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Some Intrinsic Characterizations of Besov–Triebel–Lizorkin–Morrey–Type Spaces on Lipschitz Domains

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Abstract

We give Littlewood–Paley type characterizations for Besov–Triebel–Lizorkin–type spaces $\mathscr{B}_{pq}^{s\tau}$, $\mathscr{F}_{pq}^{s\tau}$ and Besov-Morrey spaces $\mathcal{N}_{uqp}^{s}$ on a special Lipschitz domain $\Omega \subset \mathbb{R}^n$: for a suitable sequence of Schwartz functions $(\phi_j)_{j=0}^{\infty}$,

$$\|f\|_{\mathscr{B}_{pq}^{s\tau}(\Omega)} \approx \sup_{P \text{ dyadic cubes}} |P|^{-\tau} \|(2^{js}\phi_j * f)_{j=\log_2 \ell(P)}^{\infty}\|_{\ell^q(L^p(\Omega\cap P))};$$
$$\|f\|_{\mathscr{F}_{pq}^{s\tau}(\Omega)} \approx \sup_{P \text{ dyadic cubes}} |P|^{-\tau} \|(2^{js}\phi_j * f)_{j=\log_2 \ell(P)}^{\infty}\|_{L^p(\Omega\cap P;\ell^q)};$$
$$\|f\|_{\mathcal{N}_{uqp}^{s}(\Omega)} \approx \left\|\left(\sup_{P \text{ dyadic cubes}} |P|^{\frac{1}{u}-\frac{1}{p}} \cdot 2^{js}\|\phi_j * f\|_{L^p(\Omega\cap P)}\right)_{j=0}^{\infty}\right\|_{\ell^q}.$$

We also show that $\|f\|_{\mathscr{B}_{pq}^{s\tau}(\Omega)}$, $\|f\|_{\mathscr{F}_{pq}^{s\tau}(\Omega)}$ and $\|f\|_{\mathcal{N}_{uqp}^{s}(\Omega)}$ have equivalent (quasi-) norms via derivatives: for $\mathscr{X}^{\bullet} \in \{\mathscr{B}_{pq}^{\bullet,\tau}, \mathscr{F}_{pq}^{\bullet,\tau}, \mathcal{N}_{uqp}^{\bullet}\}$, we have $\|f\|_{\mathscr{X}^s(\Omega)} \approx \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{\mathscr{X}^{s-m}(\Omega)}$.
In particular $\|f\|_{\mathscr{F}_{\infty q}^{s}(\Omega)} \approx \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{\mathscr{F}_{\infty q}^{s-m}(\Omega)} \approx \sup_P |P|^{-n/q} \|(2^{js}\phi_j * f)_{j=\log_2 \ell(P)}^{\infty}\|_{\ell^q(L^q(\Omega\cap P))}$.

Keywords Rychkov's extension operator · Lipschitz domains · Besov-type space · Triebel–Lizorkin-type space · Besov–Morrey space

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a *special Lipschitz domain*, that is, Ω is of the form $\{(x', x_n) : x_n > \rho(x')\}$ where $\rho : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function such that $\|\nabla\rho\|_{L^\infty} < \infty$. (See also [14, Definition 1.103].)

In [9], based on the construction of his extension operator, Rychkov gave a Littlewood-Paley type intrinsic characterization of the Triebel-Lizorkin spaces on Ω: for $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$, $\mathscr{F}^s_{pq}(\Omega)$ has the following equivalent (quasi-)norm (see [9, Theorem 3.2]):

$$f \mapsto \|(2^{js}\phi_j * f)_{j=0}^\infty\|_{\ell^q(\mathbb{Z}_{\ge 0}; L^p(\Omega))} = \bigg(\int_\Omega \Big(\sum_{j=0}^\infty 2^{jsq}|\phi_j * f(x)|^q\Big)^{p/q} dx\bigg)^{1/p}. \quad (1)$$

We take obvious modification for $q = \infty$. Here $(\phi_j)_{j=0}^\infty$ is a carefully chosen family of Schwartz functions such that the convolution $\phi_j * f$ is defined on Ω, see Definition 4.

In [12, version 3, Proposition 6.6], we used Rychkov's construction to prove that $\|f\|_{\mathscr{F}^s_{pq}(\Omega)}$ have equivalent (quasi-)norms via their derivatives. More precisely, let $m \ge 1$, for every $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$ there is a $C = C(\Omega, p, q, s, m) > 0$ such that

$$C^{-1}\|f\|_{\mathscr{F}^s_{pq}(\Omega)} \le \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{\mathscr{F}^{s-m}_{pq}(\Omega)} \le C\|f\|_{\mathscr{F}^s_{pq}(\Omega)}, \quad \forall f \in \mathscr{F}^s_{pq}(\Omega). \quad (2)$$

Both (1) and (2) miss the endpoint: do we have the analogy of (1) and (2) for $p = \infty$? In this paper, we give the positive answers to both cases, by using the recently developed Triebel-Lizorkin-type spaces $\mathscr{F}^{s\tau}_{pq}$: we have the coincidences $\mathscr{F}^s_{\infty q} = \mathscr{F}^{s,\frac{1}{p}}_{pq} = \mathscr{B}^{s,\frac{1}{q}}_{qq}$ for $0 < p < \infty$ (see (9)).

To make the results more general, we include the discussions of Besov-type spaces $\mathscr{B}^{s\tau}_{pq}$ and the Besov-Morrey spaces $\mathscr{N}^{s\tau}_{pq}$, see Definition 6.

We denote by $\mathcal{Q}$ the set of dyadic cubes in $\mathbb{R}^n$, that is

$$\mathcal{Q} := \{Q_{J,v} : J \in \mathbb{Z}, v \in \mathbb{Z}^n\}, \quad \text{where} \quad Q_{J,v} := 2^{-J}v + (0, 2^{-J})^n. \quad (3)$$

Our result for (1) is the following:

Theorem 1 *(Littlewood-Paley type characterizations) Let $\Omega = \{(x', x_n) : x_n > \rho(x')\} \subset \mathbb{R}^n$ be a special Lipschitz domain and let $(\phi_j)_{j=0}^\infty$ be a Littlewood-Paley family associated with Ω (see Definition 4). Then for $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $\tau \ge 0$ ($p < \infty$ for $\mathscr{F}$-cases), we have the following equivalent (quasi-)norms:*

$$\|f\|_{\mathscr{B}^{s\tau}_{pq}(\Omega)} \approx_{\phi,p,q,s,\tau} \|(2^{js}\mathbf{1}_\Omega \cdot (\phi_j * f))_{j=0}^\infty\|_{\ell^q L^p_\tau}$$
$$= \sup_{Q_{J,v}\in\mathcal{Q}} 2^{nJ\tau}\Big(\sum_{j=\max(0,J)}^\infty 2^{jsq}\|\phi_j * f\|^q_{L^p(Q_{J,v}\cap\Omega)}\Big)^{\frac{1}{q}};$$

$$\begin{aligned}\|f\|_{\mathscr{F}^{s\tau}_{pq}(\Omega)} &\approx_{\phi,p,q,s,\tau} \|(2^{js}\mathbf{1}_\Omega\cdot(\phi_j*f))_{j=0}^\infty\|_{L^p_\tau\ell^q}\\ &= \sup_{Q_{J,v}\in\mathcal{Q}} 2^{nJ\tau}\Big(\int_{Q_{J,v}\cap\Omega}\Big(\sum_{j=\max(0,J)}^\infty 2^{jsq}|\phi_j*f(x)|^q\Big)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}};\\ \|f\|_{\mathscr{N}^{s\tau}_{pq}(\Omega)} &\approx_{\phi,p,q,s,\tau} \|(2^{js}\mathbf{1}_\Omega\cdot(\phi_j*f))_{j=0}^\infty\|_{\ell^q M^p_\tau}\\ &= \Big(\sum_{j=0}^\infty \sup_{Q_{J,v}\in\mathcal{Q}} 2^{(js+nJ\tau)q}\|\phi_j*f\|^q_{L^p(Q_{J,v}\cap\Omega)}\Big)^{\frac{1}{q}}.\end{aligned}$$

(See Definition 5 for $\ell^q L^p_\tau$, $L^p_\tau\ell^q$ and $\ell^q M^p_\tau$.) *In particular for* $0<q\le\infty$ *and* $s\in\mathbb{R}$,

$$\|f\|_{\mathscr{F}^s_{\infty q}(\Omega)} \approx_{\phi,q,s} \sup_{J\in\mathbb{Z},v\in\mathbb{Z}^n} 2^{J\frac{n}{q}}\int_{Q_{J,v}\cap\Omega}\Big(\sum_{j=\max(0,J)}^\infty 2^{jqs}|\phi_j*f(x)|^q dx\Big)^{\frac{1}{q}}.$$

One can also get some characterizations on bounded Lipschitz domain, whose expressions are less elegant however. See Remark 24.

Similar to [9, Theorem 2.3], we also have the corresponding characterizations using Peetre maximal functions, see Proposition 21 and Corollary 23.

Our result for (2) is the following:

Theorem 2 *(Equivalent norm characterizations via derivatives) Let* $\mathscr{A}\in\{\mathscr{B},\mathscr{F},\mathscr{N}\}$, $0<p,q\le\infty$, $s\in\mathbb{R}$ *and* $\tau\ge 0$ *(*$p<\infty$ *for* $\mathscr{F}$*-cases). Let* $\Omega\subset\mathbb{R}^n$ *be either a special Lipschitz domain or a bounded Lipschitz domain. Then for any positive integer* m*, the space* $\mathscr{A}^{s\tau}_{pq}(\Omega)$ *has the following equivalent (quasi-)norm:*

$$\|f\|_{\mathscr{A}^{s,\tau}_{p,q}(\Omega)} \approx_{p,q,s,m,\tau,\Omega} \sum_{|\alpha|\le m}\|\partial^\alpha f\|_{\mathscr{A}^{s-m,\tau}_{p,q}(\Omega)}. \tag{4}$$

In particular $\|f\|_{\mathscr{F}^s_{\infty,q}(\Omega)}\approx_{q,s,m,\Omega}\sum_{|\alpha|\le m}\|\partial^\alpha f\|_{\mathscr{F}^{s-m}_{\infty,q}(\Omega)}$ *for all* $0<q\le\infty$ *and* $s\in\mathbb{R}$.

The Besov-Morrey case $\mathscr{A}=\mathscr{N}$ of Theorem 2 was stated in [25, Proposition 4.15]. However, the key step in their proof requires [15, (4.70)] (see [25, Remark 4.14]), which cannot be achieved.

Remark 3 In the proof of [15, Proposition 4.21], Triebel claimed the following statement:

$$\begin{aligned}\|f\|_{\mathscr{A}^s_{pq}(\Omega)} &\approx \|Ef\|_{\mathscr{A}^s_{pq}(\mathbb{R}^n)} \approx \sum_{|\alpha|\le m}\|\partial^\alpha Ef\|_{\mathscr{A}^s_{pq}(\mathbb{R}^n)}\\ &= \sum_{|\alpha|\le m}\|E\partial^\alpha f\|_{\mathscr{A}^s_{pq}(\mathbb{R}^n)} \lesssim \sum_{|\alpha|\le m}\|\partial^\alpha f\|_{\mathscr{A}^s_{pq}(\Omega)}.\end{aligned} \tag{5}$$

Here $E=E_\Omega$ is an extension operator which is bounded on $\mathscr{A}^s_{pq}(\Omega)\to\mathscr{A}^s_{pq}(\mathbb{R}^n)$ and $\mathscr{A}^{s-m}_{pq}(\Omega)\to\mathscr{A}^{s-m}_{pq}(\mathbb{R}^n)$.

However, the commutativity $\partial^\alpha \circ E = E \circ \partial^\alpha$ in (5) (see [15, (4.70)]) cannot be achieved. In [12, Remark 1.6] we borrowed some facts from several complex variables to show that $\partial^\alpha \circ E = E \circ \partial^\alpha$ can never be true: if it is true (even locally) then $\overline{\partial}$-equation for Ω can gain 1 derivative. To prove Theorem 2 (also to fix the proof of [25, Proposition 4.15]), simply using the boundedness of E_Ω is not enough.

By observing (5) more carefully, the argument still works if $\partial^\alpha \circ E = E^\alpha \circ \partial^\alpha$ hold for some extension operators $E^\alpha : \mathscr{A}_{pq}^{s-m}(\Omega) \to \mathscr{A}_{pq}^{s-m}(\Omega)$. This can be done if E is the standard half space extension.[1] Using the operators E^α Triebel proved the equivalent norms via derivatives for $\mathbb{R}^n_+$ and for smooth domains, see [16, Section 3.3.5].

In our case E is Rychkov's extension operator (see (31)). Even on special Lipschitz domain, it is not known to the author whether $\partial^\alpha \circ E = E^\alpha \circ \partial^\alpha$ can be achieved (which in general should have the form (27)). Nevertheless, a weaker form $\partial^\alpha \circ E = \sum_\beta E^{\alpha,\beta} \circ \partial^\beta$ is enough to fix (5). In the proof we introduce $E^{\alpha,\beta}$ in (41) and get the proof using (42).

See also [12, Section 2.2 and Remark 6.5].

2 Function Spaces and Notations

Let $U \subseteq \mathbb{R}^n$ be an open set, we define $\mathscr{S}'(U)$ to be the space of restricted tempered distributions: $\mathscr{S}'(U) := \{\tilde{f}|_U : \tilde{f} \in \mathscr{S}'(\mathbb{R}^n)\}$. See also [9, Proposition 3.1].

We use the notation $A \lesssim B$ to mean that $A \le CB$ where C is a constant independent of A, B. We use $A \approx B$ for "$A \lesssim B$ and $B \lesssim A$". And we use $A \lesssim_x B$ to emphasize that the constant depends on the quantity x.

When p or $q < 1$, we use "norms" (for $\mathscr{A}_{pq}^{s\tau}$ etc.) as the abbreviation to the usual "quasi-norms".

In the paper we use the following Littlewood–Paley family, whose elements do not have compact supports in the Fourier side. It is crucially useful in the construction of Rychkov's extension operator.

Definition 4 Let $\Omega = \{x_n > \rho(x')\}$ be a special Lipschitz domain, a **Littlewood-Paley family** associated with Ω is a sequence $\phi = (\phi_j)_{j=0}^\infty \subset \mathscr{S}(\mathbb{R}^n)$ of Schwartz functions that satisfies the following:

- (P.a) *Moment condition*: $\int x^\alpha \phi_1(x)dx = 0$ for all multi-indices $\alpha \in \mathbb{Z}^n_{\ge 0}$.
- (P.b) *Scaling condition*: $\phi_j(x) = 2^{(j-1)n}\phi_1(2^{j-1}x)$ for all $j \ge 2$.
- (P.c) *Approximate identity*: $\sum_{j=0}^\infty \phi_j = \delta_0$ is the Direc delta measure.
- (P.d) *Support condition*: $\operatorname{supp}\phi_j \subset \{(x', x_n) : x_n < -\|\nabla\rho\|_{L^\infty} \cdot |x'|\}$ for all $j \ge 0$.

[1] The half space extension works on $\mathbb{R}^n_+ = \{x_n > 0\}$. It has the form $Ef(x', x_n) = \sum_j a_j f(x', -b_j x_n)$ when $x_n < 0$. In this case $E^\alpha f(x', x_n) = \sum_j a_j(-b_j)^{\alpha_n} f(x', -b_j x_n)$ has the similar expression to E.

In the paper we use the sequence spaces $\ell^q L^p_\tau$, $L^p_\tau \ell^q$, $\ell^q M^p_\tau$ given by the following:

Definition 5 Let $0 < p, q \leq \infty$ and $\tau \geq 0$. We denote by $\ell^q L^p_\tau(\mathbb{R}^n)$ and $L^p_\tau \ell^q(\mathbb{R}^n)$ the spaces of vector valued measurable functions $(f_j)_{j=0}^\infty \subset L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that the following (quasi-)norms are finite respectively:

$$\begin{aligned}
\|(f_j)_{j=0}^\infty\|_{\ell^q L^p_\tau} &:= \sup_{Q_{J,v}\in\mathcal{Q}} 2^{nJ\tau} \|(f_j)_{j=\max(0,J)}^\infty\|_{\ell^q(L^p(Q_{J,v}))} \\
&= \sup_{J\in\mathbb{Z}, v\in\mathbb{Z}^n} 2^{nJ\tau} \Big(\sum_{j=\max(0,J)}^\infty \|f_j\|^q_{L^p(Q_{J,v})} \Big)^{\frac{1}{q}}; \\
\|(f_j)_{j=0}^\infty\|_{L^p_\tau \ell^q} &:= \sup_{Q_{J,v}\in\mathcal{Q}} 2^{nJ\tau} \|(f_j)_{j=\max(0,J)}^\infty\|_{L^p(Q_{J,v};\ell^q)} \\
&= \sup_{J\in\mathbb{Z}, v\in\mathbb{Z}^n} 2^{nJ\tau} \Big(\int_{Q_{J,v}} \Big(\sum_{j=\max(0,J)}^\infty |f_j(x)|^q \Big)^{\frac{p}{q}} dx \Big)^{\frac{1}{p}}.
\end{aligned}$$

We define the Morrey space.[2] $M^p_\tau(\mathbb{R}^n)$ to be the set of all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ whose (quasi-)norm below is finite:

$$\|f\|_{M^p_\tau} := \sup_{Q_{J,v}\in\mathcal{Q}} 2^{nJ\tau} \|f\|_{L^p(Q_{J,v})}.$$

We define $\ell^q M^p_\tau(\mathbb{R}^n) := \ell^q(\mathbb{Z}_{\geq 0}; M^p_\tau(\mathbb{R}^n))$ with $\|(f_j)_{j=0}^\infty\|_{\ell^q M^p_\tau} := \big(\sum_{j=0}^\infty \|f_j\|^q_{M^p_\tau(\mathbb{R}^n)} \big)^{\frac{1}{q}}$.

Our Besov-type spaces $\mathscr{B}^{s\tau}_{pq}$, Triebel-Lizorkin-type spaces $\mathscr{F}^{s\tau}_{pq}$ and Besov-Morrey spaces $\mathscr{N}^{s\tau}_{pq}$ are given by the following:

Definition 6 Let $\lambda = (\lambda_j)_{j=0}^\infty$ be a sequence of Schwartz functions satisfying:

(P.a') The Fourier transform $\hat{\lambda}_0(\xi) = \int_{\mathbb{R}^n} \lambda_0(x) 2^{-2\pi i x\xi} dx$ satisfies $\operatorname{supp} \hat{\lambda}_0 \subset \{|\xi| < 2\}$ and $\hat{\lambda}_0|_{\{|\xi|<1\}} \equiv 1$.
(P.b') $\lambda_j(x) = 2^{jn}\lambda_0(2^j x) - 2^{(j-1)n}\lambda_0(2^{j-1}x)$ for $j \geq 1$.

Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $\tau \geq 0$ ($p < \infty$ for $\mathscr{F}$-cases). We define the *Besov-type Morrey space* $\mathscr{B}^{s\tau}_{pq}(\mathbb{R}^n)$, the *Triebel-Lizorkin-type Morrey space* $\mathscr{F}^{s\tau}_{pq}(\mathbb{R}^n)$ and the *Besov-Morrey space* $\mathscr{N}^{s\tau}_{pq}(\mathbb{R}^n)$, to be the sets of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ such that the following norms are finite, respectively:

$$\begin{aligned}
\|f\|_{\mathscr{B}^{s\tau}_{pq}(\mathbb{R}^n)} &:= \|(2^{js}\lambda_j * f)_{j=0}^\infty\|_{\ell^q L^p_\tau}; \\
\|f\|_{\mathscr{F}^{s\tau}_{pq}(\mathbb{R}^n)} &:= \|(2^{js}\lambda_j * f)_{j=0}^\infty\|_{L^p_\tau \ell^q}; \\
\|f\|_{\mathscr{N}^{s\tau}_{pq}(\mathbb{R}^n)} &:= \|(2^{js}\lambda_j * f)_{j=0}^\infty\|_{\ell^q M^p_\tau}.
\end{aligned} \tag{6}$$

[2] Our notation is different from the standard one, which can be found in for example [20, Definition 2.1].

Let $\mathscr{A} \in \{\mathscr{B}, \mathscr{F}, \mathscr{N}\}$. For an (arbitrary) open subset $U \subseteq \mathbb{R}^n$, we define $\mathscr{A}^{s\tau}_{pq}(U) := \{\tilde{f}|_U : \tilde{f} \in \mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n)\}$ ($p < \infty$ for $\mathscr{F}$-cases) with the norm

$$\|f\|_{\mathscr{A}^{s\tau}_{pq}(U)} := \inf\{\|\tilde{f}\|_{\mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n)} : \tilde{f} \in \mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n), \tilde{f}|_U = f\}. \tag{7}$$

The definitions of the spaces $\mathscr{A}^{s\tau}_{pq}(U)$ do not depend on the choice of $(\lambda_j)_{j=0}^{\infty}$ which satisfies (P.a') and (P.b'). See [24, Page 39, Corollary 2.1] and [20, Theorem 2.8].

Remark 7 We remark some known results and different notations for these spaces in $\mathbb{R}^n$ from the literature:

(i) Clearly $\mathscr{B}^s_{pq}(\mathbb{R}^n) = \mathscr{B}^{s0}_{pq}(\mathbb{R}^n) = \mathscr{N}^{s0}_{pq}(\mathbb{R}^n)$ and $\mathscr{F}^{s0}_{pq}(\mathbb{R}^n) = \mathscr{F}^s_{pq}(\mathbb{R}^n)$ (provided $p < \infty$).

(ii) In applications only $0 \le \tau \le \frac{1}{p}$ is interesting: by [27, Theorem 2] and [10, Lemma 3.4],

$$\begin{aligned} \mathscr{B}^{s,\tau}_{p,q}(\mathbb{R}^n) = \mathscr{F}^{s,\tau}_{p,q}(\mathbb{R}^n) = \mathscr{B}^{s+n(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^n), \\ \mathscr{N}^{s,\tau}_{p,q}(\mathbb{R}^n) = \{0\}, \quad \forall\, 0 < p, q \le \infty,\ s \in \mathbb{R},\ \tau > \tfrac{1}{p}. \end{aligned} \tag{8}$$

(iii) For the case $\tau = 1/p$, by [27, Theorem 2] and [10, Remark 11(ii)],

$$\begin{aligned} \mathscr{B}^{s,\frac{1}{p}}_{p,\infty}(\mathbb{R}^n) = \mathscr{F}^{s,\frac{1}{p}}_{p,\infty}(\mathbb{R}^n) = \mathscr{B}^s_{\infty,\infty}(\mathbb{R}^n), \\ \mathscr{N}^{s,\frac{1}{p}}_{p,q}(\mathbb{R}^n) = \mathscr{B}^s_{\infty,q}(\mathbb{R}^n), \quad \forall\, 0 < p, q \le \infty,\ s \in \mathbb{R}. \end{aligned}$$

(iv) Although $\mathscr{F}^{s\tau}_{pq}$-spaces are only defined for $p < \infty$, we have a description for $\mathscr{F}^s_{\infty q}$-spaces as the following (see [24, Page 41, Proposition 2.4(iii)] and [2, Section 5]):

$$\mathscr{F}^s_{\infty q}(\mathbb{R}^n) = \mathscr{F}^{s,\frac{1}{p}}_{p,q}(\mathbb{R}^n) = \mathscr{B}^{s,\frac{1}{q}}_{q,q}(\mathbb{R}^n), \quad \forall\, 0 < p < \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}. \tag{9}$$

(v) Our notation $\mathscr{N}^{s\tau}_{pq}$ corresponds to the $\mathcal{B}^{s\tau}_{pq}$ in [10, Definition 5]. For the classical notations[3] $\mathcal{N}^s_{uqp}$ we have correspondence (see [10, Remark 13(iii)] for example):

$$\mathcal{N}^s_{u,q,p}(\mathbb{R}^n) = \mathscr{N}^{s,\frac{1}{p}-\frac{1}{u}}_{p,q}(\mathbb{R}^n), \quad \forall\, 0 < p \le u \le \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}.$$

(vi) We do not talk about the *Triebel-Lizorkin-Morrey spaces* $\mathcal{E}^s_{uqp}$ in the paper, because they are special cases of the Triebel-Lizorkin-type spaces: we have $\mathcal{E}^s_{u,q,p}(\mathbb{R}^n) = \mathscr{F}^{s,1/p-1/u}_{p,q}(\mathbb{R}^n)$ for all $p \in (0,\infty)$, $q \in (0,\infty]$, $u \in [p,\infty]$ and $s \in \mathbb{R}$. See [24, Corollary 3.3].

[3] Some papers may have different order of the indices. For example, in [7] this is written as $\mathcal{N}^s_{upq}$.

(vii) There are also papers that use the notations $\Lambda^{\varrho}\mathscr{A}^s_{pq}$ and $\Lambda_{\varrho}\mathscr{A}^s_{pq}$ for $\mathscr{A}\in\{\mathscr{B},\mathscr{F}\}$ and $-n\le\varrho\le 0$ ($p<\infty$ for $\mathscr{F}$-cases), for example [6, 19]. These spaces describe the same collection to $\mathscr{A}^{s\tau}_{pq}$ for $\mathscr{A}\in\{\mathscr{B},\mathscr{F},\mathscr{N}\}$, see [6, Remarks 2.7 and 2.9] for example.

For more discussions, we refer the reader to [6, 18, 24].

3 Proof of the Theorems

Our proof follows from some results in [9] and [26].

The key ingredient is the **Peetre maximal operators** introduced in [8].

Definition 8 Let $N>0$, $U\subseteq\mathbb{R}^n$ be an open set and let $\eta=(\eta_j)_{j=0}^\infty$ be a sequence of Schwartz functions. The associated *Peetre maximal operators* $(\mathcal{P}^{\eta,N}_{U,j})_{j=0}^\infty$ are given by

$$\mathcal{P}^{\eta,N}_{U,j}f(x):=\sup_{y\in U}\frac{|\eta_j*f(y)|}{(1+2^j|x-y|)^N},\quad f\in\mathscr{S}'(\mathbb{R}^n),\quad x\in\mathbb{R}^n,\quad j\ge 0.$$

Lemma 9 *Let $\phi=(\phi_j)_{j=0}^\infty$ be a Littlewood–Paley family associated with a special Lipschitz domain Ω (see Definition 4). Then there is a $\psi=(\psi_j)_{j=0}^\infty\subset\mathscr{S}'(\mathbb{R}^n)$ satisfying (P.a) and (P.b) such that $(\psi_j*\phi_j)_{j=0}^\infty$ is also associated with Ω.*

Proof The assumptions $\phi_j(x)=2^{(j-1)n}\phi_1(2^{j-1}x)$ for $j\ge 1$ and $\sum_{j=0}^\infty\phi_j=\delta_0$ imply $\phi_1(x)=2^n\phi_0(2x)-\phi_0(x)$, i.e. $\hat\phi_1(\xi)=\hat\phi_0(\xi/2)-\hat\phi_0(\xi)$. We can take $\psi=(\psi_j)_{j=0}^\infty$ via the Fourier transforms:

$$\hat\psi_0(\xi):=2\hat\phi_0(\xi)-\hat\phi_0(\xi)^3;$$
$$\hat\psi_j(\xi):=(\hat\phi_0(2^{-j}\xi)+\hat\phi_0(2^{1-j}\xi))(2-\hat\phi_0(2^{-j}\xi)^2-\hat\phi_0(2^{1-j}\xi)^2),\ \text{for } j\ge 1.$$

See [9, Proposition 2.1] for details.

Lemma 10 *([1, Lemma 2.1]) Let $\eta=(\eta_j)_{j=0}^\infty$ and $\theta=(\theta_j)_{j=0}^\infty\subset\mathscr{S}(\mathbb{R}^n)$ both satisfy conditions (P.a) and (P.b). Then for any $N>0$ there exists a $C=C(\eta,\theta,N)>0$ such that*

$$\int_{\mathbb{R}^n}|\eta_j*\theta_k(x)|(1+2^k|x|)^N dx\lesssim_{\eta,\theta,N}2^{-N|j-k|},\qquad\forall j,k\ge 0.$$

Lemma 11 *Let $0<p,q\le\infty$, $\tau\ge 0$ and $\delta>n\tau$. There is a $C=C(n,p,q,\tau,\delta)>0$ such that for every $(g_j)_{j=0}^\infty\subset L^p_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\Big\|\Big(\sum_{k\ge 0}2^{-\delta|j-k|}g_k\Big)_{j=0}^\infty\Big\|_{\ell^q L^p_\tau}\le C\|(g_j)_{j=0}^\infty\|_{\ell^q L^p_\tau};\tag{10}$$

$$\Big\|\Big(\sum_{k\geq 0}2^{-\delta|j-k|}g_k\Big)_{j=0}^{\infty}\Big\|_{L^p_\tau\ell^q}\leq C\|(g_j)_{j=0}^\infty\|_{L^p_\tau\ell^q},\qquad \textit{provided } p<\infty; \tag{11}$$

$$\Big\|\Big(\sum_{k\geq 0}2^{-\delta|j-k|}g_k\Big)_{j=0}^{\infty}\Big\|_{\ell^q M^p_\tau}\leq C\|(g_j)_{j=0}^\infty\|_{\ell^q M^p_\tau}. \tag{12}$$

Proof (10) and (11) have been done in [26, Lemma 2.3]. We only prove (12).

Using the case $\tau=0$ in (10) we have

$$\begin{aligned}&\big\|\big(\textstyle\sum_{k\geq 0}2^{-\delta|j-k|}f_k\big)_{j=0}^\infty\big\|_{\ell^q(L^p)}\\ &\quad\lesssim_{p,q,\delta}\|(f_j)_{j=0}^\infty\|_{\ell^q(L^p)},\quad \forall (f_j)_{j=0}^\infty\in\ell^q(\mathbb{Z}_{\geq 0};L^p(\mathbb{R}^n)).\end{aligned}$$

Note that $\|g_k\|_{M^p_\tau}=\|\sup_{Q_{J,v}}|2^{nJ\tau}\mathbf{1}_{Q_{J,v}}\cdot g_k|\|_{L^p(\mathbb{R}^n)}$. By taking $f_k:=\sup_{Q_{J,v}}|2^{nJ\tau}\mathbf{1}_{Q_{J,v}}\cdot g_k|$ above we have

$$\begin{aligned}&\Big\|\Big(\sum_{k\geq 0}2^{-\delta|j-k|}|g_k|\Big)_{j=0}^\infty\Big\|_{\ell^q M^p_\tau}\\ &=\Big\|\Big(\sup_{Q_{J,v}\in\mathcal{Q}}2^{nJ\tau}\mathbf{1}_{Q_{J,v}}\cdot\sum_{k\geq 0}2^{-\delta|j-k|}|g_k|\Big)_j\Big\|_{\ell^q(L^p)}\\ &\leq\Big\|\Big(\sum_{k\geq 0}2^{-\delta|j-k|}\sup_{Q_{J,v}\in\mathcal{Q}}2^{nJ\tau}\mathbf{1}_{Q_{J,v}}\cdot|g_k|\Big)_j\Big\|_{\ell^q(L^p)}\\ &=\Big\|\Big(\sum_{k\geq 0}2^{-\delta|j-k|}f_k\Big)_j\Big\|_{\ell^q(L^p)}\\ &\lesssim_{p,q,\delta}\|(f_j)_{j=0}^\infty\|_{\ell^q(L^p)}=\|(g_j)_{j=0}^\infty\|_{\ell^q M^p_\tau}.\end{aligned}$$

□

Lemma 12 *Let $\Omega\subset\mathbb{R}^n$ be a special Lipschitz domain, let $\phi=(\phi_j)_{j=0}^\infty$ be a Littlewood-Paley family associated with Ω, and let $\theta=(\theta_j)_{j=0}^\infty$ satisfies conditions (P.a), (P.b) and (P.d). Then for any $N>0$ and $\gamma\in(0,\infty]$ there is a $C=C(\theta,\phi,N)>0$, such that, for every $f\in\mathscr{S}'(\mathbb{R}^n)$, $j\geq 0$ and $x\in\Omega$,*

$$\mathcal{P}^{\theta,N}_{\Omega,j}f(x)\leq C\Big(\sum_{k=0}^\infty 2^{-N\gamma|j-k|}\int_\Omega\frac{2^{kn}|\phi_k\ast f(y)|^\gamma dy}{(1+2^k|x-y|)^{N\gamma}}\Big)^{1/\gamma}. \tag{13}$$

Proof The special case $\theta=\phi$ of (13) is proved in [9, Proof of Theorem 3.2, Step 1]. Namely, we have

$$\mathcal{P}^{\phi,N}_{\Omega,j}f(x)\lesssim_{\phi,N}\Big(\sum_{k=0}^\infty 2^{-N\gamma|j-k|}\int_\Omega\frac{2^{kn}|\phi_k\ast f(y)|^\gamma dy}{(1+2^k|x-y|)^{N\gamma}}\Big)^{1/\gamma}. \tag{14}$$

Also see [21, Proof of Theorem 2.6, Step 1] for the argument. Thus it suffices to prove the case $\gamma = \infty$:

$$\mathcal{P}^{\theta,N}_{\Omega,j} f(x) \lesssim_{\theta,\phi,N} \sup_{k\geq 0} 2^{-N|j-k|}\mathcal{P}^{\phi,N}_{\Omega,k} f(x), \quad \forall f \in \mathscr{S}'(\mathbb{R}^n), \quad j \geq 0, \quad x \in \Omega. \tag{15}$$

Let $\psi = (\psi_j)_{j=0}^{\infty}$ satisfies the consequence of Lemma 9, so $\theta_j * f = \sum_{k=0}^{\infty}(\theta_j * \psi_k) * (\phi_k * f)$ for $j \geq 0$. By assumption ϕ_j, ψ_j, θ_j are supported in $K = \{x_n < -\|\nabla\rho\|_{L^\infty} \cdot |x'|\}$ where ρ is the defining function for $\Omega = \{x_n > \rho(x')\}$. Using the property $\Omega - K \subseteq \Omega$, we have

$$\begin{aligned} &\mathbf{1}_\Omega \cdot (\theta_j * f) = \mathbf{1}_\Omega \cdot \textstyle\sum_{k=0}^{\infty}(\theta_j * \psi_k) * (\mathbf{1}_\Omega \cdot (\phi_k * f)); \\ \text{and thus}\quad &\mathcal{P}^{\theta,N}_{\Omega,j} f(x) = \sup_{z\in\Omega} \frac{|\theta_j * f(z)|}{(1+2^j|x-z|)^N} \\ &\leq \sup_{z\in\Omega} \sum_{k=0}^{\infty} \int_\Omega \frac{|\theta_j * \psi_k(z-y)||\phi_k * f(y)|dy}{(1+2^j|x-z|)^N}. \end{aligned}$$

The elementary inequality yields

$$\begin{aligned} \frac{1}{(1+2^j|x-z|)^N} &\leq \frac{2^{N|j-k|}}{(1+2^k|x-z|)^N}\frac{(1+2^k|z-y|)^N}{(1+2^k|z-y|)^N} \\ &\leq 2^{N|j-k|}\frac{(1+2^k|z-y|)^N}{(1+2^k|x-y|)^N}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathcal{P}^{\theta,N}_{\Omega,j} f(x) \\ &= \sup_{z\in\Omega} \frac{|\phi_k * f(z)|}{(1+2^k|x-z|)^N} \sum_{k=0}^{\infty} \int_\Omega 2^{N|j-k|}|\theta_j * \psi_k(z-y)|(1+2^k|z-y|)^N dy \\ &\leq \sup_{k\geq 0} 2^{-N|j-k|}\mathcal{P}^{\phi,N}_{\Omega,k} f(x) \sum_{l=0}^{\infty} \int_\Omega 2^{2N|j-l|}|\theta_j * \psi_l(y)|(1+2^l|y|)^N dy \\ &\lesssim_{\theta,\phi,N} \sup_{k\geq 0} 2^{-N|j-k|}\mathcal{P}^{\phi,N}_{\Omega,k} f(x) \sum_{l=0}^{\infty} 2^{(2N-(2N+1))|j-l|} \lesssim \sup_{k\geq 0} 2^{-N|j-k|}\mathcal{P}^{\phi,N}_{\Omega,k} f(x). \end{aligned} \tag{16}$$

Here the last inequality is obtained by applying Lemma 10.

Therefore we get (15). Combining it with (14) we complete the proof.

Recall the Hardy–Littlewood maximal function $\mathcal{M}f(x) := \sup_{R>0}|B(0,R)|^{-1} \int_{B(x,R)}|f(y)|dy$ for $f \in L^1_{\mathrm{loc}}$.

Lemma 13 *Let $N > n$. There is a $C = C(N) > 0$ such that for any $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $J \in \mathbb{Z}$, $v \in \mathbb{Z}^n$, $k \geq J$ and $x \in Q_{J,v}$,*

$$\int_{\mathbb{R}^n} \frac{2^{kn}|g(y)|dy}{(1+2^k|x-y|)^N} \leq C \sum_{w\in\mathbb{Z}^n} \frac{\mathcal{M}(\mathbf{1}_{Q_{J,w}} \cdot g)(x)}{(1+|v-w|)^{N-n}}. \tag{17}$$

Our lemma here is weaker than the corresponding estimate in [26, Proof of Theorem 1.2, Step 3].

Proof By taking a translation, it suffices to prove the estimate on $x \in Q_{J,0}$, i.e for $v = 0$. Note that if $y \in Q_{J,w}$, then $|x-y| \geq \operatorname{dist}(Q_{J,w}, Q_{J,0}) \geq \frac{1}{\sqrt{n}} 2^{-J} \max(0, |w| - \sqrt{n})$ and $|x-y| \leq |w| + \sqrt{n}$. Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{2^{kn}|g(y)|dy}{(1+2^k|x-y|)^N} \\
&\leq \int_{B(x,3\sqrt{n}2^{-J})} \frac{2^{kn}|g(y)|dy}{(1+2^k|x-y|)^N} + \sum_{|w|>2\sqrt{n}} \int_{Q_{J,w}} \frac{2^{kn}|g(y)|dy}{(1+2^k|x-y|)^N} \\
&\lesssim \left\| \frac{2^{n(k-J)}}{(1+2^k|y|)^N} \right\|_{L^1(\mathbb{R}^n_y)} \mathcal{M}(\mathbf{1}_{B(0,4\sqrt{n}2^{-J})} \cdot g)(x) \\
&\quad + \sum_{|w|>2\sqrt{n}} \frac{2^{kn}}{\left(1+2^k 2^{-J}(\frac{|w|}{\sqrt{n}} - 1)\right)^N} \int_{Q_{J,w}} |g(y)|dy \\
&\lesssim \sum_{|w|<4\sqrt{n}} \mathcal{M}(\mathbf{1}_{Q_{J,w}} \cdot g)(x) \\
&\quad + \sum_{|w|>2\sqrt{n}} \frac{2^{-(k-J)(N-n)}}{|w|^{N-n}} \cdot \frac{2^{nJ}}{|w|^n} \int_{B(x,2^{-J}(|w|+\sqrt{n}))} |\mathbf{1}_{Q_{J,w}} \cdot g(y)|dy \\
&\lesssim \sum_{w\in\mathbb{Z}^n} \frac{1}{(1+|w|)^{N-n}} \cdot \mathcal{M}(\mathbf{1}_{Q_{J,w}} \cdot g)(x).
\end{aligned}$$

□

Combining Lemmas 11 - 13 we have the following Morrey–type estimates for Peetre maximal functions.

Proposition 14 *Keeping the assumptions of Lemma 12, for every $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $\tau \geq 0$ and $N > \max(2n/\min(p,q), |s| + n\tau)$, there is a $C = C(\theta, \phi, p, q, s, \tau, N) > 0$ such that for every $f \in \mathscr{S}'(\Omega)$,*

$$\left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\theta,N}_{\Omega,j} f)\right)_{j=0}^\infty \right\|_{\ell^q L^p_\tau} \leq C \left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\phi_j * f)\right)_{j=0}^\infty \right\|_{\ell^q L^p_\tau}; \tag{18}$$

$$\left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\theta,N}_{\Omega,j} f)\right)_{j=0}^\infty \right\|_{L^p_\tau \ell^q} \leq C \left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\phi_j * f)\right)_{j=0}^\infty \right\|_{L^p_\tau \ell^q}, \quad \textit{provided } p < \infty; \tag{19}$$

$$\left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\theta,N}_{\Omega,j} f)\right)_{j=0}^\infty \right\|_{\ell^q M^p_\tau} \leq C \left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\phi_j * f)\right)_{j=0}^\infty \right\|_{\ell^q M^p_\tau}. \tag{20}$$

Remark 15 It is possible that the assumption $N > \max(\frac{2n}{\min(p,q)}, |s| + n\tau)$ can be relaxed to $N > \frac{n}{\min(p,q)}$. In applications, we only need a large enough N that does not depend on f.

A similar result for (20) can be found in [20, Proposition 2.12]. Note that we require θ_j to have Fourier compact supports in that proposition.

Proof We use a convention $\phi_j :\equiv 0$ for $j \le -1$. Thus in the computations below every sequence $(a_j)_{j=J}^{\infty}$ is identical to $(a_j)_{j=\max(0,J)}^{\infty}$.

By the assumption on N we can take $\gamma \in (0, \min(p,q))$ such that $N\gamma > 2n$. We first prove (19).

Since $N > |s|+n\tau$. By Lemma 12 and using $2^{j\gamma s}2^{-N\gamma|j-k|} \le 2^{-(N-|s|)\gamma|j-k|}2^{k\gamma s}$,

$$\begin{aligned}\|(2^{js}\mathcal{P}^{\theta,N}_{\Omega,j}f)_{j=0}^{\infty}\|_{L^p_\tau\ell^q} &= \left\|\left(2^{j\gamma s}(\mathcal{P}^{\theta,N}_{\Omega,j}f)^{\gamma}\right)_{j=0}^{\infty}\right\|^{\frac{1}{\gamma}}_{L^{\frac{p}{\gamma}}_{\tau\gamma}\ell^{\frac{q}{\gamma}}} \\ &\lesssim \left\|\left(\sum_{k=0}^{\infty}2^{(|s|-N)\gamma|j-k|}\int_{\Omega}\frac{2^{kn}|2^{ks}\phi_k * f(y)|^{\gamma}dy}{(1+2^k|\cdot-y|)^{N\gamma}}\right)_{j=0}^{\infty}\right\|^{\frac{1}{\gamma}}_{L^{\frac{p}{\gamma}}_{\tau\gamma}\ell^{\frac{q}{\gamma}}}.\end{aligned}$$

By Lemma 11 and since $(N-|s|)\gamma > n\tau\gamma$,

$$\begin{aligned}&\left\|\left(\sum_{k=0}^{\infty}2^{(|s|-N)\gamma|j-k|}\int_{\Omega}\frac{2^{kn}|2^{ks}\phi_k * f(y)|^{\gamma}dy}{(1+2^k|\cdot-y|)^{N\gamma}}\right)_{j=0}^{\infty}\right\|_{L^{\frac{p}{\gamma}}_{\tau\gamma}\ell^{\frac{q}{\gamma}}} \\ &\quad\lesssim \left\|\left(\int_{\Omega}\frac{2^{kn}|2^{ks}\phi_k * f(y)|^{\gamma}dy}{(1+2^k|\cdot-y|)^{N\gamma}}\right)_{k=0}^{\infty}\right\|_{L^{\frac{p}{\gamma}}_{\tau\gamma}\ell^{\frac{q}{\gamma}}}.\end{aligned}$$

Applying Lemma 13 with $g(x) = \mathbf{1}_{\Omega}(x)\cdot|2^{ks}\phi_k * f(x)|^{\gamma}$ for each $k \ge 0$ and expanding the $L^{\frac{p}{\gamma}}_{\tau\gamma}\ell^{\frac{q}{\gamma}}$-norm,

$$\begin{aligned}&\left\|\left(\int_{\Omega}\frac{2^{kn}|2^{ks}\phi_k * f(y)|^{\gamma}dy}{(1+2^k|\cdot-y|)^{N\gamma}}\right)_{k=0}^{\infty}\right\|_{L^{\frac{p}{\gamma}}_{\tau\gamma}\ell^{\frac{q}{\gamma}}} \\ &= \sup_{J\in\mathbb{Z}, v\in\mathbb{Z}^n} 2^{nJ\tau\gamma\cdot\frac{1}{\gamma}}\left\|\left(\int_{\Omega}\frac{2^{kn}|2^{ks}\phi_k * f(y)|^{\gamma}dy}{(1+2^k|\cdot-y|)^{N\gamma}}\right)_{k=J}^{\infty}\right\|^{\frac{1}{\gamma}}_{L^{\frac{p}{\gamma}}(Q_{J,v};\ell^{\frac{q}{\gamma}})} \\ &\lesssim_{N,\gamma} \sup_{J\in\mathbb{Z}, v\in\mathbb{Z}^n} 2^{nJ\tau}\left\|\left(\sum_{w\in\mathbb{Z}^n}\frac{\mathcal{M}(\mathbf{1}_{Q_{J,w}}\cdot\mathbf{1}_{\Omega}\cdot|2^{ks}\phi_k * f|^{\gamma})}{(1+|w-v|)^{N\gamma-n}}\right)_{k=J}^{\infty}\right\|^{\frac{1}{\gamma}}_{L^{\frac{p}{\gamma}}(Q_{J,v};\ell^{\frac{q}{\gamma}})} \\ &\le \left(\sum_{v\in\mathbb{Z}^n}\frac{1}{(1+|v|)^{N\gamma-n}}\right)^{1/\gamma} \\ &\quad\times \sup_{J\in\mathbb{Z}, w\in\mathbb{Z}^n} 2^{nJ\tau}\left\|\left(\mathcal{M}(\mathbf{1}_{Q_{J,w}\cap\Omega}\cdot|2^{ks}\phi_k * f|^{\gamma})\right)_{k=J}^{\infty}\right\|^{\frac{1}{\gamma}}_{L^{\frac{p}{\gamma}}(\mathbb{R}^n;\ell^{\frac{q}{\gamma}})}.\end{aligned}$$

Since $N\gamma - n > n$ the sum $\sum_{v\in\mathbb{Z}^n}(1+|v|)^{n-N\gamma}$ is finite.

Finally, applying Fefferman-Stein's inequality to $\left(\mathcal{M}(\mathbf{1}_{Q_{J,w}\cap\Omega}\cdot|2^{ks}\phi_k*f|^\gamma)\right)_{k=J}^\infty$ in $L^{\frac{p}{\gamma}}(\mathbb{R}^n;\ell^{\frac{q}{\gamma}})$ for each $J\in\mathbb{Z}$ (see [3, Theorem 1(1)] and also [5, Remark 5.6.7]), since $1<p/\gamma<\infty$ and $1<q/\gamma\le\infty$,

$$\begin{aligned}
&\sup_{Q_{J,w}\in\mathcal{Q}} 2^{nJ\tau}\left\|\left(\mathcal{M}(\mathbf{1}_{Q_{J,w}\cap\Omega}\cdot|2^{ks}\phi_k*f|^\gamma)\right)_{k=J}^\infty\right\|_{L^{\frac{p}{\gamma}}(\mathbb{R}^n;\ell^{\frac{q}{\gamma}})}^{\frac{1}{\gamma}}\\
&\lesssim \sup_{Q_{J,w}} 2^{nJ\tau}\left\|\left(\mathbf{1}_{Q_{J,w}\cap\Omega}\cdot|2^{ks}\phi_k*f|^\gamma\right)_{k=J}^\infty\right\|_{L^{\frac{p}{\gamma}}(\mathbb{R}^n;\ell^{\frac{q}{\gamma}})}^{\frac{1}{\gamma}}\\
&= \sup_{Q_{J,w}} 2^{nJ\tau}\left\|\left(\mathbf{1}_\Omega\cdot(2^{ks}\phi_k*f)\right)_{k=J}^\infty\right\|_{L^p(Q_{J,w};\ell^q)}\\
&= \left\|\left(2^{ks}\mathbf{1}_\Omega\cdot(\phi_k*f)\right)_{k=0}^\infty\right\|_{L^p_\tau\ell^q}.
\end{aligned}$$

This completes the proof of (19).

The proof of (18) and (20) are similar but simpler: by assumption $1<p/\gamma\le\infty$ we have

$$\mathcal{M}:L^{\frac{p}{\gamma}}(\mathbb{R}^n)\to L^{\frac{p}{\gamma}}(\mathbb{R}^n). \tag{21}$$

Therefore, we prove (18) by the following:

$$\begin{aligned}
&\|(2^{js}\mathcal{P}^{\theta,N}_{\Omega,j}f)_{j=0}^\infty\|_{\ell^q L^p_\tau}\\
&\lesssim_{\theta,\phi,s,\tau,N,\gamma}\left\|\left(\sum_{k=0}^\infty 2^{-(n\tau+1)\gamma|j-k|}\int_\Omega\frac{2^{kn}|2^{ks}\phi_k*f(y)|^\gamma dy}{(1+2^k|\cdot-y|)^{N\gamma}}\right)_{j=0}^\infty\right\|_{\ell^{\frac{q}{\gamma}}L^{\frac{p}{\gamma}}_{\tau\gamma}}^{\frac{1}{\gamma}} && \text{by (13)}\\
&\lesssim_{p,q,s,\tau}\left\|\left(\int_\Omega\frac{2^{kn}|2^{ks}\phi_k*f(y)|^\gamma dy}{(1+2^k|\cdot-y|)^{N\gamma}}\right)_{k=0}^\infty\right\|_{\ell^{\frac{q}{\gamma}}L^{\frac{p}{\gamma}}_{\tau\gamma}}^{\frac{1}{\gamma}} && \text{by (10)}\\
&\lesssim_{N,\gamma}\left(\sum_{v\in\mathbb{Z}^n}\frac{1}{(1+|v|)^{N\gamma-n}}\right)^{1/\gamma}\left\|\left(\mathcal{M}(\mathbf{1}_\Omega\cdot|2^{ks}\phi_k*f|^\gamma)\right)_{k=0}^\infty\right\|_{\ell^{\frac{q}{\gamma}}L^{\frac{p}{\gamma}}_{\tau\gamma}}^{\frac{1}{\gamma}} && \text{by (17)}\\
&\lesssim_{p,\gamma}\left\|\left(\mathbf{1}_\Omega\cdot|2^{ks}\phi_k*f|^\gamma\right)_{k=0}^\infty\right\|_{\ell^{q/\gamma}L^{p/\gamma}_{\tau\gamma}}^{1/\gamma}=\left\|\left(2^{ks}\mathbf{1}_\Omega\cdot(\phi_k*f)\right)_{k=0}^\infty\right\|_{\ell^qL^p_\tau} && \text{by (21).}
\end{aligned}$$

Finally we prove (20). Using (15) and (12) (since $N>|s|+n\tau$) we have

$$\begin{aligned}
&\|(2^{js}\mathcal{P}^{\theta,N}_{\Omega,j}f)_{j=0}^\infty\|_{\ell^qM^p_\tau}\lesssim_{\theta,\phi,s,N}\\
&\left\|\left(\sum_{k=0}^\infty 2^{(N-|s|)|j-k|}2^{ks}\mathcal{P}^{\phi,N}_{\Omega,k}f\right)_{j=0}^\infty\right\|_{\ell^qM^p_\tau}\lesssim_{p,q,\tau,N}\|(2^{js}\mathcal{P}^{\phi,N}_{\Omega,j}f)_{j=0}^\infty\|_{\ell^qM^p_\tau}.
\end{aligned} \tag{22}$$

Taking $\gamma\in(n/N,\min(p,q))$, we have $2^{js}(\mathcal{P}^{\phi,N}_{\Omega,j}f)\lesssim_{N,\gamma}\mathcal{M}(|2^{js}\mathbf{1}_\Omega\cdot(\phi_j*f)|^\gamma)^{1/\gamma}$ pointwise in $\mathbb{R}^n$.

When $p<\infty$ and $\tau<1/p$, by [20, Lemma 2.5] we have

$$\|2^{js}\mathcal{P}^{\phi,N}_{\Omega,j}f\|_{M^p_\tau}\lesssim_{N,\gamma}\left\|\mathcal{M}\left(|2^{js}\mathbf{1}_\Omega\cdot(\phi_j*f)|^\gamma\right)^{1/\gamma}\right\|_{M^p_\tau}$$

$$\lesssim_{p,\gamma,\tau} \|2^{js}\mathbf{1}_\Omega \cdot (\phi_j * f)\|_{M^p_\tau}, \quad j \geq 0. \tag{23}$$

We see that (23) is valid for all $1 < p/\gamma \leq \infty$, $\tau \geq 0$.

When $\tau = 1/p$, we have $M^p_\tau = L^\infty$ by [10, Remark 11(ii)], so (23) follows from (21). When $\tau > 1/p$ we have $M^p_\tau = \{0\}$, so (23) holds trivially.

Thus by taking ℓ^q-sum of (23), we get (20), completing the proof.

Proposition 16 *Let $\theta = (\theta_j)_{j=0}^\infty$ satisfies (P.a) and (P.b), and let $\lambda = (\lambda_j)_{j=0}^\infty$ satisfies (P.a') and (P.b'). For any $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $\tau \geq 0$ and $N > \max(2n/\min(p,q), |s| + n\tau)$, there is a $C = C(\theta, \lambda, p, q, s, \tau, N) > 0$ such that for every $\tilde{f} \in \mathscr{S}'(\mathbb{R}^n)$,*

$$\|(2^{js}\mathcal{P}^{\theta,N}_{\mathbb{R}^n,j}\tilde{f})_{j=0}^\infty\|_{\ell^q L^p_\tau} \leq C\|(2^{js}\lambda_j * \tilde{f})_{j=0}^\infty\|_{\ell^q L^p_\tau}; \tag{24}$$

$$\|(2^{js}\mathcal{P}^{\theta,N}_{\mathbb{R}^n,j}\tilde{f})_{j=0}^\infty\|_{L^p_\tau \ell^q} \leq C\|(2^{js}\lambda_j * \tilde{f})_{j=0}^\infty\|_{L^p_\tau \ell^q}, \quad \textit{provided } p < \infty; \tag{25}$$

$$\|(2^{js}\mathcal{P}^{\theta,N}_{\mathbb{R}^n,j}\tilde{f})_{j=0}^\infty\|_{\ell^q M^p_\tau} \leq C\|(2^{js}\lambda_j * \tilde{f})_{j=0}^\infty\|_{\ell^q M^p_\tau}. \tag{26}$$

Proof The proof is the same as that for Proposition 14, except that we replace every Ω by $\mathbb{R}^n$ in the arguments. We leave the details to readers.

Based on Proposition 14, we can prove a boundedness result of Rychkov-type operators on $\mathscr{A}^{s\tau}_{pq}$-spaces.

Proposition 17 *Let $\Omega \subset \mathbb{R}^n$ be a special Lipschitz domain and let $\gamma \in \mathbb{R}$. Let $\eta = (\eta_j)_{j=0}^\infty$ and $\theta = (\theta_j)_{j=0}^\infty$ satisfy conditions (P.a), (P.b) and (P.d) with respect to Ω. We define an operator.*[4] *$T^{\eta,\theta,\gamma}_\Omega$ as*

$$T^{\eta,\theta,\gamma}_\Omega f := \sum_{j=0}^\infty 2^{j\gamma}\eta_j * (\mathbf{1}_\Omega \cdot (\theta_j * f)), \quad f \in \mathscr{S}'(\Omega). \tag{27}$$

Then for $\mathscr{A} \in \{\mathscr{B}, \mathscr{F}, \mathscr{N}\}$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $\tau \geq 0$ ($p < \infty$ for $\mathscr{F}$-cases), we have the boundedness

$$T^{\eta,\theta,\gamma}_\Omega : \mathscr{A}^{s,\tau}_{p,q}(\Omega) \to \mathscr{A}^{s-\gamma,\tau}_{p,q}(\mathbb{R}^n).$$

Proof Recall $\mathscr{S}'(\Omega) = \{\tilde{f}|_\Omega : \tilde{f} \in \mathscr{S}'(\mathbb{R}^n)\}$ is defined via restrictions. We see that $T^{\eta,\theta,\gamma}_\Omega : \mathscr{S}'(\Omega) \to \mathscr{S}'(\mathbb{R}^n)$ is well-defined in the sense that, for every extension $\tilde{f} \in \mathscr{S}'(\mathbb{R}^n)$ of f, the summation $\sum_{j=0}^\infty 2^{j\gamma}\eta_j * (\mathbf{1}_\Omega \cdot (\theta_j * \tilde{f}))$ converges $\mathscr{S}'(\mathbb{R}^n)$ and does not depend on the choice of $\tilde{f}$. See [12, Propositions 3.11 and 3.16] for example.

Let $\lambda = (\lambda_j)_{j=0}^\infty$ be as in Definition 6 that defines the $\mathscr{A}^{s\tau}_{pq}$-norms. By Lemma 10, for every $j, k \geq 0$, $\int_{\mathbb{R}^n} |\lambda_j * \eta_k(y)|(1 + 2^k|y|)^N dy \lesssim_{\lambda,\eta,N} 2^{-N|j-k|}$. Thus by the

4 The notation is slightly different from the one in [12, Theorem 1.2].

similar argument to (16), for every $N > |s-\gamma|$,

$$\begin{aligned}&2^{j(s-\gamma)}2^{k\gamma}|\lambda_j * \eta_k * (\mathbf{1}_\Omega\cdot(\theta_k * f))(x)|\\&\le 2^{j(s-\gamma)}2^{k\gamma}\int_\Omega|\lambda_j*\eta_k(y)|(1+2^k|y|)^N dy\cdot\sup_{t\in\Omega}\frac{|\theta_k * f(t)|}{(1+2^k|x-t|)^N}\\&\lesssim_{\lambda,\eta,N} 2^{-(N-|s-\gamma|)|j-k|}2^{ks}(\mathcal{P}^{\theta,N}_{\Omega,k}f)(x).\end{aligned}$$

Therefore, by Lemma 11, for any $N > |s-\gamma| + n\tau$,

$$\|(2^{j(s-\gamma)}\lambda_j * T^{\eta,\theta,\gamma}_\Omega f)_{j=0}^\infty\|_{\ell^q L^p_\tau} \lesssim_{\lambda,\eta,p,q,s,\gamma,\tau,N} \|(2^{ks}\mathcal{P}^{\theta,N}_{\Omega,k}f)_{k=0}^\infty\|_{\ell^q L^p_\tau}; \tag{28}$$

$$\|(2^{j(s-\gamma)}\lambda_j * T^{\eta,\theta,\gamma}_\Omega f)_{j=0}^\infty\|_{L^p_\tau \ell^q} \lesssim_{\lambda,\eta,p,q,s,\gamma,\tau,N} \|(2^{ks}\mathcal{P}^{\theta,N}_{\Omega,k}f)_{k=0}^\infty\|_{L^p_\tau \ell^q},\ (p<\infty); \tag{29}$$

$$\|(2^{j(s-\gamma)}\lambda_j * T^{\eta,\theta,\gamma}_\Omega f)_{j=0}^\infty\|_{\ell^q M^p_\tau} \lesssim_{\lambda,\eta,p,q,s,\gamma,\tau,N} \|(2^{ks}\mathcal{P}^{\theta,N}_{\Omega,k}f)_{k=0}^\infty\|_{\ell^q M^p_\tau}. \tag{30}$$

Let $\tilde f \in \mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n)$ be an extension of f. Clearly $\mathcal{P}^{\theta,N}_{\Omega,k}f(x) = \mathcal{P}^{\theta,N}_{\Omega,k}\tilde f(x) \le \mathcal{P}^{\theta,N}_{\mathbb{R}^n,k}\tilde f(x)$ holds pointwise for $x\in\mathbb{R}^n$. Therefore, by choosing $N > 2n/\min(p,q)$ and combining (28) and (24), we have

$$\begin{aligned}\|T^{\eta,\theta,\gamma}_\Omega f\|_{\mathscr{B}^{s\tau}_{pq}(\mathbb{R}^n)} &= \|(2^{j(s-\gamma)}\lambda_j * T^{\eta,\theta,\gamma}_\Omega f)_{j=0}^\infty\|_{\ell^q L^p_\tau}\\&\lesssim_{\eta,\theta,\lambda,p,q,s,\gamma,\tau} \|(2^{js}\lambda_j * \tilde f)_{j=0}^\infty\|_{\ell^q L^p_\tau} = \|\tilde f\|_{\mathscr{B}^{s\tau}_{pq}(\mathbb{R}^n)}.\end{aligned}$$

Taking the infimum over all extensions $\tilde f$ of f we get the boundedness $T^{\eta,\theta,\gamma}_\Omega : \mathscr{B}^{s,\tau}_{p,q}(\Omega) \to \mathscr{B}^{s-\gamma,\gamma}_{p,q}(\mathbb{R}^n)$. Similarly using (29), (25) and (30), (26) we get $T^{\eta,\theta,\gamma}_\Omega : \mathscr{A}^{s,\tau}_{p,q}(\Omega) \to \mathscr{A}^{s-\gamma,\gamma}_{p,q}(\mathbb{R}^n)$ for $\mathscr{A}\in\{\mathscr{F},\mathscr{N}\}$.

Remark 18 Under the definition (7), the operator norms of $T^{\eta,\theta,\gamma}_\Omega$ do not depend[5] on Ω. This is due to the same reason as mentioned in [12, Remark 3.14]:

One can see that the constants in Proposition 14 depend on everything except on Ω. The same hold for the implied constants in (28), (29) and (30). After the pointwise inequality $\mathcal{P}^{\theta,N}_{\Omega,k}f \le \mathcal{P}^{\theta,N}_{\mathbb{R}^n,k}\tilde f$, it remains to estimate $(2^{js}\mathcal{P}^{\theta,N}_{\mathbb{R}^n,j}\tilde f)_{j=0}^\infty$ (which is Proposition 16), where Ω is not involved.

Corollary 19 *([25, 28, 29]) Let $\Omega\subset\mathbb{R}^n$ be a special Lipschitz domain. Let $\phi = (\phi_j)_{j=0}^\infty$ and $\psi = (\psi_j)_{j=0}^\infty$ be as in the assumption and conclusion of Lemma 9 with respect to Ω. Then the Rychkov's extension operator*

$$E_\Omega f = E^{\psi,\phi}_\Omega f := \sum_{j=0}^\infty \psi_j * (\mathbf{1}_\Omega\cdot(\phi_j * f)),\quad f\in\mathscr{S}'(\Omega), \tag{31}$$

[5] It can depend on the upper bound of $\|\nabla\rho\|_{L^\infty}$, which is bounded by $\inf\{-\frac{x_n}{|x'|} : (x',x_n)\in\operatorname{supp}\phi_j\}$ where $\phi\in\{\eta,\theta\}$ and $j\ge 0$.

is well-defined and has boundedness $E_\Omega : \mathscr{A}^{s\tau}_{pq}(\Omega) \to \mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n)$ *for* $\mathscr{A} \in \{\mathscr{B}, \mathscr{F}, \mathscr{N}\}$ *and all* $0 < p, q \le \infty$, $s \in \mathbb{R}$, $\tau \ge 0$ *(*$p < \infty$ *for* $\mathscr{F}$*-cases).*

Proof E_Ω is an extension operator because by assumption $E_\Omega f|_\Omega = \sum_{j=0}^\infty \psi_j * \phi_j * f = f$. The boundedness is immediate since $E_\Omega = T_\Omega^{\psi,\phi,0}$ from (27).

Remark 20 Corollary 19 is not new. See [25, Proposition 4.13] for $\mathscr{A} = \mathscr{N}$, [28, Section 4] for $\mathscr{A} = \mathscr{F}$ and [29, Section 4] for $\mathscr{A} = \mathscr{B}$. For the proof we also refer [4, Theorem 3.6] to readers.

The key to prove Theorem 1 is to use the following analog of [9, Theorem 2.3].

Proposition 21 *(Characterizations via Peetre's maximal functions) Let* $\Omega \subset \mathbb{R}^n$ *be a special Lipschitz domain and let* $\phi = (\phi_j)_{j=0}^\infty$ *be a Littlewood-Paley family associated with* Ω*. Then for* $0 < p, q \le \infty$, $s \in \mathbb{R}$ *and* $\tau \ge 0$ *(*$p < \infty$ *for* $\mathscr{F}$*-cases), we have the following intrinsic characterizations: for every* $N > \max(\frac{2n}{\min(p,q)}, |s| + n\tau)$,

$$\|f\|_{\mathscr{B}^{s\tau}_{pq}(\Omega)} \approx_{\phi,p,q,s,\tau,N} \left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\phi,N}_{\Omega,j} f)\right)_{j=0}^\infty \right\|_{\ell^q L^p_\tau}; \tag{32}$$

$$\|f\|_{\mathscr{F}^{s\tau}_{pq}(\Omega)} \approx_{\phi,p,q,s,\tau,N} \left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\phi,N}_{\Omega,j} f)\right)_{j=0}^\infty \right\|_{L^p_\tau \ell^q}, \quad \textit{provided } p < \infty; \tag{33}$$

$$\|f\|_{\mathscr{N}^{s\tau}_{pq}(\Omega)} \approx_{\phi,p,q,s,\tau,N} \left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\phi,N}_{\Omega,j} f)\right)_{j=0}^\infty \right\|_{\ell^q M^p_\tau}. \tag{34}$$

Remark 22 (32) and (33) are not new as well. The case $\mathscr{A} = \mathscr{F}$ is done in [13, Theorem 1.7], where a more general setting is considered. See also [4, Proof of Theorem 3.6, Step 2] for a proof of $\mathscr{A} \in \{\mathscr{B}, \mathscr{F}\}$.

As already mentioned in Remark 15, it is possible that the assumption of N can be weakened.

Proof of Proposition 21 Let $\lambda = (\lambda_j)_{j=0}^\infty$ be as in Definition 6 that defines the $\mathscr{A}^{s\tau}_{pq}$-norms. We only prove (33) since the proof of (32) and (34) are the same by replacing $L^p_\tau \ell^q$ with $\ell^q L^p_\tau$ and $\ell^q M^p_\tau$, and including the discussion of $p = \infty$.
($\gtrsim$) For $f \in \mathscr{F}^{s\tau}_{pq}(\Omega)$, let $\tilde f \in \mathscr{F}^{s\tau}_{pq}(\mathbb{R}^n)$ be an extension of f. We see that pointwisely

$$(\mathbf{1}_\Omega \cdot \mathcal{P}^{\phi,N}_{\Omega,j} f)(x) \le \mathcal{P}^{\phi,N}_{\Omega,j} f(x) = \mathcal{P}^{\phi,N}_{\Omega,j} \tilde f(x) \le \mathcal{P}^{\phi,N}_{\mathbb{R}^n,j} \tilde f(x), \quad j \ge 0, \quad x \in \mathbb{R}^n.$$

Thus by Proposition 14,

$$\begin{aligned} \left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\phi,N}_{j} f)\right)_{j=0}^\infty \right\|_{L^p_\tau \ell^q} &\le \left\| \left(2^{js}\mathcal{P}^{\phi,N}_{\Omega,j} \tilde f\right)_{j=0}^\infty \right\|_{L^p_\tau \ell^q} \\ &\lesssim_{\lambda,\phi,p,q,s,\gamma,\tau,N} \left\| \left(2^{js}\lambda_j * \tilde f\right)_{j=0}^\infty \right\|_{L^p_\tau \ell^q} = \|\tilde f\|_{\mathscr{F}^{s\tau}_{pq}(\mathbb{R}^n)}. \end{aligned}$$

Taking infimum over all extensions $\tilde f$ of f, we get $\|f\|_{\mathscr{F}^{s\tau}_{pq}(\Omega)} \gtrsim \left\| \left(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\phi,N}_{\Omega,j} f)\right)_{j=0}^\infty \right\|_{L^p_\tau \ell^q}$.
($\lesssim$) By Corollary 19 we have $\|f\|_{\mathscr{F}^{s\tau}_{pq}(\Omega)} \approx \|E_\Omega f\|_{\mathscr{F}^{s\tau}_{pq}(\mathbb{R}^n)} = \|(2^{js}\lambda_j * E_\Omega f)_{j=0}^\infty\|_{L^p_\tau \ell^q}$. Therefore using (28) with the fact that $E_\Omega = T_\Omega^{\psi,\phi,0}$,

$$\|(2^{js}\lambda_j * E_\Omega f)_{j=0}^\infty\|_{L^p_\tau \ell^q} = \|(2^{js}\lambda_j * T_\Omega^{\psi,\phi,0} f)_{j=0}^\infty\|_{L^p_\tau \ell^q}$$

$$\lesssim_{\psi,\phi,\lambda,p,q,s,\tau} \|(2^{js}\mathcal{P}^{\phi,N}_{\Omega,j} f)_{j=0}^{\infty}\|_{L^p_\tau \ell^q}. \tag{35}$$

Write $\Omega = \{(x', x_n) : x_n > \rho(x')\}$. We define a "fold map" $L = L_\Omega : \mathbb{R}^n \twoheadrightarrow \overline{\Omega}$ as

$$L(x) := x \quad \text{if } x \in \Omega; \qquad L(x) := (x', 2\rho(x') - x_n), \quad \text{if } x \notin \Omega.$$

Recall $\Omega = \{x_n > \rho(x')\}$. By direct computation, we have

$$|L(x) - y| \le \big(\|\nabla\rho\|_{L^\infty} + \sqrt{1 + \|\nabla\rho\|^2_{L^\infty}}\big)^2 |x - y| \lesssim_\Omega |x - y| \qquad x \in \mathbb{R}^n, \quad y \in \Omega. \tag{36}$$

Therefore

$$\begin{aligned}\mathcal{P}^{\phi,N}_{\Omega,j} f(x) &= \sup_{y\in\Omega} \frac{|\phi_j * f(y)|}{(1 + 2^j|x - y|)^N} \lesssim_{\Omega,N} \sup_{y\in\Omega} \frac{|\phi_j * f(y)|}{(1 + 2^j|L(x) - y|)^N} \\ &= \big(\mathcal{P}^{\phi,N}_{\Omega,j} f\big)\big(L(x)\big), \quad x \in \mathbb{R}^n.\end{aligned}$$

Clearly for $0 < p \le \infty$ we have the following estimate for cube $Q \in \mathcal{Q}$ and function $g \in L^p_{\mathrm{loc}}(\Omega)$:

$$\|g \circ L\|_{L^p(Q)} \lesssim_p \|g\|_{L^p(\Omega\cap L^{-1}(Q))} \lesssim_p \sum_{P\in\mathcal{I}_Q} \|\mathbf{1}_\Omega \cdot g\|_{L^p(P)},$$
$$\text{where } \mathcal{I}_Q := \{P \in \mathcal{Q} : |P| = |Q|,\ P \cap L^{-1}(Q) \ne \varnothing\}.$$

By (36) we have control of the cardinality $\#\mathcal{I}_Q \lesssim_n (1 + \|\nabla\rho\|_{L^\infty})^{2n} \lesssim_\Omega 1$, which is uniform in $Q \in \mathcal{Q}$. Therefore,

$$\begin{aligned}\|(2^{js}\mathcal{P}^{\phi,N}_{\Omega,j} f)_{j=0}^{\infty}\|_{L^p_\tau\ell^q} &\lesssim_N \big\|\big(2^{js}(\mathcal{P}^{\phi,N}_{\Omega,j} f) \circ L\big)_{j=0}^{\infty}\big\|_{L^p_\tau\ell^q} \\ &\lesssim_{p,q,\Omega} \big\|\big(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\phi,N}_{\Omega,j} f)\big)_{j=0}^{\infty}\big\|_{L^p_\tau\ell^q}.\end{aligned} \tag{37}$$

Combining (35) and (37) we get $\|f\|_{\mathscr{F}^{s\tau}_{pq}(\Omega)} \lesssim \big\|\big(2^{js}\mathbf{1}_\Omega\cdot(\mathcal{P}^{\phi,N}_{\Omega,j} f)\big)_{j=0}^{\infty}\big\|_{L^p_\tau\ell^q}$, finishing the proof.

We can now prove Theorem 1:

Proof of Theorem 1 The $\mathscr{F}^s_{\infty q}$-cases follow immediately from the $\mathscr{F}^{s\tau}_{pq}$-cases using (9).

Fix a $N > \max(2n/\min(p, q), |s| + n\tau)$. We only prove the $\mathscr{F}^{s\tau}_{pq}$-cases. The proofs of the $\mathscr{B}^{s\tau}_{pq}$-cases and the $\mathscr{N}^{s\tau}_{pq}$-cases are the same, except that we replace every $L^p_\tau\ell^q$ with $\ell^q L^p_\tau$ and $\ell^q M^p_\tau$.

By Proposition 21 we have $\|f\|_{\mathscr{F}^{s\tau}_{pq}(\Omega)} \approx \big\|\big(2^{js}\mathbf{1}_\Omega\cdot(\mathcal{P}^{\phi,N}_{\Omega,j} f)\big)_{j=0}^{\infty}\big\|_{L^p_\tau\ell^q}$. Therefore, it suffices to show that $\big\|\big(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\phi,N}_{\Omega,j} f)\big)_{j=0}^{\infty}\big\|_{L^p_\tau\ell^q} \approx \big\|\big(2^{js}\mathbf{1}_\Omega \cdot (\phi_j * f)\big)_{j=0}^{\infty}\big\|_{L^p_\tau\ell^q}$.

Clearly $\big\|\big(2^{js}\mathbf{1}_\Omega \cdot (\mathcal{P}^{\phi,N}_{\Omega,j} f)\big)_{j=0}^{\infty}\big\|_{L^p_\tau\ell^q} \ge \big\|\big(2^{js}\mathbf{1}_\Omega \cdot (\phi_j * f)\big)_{j=0}^{\infty}\big\|_{L^p_\tau\ell^q}$ since $\phi_j * f(x) \le \mathcal{P}^{\phi,N}_{\Omega,j} f(x)$ holds for all $f \in \mathscr{S}'(\Omega)$, $x \in \Omega$ and $j \ge 0$. The converse

$\left\|\left(2^{js}\mathbf{1}_\Omega\cdot(\mathcal{P}^{\phi,N}_{\Omega,j}f)\right)_{j=0}^\infty\right\|_{L^p_\tau\ell^q}\lesssim_{\phi,p,q,s,\tau,N}\left\|\left(2^{js}\mathbf{1}_\Omega\cdot(\phi_j*f)\right)_{j=0}^\infty\right\|_{L^p_\tau\ell^q}$ follows from (18). Thus, we prove the $\mathscr{F}^{s\tau}_{pq}$-cases.

We have the immediate analogy of [26, Theorem 1.1] on Lipschitz domains:

Corollary 23 *Keeping the assumptions in Proposition 21, we have the following intrinsic characterizations: for every* $N>\max(2n/\min(p,q),|s|+n\tau)$,

$$\|f\|_{\mathscr{B}^{s\tau}_{pq}(\Omega)}\approx_{\phi,p,q,s,\tau,N}\sup_{Q_{J,v}\in\mathcal{Q}}2^{nJ\tau}\Bigg(\sum_{j=\max(0,J)}^\infty 2^{jsq}\|\mathcal{P}^{\phi,N}_{(Q_{J,v}\cap\Omega),j}f\|^q_{L^p(Q_{J,v}\cap\Omega)}\Bigg)^{\frac{1}{q}};$$

$$\|f\|_{\mathscr{F}^{s\tau}_{pq}(\Omega)}\approx\sup_{Q_{J,v}}2^{nJ\tau}\Bigg(\int_{Q_{J,v}\cap\Omega}\Bigg(\sum_{j=\max(0,J)}^\infty 2^{jsq}|\mathcal{P}^{\phi,N}_{(Q_{J,v}\cap\Omega),j}f(x)|^q\Bigg)^{\frac{p}{q}}dx\Bigg)^{\frac{1}{p}},\ (p<\infty);$$

$$\|f\|_{\mathscr{N}^{s\tau}_{pq}(\Omega)}\approx_{\phi,p,q,s,\tau,N}\Bigg(\sum_{j=0}^\infty\sup_{Q_{J,v}\in\mathcal{Q}}2^{nJ\tau q+jsq}\|\mathcal{P}^{\phi,N}_{(Q_{J,v}\cap\Omega),j}f\|^q_{L^p(Q_{J,v}\cap\Omega)}\Bigg)^{\frac{1}{q}}.$$

Proof Since $|\phi_j*f(x)|\le\mathcal{P}^{\phi,N}_{(Q_{J,v}\cap\Omega),j}f(x)\le\mathcal{P}^{\phi,N}_{\Omega,j}f(x)$ pointwisely for every $Q_{J,v}\in\mathcal{Q}$ and $x\in Q_{J,v}\cap\Omega$, the results follow immediately by combining Theorem 1 and Proposition 21.

Remark 24 By the standard partition of unity argument, we can give the analogy of Theorem 1 on a bounded Lipschitz domain. An example is the following:

$$\|f\|_{\mathscr{B}^{s\tau}_{pq}(\Omega)}\approx\sum_{\nu=1}^N\|(2^{js}\mathbf{1}_{\Omega\cap U_\nu}\cdot(\phi^\nu_j*(\chi_\nu f)))_{j=0}^\infty\|_{\ell^qL^p_\tau};\tag{38}$$

$$\|f\|_{\mathscr{F}^{s\tau}_{pq}(\Omega)}\approx\sum_{\nu=1}^N\|(2^{js}\mathbf{1}_{\Omega\cap U_\nu}\cdot(\phi^\nu_j*(\chi_\nu f)))_{j=0}^\infty\|_{L^p_\tau\ell^q};\tag{39}$$

$$\|f\|_{\mathscr{N}^{s\tau}_{pq}(\Omega)}\approx\sum_{\nu=1}^N\|(2^{js}\mathbf{1}_{\Omega\cap U_\nu}\cdot(\phi^\nu_j*(\chi_\nu f)))_{j=0}^\infty\|_{\ell^qM^p_\tau};\tag{40}$$

Here $\{U_\nu,(\phi^\nu_j)_{j=0}^\infty,\chi_\nu\}_{\nu=1}^N$ satisfy the following:

- $\{U_\nu\}_{\nu=1}^N$ is an open cover of $\overline{\Omega}$, and there are cones $K_\nu\subset\mathbb{R}^n$ such that $U_\nu\cap(\Omega-K_\nu)\subseteq U_\nu\cap\Omega$ for each $\nu=1,\dots,N$.
- For $\nu=1,\dots,N$, $(\phi^\nu_j)_{j=0}^\infty$ satisfies (P.a) - (P.c) in Definition 4, with support condition $\operatorname{supp}\phi^\nu_j\subset K_\nu$ for $j\ge0$.
- $\chi_\nu\in C_c^\infty(U_\nu)$ for $\nu=1,\dots,N$, and satisfy[6] $\sum_{\nu=1}^N\chi_\nu|_{\overline{\Omega}}\equiv1$.

To prove (38), (39) and (40) the only thing we need are the following standard results ($p<\infty$ for $\mathscr{F}$-cases):

(Ψ.a) Let $\chi\in C_c^\infty(\mathbb{R}^n)$. Then $[\tilde f\mapsto\chi\tilde f]:\mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n)\to\mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n)$ is bounded.

6 In fact we can relax the condition to $\sum_{\nu=1}^N\chi_\nu|_{\overline{\Omega}}>c$ for some $c>0$.

(Ψ.b) Let Φ be an invertible affine linear transform. Then $[\tilde f\mapsto \tilde f\circ\Phi]:\mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n)\to\mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n)$ is bounded.

(Ψ.c) For every $m\geq 1$, we have equivalent norms $\|f\|_{\mathscr{A}^{s,\tau}_{p,q}(\mathbb{R}^n)}\approx_{p,q,s,\tau,m}\sum_{|\alpha|\leq m}\|\partial^\alpha f\|_{\mathscr{A}^{s-m,\tau}_{p,q}(\mathbb{R}^n)}$.

One can see [24, Sections 6.1.1 and 6.2], [22, Theorem 1.6] and [11, Theorem 3.3] for their proof. See also [6, Sections 3.4, 4.2 and 4.3]. We remark that because of (8) it is enough to consider the case $0\leq\tau\leq\frac{1}{p}$. We leave the details to the readers.

One can also write down the analogy of Proposition 21 and Corollary 23 similar to (38), (39) and (40), we leave the details to the readers as well.

Finally, we prove Theorem 2 using the following fact:

Proposition 25 ([12, Theorem 1.5 (ii)]) *Let $(\phi_j)_{j=1}^\infty$ be a family*[7] *of Schwartz functions satisfying (P.a), (P.b) and (P.d). Recall that for every $j\geq 1$, $\phi_j(x)=2^{(j-1)n}\phi_1(2^{j-1}x)$, $\int x^\alpha\phi_j(x)dx=0$ for all α, and $\operatorname{supp}\phi_j\subset\{x_n<-A|x'|\}$ for some $A>0$.*

Then for any $m\geq 1$, there are families of Schwartz functions $\tilde\phi^\beta=(\tilde\phi^\beta_j)_{j=1}^\infty$ for $|\beta|=m$ that also satisfy (P.a), (P.b) and (P.d), such that

$$\phi_j=2^{-jm}\sum_{|\beta|=m}\partial^\beta\tilde\phi^\beta_j,\quad \textit{for every } j\geq 1.$$

Proof of Theorem 2 Once the case of special Lipschitz domains is done, the proof of the case of bounded Lipschitz domains follows from the standard partition of unity argument (one can read [12, Section 6] for details) along with the facts (Ψ.a), (Ψ.b) and (Ψ.c) mentioned in Remark 24.

Let $\Omega\subset\mathbb{R}^n$ be a special Lipschitz domain. Let $f\in\mathscr{A}^{s\tau}_{pq}(\Omega)$ and let $\tilde f\in\mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n)$ be an extension of f. By (Ψ.c) we have $\|\partial^\alpha\tilde f\|_{\mathscr{A}^{s-|\alpha|,\tau}_{p,q}(\mathbb{R}^n)}\lesssim_{p,q,s,\tau,\alpha}\|\tilde f\|_{\mathscr{A}^{s\tau}_{pq}(\mathbb{R}^n)}$. Since $\partial^\alpha\tilde f$ is also an extension of $\partial^\alpha f$, by (7) in Definition 5, taking the infimum over all extensions $\tilde f$ of f we get $\sum_{|\alpha|\leq m}\|\partial^\alpha f\|_{\mathscr{A}^{s-m,\tau}_{p,q}(\Omega)}\lesssim\|f\|_{\mathscr{A}^{s\tau}_{pq}(\Omega)}$.

To prove the converse inequality $\|f\|_{\mathscr{A}^{s\tau}_{pq}(\Omega)}\lesssim\sum_{|\alpha|\leq m}\|\partial^\alpha f\|_{\mathscr{A}^{s-m,\tau}_{p,q}(\Omega)}$, let $(\phi_j,\psi_j)_{j=0}^\infty$ be as in (31).

We let $(\tilde\phi^\beta_j)_{j=1}^\infty\subset\mathscr{S}(\mathbb{R}^n)$ ($|\beta|>0$) be given in Proposition 25. Thus $\phi_j=2^{-jq}\sum_{\beta:|\beta|=q}\partial^\beta\tilde\phi^\beta_j$ for all $j,q\geq 1$.

For $\alpha\neq 0$, we define $\psi^\alpha=(\psi^\alpha_j)_{j=1}^\infty$ by $\psi^\alpha_j(x):=2^{-j|\alpha|}\partial^\alpha\psi_j(x)$ (for $j\geq 1$). Thus the sequences ψ^α (for $\alpha\neq 0$) all satisfy (P.a), (P.b) and (P.d).

We define a family of linear operators,

$$\begin{aligned} E^{\alpha,0}f&=E^{\alpha,0}_\Omega f:=\partial^\alpha\psi_0*(\mathbf{1}_\Omega\cdot(\phi_0*f)),\\ E^{\alpha,\beta}f&=E^{\alpha,\beta}_\Omega f:=\sum_{j=1}^\infty\psi^\alpha_j*(\mathbf{1}_\Omega\cdot(\tilde\phi^\beta_j*f)),\ \text{for } |\alpha|=|\beta|>0. \end{aligned}\tag{41}$$

[7] Here the index of the Schwartz family start from $j=1$. In Definition 5 we start with $j=0$.

For every $f \in \mathscr{S}'(\Omega)$ and for every multi-index $\alpha \neq 0$, we see that

$$\begin{aligned} \partial^\alpha Ef &= \sum_{j=0}^{\infty} \partial^\alpha \psi_j * (\mathbf{1}_\Omega \cdot (\phi_j * f)) = \partial^\alpha \psi_0 * (\mathbf{1}_\Omega \cdot (\phi_0 * f)) \\ &\quad + \sum_{j=1}^{\infty} \sum_{\beta:|\beta|=|\alpha|} 2^{j|\alpha|} \psi_j^\alpha * (\mathbf{1}_\Omega \cdot 2^{-j|\alpha|}(\partial^\beta \tilde{\phi}_j^\beta * f)) \\ &= \partial^\alpha \psi_0 * (\mathbf{1}_\Omega \cdot (\phi_0 * f)) + \sum_{\beta:|\beta|=|\alpha|} \sum_{j=1}^{\infty} \psi_j^\alpha * (\mathbf{1}_\Omega \cdot (\tilde{\phi}_j^\beta * \partial^\beta f)) \\ &= E^{\alpha,0} f + \sum_{\beta:|\beta|=|\alpha|} E^{\alpha,\beta}[\partial^\beta f]. \end{aligned} \tag{42}$$

By Proposition 17, $E^{\alpha,0}, E^{\alpha,\beta} : \mathscr{A}_{p,q}^{s-m,\tau}(\Omega) \to \mathscr{A}_{p,q}^{s-m,\tau}(\mathbb{R}^n)$ are all bounded. Therefore

$$\begin{aligned} \|f\|_{\mathscr{A}_{pq}^{s\tau}(\Omega)} &\approx \|Ef\|_{\mathscr{A}_{pq}^{s\tau}(\mathbb{R}^n)} \overset{(\Psi.c)}{\approx} \sum_{|\alpha|\le m} \|\partial^\alpha Ef\|_{\mathscr{A}_{pq}^{s-m,\tau}(\mathbb{R}^n)} \\ &\overset{(42)}{\lesssim} \|Ef\|_{\mathscr{A}_{pq}^{s-m,\tau}} + \sum_{0<|\alpha|\le m} \left(\|E^{\alpha,0} f\|_{\mathscr{A}_{pq}^{s-m,\tau}} + \sum_{\beta:|\beta|=|\alpha|} \|E^{\alpha,\beta}[\partial^\beta f]\|_{\mathscr{A}_{pq}^{s-m,\tau}} \right) \\ &\lesssim \|f\|_{\mathscr{A}_{pq}^{s-m,\tau}(\Omega)} + \sum_{0<|\alpha|\le m} \left(\|f\|_{\mathscr{A}_{pq}^{s-m,\tau}(\Omega)} + \sum_{\beta:|\beta|=|\alpha|} \|\partial^\beta f\|_{\mathscr{A}_{pq}^{s-m,\tau}(\Omega)} \right) \\ &\lesssim \sum_{|\beta|\le m} \|\partial^\beta f\|_{\mathscr{A}_{pq}^{s-m,\tau}(\Omega)}. \end{aligned}$$

This completes the proof of (4) for the case of special Lipschitz domains.

The $\mathscr{F}_{\infty q}^s$-cases follow immediately from (9) since we have $\mathscr{F}_{\infty q}^s(\mathbb{R}^n) = \mathscr{F}_{qq}^{s,\frac{1}{q}}(\mathbb{R}^n)$.

4 Further Open Questions

By the same method, using Lemma 10 - Proposition 14, it is possible for us to get the analogs of Theorems 1 and 2 on the so-called *local spaces*.

The local version of $\mathscr{A}_{pq}^{s\tau}(\mathbb{R}^n)$ for $\mathscr{A} \in \{\mathscr{B}, \mathscr{F}, \mathscr{N}\}$, denoted by $\mathscr{A}_{p,q,\mathrm{unif}}^{s,\tau}(\mathbb{R}^n)$, is defined by replacing the supremum among the set of dyadic cubes $\mathscr{Q}$ with $\{Q_{J,v} \in \mathscr{Q} : J \geq 0\}$. See [10, Section 3.4] for example. For an open subset $\Omega \subseteq \mathbb{R}^n$ we use $\mathscr{A}_{p,q,\mathrm{unif}}^{s,\tau}(\Omega) := \{\tilde{f}|_\Omega : \tilde{f} \in \mathscr{A}_{p,q,\mathrm{unif}}^{s,\tau}(\mathbb{R}^n)\}$ similarly. For more details we refer [17] to readers.

One can also consider the analog of Theorems 1 and 2 on $\mathscr{A}_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}$, the spaces with variable exponents. For example [13], which may require certain assumptions on the exponents.

In Definition 6, it is known that the norms are equivalent if $(\lambda_j)_{j=0}^{\infty}$ only satisfies the scaling condition (P.b) and the *Tauberian condition*:

$$\text{There exist } \varepsilon_0, c > 0 \quad \text{such that} \quad |\hat{\lambda}_0(\xi)| > c \text{ for } |\xi| < \varepsilon_0,$$
$$\text{and } |\hat{\lambda}_1(\xi)| > c \text{ for } \varepsilon_0/2 < |\xi| < 2\varepsilon_0. \tag{43}$$

See [22, Theorems 2.5 and 2.6] and [23, Theorem 1] for example.

It is not known to the author whether we can replace the assumption (P.c) for $(\phi_j)_{j=0}^{\infty}$ in Theorem 1 with the Tauberian condition (43).

For Theorem 2, we do not know whether (4) has the following improvement:

Question 26 *Keeping the assumptions of Theorem 2, can we find a* $C = C(\Omega, p, q, s, \tau, m) > 0$ *such that the following holds?*

$$\|f\|_{\mathscr{A}^{s\tau}_{pq}(\Omega)} \le C\left(\|f\|_{\mathscr{A}^{s-m,\tau}_{p,q}(\Omega)} + \sum_{k=0}^{n}\left\|\frac{\partial^m f}{\partial x_k^m}\right\|_{\mathscr{A}^{s-m,\tau}_{p,q}(\Omega)}\right), \quad \forall f \in \mathscr{A}^{s\tau}_{pq}(\Omega).$$

Cf. [22, Theorem 1.6]. The question is open even for the classical Besov and Triebel-Lizorkin spaces $\mathscr{A}^s_{pq}(\Omega)$ when Ω is a (special or bounded) Lipschitz domain.

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