\sigma = \sigma_A$ (removed the subscript A). We first consider the latter inequality in (6.10) for $\mathbf{k} \in \mathbb{N}^{N+1}$. Recalling that $\psi_k = \psi(2^{-k}\cdot)$ for $k \in \mathbb{N}$ with $\text{supp } \psi \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$, we see that $\mathcal{F}^{-1}\psi$ satisfies the moment condition $\int x^\alpha \mathcal{F}^{-1}\psi(x) dx = i^{|\alpha|} \partial^\alpha \psi(0) = 0$. Then, using the Taylor expansion with respect to the ξ_N variable of the symbol, we have

$$\begin{aligned} \Delta_k \sigma(x, \xi) &= \psi_{k_0}(D_x) \psi_{k_1}(D_{\xi_1}) \dots \psi_{k_N}(D_{\xi_N}) \sigma(x, \xi) \\ &= \int_{(\mathbb{R}^n)^{N+1}} \prod_{i=0}^N 2^{nk_i} (\mathcal{F}^{-1}\psi)(2^{k_i} \eta_i) \\ &\quad \times \left\{ \sigma(x - \eta_0, \xi - \eta) - \sum_{|\alpha_N| < L_N} \frac{(-\eta_N)^{\alpha_N}}{\alpha_N!} (\partial_{\xi_N}^{\alpha_N} \sigma)(x - \eta_0, \bar{\xi} - \bar{\eta}, \xi_N) \right\} d\eta \\ &= \int_{(\mathbb{R}^n)^{N+1}} \prod_{i=0}^N 2^{nk_i} (\mathcal{F}^{-1}\psi)(2^{k_i} \eta_i) \\ &\quad \times \sum_{|\alpha_N|=L_N} \frac{(-\eta_N)^{\alpha_N}}{\alpha_N!} \int_0^1 L_N(1-t_N)^{L_N-1} (\partial_{\xi_N}^{\alpha_N} \sigma)(x - \eta_0, \bar{\xi} - \bar{\eta}, \xi_N - t_N \eta_N) dt_N d\eta, \end{aligned}$$

where we wrote $d\eta = d\eta_0 d\eta_1 \dots d\eta_N$ and $\bar{\xi} = (\xi_1, \dots, \xi_{N-1}) \in (\mathbb{R}^n)^{N-1}$. Repeating the same argument to the remaining variables, we obtain

$$\begin{aligned} \Delta_k \sigma(x, \xi) &= \sum_{|\alpha_0|=L_0} \frac{1}{\alpha_0!} \dots \sum_{|\alpha_N|=L_N} \frac{1}{\alpha_N!} \\ &\quad \times \int_{(\mathbb{R}^n)^{N+1}} \prod_{i=0}^N 2^{nk_i} (\mathcal{F}^{-1}\psi)(2^{k_i} \eta_i) (-\eta_i)^{\alpha_i} \end{aligned}$$

$$\times \int_{[0,1]^{N+1}} \left(\prod_{i=0}^N L_i(1-t_i)^{L_i-1} \right) (\partial_x^{\alpha_0} \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \sigma)(x-t_0\eta_0, \xi-t\eta) dt d\eta, \tag{6.12}$$

where we wrote $dt = dt_0 dt_1 \dots dt_N$ and $\xi - t\eta = (\xi_1 - t_1\eta_1, \dots, \xi_N - t_N\eta_N)$. Then, using the fact (6.9) with the inequality

$$\langle \xi - t\eta \rangle^{m-s_0+L_0} \lesssim \langle \xi \rangle^{m-s_0+L_0} \langle \eta \rangle^{|m-s_0+L_0|}, \quad t \in [0, 1]^N,$$

we have

$$\begin{aligned} |\Delta_k \sigma(x, \xi)| &\lesssim \langle \xi \rangle^{m-s_0+L_0} \int_{(\mathbb{R}^n)^{N+1}} \prod_{i=0}^N 2^{nk_i} |(\mathcal{F}^{-1} \psi)(2^{k_i} \eta_i)| |\eta_i|^{L_i} \langle \eta \rangle^{|m-s_0+L_0|} d\eta \\ &\lesssim \langle \xi \rangle^{m-s_0+L_0} 2^{-k_0 L_0 - k_1 L_1 - \dots - k_N L_N} \end{aligned}$$

for all $k \in \mathbb{N}^{N+1}$. If one of k_i is zero, then by avoiding usage of the moment condition and the Taylor expansion for the corresponding variables, we also obtain the same conclusion as above. Hence, we see that the latter part in (6.10) holds for all $k \in (\mathbb{N}_0)^{N+1}$.

We next consider the former part in (6.10). Using (6.9) with $\alpha_0 = 0$ and the expression

$$\begin{aligned} \Delta_k \sigma(x, \xi) &= \sum_{|\alpha_1|=L_1} \frac{1}{\alpha_1!} \dots \sum_{|\alpha_N|=L_N} \frac{1}{\alpha_N!} \\ &\times \int_{(\mathbb{R}^n)^{N+1}} 2^{nk_0} (\mathcal{F}^{-1} \psi)(2^{k_0} \eta_0) \prod_{i=1}^N 2^{nk_i} (\mathcal{F}^{-1} \psi)(2^{k_i} \eta_i) (-\eta_i)^{\alpha_i} \\ &\times \int_{[0,1]^N} \left(\prod_{i=1}^N L_i(1-t_i)^{L_i-1} \right) (\partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \sigma)(x-\eta_0, \xi-t\eta) dt_1 \dots dt_N d\eta, \end{aligned}$$

instead of (6.12), we have

$$|\Delta_k \sigma(x, \xi)| \lesssim \langle \xi \rangle^{m-s_0} 2^{-k_1 L_1 - \dots - k_N L_N}$$

for $k \in \mathbb{N}^{N+1}$. Now, it is also easy to see that the above estimates actually hold for all $k \in (\mathbb{N}_0)^{N+1}$, which concludes the former inequality in (6.10).

6.3 Sharpness of s_1, \dots, s_N of Theorem 1.2

We show that the conditions on s_1, \dots, s_N stated in Theorems 1.2 are sharp. See also [26, Proposition 5.2] and [27, Proposition 7.4].

Lemma 6.6 *Let $N \geq 2$, $p, p_1, \dots, p_N \in (0, \infty]$, $1/p = 1/p_1 + \dots + 1/p_N$, $s_0, s_1, \dots, s_N \in [0, \infty)$, $t \in (0, \infty]$, and $m \in \mathbb{R}$. Suppose that the estimate*

$$\|T_\sigma\|_{H^{p_1} \times \dots \times H^{p_N} \rightarrow L^p} \lesssim \|2^{k \cdot s} \langle \xi \rangle^{-m} \Delta_k \sigma(x, \xi)\|_{L_{x, \xi}^\infty((\mathbb{R}^n)^{N+1})} \| \ell_k^{t((\mathbb{N}_0)^{N+1})} \| \tag{6.13}$$

holds for all smooth functions σ with the right hand side finite, where L^p is replaced by BMO for $p = \infty$. Then $s_j \geq n/p_j$, $j = 1, \dots, N$.

Proof We only prove $s_1 \geq n/p_1$ and the rest parts for s_2, \dots, s_N follow by symmetry.

As stated in the proof of Proposition 6.3, we may assume the assumption (6.13) with $t = \infty$. Take real-valued radial functions $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\int \varphi^2 \neq 0$ and

$$\begin{aligned} \text{supp } \varphi &\subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}, \quad \varphi = 1 \text{ on } \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}, \\ \text{supp } \psi &\subset \{x \in \mathbb{R}^n : 1/2 \leq |x| \leq 2\}. \end{aligned} \tag{6.14}$$

We set for $A \in \mathbb{N}$

$$\begin{aligned} \sigma_A(x, \xi) &= 2^{-As_1} \psi(2^{-A}x) e^{-ix \cdot \xi_1} \varphi(\xi_1) \dots \varphi(\xi_N), \\ f_1(x) &= (\mathcal{F}^{-1}\varphi)(x), \\ f_{j,A}(x) &= 2^{-An/p_j} (\mathcal{F}^{-1}\psi)(2^{-A}x), \quad j = 2, \dots, N. \end{aligned}$$

Firstly we shall prove

$$\|2^{k \cdot s} \langle \xi \rangle^{-m} \Delta_k \sigma_A(x, \xi)\|_{L_{x, \xi}^\infty \ell_k^\infty} \lesssim 1, \tag{6.15}$$

$$\|f_1\|_{H^{p_1}} \approx \|f_{j,A}\|_{H^{p_j}} \approx 1, \quad j = 2, \dots, N, \tag{6.16}$$

for $A \in \mathbb{N}$. By a scaling property of Hardy spaces (6.16) obviously follows. Let L_j be a nonnegative integer satisfying $L_j \geq s_j$ for $j = 0, 1, \dots, N$. Observing that

$$|\partial_x^{\alpha_0} \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_N}^{\alpha_N} \sigma_A(x, \xi)| \leq C_{\alpha_0, \alpha_1, \dots, \alpha_N} \langle x \rangle^{-s_1 + |\alpha_1|} \langle \xi \rangle^m, \tag{6.17}$$

we see that

$$|\Delta_k \sigma_A(x, \xi)| \lesssim \begin{cases} \langle x \rangle^{-s_1} \langle \xi \rangle^m 2^{-k_0 L_0 - k_2 L_2 - \dots - k_N L_N}, \\ \langle x \rangle^{-s_1 + L_1} \langle \xi \rangle^m 2^{-k_0 L_0 - k_1 L_1 - k_2 L_2 - \dots - k_N L_N}, \end{cases} \tag{6.18}$$

for any $k \in (\mathbb{N}_0)^{N+1}$. (See also Remark 6.7 below.) As was done in (6.11), by taking $0 \leq \theta_1 \leq 1$ such that $s_1 = L_1 \theta_1$,

$$|\Delta_k \sigma_A(x, \xi)| \lesssim \langle \xi \rangle^m 2^{-k_0 L_0 - k_1 s_1 - k_2 L_2 - \dots - k_N L_N},$$

and thus, (6.15) follows with the implicit constant independent of $A \in \mathbb{N}$.

From the condition of φ , since $\psi(2^A \cdot)\varphi = \psi(2^A \cdot)$ for $A \in \mathbb{N}$, we have

$$T_{\sigma_A}(f_1, f_{2,A}, \dots, f_{N,A})(x) = C 2^{-As_1} \psi(2^{-A}x) 2^{-An(1/p_2+\dots+1/p_N)} \{(\mathcal{F}^{-1}\psi)(2^{-A}x)\}^{N-1}$$

for $A \in \mathbb{N}$, which implies, with the assumption $1/p = 1/p_1 + \dots + 1/p_N$, that

$$\|T_{\sigma_A}(f_1, f_{2,A}, \dots, f_{N,A})\|_{L^p} \approx 2^{-As_1} 2^{-An(1/p_2+\dots+1/p_N)} 2^{An/p} = 2^{-A(s_1-n/p_1)}, \tag{6.19}$$

where we should use that *BMO* is scaling invariant when $p = \infty$.

Thus, collecting (6.15), (6.16), (6.19), and (6.13) with $t = \infty$, we see that $2^{-A(s_1-n/p_1)} \lesssim 1$. Since this holds for all $A \in \mathbb{N}$, we obtain $s_1 \geq n/p_1$, which completes the proof. \square

Remark 6.7 We first consider the former part of (6.18). In this remark, we simply write $\sigma = \sigma_A$. As in Remark 6.5, we observe the expression

$$\begin{aligned} \Delta_k \sigma(x, \xi) &= \sum_{|\alpha_0|=L_0} \frac{1}{\alpha_0!} \sum_{|\alpha_2|=L_2} \frac{1}{\alpha_2!} \dots \sum_{|\alpha_N|=L_N} \frac{1}{\alpha_N!} \\ &\times \int_{(\mathbb{R}^n)^{N+1}} 2^{nk_1} (\mathcal{F}^{-1}\psi)(2^{k_1}\eta_1) \prod_{i \neq 1} 2^{nk_i} (\mathcal{F}^{-1}\psi)(2^{k_i}\eta_i) (-\eta_i)^{\alpha_i} \\ &\times \int_{[0,1]^N} \left(\prod_{i \neq 1} L_i(1-t_i)^{L_i-1} \right) \\ &\times (\partial_x^{\alpha_0} \partial_{\xi_2}^{\alpha_2} \dots \partial_{\xi_N}^{\alpha_N} \sigma)(x - t_0\eta_0, \xi_1 - \eta_1, \xi' - t'\eta') dt_0 dt' d\eta, \end{aligned}$$

where we wrote $\xi' - t'\eta' = (\xi_2 - t_2\eta_2, \dots, \xi_N - t_N\eta_N)$ and $dt' = dt_2 \dots dt_N$. Then, using (6.17) with $\alpha_1 = 0$ and the inequality

$$\langle x - t_0\eta_0 \rangle^{-s_1} \langle (\xi_1 - \eta_1, \xi' - t'\eta') \rangle^m \lesssim \langle x \rangle^{-s_1} \langle \xi \rangle^m \langle \eta_0 \rangle^{s_1} \langle \eta \rangle^{|m|},$$

we have

$$|\Delta_k \sigma(x, \xi)| \lesssim \langle x \rangle^{-s_1} \langle \xi \rangle^m 2^{-k_0L_0-k_2L_2-\dots-k_NL_N}$$

for $k \in \mathbb{N}^{N+1}$. We can see again that the above estimates actually hold for all $k \in (\mathbb{N}_0)^{N+1}$, which concludes the former inequality in (6.18).

The latter inequality in (6.18) can be obtained from (6.12) with (6.17) by the same way.

Lemma 6.8 *Let $N \geq 2$, $p, p_1, \dots, p_N \in (0, \infty]$, $1/p = 1/p_1 + \dots + 1/p_N$, $s_0, s_1, \dots, s_N \in [0, \infty)$, $t \in (0, \infty]$, and $m \in \mathbb{R}$. Suppose that the estimate*

$$\|T_\sigma\|_{H^{p_1} \times \dots \times H^{p_N} \rightarrow L^p} \lesssim \|\sigma\|_{S_{0,0}^m(s, t; \mathbb{R}^n, N)} \tag{6.20}$$

holds for all smooth functions σ with the right hand side finite, where L^p is replaced by BMO for $p = \infty$. Then $s_j \geq n/2, j = 1, \dots, N$.

Proof We only prove $s_1 \geq n/2$ and the rest parts for s_2, \dots, s_N follow by symmetry.

We may assume the assumption (6.20) with $t = \infty$. We take real-valued radial functions $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (6.14) and set for $A \in \mathbb{N}$

$$\begin{aligned} \sigma_A(x, \xi) &= \sigma_A(\xi) = \mathcal{F}^{-1}[\psi(2^{-A}\cdot)](\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N), \\ f_{j,A}(x) &= (\mathcal{F}^{-1}\psi)(2^{-A}x), \quad j = 1, \dots, N. \end{aligned}$$

For these functions, the following hold:

$$\|\sigma_A\|_{S_{0,0}^m(s,\infty;\mathbb{R}^n,N)} \lesssim 2^{A(s_1+n/2)}, \tag{6.21}$$

$$\|f_{j,A}\|_{H^{p_j}} \approx 2^{An/p_j}, \quad j = 1, \dots, N, \tag{6.22}$$

for $A \in \mathbb{N}$. As in the previous proof, (6.22) is obvious. We shall consider (6.21). Using the fact that $\psi_{k_0}(D_x)[1]$ is equal to 1 if $k_0 = 0$ and to 0 if $k_0 \geq 1$, we have

$$\Delta_k \sigma_A(x, \xi) = \mathcal{F}^{-1}[\psi_{k_1}\psi(2^{-A}\cdot)](\xi_1) \prod_{j=2}^N \psi_{k_j}(D)\varphi(\xi_j).$$

Moreover, $\mathcal{F}^{-1}[\psi_{k_1}\psi(2^{-A}\cdot)]$ vanishes unless $|k_1 - A| \leq 1$. Hence,

$$\begin{aligned} \|\sigma_A\|_{S_{0,0}^m(s,\infty;\mathbb{R}^n,N)} &= \sup_{k=(k_0,k_1,\dots,k_N)} 2^{s\cdot k} \|\langle \xi \rangle^{-m} \Delta_k \sigma_A(x, \xi)\|_{L_{ul,\xi}^2 L_x^\infty} \\ &= \sup_{\substack{k_1,\dots,k_N \\ |k_1-A|\leq 1}} 2^{s_1 k_1 + \dots + s_N k_N} \left\| \langle \xi \rangle^{-m} \mathcal{F}^{-1}[\psi_{k_1}\psi(2^{-A}\cdot)](\xi_1) \prod_{j=2}^N \psi_{k_j}(D)\varphi(\xi_j) \right\|_{L_{ul,\xi}^2}. \end{aligned} \tag{6.23}$$

Here, if $|k_1 - A| \leq 1$, we have for sufficiently large $L_1 \geq 0$

$$|\mathcal{F}^{-1}[\psi_{k_1}\psi(2^{-A}\cdot)](\xi_1)| \lesssim 2^{An} \langle 2^A \xi_1 \rangle^{-L_1}. \tag{6.24}$$

In fact, observing that

$$\begin{aligned} \mathcal{F}^{-1}[\psi_{k_1}\psi(2^{-A}\cdot)](\xi_1) &= 2^{An} \mathcal{F}^{-1}[\psi(2^{A-k_1}\cdot)]\psi(2^A \xi_1) \\ &= 2^{An} \int_{\mathbb{R}^n} \mathcal{F}^{-1}[\psi(2^{A-k_1}\cdot)](\eta) (\mathcal{F}^{-1}\psi)(2^A \xi_1 - \eta) d\eta, \end{aligned}$$

since $\mathcal{F}^{-1}\psi \in \mathcal{S}(\mathbb{R}^n)$, we have for large $L_1 \geq 0$

$$|\mathcal{F}^{-1}[\psi_{k_1}\psi(2^{-A}\cdot)](\xi_1)| \lesssim 2^{An} \langle 2^A \xi_1 \rangle^{-L_1} \int_{\mathbb{R}^n} |\mathcal{F}^{-1}[\psi(2^{A-k_1}\cdot)](\eta)| \langle \eta \rangle^{L_1} d\eta$$

$$\approx 2^{An} \langle 2^A \xi_1 \rangle^{-L_1}.$$

Moreover, we have for sufficient large $M_j \in \mathbb{N}_0$ and $L_j \geq 0$

$$|\psi_{k_j}(D)\varphi(\xi_j)| \lesssim 2^{-k_j M_j} \langle \xi_j \rangle^{-L_j}, \quad j = 2, \dots, N. \tag{6.25}$$

In fact, from the argument used in Remark 6.5, the Taylor expansion yields that for $M_j \in \mathbb{N}_0$

$$\begin{aligned} \psi_{k_j}(D)\varphi(\xi_j) &= \sum_{|\alpha|=M_j} \frac{1}{\alpha!} \int_{\mathbb{R}^n} 2^{nk_j} (\mathcal{F}^{-1}\psi)(2^{k_j}\eta) (-\eta)^\alpha \\ &\quad \times \int_0^1 M_j(1-t)^{M_j-1} (\partial^\alpha \varphi)(\xi_j - t\eta) dt d\eta \end{aligned}$$

for any $k_j \in \mathbb{N}$. Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have for large $L_j \geq 0$

$$\begin{aligned} |\psi_{k_j}(D)\varphi(\xi_j)| &\lesssim \langle \xi_j \rangle^{-L_j} \int_{\mathbb{R}^n} 2^{nk_j} |(\mathcal{F}^{-1}\psi)(2^{k_j}\eta)| |\eta|^{M_j} \langle \eta \rangle^{L_j} d\eta \\ &\lesssim 2^{-k_j M_j} \langle \xi_j \rangle^{-L_j}. \end{aligned}$$

Since the case $k_j = 0$ is clear, we see that (6.25) holds for any $k_j \in \mathbb{N}_0$. Collecting (6.23), (6.24), and (6.25), we have

$$\begin{aligned} &\|\sigma_A\|_{S_{0,0}^m(s,\infty;\mathbb{R}^n,N)} \\ &\lesssim \sup_{\substack{k_1,\dots,k_N \\ |k_1-A|\leq 1}} 2^{s_1 k_1 + \dots + s_N k_N} \left\| \langle \xi \rangle^{-m} 2^{An} \langle 2^A \xi_1 \rangle^{-L_1} \left(\prod_{j=2}^N 2^{-k_j M_j} \langle \xi_j \rangle^{-L_j} \right) \right\|_{L_{ul,\xi}^2}. \end{aligned}$$

Hence, by the embedding $L^2 \hookrightarrow L_{ul}^2$ and the inequality $\langle \xi \rangle^m \lesssim \prod_{j=1}^N \langle \xi_j \rangle^{|m|}$, we see that

$$\begin{aligned} &\|\sigma_A\|_{S_{0,0}^m(s,\infty;\mathbb{R}^n,N)} \\ &\lesssim \sup_{\substack{k_1,\dots,k_N \\ |k_1-A|\leq 1}} 2^{s_1 k_1 + \dots + s_N k_N} \left\| \langle \xi_1 \rangle^{|m|} 2^{An} \langle 2^A \xi_1 \rangle^{-L_1} \left(\prod_{j=2}^N 2^{-k_j M_j} \langle \xi_j \rangle^{|m|-L_j} \right) \right\|_{L_{\xi}^2} \\ &\leq \sup_{\substack{k_1,\dots,k_N \\ |k_1-A|\leq 1}} 2^{s_1 k_1} 2^{An} \|\langle 2^A \xi_1 \rangle^{|m|-L_1}\|_{L_{\xi_1}^2} \left(\prod_{j=2}^N 2^{(s_j - M_j)k_j} \|\langle \xi_j \rangle^{|m|-L_j}\|_{L_{\xi_j}^2} \right) \\ &\lesssim 2^{A(s_1+n/2)}, \end{aligned}$$

which gives (6.21). Moreover, since the conditions of φ , ψ imply $\psi(2^A \cdot)\varphi = \psi(2^A \cdot)$, $A \in \mathbb{N}$,

$$T_{\sigma_A}(f_{1,A}, \dots, f_{N,A})(x) = 2^{An}(\psi * \widehat{\psi})(2^{-A}x)\{\widehat{\psi}(2^{-A}x)\}^{N-1},$$

which implies that

$$\|T_{\sigma_A}(f_{1,A}, \dots, f_{N,A})\|_{L^p} \approx 2^{An} 2^{An/p}, \quad (6.26)$$

where we should use that *BMO* is scaling invariant when $p = \infty$.

Thus, collecting (6.21), (6.22), (6.26), and (6.20) with $t = \infty$ and using the assumption $1/p = 1/p_1 + \dots + 1/p_N$, we obtain $s_1 \geq n/2$. This completes the proof. \square

The following immediately follows from Lemmas 6.6 and 6.8.

Corollary 6.9 *Let $N \geq 2$, $p, p_1, \dots, p_N \in (0, \infty]$, $1/p = 1/p_1 + \dots + 1/p_N$, $s_0, s_1, \dots, s_N \in [0, \infty]$, $t \in (0, \infty]$, and $m \in \mathbb{R}$. Suppose that the estimate*

$$\|T_\sigma\|_{H^{p_1} \times \dots \times H^{p_N} \rightarrow L^p} \lesssim \|\sigma\|_{S_{0,0}^m(s,t;\mathbb{R}^n, N)}$$

*holds for all smooth functions σ with the right hand side finite, where L^p is replaced by *BMO* for $p = \infty$. Then $s_j \geq \max\{n/p_j, n/2\}$, $j = 1, \dots, N$.*

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