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Sparse Grid Approximation in Weighted Wiener Spaces

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Abstract
We study approximation properties of multivariate periodic functions from weighted Wiener spaces by sparse grid methods constructed with the help of quasi-interpolation operators. The class of such operators includes classical interpolation and sampling operators, Kantorovich-type operators, scaling expansions associated with wavelet constructions, and others. We obtain the rate of convergence of the corresponding sparse grid methods in weighted Wiener norms as well as analogues of the Littlewood–Paley-type characterizations in terms of families of quasi-interpolation operators.

Keywords Sparse grid · Weighted Wiener spaces · Quasi-interpolation operators · Kantorovich operators · Smolyak algorithm · Littlewood–Paley-type characterizations

Mathematics Subject Classification 41A25 · 41A63 · 42A10 · 42A15 · 41A58 · 41A17 · 42B25 · 42B35

Dedicated to the 80th anniversary of Professor Stefan Samko.

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1 Introduction

In many applied problems one needs to approximate high-dimensional functions in smooth function spaces. As it is known from previous research, traditional numerical methods such as the interpolation with tensor product grids suffer from the so-called "curse of dimensionality". In other words, the computation time of such methods grows dramatically with the number of variables and the problem becomes intractable already for mild dimensions. One of the means to overcome these obstacles is to employ different sparse grids approximation methods and to impose additional assumption on smoothness. Typically, one assumes that a function belongs to a certain mixed smoothness Sobolev or Besov space (see, e.g., [6, 17]).

In this paper, we consider approximation methods that are based on generalized sparse grids (see, e.g., [6, 26]). Recall that for given parameters $T \in [-\infty, 1)$, $n \in \mathbb{N}$, and a family of univariate operators $Y = (Y_j)_{j \in \mathbb{Z}_+}$, a sparse grid approximation method is defined as follows:

$$P_{n,T}^{Y} = \sum_{\boldsymbol{j} \in \Delta(n,T)} \eta_{\boldsymbol{j}}^{Y}, \quad \eta_{\boldsymbol{j}^Y} = \prod_{i=1}^{d} \left(Y_{j_i}^{i} - Y_{j_i - 1}^{i} \right), \tag{1.1}$$

where

$$\Delta(n, T) = \left\{ \boldsymbol{k} \in \mathbb{Z}_+^d \ : \ |\boldsymbol{k}|_1 - T |\boldsymbol{k}|_\infty \le (1 - T)n \right\}$$

and Y_j^i denotes the univariate operator Y_j acting on functions in the variable x_i and $Y_{-1} = 0$. The most well studied case of the family Y is the classical Lagrange interpolation operators $I = (I_j)_{j \in \mathbb{Z}_+}$ given by

$$I_j(f)(x) = 2^{-j} \sum_{k=-2^{j-1}}^{2^{j-1}-1} f\left(x_k^j\right) \mathcal{D}_j\left(x - x_k^j\right),$$

where $x_k^j = \frac{\pi k}{2^{j-1}}$ and $\mathcal{D}_j(x) = \sum_{\ell=-2^{j-1}}^{2^{j-1}-1} e^{\mathrm{i}\ell x}$ is the Dirichlet kernel. The corresponding sparse grid for a given level n is then

$$\Gamma(n, T) = \bigcup_{\boldsymbol{j} \in \Delta(n,T)} \mathcal{I}_{j_1} \times \cdots \times \mathcal{I}_{j_d},$$

where $\mathcal{I}_j = \{x_k^j \ : \ k = -2^j, \ldots, 2^j - 1\}$, i.e., $P_{n,T}^{I} f(y) = f(y)$ for all $y \in \Gamma(n, T)$ and $f \in C(\mathbb{T}^d)$. Here, the case $T = -\infty$ corresponds to the interpolation on the full tensor grid; the case $T = 0$ represents interpolation on the Smolyak grid, which is also called the regular sparse grid (see [46], see also [17, Ch. 5]); and the case $0 < T < 1$ resembles the so-called energy-norm based sparse grids (see [6, 24]).

One of the important characteristics of a sparse grid is its cardinality. Note that (see, e.g., [26])

$$\operatorname{card}\Gamma(n,T) \lesssim \sum_{\boldsymbol{k}\in\Delta(T,n)} 2^{|\boldsymbol{k}|_1} \lesssim \begin{cases} 2^n, & \text{if } 0<T<1,\\ 2^n n^{d-1}, & \text{if } T=0,\\ 2^{\left(\frac{1-T}{1-T/d}\right)n}, & \text{if } T<0,\\ 2^{dn}, & \text{if } T=-\infty \end{cases} \tag{1.2}$$

and the same upper bound holds for the number of frequencies of the polynomial $P^I_{n,T}f$. Thus, the most interesting cases for approximation with the algorithm $P^I_{n,T}$ is when $0\le T<1$ since in this case, for the number of elements needed to construct the corresponding algorithm, there is no exponential growth with the dimension d. Let us consider this case in more detail.

Approximation properties of the operators $P^I_{n,T}$ have been mainly studied in the case $T=0$, which corresponds to the classical Smolyak grids (see, e.g., [1, 2, 8, 12, 27, 45, 47–49]; see also the book [17, Ch. 4 and Ch. 5]). As an example, we mention the following L_q-error estimates for the Smolyak algorithm $P^I_{n,0}$ in the case of approximation of functions from the Sobolev space $\mathbf{W}^\alpha_p(\mathbb{T}^d)$ of dominating mixed smoothness $\alpha>0$ (see, e.g., [45] and [17, Chapters 4 and 5]): if $1<p,q<\infty$ and $\alpha>\max\{1/p,1/2\}$, then

$$\sup_{f\in U\mathbf{W}^\alpha_p} \|f-P^I_{n,0}(f)\|_{L_q(\mathbb{T}^d)} \asymp \begin{cases} 2^{-\alpha n} n^{\frac{d-1}{2}}, & \text{if } p\ge q,\\ 2^{-(\alpha-1/p+1/q)n}, & \text{if } q>p, \end{cases} \tag{1.3}$$

where $U\mathbf{W}^\alpha_p$ denotes the unit ball in the space $\mathbf{W}^\alpha_p(\mathbb{T}^d)$. Similar estimates (for $T=0$) in the weighted Wiener spaces (or Korobov spaces) were obtained in [11, 27, 47].

In the case $0<T<1$, the approximations by operators $P^I_{n,T}$ have been mostly investigated for functions from the so-called generalized mixed smoothness (or hybrid smoothness) Sobolev space

$$H^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f\in L_2(\mathbb{T}^d) \,:\, \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \prod_{j=1}^d (1+|k_j|)^{2\alpha}(1+|\boldsymbol{k}|)^{2\beta}|\widehat{f}(\boldsymbol{k})|^2<\infty \right\},$$

where the parameter β governs for the isotropic smoothness, whereas α reflects the smoothness in the dominating mixed sense. Herewith, the approximation error is estimated in the metric of the classical isotropic Sobolev space $H^\gamma(\mathbb{T}^d)=H^{0,\gamma}(\mathbb{T}^d)$, see, e.g., [6, 7, 24, 25]. In particular, we mention a general result obtained in the recent paper [25]: let $\alpha\ge 0$, $\beta\ge 0$, $\gamma-\beta<\alpha$, $\alpha+\frac{\beta}{d}>\frac12$. Then, for all $f\in H^{\alpha,\beta}(\mathbb{T}^d)$ and $n\in\mathbb{N}$, one has

$$\|f-P^I_{n,T}f\|_{H^\gamma(\mathbb{T}^d)} \lesssim \Omega_I(n)\|f\|_{H^{\alpha,\beta}(\mathbb{T}^d)}, \tag{1.4}$$

where[1]

$$\Omega_I(n) = \begin{cases} 2^{-\left(\alpha-(\gamma-\beta)-(\alpha T-(\gamma-\beta))\frac{d-1}{d-T}\right)n} n^{\frac{d-1}{2}}, & T \ge \frac{\gamma-\beta}{\alpha}, \\ 2^{-(\alpha-(\gamma-\beta))n}, & T < \frac{\gamma-\beta}{\alpha}, \end{cases}$$

which again shows the importance of the case $0 \le T \le \frac{\gamma-\beta}{\alpha} < 1$, cf. (1.2).

Along with the classical interpolation operators $I = (I_j)_{j\in\mathbb{Z}_+}$ one also considers the family of the partial sums of Fourier series (see, e.g., [1, 2, 13]), families of convolution type operators (see, e.g., [45, 54]), quasi-interpolation operators based on scaled B-splines with integer knots (see, e.g., [14–16]).

In this paper, as a family Y in (1.1), we make use of the general quasi-interpolation operators defined by

$$Q_j(f, \varphi_j, \widetilde{\varphi}_j)(x) = 2^{-j} \sum_{k=-2^{j-1}}^{2^{j-1}-1} (f * \widetilde{\varphi}_j)(x_k^j)\varphi_j(x - x_k^j), \quad j \in \mathbb{Z}_+, \tag{1.5}$$

where $(\varphi_j)_{j\in\mathbb{Z}_+}$ is a family of univariate trigonometric polynomials and $(\widetilde{\varphi}_j)_{j\in\mathbb{Z}_+}$ is a family of functions/distributions on $\mathbb{T}$. Note that in the non-periodic case approximation properties of operators of such type in various function spaces (classical and weighted L_p, Sobolev, Besov, Wiener) have been studied, for example, in the works [21, 29, 30, 38, 39]. It worths noting that the operators (1.5) can be successfully employed in such applied problems, where the data contains some noise and the functional information is provided by other means than point evaluation (averages, divided differences, etc.), see, e.g., [10, 56].

In the periodic case, an analog of estimate (1.3) for any $\alpha > 0$ has been recently established in [32] for the approximation processes $P_{n,0}^Q$ with $Q = (Q_j)_{j\in\mathbb{Z}_+}$ defined by (1.5). The corresponding proof is essentially based on the results from [35], where under different compatibility conditions on $(\varphi_j)_{j\in\mathbb{Z}_+}$ and $(\widetilde{\varphi}_j)_{j\in\mathbb{Z}_+}$, L_p-error of approximation by the operators Q_j were obtained. Similar results in weighted Wiener spaces and $L_2(\mathbb{T})$ have been derived in [28, 34], correspondingly.

The goal of the present work is to establish analogues of error estimate (1.4) for sparse grid approximation methods constructed using general quasi-interpolation operators. Comparing our findings with the previously known results, we stress two important differences. Firstly, we build the approximation schemes using quasi-interpolation operators (1.5) rather than the classical interpolation operators $(I_j)_{j\in\mathbb{Z}_+}$ constructed using the Dirichlet kernel and the values of a function at sets of equidistant interpolation nodes as in [7, 24, 25]. Secondly, we work with a more general scale of spaces, namely with the weighted Wiener spaces $A_p^{\alpha,\beta}(\mathbb{T}^d)$ rather than with the Sobolev spaces $H^{\alpha,\beta}(\mathbb{T}^d)$, which correspond to the case $p = 2$. We would like to stress that possibility to vary the family $(\widetilde{\varphi}_j)_{j\in\mathbb{Z}_+}$ allows us to prove the results under essentiality less restrictive conditions on the parameters α and β.

The paper is organized as follows. In Sect. 2 we introduce basic notations, define isotropic, mixed, and hybrid weighted Wiener spaces and general quasi-interpolation

[1] Note that there are typos in formula (21) and related estimates in [25]. See also [24, Lemma 8].

operators. Section 3 is devoted to auxiliary results. In particular, we prove the following useful estimate

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_{A_q^\gamma(\mathbb{T})} \lesssim 2^{-j\min(\alpha-\gamma,s)} \|f\|_{A_q^\alpha(\mathbb{T})},$$

see Lemma 3.1. In Sect. 4 we establish our main tools, the so-called "discrete" Littlewood-Paley type characterizations. In Sect. 5 we prove our main results: we consider approximation in the isotropic Wiener space $A_q^\gamma(\mathbb{T}^d)$ (in Subsect. 5.1) and in the mixed Wiener spaces $A_{q,\mathrm{mix}}^\gamma(\mathbb{T}^d)$ (in Subsect. 5.2). In Subsect. 5.3 we discuss the sharpness of the obtained results. In Sect. 6 we consider the specific sets of parameters, where our main results (Theorems 5.1 and 5.3) provide the most effective error estimates with respect to the approximation rate and the number of degrees of freedom.

2 Weighted Wiener Spaces and Quasi-interpolation Operators

2.1 Basic Notation

In what follows, $\mathbb{Z}_+^d = \{\boldsymbol{x} \in \mathbb{Z}^d : x_i \geq 0, i = 1, \dots, d\}$ and $\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$ is the d-dimensional torus. Further, for vectors $\boldsymbol{x} = (x_1, \dots, x_d)$ and $\boldsymbol{k} = (k_1, \dots, k_d)$ in $\mathbb{R}^d$, we denote $(\boldsymbol{x}, \boldsymbol{k}) = x_1 k_1 + \dots + x_d k_d$.

If $\boldsymbol{j} \in \mathbb{Z}_+^d$, we set $|\boldsymbol{j}|_1 = \sum_{k=1}^d j_k$, $|\boldsymbol{j}|_\infty = \max_{k=1}^d j_k$, and $2^{\boldsymbol{j}} = (2^{j_1}, \dots, 2^{j_d})$. For $1 \leq p \leq \infty$, p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$. For $1 \leq p, q \leq \infty$, we set $\sigma_{p,q} = \left(\frac{1}{q} - \frac{1}{p}\right)_+$.

If $f \in L_1(\mathbb{T}^d)$, then

$$\widehat{f}(\boldsymbol{k}) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\boldsymbol{x}) e^{-\mathrm{i}(\boldsymbol{k},\boldsymbol{x})} \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{k} \in \mathbb{Z}^d,$$

denotes the k-th Fourier coefficient of f. As usual, the convolution of integrable functions f and g is given by

$$(f * g)(\boldsymbol{x}) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\boldsymbol{x} - \boldsymbol{t}) g(\boldsymbol{t}) \mathrm{d}\boldsymbol{t}.$$

By $\mathcal{T}_{\boldsymbol{j}}^d$, $\boldsymbol{j} \in \mathbb{Z}_+^d$, we denote the following set of trigonometric polynomials:

$$\mathcal{T}_{\boldsymbol{j}}^d = \mathrm{span}\left\{ e^{\mathrm{i}(\boldsymbol{k},\boldsymbol{x})} : \boldsymbol{k} \in D_{j_1} \times \dots \times D_{j_d} \right\},$$

where

$$D_j = [-2^{j-1}, 2^{j-1}) \cap \mathbb{Z}.$$

Let $\mathcal{D} = C^\infty(\mathbb{T})$ be the space of infinitely differentiable functions on $\mathbb{T}$. The linear space of periodic distributions (continuous linear functionals on $\mathcal{D}$) is denoted by $\mathcal{D}'$.

It is known (see, e.g., [44, p. 144]) that any periodic distribution $\widetilde{\varphi}$ can be expanded in a weakly convergent (in $\mathcal{D}'$) Fourier series

$$\widetilde{\varphi}(x) = \sum_{k\in\mathbb{Z}} \widehat{\widetilde{\varphi}}(k)e^{\mathrm{i}kx},$$

where the sequence $(\widehat{\widetilde{\varphi}}(k))_k$ has at most polynomial growth.

Throughout the paper, we use the notation $A \lesssim B$, with $A, B \geq 0$, for the estimate $A \leq C\,B$, where C is a positive constant independent of the essential variables in A and B (usually, f, j, and n). If $A \lesssim B$ and $B \lesssim A$ simultaneously, we write $A \asymp B$ and say that A is equivalent to B. For two function spaces X and Y, we will use the notation $Y \hookrightarrow X$ if $Y \subset X$ and $\|f\|_X \lesssim \|f\|_Y$ for all $f \in Y$. The unit ball in some normed vector space X is denoted by UX.

2.2 Weighted Wiener Spaces

We will employ the following function spaces with the parameters $\alpha, \beta \in \mathbb{R}$, and $0 < q \leq \infty$.

• *The periodic (isotropic) Wiener space* $A_q^\alpha(\mathbb{T}^d)$ is the collection of all $f \in L_1(\mathbb{T}^d)$ such that

$$\|f\|'_{A_q^\alpha(\mathbb{T}^d)} := \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} (1+|\boldsymbol{k}|)^{q\alpha}|\widehat{f}(\boldsymbol{k})|^q\right)^{1/q} < \infty, \quad q < \infty,$$

$$\|f\|'_{A_\infty^\alpha(\mathbb{T}^d)} := \sup_{\boldsymbol{k}\in\mathbb{Z}^d} (1+|\boldsymbol{k}|)^{\alpha}|\widehat{f}(\boldsymbol{k})| < \infty, \quad q = \infty.$$

In the case $\alpha = 0$, we use the following standard notation $A_q(\mathbb{T}^d) = A_q^0(\mathbb{T}^d)$.

• *The periodic (mixed) Wiener space* $A_{q,\mathrm{mix}}^\alpha(\mathbb{T}^d)$ is the collection of all $f \in L_1(\mathbb{T}^d)$ such that

$$\|f\|'_{A_{q,\mathrm{mix}}^\alpha(\mathbb{T}^d)} := \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \prod_{j=1}^d (1+|k_j|)^{q\alpha}|\widehat{f}(\boldsymbol{k})|^q\right)^{1/q} < \infty, \quad q < \infty,$$

$$\|f\|'_{A_{\infty,\mathrm{mix}}^\alpha(\mathbb{T}^d)} := \sup_{\boldsymbol{k}\in\mathbb{Z}^d} \prod_{j=1}^d (1+|k_j|)^{\alpha}|\widehat{f}(\boldsymbol{k})| < \infty, \quad q = \infty.$$

• *The periodic (hybrid) Wiener space* $A_q^{\alpha,\beta}(\mathbb{T}^d)$ is the collection of all $f \in L_1(\mathbb{T}^d)$ such that

$$\|f\|'_{A_q^{\alpha,\beta}(\mathbb{T}^d)} := \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \prod_{j=1}^d (1+|k_j|)^{q\alpha}(1+|\boldsymbol{k}|)^{q\beta}|\widehat{f}(\boldsymbol{k})|^q\right)^{1/q} < \infty, \quad q < \infty,$$

$$\|f\|'_{A^{\alpha,\beta}_\infty(\mathbb{T}^d)} := \sup_{\boldsymbol{k}\in\mathbb{Z}^d} \prod_{j=1}^d (1+|k_j|)^\alpha (1+|\boldsymbol{k}|)^\beta |\widehat{f}(\boldsymbol{k})| < \infty, \quad q=\infty.$$

Remark 2.1 (i) It is easy to see that for any $\alpha > 0$ and $1 \le q \le \infty$, the following embeddings hold:

$$A^{d\alpha}_q(\mathbb{T}^d) \hookrightarrow A^\alpha_{q,\text{mix}}(\mathbb{T}^d) \hookrightarrow A^\alpha_q(\mathbb{T}^d).$$

(ii) Note that the isotropic Wiener space $A^\alpha_q(\mathbb{T}^d)$ coincides with the corresponding periodic Sobolev space $H^\alpha(\mathbb{T}^d)$ in the case $q=2$. The same holds for the mixed or hybrid Wiener spaces.

(iii) For more information about the weighted Wiener spaces $A^{\alpha,\beta}_q(\mathbb{T}^d)$, see papers [5, 19, 20, 40, 41, 43].

(iv) Note that according to the terminology suggested by H. Feichtinger, the spaces $A^{\alpha,\beta}_q(\mathbb{T}^d)$ can be also called Fourier-Wermer spaces, see [55] for the motivation.

As usual, for $f \in L_1(\mathbb{T}^d)$, we define the diadic blocks $\delta_{\boldsymbol{k}}(f)$, $\boldsymbol{k}\in\mathbb{Z}^d_+$, by

$$\delta_{\boldsymbol{k}}(f)(x) = \sum_{\boldsymbol{k}\in\mathcal{P}_{\boldsymbol{k}}} \widehat{f}(\boldsymbol{k}) e^{\mathrm{i}(\boldsymbol{k},\boldsymbol{x})},$$

where

$$\mathcal{P}_{\boldsymbol{k}} := P_{k_1} \times \cdots \times P_{k_d},$$

$P_j = \{\ell \in \mathbb{Z} : 2^{j-1} \le |\ell| < 2^j\}$ for $j>0$, and $P_0 = \{0\}$.

Recall that for all $f \in L_p(\mathbb{T}^d)$, $1<p<\infty$, the Littlewood-Paley decomposition reads as follows

$$f = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d_+} \delta_{\boldsymbol{\ell}}(f).$$

The next lemma is a simple consequence of the definition of the space $A^{\alpha,\beta}_q(\mathbb{T}^d)$.

Lemma 2.2 *Let $0<q\le\infty$, $\alpha\ge 0$, and let $\beta\in\mathbb{R}$ be such that $\alpha+\beta\ge 0$. Then*

$$A^{\alpha,\beta}_q(\mathbb{T}^d) = \left\{ f \in L_1(\mathbb{T}^d) : \|f\|_{A^{\alpha,\beta}_q(\mathbb{T}^d)} := \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d_+} 2^{q(\alpha|\boldsymbol{k}|_1+\beta|\boldsymbol{k}|_\infty)} \|\delta_{\boldsymbol{k}}(f)\|^q_{A_q(\mathbb{T}^d)} \right)^{1/q} < \infty \right\}$$

with the usual modification in the case $q=\infty$ in the sense of equivalent norms.

2.3 Quasi-interpolation Operators

Consider a family of general univariate quasi-interpolation operators $Q = (Q_j)_{j\in\mathbb{Z}_+}$ given by

$$Q_j(f, \varphi_j, \widetilde{\varphi}_j)(x) = 2^{-j} \sum_{k\in D_j} (f * \widetilde{\varphi}_j)(x_k^j)\varphi_j(x - x_k^j), \quad x_k^j = \frac{\pi k}{2^{j-1}},$$

where $(\varphi_j)_{j\in\mathbb{Z}_+}$ is a family of univariate trigonometric polynomials in $\mathcal{T}_j^1$, $(\widetilde{\varphi}_j)_{j\in\mathbb{Z}_+}$ is a family of functions/distributions, and the convolution $f * \widetilde{\varphi}_j$ is defined in some suitable way for any $j \in \mathbb{Z}_+$.

Below, we assume that the following conditions on $(\varphi_j)_{j\in\mathbb{Z}_+}$ and $(\widetilde{\varphi}_j)_{j\in\mathbb{Z}_+}$ hold:

- *The growth condition of order* $N \geq 0$ for the Fourier coefficients of $\widetilde{\varphi}_j$:

$$|\widehat{\widetilde{\varphi}_j}(\ell)| \leq C_{\widetilde{\varphi}}(1 + |2^{-j}\ell|^N) \quad \text{for all} \quad \ell \in \mathbb{Z}, \quad j \in \mathbb{Z}_+. \tag{2.1}$$

- *The uniform boundedness condition* for the Fourier coefficients of φ_j:

$$|\widehat{\varphi_j}(\ell)| \leq C_\varphi \quad \text{for all} \quad \ell \in \mathbb{Z}, \quad j \in \mathbb{Z}_+. \tag{2.2}$$

- *The compatibility condition of order* $s > 0$ for φ_j and $\widetilde{\varphi}_j$:

$$|1 - \widehat{\varphi_j}(\ell)\widehat{\widetilde{\varphi}_j}(\ell)| \leq C_{\varphi,\widetilde{\varphi},s}|2^{-j}\ell|^s \quad \text{for all} \quad \ell \in D_j, \quad j \in \mathbb{Z}_+. \tag{2.3}$$

Let us consider two important classes of quasi-interpolation operators and examine the above conditions.

Example 2.3 *Quasi-interpolation sampling operators* are defined by

$$S_j(f, \varphi_j)(x) = 2^{-j} \sum_{k\in D_j} \left(\sum_{|\nu|\leq m} a_{\nu,j} f^{(r_\nu)}(x_{k-\nu}^j) \right) \varphi_j(x - x_k^j), \tag{2.4}$$

where $a_{\nu,j} \in \mathbb{C}$, $r_\nu \in \mathbb{Z}_+$, and $\varphi_j \in \mathcal{T}_j^1$. Note that $S_j(f, \varphi_j) = Q_j(f, \varphi_j, \widetilde{\varphi}_j)$ with

$$\widetilde{\varphi}_j(x) = \sum_{|\nu|\leq m} a_{\nu,j}\delta^{(r_\nu)}(x - x_\nu^j) \sim \sum_{\ell\in\mathbb{Z}} \left(\sum_{|\nu|\leq m} a_{\nu,j}(\mathrm{i}\ell)^{r_\nu} e^{-\mathrm{i}\ell x_\nu^j} \right) e^{\mathrm{i}\ell x}.$$

One can see that condition (2.1) with $N = \max_{|\nu|\leq m} r_\nu$ is satisfied if

$$\sup_{j\in\mathbb{Z}_+} \sum_{|\nu|\leq m} 2^{r_\nu j}|a_{\nu,j}| < \infty$$

and condition (2.3) with $s > 0$ is fulfilled if

$$\left|1 - \widehat{\varphi}_j(\ell) \sum_{|\nu|\le m} \overline{a_{\nu,j}}(-\mathrm{i}\ell)^{r_\nu} e^{\mathrm{i}\ell x_\nu^j}\right| \le c|2^{-j}\ell|^s \quad \text{for all} \quad \ell \in D_j, \quad j \in \mathbb{Z}_+.$$

(i) A particular example of (2.4) is the classical Lagrange interpolation operator

$$I_j(f)(x) = 2^{-j} \sum_{k\in D_j} f(x_k^j)\mathcal{D}_j(x - x_k^j), \tag{2.5}$$

where

$$\mathcal{D}_j(x) = \sum_{\ell\in D_j} e^{\mathrm{i}\ell x}$$

is the Dirichlet kernel. Note that $I_j(f) = S_j(f, \varphi_j)$ with $\varphi_j = \mathcal{D}_j$, $m = 0$, $a_{0,j} = 1$, and $r_0 = 0$. In this case it is easy to see that conditions (2.1)–(2.3) are fulfilled for $N = 0$ and any $s > 0$.

(ii) As an example of quasi-interpolation operators that are generated by an average sampling instead of the exact samples of f, we consider

$$A_j(f)(x) = 2^{-j} \sum_{k\in D_j} \lambda_j(f)(x_k^j)\mathcal{D}_j(x - x_k^j), \tag{2.6}$$

where

$$\lambda_j(f)(x) = \frac{1}{4}\left(f\left(x - \frac{\pi}{2^j}\right) + 2f(x) + f\left(x + \frac{\pi}{2^j}\right)\right).$$

We have that $A_j(f) = Q_j(f, \varphi_j, \widetilde{\varphi}_j)$ with $\varphi_j(x) = \mathcal{D}_j(x)$ and $\widetilde{\varphi}_j(x) \sim \sum_{\ell\in\mathbb{Z}} \cos^2\left(\frac{\pi\ell}{2^{j+1}}\right) e^{\mathrm{i}\ell x}$ and conditions (2.1)–(2.3) are fulfilled with $N = 0$ and $s = 2$. Note that the operators of such type are used in applications, for example, in order to reduce noise contribution (see, e.g., [56]).

(iii) At the same time if in (2.6), we replace the Dirichlet kernel $\mathcal{D}_j$ by

$$\mathcal{D}_j^*(x) = \sum_{\ell\in D_j} \frac{1}{\cos^2\left(\frac{\pi\ell}{2^{j+1}}\right)} e^{\mathrm{i}\ell x},$$

then condition (2.3) will hold for arbitrary $s > 0$.

(iv) We also consider the following type of operators:

$$B_j(f, \varphi_j)(x) = 2^{-j} \sum_{k\in D_j} \left(f\left(x_k^j\right) + a2^{-j} f'\left(x_k^j\right) + b2^{-2j} f''\left(x_k^j\right)\right) \mathcal{D}_j\left(x - x_k^j\right). \tag{2.7}$$

We have that $B_j(f,\varphi_j) = Q_j(f,\varphi_j,\widetilde{\varphi}_j)$ if $\varphi_j = \mathcal{D}_j$ and

$$\widetilde{\varphi}_j(x) = \delta(x) + a2^{-j}\delta'(x) + b2^{-2j}\delta''(x).$$

It is not difficult to see that if $b \neq 0$, then condition (2.1) holds with $N = 2$. At the same time if $a \neq 0$, then compatibility condition (2.3) holds with $s = 1$. Note that in the non-periodic case operators of type (2.7) have been studied, e.g., in [4, 33].

Example 2.4 *Kantorovich-type operators* are defined by

$$K_j(f,\varphi_j)(x) = \sum_{k\in D_j} \frac{2^{\sigma-1}}{\pi} \int_{-\pi 2^{-j-\sigma}}^{\pi 2^{-j-\sigma}} f\left(t + x_k^j\right) \mathrm{d}t \, \varphi_j\left(x - x_k^j\right), \tag{2.8}$$

where $\sigma \geq 1$ and as above $\varphi_j \in \mathcal{T}_j$. It is clear that by taking $\widetilde{\varphi}_j(x) = 2^{j+\sigma}\chi_{[-\pi 2^{-j-\sigma},\pi 2^{-j-\sigma}]}(x)$, i.e., the normalized characteristic function of $[-\pi 2^{-j-\sigma}, \pi 2^{-j-\sigma}]$, we have that $K_j(f,\varphi_j) = Q_j(f,\varphi_j,\widetilde{\varphi}_j)$. Next, since

$$\widehat{\widetilde{\varphi}}_j(\ell) = \frac{\sin \pi 2^{-j-\sigma}\ell}{\pi 2^{-j-\sigma}\ell}, \quad \ell \in \mathbb{Z},$$

it is not difficult to see that (2.1) holds for $N = 0$. Concerning condition (2.3), we have that in this case it has the following form:

$$\left| 1 - \widehat{\varphi}_j(k) \frac{\sin \pi 2^{-j-\sigma} k}{\pi 2^{-j-\sigma} k} \right| \lesssim |2^{-j}k|^s, \quad \forall k \in D_j, \quad \forall j \in \mathbb{Z}_+. \tag{2.9}$$

(i) If $\varphi_j = \mathcal{D}_j$, we have that (2.9) holds for $s = 2$.
(ii) At the same time, for

$$\varphi_j(x) = \mathcal{D}_j^*(x) = \sum_{\ell\in D_j} \frac{\pi 2^{-j-\sigma}\ell}{\sin \pi 2^{-j-\sigma}\ell} e^{\mathrm{i}\ell x},$$

condition (2.9) holds for any $s > 0$.

Note that in recent years, the Kantorovich type operators (2.8) have been intensively studied in many works, see, e.g., [3, 9, 10, 33, 36, 38] in the non-periodic case and [28, 34, 35] in the periodic case. It is worth noting that operators of this type have several advantages over the interpolation and sampling operators. Particularly, using the averages of a function instead of the sampled values $f(x_k^j)$ allows to deal with discontinues signals and to reduce the so-called time-jitter errors, which is an important issue in digital image processing.

3 Auxiliary Results

For $j \in \mathbb{Z}_+$ and $\psi \in L_1(\mathbb{T})$, we define the following amalgam-type norm:

$$\|\psi\|_{\widetilde{A}_{p,j}(\mathbb{T})} = \sup_{\ell \in D_j} \left(\sum_{\mu \in \mathbb{Z}} |\widehat{\psi}(\ell + 2^j \mu)|^p \right)^{1/p} \quad \text{if} \quad 1 \le p < \infty$$

and

$$\|\psi\|_{\widetilde{A}_{\infty,j}(\mathbb{T})} = \|\psi\|_{A_\infty(\mathbb{T})} \quad \text{if} \quad p = \infty.$$

Lemma 3.1 *Let $1 \le q \le \infty$, $0 \le \gamma < \alpha$, and let $(\varphi_j)_{j \in \mathbb{Z}_+}$ and $(\widetilde{\varphi}_j)_{j \in \mathbb{Z}_+}$ be such that $\widetilde{\varphi}_j \in \mathcal{D}'(\mathbb{T})$ and $\varphi_j \in \mathcal{T}_j^1$ for each $j \in \mathbb{Z}_+$. Suppose conditions (2.1), (2.2), and (2.3) are fulfilled with $N \ge 0$ and $s > 0$. Further suppose that*

(i) $\alpha > N + 1/q'$ *if* $q \ne 1$ *and* $\alpha \ge N$ *if* $q = 1$ *or*
(ii) $N = 0$ *and* $\sup_{j \in \mathbb{Z}_+} \|\widetilde{\varphi}_j\|_{\widetilde{A}_{q',j}(\mathbb{T})} < \infty$.

Then, for all $f \in A_q^\alpha(\mathbb{T})$ and $j \in \mathbb{Z}_+$, we have

$$\|f - Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_{A_q^\gamma(\mathbb{T})} \lesssim 2^{-j \min(\alpha-\gamma, s)} \|f\|_{A_q^\alpha(\mathbb{T})}. \tag{3.1}$$

Proof The proof of the lemma under conditions in (i) can be found in [34, Remark 7]. In what follows, we prove (3.1) assuming that condition (ii) holds. We consider only the case $1 \le q < \infty$. The case $q = \infty$ can be treated similarly. First we show that

$$\|Q_j(f, \varphi_j, \widetilde{\varphi}_j)\|_{A_q^\gamma(\mathbb{T})} \lesssim \|f\|_{A_q^\gamma(\mathbb{T})}. \tag{3.2}$$

Indeed, using the representation

$$\begin{aligned} Q_j(f, \varphi_j, \widetilde{\varphi}_j)(x) &= \sum_{\ell \in D_j} \widehat{\varphi_j}(\ell) \left(2^{-j} \sum_{k \in D_j} (f * \widetilde{\varphi}_j)(x_k^j) e^{-\mathrm{i}\ell x_k^j} \right) e^{\mathrm{i}\ell x} \\ &= \sum_{\ell \in D_j} \widehat{\varphi_j}(\ell) \left(\sum_{\nu \in \mathbb{Z}} \widehat{f}(\nu) \widehat{\widetilde{\varphi}_j}(\nu) 2^{-j} \sum_{k \in D_j} e^{2\pi \mathrm{i} \frac{\nu - \ell}{2^j}} \right) e^{\mathrm{i}\ell x} \\ &= \sum_{\ell \in D_j} \widehat{\varphi_j}(\ell) \left(\sum_{\mu \in \mathbb{Z}} \widehat{f}(\ell + 2^j \mu) \widehat{\widetilde{\varphi}_j}(\ell + 2^j \mu) \right) e^{\mathrm{i}\ell x}, \end{aligned}$$

condition (2.2), and Hölder's inequality, we derive

$$
\begin{aligned}
\|Q_j(f,\varphi_j,\widetilde{\varphi}_j)\|^q_{A^\gamma_q(\mathbb{T})} &\lesssim \sum_{\ell\in D_j}(1+|\ell|)^{q\gamma}\left|\sum_{\mu\in\mathbb{Z}}\widehat{f}(\ell+2^j\mu)\widehat{\widetilde{\varphi}}_j(\ell+2^j\mu)\right|^q\\
&\lesssim \sum_{\ell\in D_j}(1+|\ell|)^{q\gamma}\sum_{\mu\in\mathbb{Z}}|\widehat{f}(\ell+2^j\mu)|^q\left(\sum_{\mu\in\mathbb{Z}}|\widehat{\widetilde{\varphi}}_j(\ell+2^j\mu)|^{q'}\right)^{q/q'}\\
&\lesssim \sup_{\ell\in D_j}\left(\sum_{\mu\in\mathbb{Z}}|\widehat{\widetilde{\varphi}}_j(\ell+2^j\mu)|^{q'}\right)^{q/q'}\|f\|^q_{A^\gamma_q(\mathbb{T})}\lesssim\|f\|^q_{A^\gamma_q(\mathbb{T})},
\end{aligned}
$$

which gives (3.2).

Now, we prove inequality (3.1). Let

$$
t_j(x)=\sum_{k\in D_j}\widehat{f}(k)e^{\mathrm{i}kx}.
$$

Applying (3.2), we obtain

$$
\begin{aligned}
&\|f-Q_j(f,\varphi_j,\widetilde{\varphi}_j)\|_{A^\gamma_q(\mathbb{T})}\\
&\le\|f-t_j\|_{A^\gamma_q(\mathbb{T})}+\|t_j-Q_j(t_j,\varphi_j,\widetilde{\varphi}_j)\|_{A^\gamma_q(\mathbb{T})}+\|Q_j(f-t_j,\varphi_j,\widetilde{\varphi}_j)\|_{A^\gamma_q(\mathbb{T})}\\
&\lesssim\|f-t_j\|_{A^\gamma_q(\mathbb{T})}+\|t_j-Q_j(t_j,\varphi_j,\widetilde{\varphi}_j)\|_{A^\gamma_q(\mathbb{T})}\\
&\lesssim 2^{-j(\alpha-\gamma)}\|f\|_{A^\alpha_q(\mathbb{T})}+\|t_j-Q_j(t_j,\varphi_j,\widetilde{\varphi}_j)\|_{A^\gamma_q(\mathbb{T})}.
\end{aligned}
\tag{3.3}
$$

Next, using inequality (2.3), we get

$$
\begin{aligned}
\|t_j-Q_j(t_j,\varphi_j,\widetilde{\varphi}_j)\|_{A^\gamma_q(\mathbb{T})}&=\left\|\sum_{k\in D_j}(1-\widehat{\varphi}_j(k)\widehat{\widetilde{\varphi}}_j(k))\widehat{f}(k)e^{ikx}\right\|_{A^\gamma_q(\mathbb{T})}\\
&\lesssim\left(\sum_{k\in D_j}(1+|k|)^{\gamma q}|2^{-j}k|^{sq}|\widehat{f}(k)|^q\right)^{1/q}\\
&\lesssim 2^{-js}\left(\sum_{k\in D_j}(1+|k|)^{(s-(\alpha-\gamma))q}(1+|k|)^{\alpha q}|\widehat{f}(k)|^q\right)^{1/q}\\
&\lesssim 2^{-\min(\alpha-\gamma,s)j}\|f\|_{A^\alpha_q(\mathbb{T})}.
\end{aligned}
\tag{3.4}
$$

Finally, combining (3.4) and (3.3), we arrive at (3.1). □

Remark 3.2 With regard to (i) and (ii) in Lemma 3.1, we note that the condition (i) can be applied to the operators $S_j(f,\varphi_j)$ in Example 2.3. Condition (ii), unlike condition (i), allows any parameter $\alpha>\gamma$ and is especially beneficial for the Kantorovich type operators $K_j(f,\varphi_j)$, see Example 2.4. Indeed, for $\widetilde{\varphi}_j(x) = 2^{j+\sigma}\chi_{[-\pi 2^{-j-\sigma},\pi 2^{-j-\sigma}]}(x) \sim \sum_{\ell\in\mathbb{Z}} \frac{\sin \pi 2^{-j-\sigma}\ell}{\pi 2^{-j-\sigma}\ell} e^{i\ell x}$, we have

$$\sup_{j\in\mathbb{Z}_+} \|\widetilde{\varphi}_j\|_{\widetilde{A}_{q',j}(\mathbb{T})} = \sup_{j\in\mathbb{Z}_+,\, \ell\in D_j} \left(\sum_{\mu\in\mathbb{Z}} \left|\frac{\sin \pi 2^{-\sigma}(2^{-j}\ell+\mu)}{\pi 2^{-\sigma}(2^{-j}\ell+\mu)}\right|^{q'}\right)^{1/q'} < \infty$$

in the case $1 < q' < \infty$. The case $q' = \infty$ is clear.

We will need the following Bernstein inequality.

Lemma 3.3 *Let* $1 \le q \le \infty$, $\min\{\alpha, \alpha+\beta-\gamma\} > 0$, *and* $\boldsymbol{\ell} \in \mathbb{Z}_+^d$. *Then, for any* $f \in \mathcal{T}_{\boldsymbol{\ell}}^d$, *we have*

$$\|f\|_{A_q^{\alpha,\beta}(\mathbb{T}^d)} \le 2^{\alpha|\boldsymbol{\ell}|_1+(\beta-\gamma)|\boldsymbol{\ell}|_\infty} \|f\|_{A_q^{\gamma}(\mathbb{T}^d)}.$$

Proof The proof is similar to the proof of Lemma 2.10 in [7]. □

Lemma 3.4 (See [7]) *Let* $\alpha > 0$, $\beta \in \mathbb{R}$, $\varepsilon = \min(\alpha, \alpha+\beta) > 0$, *and*

$$\psi(\boldsymbol{k}) := \alpha|\boldsymbol{k}|_1 + \beta|\boldsymbol{k}|_\infty, \quad \boldsymbol{k} \in \mathbb{Z}_+^d.$$

Then the inequality

$$\psi(\boldsymbol{k}) \le \psi(\boldsymbol{k}') - \varepsilon|\boldsymbol{k}'-\boldsymbol{k}|_1$$

holds for all $\boldsymbol{k}, \boldsymbol{k}' \in \mathbb{Z}_+^d$ *with* $\boldsymbol{k}' \ge \boldsymbol{k}$ *componentwise.*

Lemma 3.5 *Let* $T < 1$, $r < t$, *and* $t \ge 0$. *Then, for all* $n \in \mathbb{N}$,

$$\sum_{\boldsymbol{k}\notin\Delta(n,T)} 2^{-t|\boldsymbol{k}|_1+r|\boldsymbol{k}|_\infty} \lesssim \begin{cases} 2^{-\left(t-r-(tT-r)\frac{d-1}{d-T}\right)n} n^{d-1}, & T \ge \frac{r}{t}, \\ 2^{-(t-r)n}, & T < \frac{r}{t} \end{cases} \tag{3.5}$$

and

$$\sup_{\boldsymbol{k}\notin\Delta(n,T)} 2^{-t|\boldsymbol{k}|_1+r|\boldsymbol{k}|_\infty} \lesssim \begin{cases} 2^{-\left(t-r-(tT-r)\frac{d-1}{d-T}\right)n}, & T \ge \frac{r}{t}, \\ 2^{-(t-r)n}, & T < \frac{r}{t}. \end{cases} \tag{3.6}$$

Proof Estimate (3.5) can be found in the proof of [31, Theorem 4]. Estimate (3.6) can be proved by standard arguments using the method of Lagrange multipliers and Kuhn-Tacker conditions. □

4 Littlewood–Paley Type Characterizations

Proposition 4.1 *Let* $1 \le q \le \infty$*,* $\alpha > 0$*,* $\beta \in \mathbb{R}$*,* $\alpha + \beta > 0$*, and let* $Q = (Q_j(\cdot, \varphi_j, \widetilde{\varphi}_j))_{j\in\mathbb{Z}_+}$*, where* $(\varphi_j)_{j\in\mathbb{Z}_+}$ *and* $(\widetilde{\varphi}_j)_{j\in\mathbb{Z}_+}$ *be such that* $\widetilde{\varphi}_j \in \mathcal{D}'(\mathbb{T})$ *and* $\varphi_j \in \mathcal{T}_j^1$ *for each* $j \in \mathbb{Z}_+$*. Suppose conditions* (2.1), (2.2)*, and* (2.3) *are satisfied with the parameters* $N \ge 0$ *and* $s > \max(\alpha + \beta, \alpha)$*. Assume also that*

(i) $\min(\alpha + \beta, \alpha) > N + 1/q'$ *or*
(ii) $N = 0$ *and* $\sup_{j\in\mathbb{Z}_+} \|\widetilde{\varphi}_j\|_{\widetilde{A}_{q',j}(\mathbb{T})} < \infty$.

Then every function $f \in A_q^{\alpha,\beta}(\mathbb{T}^d)$ *can be represented by the series*

$$f = \sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} \eta_{\boldsymbol{j}}^Q(f), \tag{4.1}$$

which converges unconditionally in $A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)$ *with* $0 \le \widetilde{\alpha} < \alpha$ *and satisfies*

$$\left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)} \|\eta_{\boldsymbol{j}}^Q(f)\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q} \lesssim \|f\|_{A_q^{\alpha,\beta}(\mathbb{T}^d)}. \tag{4.2}$$

Proof *Step 1* First we prove the proposition assuming that the condition in (i) holds. Let $f \in A_q^{\alpha,\beta}(\mathbb{T}^d)$ and $\boldsymbol{j} \in \mathbb{Z}_+^d$. We have

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f)(\boldsymbol{x}), \tag{4.3}$$

where we set $\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f) = 0$ for $\boldsymbol{j} + \boldsymbol{\ell} \in \mathbb{Z}^d \setminus \mathbb{Z}_+^d$. In light of (4.3), we get

$$|\eta_{\boldsymbol{j}}^Q(f)(\boldsymbol{x})| \le \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} |\eta_{\boldsymbol{j}}^Q\left(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f)\right)(\boldsymbol{x})|$$

and, therefore,

$$\|\eta_{\boldsymbol{j}}^Q(f)\|_{A_q(\mathbb{T}^d)} \le \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \|\eta_{\boldsymbol{j}}^Q\left(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f)\right)\|_{A_q(\mathbb{T}^d)}. \tag{4.4}$$

In what follows, for simplicity we consider only the case $q < \infty$. The case $q = \infty$ can be treated similarly. Multiplying by $2^{\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty}$ and taking ℓ_q-norm on both sides of (4.4), we obtain

$$\left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\eta_{\boldsymbol{j}}^Q(f)\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q}$$

$$\leq \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d}\left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\eta_{\boldsymbol{j}}^Q(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q} \tag{4.5}$$

$$= \sum_{\boldsymbol{\ell}_2^d\in\mathbb{Z}^{d-1}}\sum_{-j_1\leq\ell_1<-1}(\dots)+\sum_{\boldsymbol{\ell}_2^d\in\mathbb{Z}^{d-1}}\sum_{\ell_1\geq-1}(\dots)=S_1+S_2,$$

where $\boldsymbol{\ell}_k^d=(\ell_k,\dots,\ell_d)$, $k=2,\dots,d$.

Consider the sum S_1. Denoting

$$\eta_{\boldsymbol{j}_k^d}^Q=\eta_{\boldsymbol{j}_k^d,k}^Q=\prod_{i=k}^d(Q_{j_i}^i-Q_{j_i-1}^i),\quad k=2,\dots,d,$$

where $Q_{j_i}^i$ is the univariate operator $Q_{j_i}(\cdot,\varphi_{j_i},\widetilde{\varphi}_{j_i})$ acting on functions in the variable x_i, we obtain

$$S_1=\sum_{\boldsymbol{\ell}_2^d\in\mathbb{Z}^{d-1}}\sum_{-j_1\leq\ell_1<-1}\left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|(Q_{j_1}^1-Q_{j_1-1}^1)\eta_{\boldsymbol{j}_2^d}^Q(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q}$$

$$\leq\sum_{b\in\{-1,0\}}\sum_{\boldsymbol{\ell}_2^d\in\mathbb{Z}^{d-1}}\sum_{-j_1\leq\ell_1<-1}\left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|(Q_{j_1+b}^1-I)\eta_{\boldsymbol{j}_2^d}^Q(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q}, \tag{4.6}$$

where I is the identity operator. Taking into account that $Q_j(t,\varphi,\widetilde{\varphi})=\widetilde{\varphi}_j*\varphi_j*t$ for any trigonometric polynomial $t\in\mathcal{T}_{j-1}^1$ and using condition (2.3) and Bernstein's inequality, we derive that

$$\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|(Q_{j_1+b}^1-I)\eta_{\boldsymbol{j}_2^d}^Q(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|_{A_q(\mathbb{T}^d)}^q$$

$$\lesssim\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty-sj_1)}\left\|\frac{\partial^s}{\partial x_1^s}\eta_{\boldsymbol{j}_2^d}^Q(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\right\|_{A_q(\mathbb{T}^d)}^q$$

$$\lesssim\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty+s\ell_1)}\|\eta_{\boldsymbol{j}_2^d}^Q(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|_{A_q(\mathbb{T}^d)}^q \tag{4.7}$$

$$\lesssim 2^{(s-\alpha)\ell_1 q}\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}+\ell_1\boldsymbol{e}_1|_1+\beta|\boldsymbol{j}|_\infty)}\|\eta_{\boldsymbol{j}_2^d}^Q(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|_{A_q(\mathbb{T}^d)}^q$$

$$\lesssim 2^{(s-\max(\alpha+\beta,\alpha))\ell_1 q}\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}+\ell_1\boldsymbol{e}_1|_1+\beta|\boldsymbol{j}+\ell_1\boldsymbol{e}_1|_\infty)}\|\eta_{\boldsymbol{j}_2^d}^Q(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|_{A_q(\mathbb{T}^d)}^q,$$

where in the last inequality we use the estimates

$$|\boldsymbol{j}|_\infty \le |\boldsymbol{j}+\ell_1\boldsymbol{e}_1|_\infty + |\ell_1\boldsymbol{e}_1|_\infty = |\boldsymbol{j}+\ell_1\boldsymbol{e}_1|_\infty - \ell_1 \quad \text{in the case} \quad \beta \ge 0$$

and

$$|\boldsymbol{j}|_\infty \ge |\boldsymbol{j}+\ell_1\boldsymbol{e}_1|_\infty \quad \text{in the case} \quad \beta < 0.$$

Next, combining (4.6) and (4.7) and using the fact that $\sum_{\ell_1<-1} 2^{(s-\max(\alpha+\beta,\alpha))\ell_1} < \infty$, we obtain

$$S_1 \lesssim \sum_{\boldsymbol{\ell}_2^d \in \mathbb{Z}^{d-1}} \left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)} \|\eta^Q_{\boldsymbol{j}_2^d}(\delta_{j_1,j_2+\ell_2,\dots,j_d+\ell_d}(f))\|^q_{A_q(\mathbb{T}^d)} \right)^{1/q}. \tag{4.8}$$

Now, we consider the sum S_2. Similar to (4.6), we have

$$\begin{aligned} S_2 &= \sum_{\boldsymbol{\ell}_2^d \in \mathbb{Z}^{d-1}} \sum_{\ell_1 \ge -1} \left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)} \|(Q^1_{j_1} - Q^1_{j_1-1})\eta^Q_{\boldsymbol{j}_2^d}(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|^q_{A_q(\mathbb{T}^d)} \right)^{1/q} \\ &\le \sum_{b\in\{-1,0\}} \sum_{\boldsymbol{\ell}_2^d \in \mathbb{Z}^{d-1}} \sum_{\ell_1 \ge -1} \left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)} \|(Q^1_{j_1+b} - I)\eta^Q_{\boldsymbol{j}_2^d}(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|^q_{A_q(\mathbb{T}^d)} \right)^{1/q}. \end{aligned} \tag{4.9}$$

Choosing ζ such that $N + 1/q' < \zeta < \min(\alpha, \alpha+\beta)$ and applying Lemma 3.1(i), Bernstein's inequality, and Lemma 3.4, we obtain for $\ell_1 \ge 0$ that

$$\begin{aligned} &\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)} \|(Q^1_{j_1+b} - I)\eta^Q_{\boldsymbol{j}_2^d}(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|^q_{A_q(\mathbb{T}^d)} \\ &\lesssim \sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty-\zeta j_1)} \|\eta^Q_{\boldsymbol{j}_2^d}(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|^q_{A_q^{(\zeta,0,\dots,0)}(\mathbb{T}^d)} \\ &\lesssim \sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty+\zeta\ell_1)} \|\eta^Q_{\boldsymbol{j}_2^d}(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|^q_{A_q(\mathbb{T}^d)} \\ &\lesssim 2^{-q(\min(\alpha,\alpha+\beta)-\zeta)\ell_1} \sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}+\ell_1\boldsymbol{e}_1|_1+\beta|\boldsymbol{j}+\ell_1\boldsymbol{e}_1|_\infty)} \|\eta^Q_{\boldsymbol{j}_2^d}(\delta_{\boldsymbol{j}+\boldsymbol{\ell}}(f))\|^q_{A_q(\mathbb{T}^d)} \\ &\lesssim 2^{-q(\min(\alpha,\alpha+\beta)-\zeta)\ell_1} \sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)} \|\eta^Q_{\boldsymbol{j}_2^d}(\delta_{j_1,j_2+\ell_2,\dots,j_d+\ell_d}(f))\|^q_{A_q(\mathbb{T}^d)}. \end{aligned} \tag{4.10}$$

A similar estimate clearly holds for $\ell_1 = -1$. Thus, combining (4.9) and (4.10) and taking into account that $\sum_{\ell_1 \ge -1} 2^{-(\min(\alpha,\alpha+\beta)-\zeta)\ell_1} < \infty$, we get

$$S_2 \lesssim \sum_{\boldsymbol{\ell}_2^d \in \mathbb{Z}^{d-1}} \left(\sum_{\boldsymbol{j} \in \mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)} \|\eta_{\boldsymbol{j}_2^d}^Q(\delta_{j_1, j_2+\ell_2,\dots,j_d+\ell_d}(f))\|_{A_q(\mathbb{T}^d)}^q \right)^{1/q} \tag{4.11}$$

In the next step, collecting (4.5), (4.8), and (4.11) implies

$$\begin{aligned} &\left(\sum_{\boldsymbol{j} \in \mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)} \|\eta_{\boldsymbol{j}}^Q(f)\|_{A_q(\mathbb{T}^d)}^q \right)^{1/q} \\ &\qquad \lesssim \sum_{\boldsymbol{\ell}_2^d \in \mathbb{Z}^{d-1}} \left(\sum_{\boldsymbol{j} \in \mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)} \|\eta_{\boldsymbol{j}_2^d}^Q(\delta_{j_1, j_2+\ell_2,\dots,j_d+\ell_d}(f))\|_{A_q(\mathbb{T}^d)}^q \right)^{1/q}. \end{aligned}$$

Then, repeating the above procedure for the parameters $\ell_2, \dots, \ell_d$, we prove (4.2) by Lemma 2.2.

Step 2 Let us prove representation (4.1). Applying Lemma 3.3 (here without loss of generality, we can assume that $\min(\widetilde{\alpha}+\beta, \widetilde{\alpha}) > 0$), Hölder's inequality, and (4.2), we obtain

$$\begin{aligned} &\sum_{\boldsymbol{k} \in \mathbb{Z}_+^d} \|\eta_{\boldsymbol{k}}^Q(f)\|_{A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)} \\ &\lesssim \sum_{\boldsymbol{k} \in \mathbb{Z}_+^d} 2^{\widetilde{\alpha}|\boldsymbol{k}|_1+\beta|\boldsymbol{k}|_\infty} \|\eta_{\boldsymbol{k}}^Q(f)\|_{A_q(\mathbb{T}^d)} \\ &= \sum_{\boldsymbol{k} \in \mathbb{Z}_+^d} 2^{-(\alpha-\widetilde{\alpha})|\boldsymbol{k}|_1} \cdot 2^{\alpha|\boldsymbol{k}|_1+\beta|\boldsymbol{k}|_\infty} \|\eta_{\boldsymbol{k}}^Q(f)\|_{A_q(\mathbb{T}^d)} \\ &\le \left(\sum_{\boldsymbol{k} \in \mathbb{Z}_+^d} 2^{-q'(\alpha-\widetilde{\alpha})|\boldsymbol{k}|_1} \right)^{1/q'} \left(\sum_{\boldsymbol{k} \in \mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{k}|_1+\beta|\boldsymbol{k}|_\infty)} \|\eta_{\boldsymbol{k}}^Q(f)\|_{A_q(\mathbb{T}^d)}^q \right)^{1/q} \\ &\lesssim \|f\|_{A_q^{\alpha,\beta}(\mathbb{T}^d)}. \end{aligned} \tag{4.12}$$

Therefore, $\sum_{\boldsymbol{k} \in \mathbb{Z}_+^d} \eta_{\boldsymbol{k}}^Q(f)$ converges unconditionally in $A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)$.

Now we show that for any trigonometric polynomial g,

$$g = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^d} \eta_{\boldsymbol{k}}^Q(g). \tag{4.13}$$

It is clear that it suffices to verify (4.13) for $t_{\boldsymbol{j}}(\boldsymbol{x}) = e^{\mathrm{i}(\boldsymbol{j},\boldsymbol{x})}$ with arbitrary $\boldsymbol{j} \in \mathbb{Z}^d$. By the triangle inequality, we have

$$\begin{aligned} &\Big\| t_{\boldsymbol{j}} - \sum_{\boldsymbol{k}\in\mathbb{Z}_+^d} \eta_{\boldsymbol{k}}^Q(t_{\boldsymbol{j}}) \Big\|_{A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)} \\ &\le \Big\| t_{\boldsymbol{j}} - \sum_{\boldsymbol{k}\in\mathbb{Z}_+^d,\, |\boldsymbol{k}|_\infty\le m} \eta_{\boldsymbol{k}}^Q(t_{\boldsymbol{j}}) \Big\|_{A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)} + \sum_{\boldsymbol{k}\in\mathbb{Z}_+^d,\, |\boldsymbol{k}|_\infty> m} \|\eta_{\boldsymbol{k}}^Q(t_{\boldsymbol{j}})\|_{A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)} \\ &:= I_1(m) + I_2(m). \end{aligned} \tag{4.14}$$

We obviously have that

$$I_2(m) = 0 \quad \text{for } m \text{ large enough.} \tag{4.15}$$

Since

$$\sum_{k_i=0}^{m} (Q_{k_i}^i - Q_{k_i-1}^i) = Q_m^i, \quad i = 1, \dots, m,$$

we obtain

$$\sum_{\boldsymbol{k}\in\mathbb{Z}_+^d,\, |\boldsymbol{k}|_\infty\le m} \eta_{\boldsymbol{k}}^Q = \prod_{i=1}^d Q_m^i.$$

Thus, for any $\boldsymbol{j} \in (-2^{m-1}, 2^{m-1})^d \cap \mathbb{Z}^d$, we have

$$\sum_{\boldsymbol{k}\in\mathbb{Z}_+^d,\, |\boldsymbol{k}|_\infty\le m} \eta_{\boldsymbol{k}}^Q(t_{\boldsymbol{j}})(\boldsymbol{x}) = \prod_{i=1}^d \widehat{\varphi_m}(j_i)\widehat{\widetilde{\varphi}_m}(j_i) e^{\mathrm{i}(\boldsymbol{j},\boldsymbol{x})}.$$

Using this and conditions (2.1), (2.2), (2.3), we find

$$\begin{aligned} I_1(m) &= \Big| 1 - \prod_{i=1}^d \widehat{\varphi_m}(j_i)\widehat{\widetilde{\varphi}_m}(j_i) \Big| \\ &= \Big| 1 - \widehat{\varphi_m}(j_1)\widehat{\widetilde{\varphi}_m}(j_1) + \sum_{\nu=2}^d \prod_{i=1}^{\nu-1} \widehat{\varphi_m}(j_i)\widehat{\widetilde{\varphi}_m}(j_i) \big(1 - \widehat{\varphi_m}(j_\nu)\widehat{\widetilde{\varphi}_m}(j_\nu)\big) \Big| \\ &\lesssim \sum_{\nu=1}^d \Big| 1 - \widehat{\varphi_m}(j_\nu)\widehat{\widetilde{\varphi}_m}(j_\nu) \Big| \lesssim 2^{-ms} \sum_{\nu=1}^d |j_\nu|^s \to 0 \quad \text{as} \quad m \to \infty. \end{aligned} \tag{4.16}$$

Therefore, combining (4.14), (4.15), and (4.16), we arrive at (4.13).

The rest of the proof is quite standard. Denote $F := \sum_{\boldsymbol{k}\in\mathbb{Z}_+^d} \eta_{\boldsymbol{k}}^Q(f)$. Using (4.13), we have for every trigonometric polynomial g that

$$F - g = \sum_{\boldsymbol{k}\in\mathbb{Z}_+^d} \eta_{\boldsymbol{k}}^Q(f-g)$$

with convergence in $A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)$. Hence, by (4.12), we derive

$$\|F - f\|_{A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)} \le \|F - g\|_{A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)} + \|g - f\|_{A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)} \lesssim \|f - g\|_{A_q^{\alpha,\beta}(\mathbb{T}^d)}.$$

Choosing g close enough to f yields $\|F - f\|_{A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)} < \varepsilon$ for all $\varepsilon > 0$ and hence $\|F - f\|_{A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)} = 0$, which implies (4.1).

By the same scheme, using Lemma 3.1 (ii), the proof of the proposition under condition (ii) also follows. □

We will also need the following modification of inequality (4.2).

Lemma 4.2 *Let $f \in A_q(\mathbb{T}^d)$, $1 \le q < p \le \infty$, and $1 \le \theta \le \infty$. Under conditions of Proposition 4.1, there exists a constant $C = C(\alpha, \beta, q, \theta, d) > 0$ such that*

$$\begin{aligned}&\Big(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{\theta(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\eta_{\boldsymbol{j}}^Q(f)\|_{A_q(\mathbb{T}^d)}^\theta\Big)^{1/\theta}\\ &\quad\le C\Big(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{\theta((\alpha+\frac{1}{q}-\frac{1}{p})|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\delta_{\boldsymbol{j}}(f)\|_{A_p(\mathbb{T}^d)}^\theta\Big)^{1/\theta} \qquad (4.17)\end{aligned}$$

whenever the sum in the right-hand side is finite.

Proof First, repeating the same procedure as in the proof of Step 1 of Proposition 4.1, we obtain

$$\begin{aligned}&\Big(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{\theta(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\eta_{\boldsymbol{j}}^Q(f)\|_{A_q(\mathbb{T}^d)}^\theta\Big)^{1/\theta}\\ &\lesssim \Big(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{\theta(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\delta_{\boldsymbol{j}}(f)\|_{A_q(\mathbb{T}^d)}^\theta\Big)^{1/\theta}.\end{aligned}$$

Then, applying the inequality

$$\|\delta_{\boldsymbol{j}}(f)\|_{A_q(\mathbb{T}^d)} \lesssim 2^{(\frac{1}{q}-\frac{1}{p})|\boldsymbol{j}|_1}\|\delta_{\boldsymbol{j}}(f)\|_{A_p(\mathbb{T}^d)},$$

which easily follows from Hölder's inequality and the fact that $\operatorname{spec}\delta_{\boldsymbol{j}}(f) \subset P_{j_1} \times \cdots \times P_{j_d}$, $P_j = \{\ell \in \mathbb{Z} : 2^{j-1} \le |\ell| < 2^j\}$, we arrive at (4.17). □

A reverse statement to Proposition 4.1 is written as follows.

Proposition 4.3 *Let $\alpha > 0$, $\beta \in \mathbb{R}$, $\alpha+\beta > 0$, $1 \le q \le \infty$, and let $(f_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{Z}_+^d}$ be such that $f_{\boldsymbol{j}} \in \mathcal{T}_{\boldsymbol{j}}^d$ and*

$$\left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|f_{\boldsymbol{j}}\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q} < \infty.$$

Suppose that the series $\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} f_{\boldsymbol{j}}$ converges to a function f in $A_q(\mathbb{T}^d)$. Then $f \in A_q^{\alpha,\beta}(\mathbb{T}^d)$ and moreover, there is a constant $C = C(\alpha,\beta,q,d)$ such that

$$\|f\|_{A_q^{\alpha,\beta}(\mathbb{T}^d)} \le C\left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|f_{\boldsymbol{j}}\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q}.$$

Proof The proposition can be proved repeating step by step the proof of Proposition 3.4 in [7]. For completeness we present a detailed proof.

For $\boldsymbol{\ell} \in \mathbb{Z}_+^d$, we write f as the series

$$f = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} f_{\boldsymbol{\ell}+\boldsymbol{j}}$$

with $f_{\boldsymbol{\ell}+\boldsymbol{j}} := 0$ for $\boldsymbol{j}+\boldsymbol{\ell} \in \mathbb{Z}^d \setminus \mathbb{Z}_+^d$. Using the triangle inequality and taking into account that $\delta_{\boldsymbol{\ell}}(f_{\boldsymbol{\ell}+\boldsymbol{j}}) = 0$ for $\boldsymbol{j} \notin \mathbb{Z}_+^d$, we obtain

$$\begin{aligned}\|\delta_{\boldsymbol{\ell}}(f)\|_{A_q(\mathbb{T}^d)} &= \left\|\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} \delta_{\boldsymbol{\ell}}(f_{\boldsymbol{\ell}+\boldsymbol{j}})\right\|_{A_q(\mathbb{T}^d)} \\ &\le \sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} \|\delta_{\boldsymbol{\ell}}(f_{\boldsymbol{\ell}+\boldsymbol{j}})\|_{A_q(\mathbb{T}^d)} \le \sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} \|f_{\boldsymbol{\ell}+\boldsymbol{j}}\|_{A_q(\mathbb{T}^d)}.\end{aligned}$$

This inequality together with Lemma 3.4 yields

$$2^{\alpha|\boldsymbol{\ell}|_1+\beta|\boldsymbol{\ell}|_\infty}\|\delta_{\boldsymbol{\ell}}(f)\|_{A_q(\mathbb{T}^d)} \lesssim \sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{-\min\{\alpha,\alpha+\beta\}|\boldsymbol{j}|_1} \cdot 2^{\alpha|\boldsymbol{\ell}+\boldsymbol{j}|_1+\beta|\boldsymbol{\ell}+\boldsymbol{j}|_\infty}\|f_{\boldsymbol{\ell}+\boldsymbol{j}}\|_{A_q(\mathbb{T}^d)}.$$

Then, by Minkowski's inequality, we obtain

$$\left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{\ell}|_1+\beta|\boldsymbol{\ell}|_\infty)}\|\delta_{\boldsymbol{\ell}}(f)\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q}$$
$$\lesssim \sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{-\min\{\alpha,\alpha+\beta\}|\boldsymbol{j}|_1}\left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{\ell}+\boldsymbol{j}|_1+\beta|\boldsymbol{\ell}+\boldsymbol{j}|_\infty)}\|f_{\boldsymbol{\ell}+\boldsymbol{j}}\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q}$$
$$\lesssim \left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{\ell}|_1+\beta|\boldsymbol{\ell}|_\infty)}\|f_{\boldsymbol{\ell}}\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q}.$$

Thus, Lemma 2.2 concludes the proof. □

Propositions 4.1 and 4.3 suggest the following useful necessary and sufficient conditions for $f \in A_q^{\alpha,\beta}(\mathbb{T}^d)$ to be represented as $f = \sum_{\boldsymbol{j}\in\mathbb{Z}_+^d}\eta_{\boldsymbol{j}}^Q(f)$. This generalizes Theorem 3.6 in [7].

Theorem 4.4 *Let* $1 \le q \le \infty$, $\alpha > 0$, $\beta \in \mathbb{R}$, $\alpha+\beta > 0$, *and let* $Q = (Q_j(\cdot,\varphi_j,\widetilde{\varphi}_j))_{j\in\mathbb{Z}_+}$, *where* $(\varphi_j)_{j\in\mathbb{Z}_+}$ *and* $(\widetilde{\varphi}_j)_{j\in\mathbb{Z}_+}$ *be such that* $\widetilde{\varphi}_j \in \mathcal{D}'(\mathbb{T})$ *and* $\varphi_j \in \mathcal{T}_j^1$ *for each* $j \in \mathbb{Z}_+$. *Suppose conditions* (2.1), (2.2), *and* (2.3) *are satisfied with parameters* $N \ge 0$ *and* $s > \max(\alpha+\beta,\alpha)$. *Assume also that*

(i) $\min(\alpha+\beta,\alpha) > N + 1/q'$ *or*
(ii) $N = 0$ *and* $\sup_{j\in\mathbb{Z}_+}\|\widetilde{\varphi}_j\|_{\widetilde{A}_{q',j}(\mathbb{T})} < \infty$.

Then a function f *belongs to* $A_q^{\alpha,\beta}(\mathbb{T}^d)$ *if and only if it can be represented by the series* (4.1) *converging unconditionally in* $A_q^{\widetilde{\alpha},\beta}(\mathbb{T}^d)$ *with* $\widetilde{\alpha} < \alpha$ *and satisfying* $\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\eta_{\boldsymbol{j}}^Q(f)\|_{A_q(\mathbb{T}^d)}^q < \infty$. *Moreover, the norm* $\|f\|_{A_q^{\alpha,\beta}(\mathbb{T}^d)}$ *is equivalent to the norm*

$$\|f\|_{A_q^{\alpha,\beta}(\mathbb{T}^d)}^+ := \left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{q(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\eta_{\boldsymbol{j}}^Q(f)\|_{A_q(\mathbb{T}^d)}^q\right)^{1/q}.$$

5 Error Estimates

In this section, we obtain estimates for the error of approximation by quasi-interpolation operators

$$P_{n,T}^Q = \sum_{\boldsymbol{j}\in\Delta(n,T)}\eta_{\boldsymbol{j}}^Q, \quad n\in\mathbb{N}, \quad T<1,$$

where

$$\Delta(n, T) = \{\boldsymbol{k} \in \mathbb{Z}_+^d \,:\, |\boldsymbol{k}|_1 - T|\boldsymbol{k}|_\infty \le (1-T)n\}.$$

In what follows, we distinguish between approximation of a function $f \in A_p^{\alpha,\beta}(\mathbb{T}^d)$ in the isotropic space $A_q^\gamma(\mathbb{T}^d)$ (Theorem 5.1) and in the mixed space $A_{q,\mathrm{mix}}^\gamma(\mathbb{T}^d)$ (Theorem 5.3) since we use slightly different ingredients in the corresponding proofs.

Recall that

$$\sigma_{p,q} = \left(\frac{1}{q} - \frac{1}{p}\right)_+ .$$

5.1 Error Estimates in $A_q^\gamma(\mathbb{T}^d)$

Theorem 5.1 *Let $1 \le p, q \le \infty$, $\alpha > \sigma_{p,q}$, $\beta \in \mathbb{R}$, $\gamma \ge 0$, $\gamma - \beta < \alpha - \sigma_{p,q}$, and let $Q = (Q_j(\cdot, \varphi_j, \widetilde{\varphi}_j))_{j\in\mathbb{Z}_+}$, where $\widetilde{\varphi}_j \in \mathcal{D}'(\mathbb{T})$ and $\varphi_j \in \mathcal{T}_j^1$ for each $j \in \mathbb{Z}_+$. Suppose conditions (2.1), (2.2), and (2.3) are satisfied with parameters $N \ge 0$ and $s > \max(\alpha+\beta, \alpha)$. Assume also that*

(i) $\min(\alpha+\beta, \alpha) > N + 1/p'$ *or*
(ii) $N = 0$ *and* $\sup_{j\in\mathbb{Z}_+} \|\widetilde{\varphi}_j\|_{\widetilde{A}_{q',j}(\mathbb{T})} < \infty$.

Then, for all $f \in A_p^{\alpha,\beta}(\mathbb{T}^d)$ and $n \in \mathbb{N}$, we have

$$\|f - P_{n,T}^Q f\|_{A_q^\gamma(\mathbb{T}^d)} \le C\Omega(n)\|f\|_{A_p^{\alpha,\beta}(\mathbb{T}^d)}, \tag{5.1}$$

where

$$\Omega(n) = \begin{cases} 2^{-\left(\alpha+\beta-\gamma-\sigma_{p,q}-((\alpha-\sigma_{p,q})T-(\gamma-\beta))\frac{d-1}{d-T}\right)n} n^{(d-1)(1-\frac{1}{p})}, & \frac{\gamma-\beta}{\alpha-\sigma_{p,q}} \le T < 1, \\ 2^{-(\alpha+\beta-\gamma-\sigma_{p,q})n}, & T < \frac{\gamma-\beta}{\alpha-\sigma_{p,q}}, \end{cases}$$

and the constant C does not depend on f and n.

Remark 5.2 (i) In the case $p = q = 2$ and $Q = (I_j)_{j\in\mathbb{Z}_+}$, where I_j is the Lagrange interpolation operator defined in (1.3), Theorem 5.1 was proved in [25], see also [7] and [24]. For similar results in the case $p = q = 2$, $\gamma = T = 0$, and $Q = (K_j)_{j\in\mathbb{Z}_+}$, where K_j is defined in (2.8), see [32]. (ii) Under conditions of Theorem 5.1 with $1 \le q \le 2$ and $\gamma = 0$, by the Hausdorff-Young inequality, estimate (5.1) implies that

$$\|f - P_{n,T}^Q f\|_{L_{q'}(\mathbb{T}^d)} \le C\Omega(n)\|f\|_{A_p^{\alpha,\beta}(\mathbb{T}^d)}. \tag{5.2}$$

We can further extend this result considering a more general Pitt's inequality [23, 52, 53]

$$\|f\|_{L_\xi^\eta(\mathbb{T}^d)} \lesssim \|f\|_{A_q^\gamma(\mathbb{T}^d)}, \qquad 1 \le q \le \xi \le \infty, \tag{5.3}$$

where $\|f\|_{L_\xi^\eta(\mathbb{T}^d)} = (\int_{\mathbb{T}^d} |f(x)|^\xi |x|^\eta dx)^{1/\xi}$ under the suitable conditions on ξ and η. In particular, using [22, (5.4)] we have

$$\|f - P_{n,T}^Q f\|_{L_\xi(\mathbb{T}^d)} \le C\Omega(n)\|f\|_{A_p^{\alpha,\beta}(\mathbb{T}^d)} \tag{5.4}$$

for $\xi \ge 2$, $\max(q, q') \le \xi$, and $\gamma = d(1 - \frac{1}{\xi} - \frac{1}{q})$. Taking $\xi = q'$ we see that (5.4) coincides with (5.2).

(iii) It is clear that the case $\gamma - \beta = \alpha - \sigma_{p,q}$ is degenerate since then we have $\Omega(n) = 1$ and hence the right-hand side of (5.1) does not tend to zero.

Proof of Theorem 5.1 First, we consider the case $1 < p \le q \le \infty$.

Using the estimate $\|\cdot\|_{\ell_q} \le \|\cdot\|_{\ell_p}$, Proposition 4.1, Lemma 3.3, and Hölder's inequality, we obtain

$$\begin{aligned}
\|f - P_{n,T}^Q f\|_{A_q^\gamma(\mathbb{T}^d)} &\le \|f - P_{n,T}^Q f\|_{A_p^\gamma(\mathbb{T}^d)} \\
&= \Big\| \sum_{\boldsymbol{j}\notin\Delta(n,T)} \eta_{\boldsymbol{j}}^Q(f) \Big\|_{A_p^\gamma(\mathbb{T}^d)} \le \sum_{\boldsymbol{j}\notin\Delta(n,T)} \|\eta_{\boldsymbol{j}}^Q(f)\|_{A_p^\gamma(\mathbb{T}^d)} \\
&\le \sum_{\boldsymbol{j}\notin\Delta(n,T)} 2^{\gamma|\boldsymbol{j}|_\infty} \|\eta_{\boldsymbol{j}}^Q(f)\|_{A_p(\mathbb{T}^d)} \\
&= \sum_{\boldsymbol{j}\notin\Delta(n,T)} 2^{-\alpha|\boldsymbol{j}|_1 - (\beta-\gamma)|\boldsymbol{j}|_\infty} 2^{\alpha|\boldsymbol{j}|_1 + \beta|\boldsymbol{j}|_\infty} \|\eta_{\boldsymbol{j}}^Q(f)\|_{A_p(\mathbb{T}^d)} \\
&\le \Bigg(\sum_{\boldsymbol{j}\notin\Delta(n,T)} 2^{-p'(\alpha|\boldsymbol{j}|_1 + (\beta-\gamma)|\boldsymbol{j}|_\infty)} \Bigg)^{1/p'} \\
&\quad \times \Bigg(\sum_{\boldsymbol{j}\notin\Delta(n,T)} 2^{p(\alpha|\boldsymbol{j}|_1 + \beta|\boldsymbol{j}|_\infty)} \|\eta_{\boldsymbol{j}}^Q(f)\|_{A_p(\mathbb{T}^d)}^p \Bigg)^{1/p}
\end{aligned} \tag{5.5}$$

Thus, Proposition 4.1 implies

$$\|f - P_{n,T}^Q f\|_{A_q^\gamma(\mathbb{T}^d)} \le \Bigg(\sum_{\boldsymbol{j}\notin\Delta(n,T)} 2^{-p'(\alpha|\boldsymbol{j}|_1 + (\beta-\gamma)|\boldsymbol{j}|_\infty)} \Bigg)^{1/p'} \|f\|_{A_p^{\alpha,\beta}(\mathbb{T}^d)}. \tag{5.6}$$

Next, combining (5.6) and (3.5), we derive (5.1) in the case $p > 1$. The case $p = 1$ is treated similarly using (3.6).

Now, we consider the case $1 \le q < p \le \infty$. Since $1/p' > 1/q'$, we can apply the intermediate estimate in (5.5) with $p = q$ given by

$$\|f - P_{n,T}^Q f\|_{A_q^\gamma(\mathbb{T}^d)} \le \sum_{\boldsymbol{j}\notin\Delta(n,T)} 2^{\gamma|\boldsymbol{j}|_\infty} \|\eta_{\boldsymbol{j}}^Q(f)\|_{A_q(\mathbb{T}^d)}.$$

Then, using Hölder's inequality, Lemmas 3.5, and 4.2 (note that condition (i) implies that $\min(\alpha - 1/q + 1/p + \beta, \alpha - 1/q + 1/p) > N + 1/q')$, we get

$$\begin{aligned}
\|f - P_{n,T}^Q f\|_{A_q^\gamma(\mathbb{T}^d)} &\le \left(\sum_{\boldsymbol{j}\notin\Delta(n,T)} 2^{-p'((\alpha-\sigma_{p,q})|\boldsymbol{j}|_1+(\beta-\gamma)|\boldsymbol{j}|_\infty)}\right)^{1/p'} \\
&\quad\times\left(\sum_{\boldsymbol{j}\notin\Delta(n,T)} 2^{p((\alpha-\sigma_{p,q})|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\eta_{\boldsymbol{j}}^Q(f)\|_{A_q(\mathbb{T}^d)}^p\right)^{1/p} \\
&\lesssim \Omega(n)\left(\sum_{\boldsymbol{j}\in\mathbb{Z}_+^d} 2^{p(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\delta_{\boldsymbol{j}}(f)\|_{A_p(\mathbb{T}^d)}^p\right)^{1/p} \\
&\lesssim \Omega(n)\|f\|_{A_p^{\alpha,\beta}(\mathbb{T}^d)}, \qquad (5.7)
\end{aligned}$$

where in the last estimate we have taken into account Lemma 2.2. □

5.2 Error Estimates in $A_{q,\mathrm{mix}}^{\gamma}(\mathbb{T}^d)$

Theorem 5.3 *Let* $1 \le p, q \le \infty$, $\beta \in \mathbb{R}$, $\gamma > 0$, $\gamma - \beta + \sigma_{p,q} < \alpha$, $\gamma + \sigma_{p,q} \le \alpha$, *and let* $Q = (Q_j(\cdot, \varphi_j, \widetilde{\varphi}_j))_{j\in\mathbb{Z}_+}$, *where* $\widetilde{\varphi}_j \in \mathcal{D}'(\mathbb{T})$ *and* $\varphi_j \in \mathcal{T}_j^1$ *for each* $j \in \mathbb{Z}_+$. *Suppose conditions* (2.1), (2.2), *and* (2.3) *are satisfied with parameters* $N \ge 0$ *and* $s > \max(\alpha + \beta, \alpha)$. *Assume also that*

(i) $\min(\alpha + \beta, \alpha) > N + 1/p'$ *or*
(ii) $N = 0$ *and* $\sup_{j\in\mathbb{Z}_+} \|\widetilde{\varphi}_j\|_{\widetilde{A}_{q',j}(\mathbb{T})} < \infty$.

Then, for all $f \in A_p^{\alpha,\beta}(\mathbb{T}^d)$ *and* $n \in \mathbb{N}$, *we have*

$$\|f - P_{n,T}^Q f\|_{A_{q,\mathrm{mix}}^\gamma(\mathbb{T}^d)} \le C\Omega_{\mathrm{mix}}(n)\|f\|_{A_p^{\alpha,\beta}(\mathbb{T}^d)}, \qquad (5.8)$$

where

$$\Omega_{\mathrm{mix}}(n) = \begin{cases} 2^{-\left(\alpha+\beta-\gamma-\sigma_{p,q}-((\alpha-\gamma-\sigma_{p,q})T+\beta)\frac{d-1}{d-T}\right)n} n^{(d-1)\sigma_{p,q}}, & \frac{-\beta}{\alpha-\gamma-\sigma_{p,q}} \le T < 1, \\ 2^{-(\alpha+\beta-\gamma-\sigma_{p,q})n}, & T < \frac{-\beta}{\alpha-\gamma-\sigma_{p,q}}, \end{cases}$$

where the constant C *does not depend on* f *and* n.

Remark 5.4 (i) This result generalizes Theorem 5.1 in [7], which corresponds to the case $p = q = 2$, $T = \beta = 0$, and $Q = (I_j)_{j\in\mathbb{Z}_+}$, where I_j is defined in (1.3).

(ii) Using the inequality (see, e.g., [7, Lemma 5.7])

$$\|f\|_{L_r(\mathbb{T}^d)} \lesssim \|f\|_{A_{2,\mathrm{mix}}^{\frac12-\frac1r}(\mathbb{T}^d)}, \quad 2 < r < \infty,$$

we easily obtain that under conditions of Theorem 5.3 with $q = 2$ and $\gamma = \frac{1}{2} - \frac{1}{r}$, inequality (5.8) implies the following error estimate:

$$\|f - P^Q_{n,T} f\|_{L_r(\mathbb{T}^d)} \le C\widetilde{\Omega}_{\text{mix}}(n)\|f\|_{A^{\alpha,\beta}_p(\mathbb{T}^d)}, \tag{5.9}$$

where

$$\widetilde{\Omega}_{\text{mix}}(n) = \begin{cases} 2^{-\left(\alpha+\beta-\widetilde{\sigma}_{p,r}-((\alpha-\widetilde{\sigma}_{p,r})T+\beta)\frac{d-1}{d-T}\right)n} n^{(d-1)\sigma_{p,2}}, & T \ge \frac{-\beta}{\alpha-\widetilde{\sigma}_{p,r}}, \\ 2^{-(\alpha+\beta-\widetilde{\sigma}_{p,r})n}, & T < \frac{-\beta}{\alpha-\widetilde{\sigma}_{p,r}}, \end{cases}$$

and

$$\widetilde{\sigma}_{p,r} = \left(\frac{1}{2} - \frac{1}{r}\right) + \left(\frac{1}{2} - \frac{1}{p}\right)_+.$$

Comparing inequalities (5.2) and (5.9) with $r = q'$ and $1 < q < 2$, we see that (5.9) provides better approximation order in the case $1 < q < 2 \le p \le \infty$.

Proof of Theorem 5.3 First, we consider the case $1 \le p \le q \le \infty$. Using Proposition 4.3 with

$$f_{\boldsymbol{j}} = \begin{cases} \eta^Q_{\boldsymbol{j}}(f), & \boldsymbol{j} \notin \Delta(n,T), \\ 0, & \boldsymbol{j} \in \Delta(n,T), \end{cases}$$

and taking into account that $A^{\gamma}_{q,\text{mix}}(\mathbb{T}^d) = A^{\gamma,0}_q(\mathbb{T}^d)$ and $f - P^Q_{n,T} f = \sum_{\boldsymbol{j}\in\mathbb{Z}^d_+} f_{\boldsymbol{j}}$, we obtain

$$\begin{aligned} &\|f - P^Q_{n,T} f\|_{A^{\gamma}_{q,\text{mix}}(\mathbb{T}^d)} \\ &\le \|f - P^Q_{n,T} f\|_{A^{\gamma}_{p,\text{mix}}(\mathbb{T}^d)} \lesssim \left(\sum_{\boldsymbol{j}\in\mathbb{Z}^d_+} 2^{p\gamma|\boldsymbol{j}|_1}\|f_{\boldsymbol{j}}\|^p_{A_p(\mathbb{T}^d)}\right)^{1/p} \\ &= \left(\sum_{\boldsymbol{j}\notin\Delta(n,T)} 2^{-p(\alpha-\gamma)|\boldsymbol{j}|_1-p\beta|\boldsymbol{j}|_\infty} 2^{p(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\eta^Q_{\boldsymbol{j}}(f)\|^p_{A_p(\mathbb{T}^d)}\right)^{1/p} \qquad (5.10) \\ &\lesssim \max_{\boldsymbol{j}\notin\Delta(n,T)} 2^{-p(\alpha-\gamma)|\boldsymbol{j}|_1-p\beta|\boldsymbol{j}|_\infty}\left(\sum_{\boldsymbol{j}\in\mathbb{Z}^d_+} 2^{p(\alpha|\boldsymbol{j}|_1+\beta|\boldsymbol{j}|_\infty)}\|\eta^Q_{\boldsymbol{j}}(f)\|^p_{A_p(\mathbb{T}^d)}\right)^{1/p} \\ &\lesssim \Omega_{\text{mix}}(n)\|f\|_{A^{\alpha,\beta}_p(\mathbb{T}^d)}, \end{aligned}$$

where the last inequality follows from Proposition 4.1.

Second, let $1 \le q < p \le \infty$. Similarly to the proof of (5.7), using estimates (5.10) with $p = q$, Hölder's inequality, and Lemmas 3.5 and 4.2, we have

$$\begin{aligned}
&\|f - P^Q_{n,T} f\|_{A^\gamma_{q,\mathrm{mix}}(\mathbb{T}^d)} \\
&\quad \le \left(\sum_{\boldsymbol{j} \notin \Delta(n,T)} 2^{q\gamma|\boldsymbol{j}|_1} \|\eta^Q_{\boldsymbol{j}}(f)\|^q_{A_q(\mathbb{T}^d)} \right)^{1/q} \\
&\quad \le \left(\left(\sum_{\boldsymbol{j} \notin \Delta(n,T)} 2^{-\frac{qp}{p-q}((\alpha-\gamma-\sigma_{p,q})|\boldsymbol{j}|_1 + \beta|\boldsymbol{j}|_\infty)} \right)^{1-q/p} \right. \\
&\qquad \left. \times \left(\sum_{\boldsymbol{j} \notin \Delta(n,T)} 2^{p((\alpha-\sigma_{p,q})|\boldsymbol{j}|_1 + \beta|\boldsymbol{j}|_\infty)} \|\eta^Q_{\boldsymbol{j}}(f)\|^p_{A_q(\mathbb{T}^d)} \right)^{q/p} \right)^{1/q} \\
&\quad \lesssim \Omega_{\mathrm{mix}}(n) \left(\sum_{\boldsymbol{j} \in \mathbb{Z}^d_+} 2^{p((\alpha-\sigma_{p,q})|\boldsymbol{j}|_1 + \beta|\boldsymbol{j}|_\infty)} \|\eta^Q_{\boldsymbol{j}}(f)\|^p_{A_q(\mathbb{T}^d)} \right)^{1/p} \\
&\quad \lesssim \Omega_{\mathrm{mix}}(n) \left(\sum_{\boldsymbol{j} \in \mathbb{Z}^d_+} 2^{p(\alpha|\boldsymbol{j}|_1 + \beta|\boldsymbol{j}|_\infty)} \|\delta_{\boldsymbol{j}}(f)\|^p_{A_p(\mathbb{T}^d)} \right)^{1/p} \\
&\quad \lesssim \Omega_{\mathrm{mix}}(n) \|f\|_{A^{\alpha,\beta}_p(\mathbb{T}^d)},
\end{aligned}$$

which proves (5.8) for $1 \le q < p \le \infty$ by Lemma 2.2. □

It is not difficult to see that Theorems 5.1 and 5.3 can also be established for more general operators

$$P^Q_\Gamma = \sum_{\boldsymbol{j} \in \Gamma} \eta^Q_{\boldsymbol{j}},$$

where Γ is some arbitrary set of indices in $\mathbb{Z}^d_+$. More precisely, we obtain the following remarks.

Remark 5.5 Suppose that conditions of Theorem 5.1 hold with Γ instead of $\Delta(n, T)$. Then, for all $f \in A^{\alpha,\beta}_p(\mathbb{T}^d)$, we have

$$\|f - P^Q_\Gamma f\|_{A^\gamma_q(\mathbb{T}^d)} \le C \left(\sum_{\boldsymbol{j} \notin \Gamma} 2^{-p'((\alpha-\sigma_{p,q})|\boldsymbol{j}|_1 + (\beta-\gamma)|\boldsymbol{j}|_\infty)} \right)^{1/p'} \|f\|_{A^{\alpha,\beta}_p(\mathbb{T}^d)},$$

where the constant C does not depend on f and Γ.

Remark 5.6 Suppose that conditions of Theorem 5.3 hold with Γ instead of $\Delta(n, T)$. Then, for all $f \in A^{\alpha,\beta}_p(\mathbb{T}^d)$, we have

$$\|f - P^Q_\Gamma f\|_{A^\gamma_{q,\mathrm{mix}}(\mathbb{T}^d)} \le C\Omega_\Gamma \|f\|_{A^{\alpha,\beta}_p(\mathbb{T}^d)},$$

where

$$\Omega_\Gamma = \begin{cases} \max_{\boldsymbol{j}\notin\Gamma} 2^{-p(\alpha-\gamma)|\boldsymbol{j}|_1 - p\beta|\boldsymbol{j}|_\infty}, & 1 \le p \le q \le \infty, \\ \left(\sum_{\boldsymbol{j}\notin\Gamma} 2^{-\frac{qp}{p-q}((\alpha-\gamma-\sigma_{p,q})|\boldsymbol{j}|_1 + \beta|\boldsymbol{j}|_\infty)}\right)^{\frac{1}{q}-\frac{1}{p}}, & 1 \le q < p \le \infty, \end{cases}$$

and the constant C does not depend on f and Γ.

5.3 Sharpness

Here, we show that inequalities (5.1) and (5.8) given in Theorems 5.1 and 5.3 are sharp for specific cases of parameters. Those cases are of special interest since they provide the best order of approximation and simultaneously are optimal with respect to the computation time (cf. (1.2)).

Theorem 5.7 (i) *Under conditions of Theorem 5.1, if* $0 < T < \frac{\gamma-\beta}{\alpha}$, *we have*

$$\sup_{f\in UA_p^{\alpha,\beta}(\mathbb{T}^d)} \|f - P_{n,T}^Q f\|_{A_p^\gamma(\mathbb{T}^d)} \asymp 2^{-(\alpha+\beta-\gamma)n} \tag{5.11}$$

for sufficiently large n.

(ii) *Under conditions of Theorem 5.3, if* $0 < T < \frac{-\beta}{\alpha-\gamma}$, *we have*

$$\sup_{f\in UA_p^{\alpha,\beta}(\mathbb{T}^d)} \|f - P_{n,T}^Q f\|_{A_{p,\mathrm{mix}}^\gamma(\mathbb{T}^d)} \asymp 2^{-(\alpha+\beta-\gamma)n}$$

for sufficiently large n.

Proof In view of Theorems 5.1 and 5.3, it is enough to consider only estimates from below. We prove only (i). The assertion (ii) can be treated similarly. We follow the idea of the proof of [7, Theorem 6.7] (see also [18]) taking into account the following lemma on lower estimates for linear widths (see, e.g., [Theorem 1][50, 51]).

Lemma 5.8 *Let* L_{m+1} *be* $(m+1)$*-dimensional subspace in a Banach space* X*, and let* $B_{m+1}(r){:=}\{f \in L_{m+1} : \|f\|_X \le r\}$. *Then*

$$\lambda_m(B_{m+1}(r), X){:=}\inf_{\mathcal{A}_m} \sup_{f\in B_{m+1}(r)} \|f - \mathcal{A}_m f\|_X \ge r,$$

where infimum is taken over all continuous linear operators A_m *in* X *with rank at most* m.

We use this lemma with $X = A_p^\gamma(\mathbb{T}^d)$ and $L_{2^n+1} = \mathrm{span}\,\{e^{\mathrm{i}kx_1} : k = 0, \dots, 2^n\}$. Let also $n_0 \in \mathbb{N}$ be such that rank $P_{n-n_0,T}^Q \le 2^n$ (we can always find such n_0 in view of (1.2)). For any $f \in L_{2^n+1}$, we have

$$\begin{aligned}\|f\|_{A_p^{\alpha,\beta}(\mathbb{T}^d)} &= \left(\sum_{k_1=0}^{n} 2^{p(\alpha+\beta)k_1}\|\delta_{k_1,0,\dots,0}(f)\|_{A_p(\mathbb{T}^d)}^p\right)^{1/p}\\ &\le \max_{k_1\in[0,n]} 2^{(\alpha+\beta-\gamma)k_1}\left(\sum_{k_1=0}^{n} 2^{p\gamma k_1}\|\delta_{k_1,0,\dots,0}(f)\|_{A_p(\mathbb{T}^d)}^p\right)^{1/p}\\ &\le 2^{(\alpha+\beta-\gamma)n}\|f\|_{A_p^{\gamma}(\mathbb{T}^d)}.\end{aligned}$$

Thus, by choosing $r = 2^{-(\alpha+\beta-\gamma)n}$, we get that $B_{2^n+1}(r) \subset UA_p^{\alpha,\beta}(\mathbb{T}^d)$. Using this embedding and Lemma 5.8, we obtain

$$\sup_{f\in UA_p^{\alpha,\beta}(\mathbb{T}^d)} \|f - P_{n-n_0,T}^{Q} f\|_{A_p^{\gamma}(\mathbb{T}^d)} \ge \lambda_{2^n}(B_{2^n+1}(r), A_p^{\gamma}(\mathbb{T}^d)) \ge 2^{-(\alpha+\beta-\gamma)n},$$

which implies (5.11). □

Remark 5.9 Note that the sharpness of Theorem 5.1 under certain natural restrictions on distributions $\widetilde{\varphi}_j$ in the case $T = 0$, $\beta = \gamma = 0$, and $p = 2 \le q \le \infty$ follows from the proof of [32, Theorem 4]. For particular cases of the parameters (mainly for the cases $T = 0$, $p, q \in \{1, 2\}$, $p \le q$, $\gamma \in \{0, 1\}$, $\beta = 0$), the sharpness of Theorem 5.1 can be also established using general estimates of linear widths (see, e.g., [7, 42]).

6 Effective Error Estimates

6.1 Energy-Norm Based Sparse Grids

Along with the general operators $P_{n,T}^{Q}$, in [7] and [15] the authors studied quasi-interpolation operators

$$P_{\Delta(\xi)}^{Q} = \sum_{\boldsymbol{j}\in\Delta(\xi)} \eta_{\boldsymbol{j}}^{Q}, \quad \xi > 0,$$

with specific choice of the family Q, where

$$\Delta(\xi) = \{\boldsymbol{k} \in \mathbb{Z}_+^d \,:\, (\alpha - \sigma_{p,q} - \varepsilon)|\boldsymbol{k}|_1 - (\gamma - \beta - \varepsilon)|\boldsymbol{k}|_\infty \le \xi\}.$$

For the reader's convenience, we reformulate Theorem 5.1 for $P_{\Delta(\xi)}^{Q}$ noting that $\Delta(\xi)$ corresponds to the set $\Delta(n, T)$ with $T = \frac{\gamma-\beta-\varepsilon}{\alpha-\sigma_{p,q}-\varepsilon}$ and $n = \frac{\xi}{\alpha-\sigma_{p,q}-\gamma+\beta}$.

Corollary 6.1 *Under conditions of Theorem 5.1, if* $0 < \varepsilon < \gamma - \beta < \alpha - \sigma_{p,q}$*, then, for all* $f \in A_p^{\alpha,\beta}(\mathbb{T}^d)$ *and* $\xi \in \mathbb{N}$*, we have*

$$\|f - P_{\Delta(\xi)}^Q f\|_{A_q^\gamma(\mathbb{T}^d)} \le C2^{-\xi} \|f\|_{A_p^{\alpha,\beta}(\mathbb{T}^d)},$$

where the constant C *does not depend on* f *and* ξ*. In particular, in the case* $p = q = 2$ *and* $\gamma = 0$*, for all* $f \in H^{\alpha,\beta}(\mathbb{T}^d)$ *and* $\xi \in \mathbb{N}$*, we have*

$$\|f - P_{\Delta(\xi)}^Q f\|_{L_2(\mathbb{T}^d)} \le C2^{-\xi} \|f\|_{H^{\alpha,\beta}(\mathbb{T}^d)}.$$

Proof The proof directly follows from Theorem 5.1 by taking $T = \frac{\gamma-\beta-\varepsilon}{\alpha-\sigma_{p,q}-\varepsilon}$. □

Remark 6.2 (i) It follows from the proof of Theorem 5.1 that in the case $p = 1$, the assertion of Corollary 6.1 remains true for $\varepsilon = 0$.

(ii) Corollary 6.1 extends Theorem 4.1 in [7], cf. [7, Remark 4.4], which corresponds to the case $p = q = 2$ and $Q = (I_j)_{j\in\mathbb{Z}_+}$, where I_j is defined in (2.5).

6.2 Smolyak Grids

In some special cases of parameters in Theorems 5.1 and 5.3, the Smolyak algorithm, i.e., the operators $P_{n,T}^Q$ with $T = 0$, provides more effective error estimates with respect to the number of frequencies than the operators $P_{n,T}^Q$, $0 < T < 1$, which correspond to the energy norm based grids. In particular, applying Theorem 5.1 with $T = 0$ and $\beta = \gamma$, we obtain the following corollary about approximation in the space $A_q^\beta(\mathbb{T}^d)$.

Corollary 6.3 *Under conditions of Theorem 5.1, for all* $f \in A_p^{\alpha,\beta}(\mathbb{T}^d)$ *and* $n \in \mathbb{N}$*, we have*

$$\|f - P_{n,0}^Q f\|_{A_q^\beta(\mathbb{T}^d)} \le C2^{-(\alpha-\sigma_{p,q})n} n^{(d-1)(1-\frac{1}{p})} \|f\|_{A_p^{\alpha,\beta}(\mathbb{T}^d)}, \tag{6.1}$$

where the constant C *does not depend on* f *and* n*. In particular, if* $p = q = 2$*, we have*

$$\|f - P_{n,0}^Q f\|_{H^\beta(\mathbb{T}^d)} \le C2^{-\alpha n} n^{\frac{(d-1)}{2}} \|f\|_{H^{\alpha,\beta}(\mathbb{T}^d)}.$$

Remark 6.4 (i) By the same arguments as in Remark 5.2(ii), we have that inequality (6.1) with $1 \le q \le 2$ and $\beta = 0$ implies that

$$\|f - P_{n,0}^Q f\|_{L_{q'}(\mathbb{T}^d)} \le C2^{-(\alpha-\sigma_{p,q})n} n^{(d-1)(1-\frac{1}{p})} \|f\|_{A_{p,\mathrm{mix}}^\alpha(\mathbb{T}^d)}.$$

In particular, if $q = 1$ and $Q = (I_j)_{j\in\mathbb{Z}_+}$, the above inequality generalizes [7, Theorem 5.6] (the case $p = 2$) and the main results of [27] (the case $p = \infty$, which corresponds to the Korobov space).

(ii) It follows from (6.1) and [42, Corollary 4.3 and (1.1)] that in the cases $p = q = 1$ and $p = 1, q = 2$ the Smolyak algorithm $P^{Q}_{n,0}$ provides optimal in order approximation among all continuous linear operator of finite rank.

In a similar way, applying Theorem 5.3 with $T = \beta = 0$, we get the following result concerning approximation by $P^{Q}_{n,0} f$ in the space $A^{\gamma}_{q,\mathrm{mix}}(\mathbb{T}^d)$.

Corollary 6.5 *Under the conditions of Theorem 5.3, for all $f \in A^{\alpha}_{p,\mathrm{mix}}(\mathbb{T}^d)$ and $n \in \mathbb{N}$, we have*

$$\|f - P^{Q}_{n,0} f\|_{A^{\gamma}_{q,\mathrm{mix}}(\mathbb{T}^d)} \leq C 2^{-(\alpha-\gamma-\sigma_{p,q})n} n^{(d-1)\sigma_{p,q}} \|f\|_{A^{\alpha}_{p,\mathrm{mix}}(\mathbb{T}^d)},$$

where the constant C does not depend on f and n. In particular, if $p = q = 2$, we have

$$\|f - P^{Q}_{n,0} f\|_{H^{\gamma}_{\mathrm{mix}}(\mathbb{T}^d)} \leq C 2^{-(\alpha-\gamma)n} \|f\|_{H^{\alpha}_{\mathrm{mix}}(\mathbb{T}^d)}.$$

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