Sampling at Twice the Nyquist Rate in Two Frequency Bins Guarantees Uniqueness in Gabor Phase Retrieval

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Abstract

We show that bandlimited signals can be uniquely recovered (up to a constant global phase factor) from Gabor transform magnitudes sampled at twice the Nyquist rate in two frequency bins.

1 Introduction

We consider the recovery of square-integrable signals $f \in L^2(\mathbb{R})$ from the magnitude of their *Gabor transforms*

$$\mathcal{G}f(x, \omega) := \sqrt[4]{2} \int_{\mathbb{R}} f(t) e^{-\pi(t-x)^2} e^{-2\pi i t \omega}, \quad (x, \omega) \in \mathbb{R}^2, \tag{1}$$

also called the *Gabor phase retrieval problem*. Specifically, we are interested in questions related to the reconstruction of f from $|\mathcal{G}f|$ measured on lattices[1] $\Lambda \subset \mathbb{R}^2$, which we refer to as *sampled Gabor phase retrieval*.

The first uniqueness result for sampled Gabor phase retrieval was presented in [1] where it is shown that real-valued, bandlimited, square-integrable signals f with

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[1] A *lattice* $\Lambda \subset \mathbb{R}^2$ is a discrete subset of the time–frequency plane that can be written as $\Lambda = L\mathbb{R}^k$, where $L \in \mathbb{R}^{2 \times k}$ is a matrix with linearly independent columns and $k \in \{1, 2\}$.

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bandwidth $B > 0$ can be recovered (up to a constant global sign factor) from $|\mathcal{G}f|$ sampled on $\frac{1}{4B}\mathbb{Z} \times \{0\}$. Just 2 months later, the paper [2] appeared in which it is revealed that the real-valuedness assumption from [1] can be dropped: specifically, the bandlimited square-integrable signals f whose Fourier transform is in L^4 can be recovered (up to a constant global phase factor) from $|\mathcal{G}f|$ sampled on $\frac{1}{4B}\mathbb{Z} \times \mathbb{Z}$. Another 2 months later, the paper [3] appeared, showing that the bandlimitedness assumption cannot be dropped. This is achieved by constructing two square-integrable functions that do not agree (up to a constant global phase factor) but whose Gabor transform magnitudes agree on Λ, where $\Lambda \subset \mathbb{R}^2$ is a general lattice in the time–frequency plane.

Finally, we mention the recent preprint [4] in which results from [2] are extended to show that all bandlimited square-integrable signals f can be recovered (up to a constant global phase factor) from $|\mathcal{G}f|$ sampled on $\frac{1}{4B}\mathbb{Z} \times \mathbb{N}$. In this paper, we further strengthen this result and prove that in fact bandlimited square-integrable signals f, with bandwidth $B > 0$, can be recovered (up to a constant global phase factor) from $|\mathcal{G}f|$ sampled on $\frac{1}{4B}\mathbb{Z} \times \{\omega_0, \omega_1\}$, where $\omega_0 \neq \omega_1$. As $\frac{1}{4B}$ is exactly twice the Nyquist rate, we therefore show that sampling at twice the Nyquist rate in two frequency bins guarantees uniqueness in Gabor phase retrieval as advertised in the title. We point out that a similar result was already known for the Cauchy wavelet transform [5]: more precisely, sampling at twice the Nyquist rate at two scales guarantees uniqueness in Cauchy wavelet phase retrieval.

Remark 1 The original motivation for this paper stems from a resemblance of the result in [1] on Gabor sign retrieval with the work in [6] on finite-dimensional sign retrieval: in the prior, it is shown that sampling at twice the Nyquist rate in a single frequency bin guarantees uniqueness in Gabor sign retrieval while, in the latter, it is shown that $2n - 1$ generic measurement vectors are necessary and sufficient for uniqueness in finite-dimensional sign retrieval. As it is also known that on the order of $4n$ generic measurement vectors are necessary and sufficient for uniqueness in finite-dimensional phase retrieval [6, 7], it seems natural to ask whether sampling at four times the Nyquist rate in one frequency bin or at twice the Nyquist rate in two frequency bins would guarantee uniqueness in Gabor phase retrieval. The former is clearly untrue as can be seen from considering $f, g \in \mathrm{PW}_B^2$ real-valued and

$$|\mathcal{G}(f + \mathrm{i}g)| = |\mathcal{G}(f - \mathrm{i}g)| \text{ on } \mathbb{R} \times \{0\}.$$

The latter is shown to be true in this paper.

1.1 Notation

We denote the normalised Gaussian by $\phi(t) = \sqrt[4]{2}\exp(-\pi t^2)$, where $t \in \mathbb{R}$. For $-\infty < a < b < \infty$ and $f \in L^p(\mathbb{R})$, with $p \in [1, \infty)$, we write $\mathrm{supp}\, f \subset [a, b]$ if $f(t) = 0$, for a.e. $t \notin [a, b]$. Moreover, we define the families of *translation* and *modulation* operators $(\mathrm{T}_x)_{x \in \mathbb{R}} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ and $(\mathrm{M}_\omega)_{\omega \in \mathbb{R}} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$

by

$$\mathrm{T}_x f(t) := f(t-x), \quad \mathrm{M}_\omega f(t) := f(t)\mathrm{e}^{2\pi \mathrm{i}t\omega}.$$

We furthermore use the convention

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi \mathrm{i}t\xi}\,\mathrm{d}t, \quad \xi \in \mathbb{R},$$

for the Fourier transform of $f \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$ and note that the Fourier transform of the normalised Gaussian is the normalised Gaussian itself, i.e. $\mathcal{F}\phi = \phi$. Additionally, as is usual in the phase retrieval literature, we will introduce an equivalence relation on the set of complex-valued functions via

$$f \sim g : \iff \exists \alpha \in \mathbb{R} : f = \mathrm{e}^{\mathrm{i}}g$$

and say that f and g *agree up to global phase* if $f \sim g$. The space of polynomials with complex argument and complex coefficients is denoted by $\mathbb{C}[z]$. Similarly, the subspace of degree $n \in \mathbb{N}$ polynomials in $\mathbb{C}[z]$ is denoted by $\mathbb{C}_n[z]$. Finally, for an entire function $F : \mathbb{C} \to \mathbb{C}$, we denote its zero set by $Z(F) \subset \mathbb{C}$ and define a function $m_F : \mathbb{C} \to \mathbb{N}_0$ which assigns the multiplicity of z as a zero of F to every $z \in \mathbb{C}$. Note that we use the convention $m_F(z) = 0$, for $z \notin Z(F)$.

1.2 Definitions and Basic Notions

We will work with the *Paley–Wiener spaces* of bandlimited functions

$$\mathrm{PW}_B^p := \{f \in L^p(\mathbb{R}); \ \mathrm{supp}(\mathcal{F}f) \subset [-B, B]\}, \quad p \in [1, \infty],$$

where $B > 0$. With this definition, it is well known that $\mathrm{PW}_B^1 \subset \mathrm{PW}_B^2$. It turns out to be useful to consider the *Bargmann transform* of square-integrable signals $f \in L^2(\mathbb{R})$ given by

$$\mathcal{B}f(z) := \sqrt[4]{2} \int_{\mathbb{R}} f(t)\mathrm{e}^{2\pi tz - \pi t^2 - \frac{\pi}{2}z^2}\,\mathrm{d}t, \quad z \in \mathbb{C}.$$

One of the two main reasons for this is that the Bargmann transform and the Gabor transform [as defined in Eq. (1)] are related via the formula [8, Proposition 3.4.1 on p. 54]

$$\mathcal{G}f(x, -\omega) = \mathrm{e}^{\pi \mathrm{i}x\omega}\mathcal{B}f(x + \mathrm{i}\omega)\mathrm{e}^{-\frac{\pi}{2}(x^2 + \omega^2)}, \quad (x, \omega) \in \mathbb{R}^2.$$

The other main reason is that the Bargmann transform of a square-integrable signal is an entire function of finite order. More precisely, we define the *order* of an entire

function $F : \mathbb{C} \to \mathbb{C}$ to be

$$\rho := \limsup_{r \to \infty} \frac{\log(\log(\sup_{|z|=r} |F(z)|))}{\log r}$$

and say that F is of *finite order* if $\rho < \infty$. For $F = \mathcal{B}f$, with $f \in L^2(\mathbb{R})$, one can use [8, Theorem 3.4.2 on p. 54] to show that $\rho \leq 2$: we say that $\mathcal{B}f$ is of *second-order*.

2 Preliminaries

Our goal is to recover a bandlimited function f with bandwidth B from the magnitudes of its Gabor transform $|\mathcal{G}f|$ sampled on the set $\frac{1}{4B}\mathbb{Z} \times \{\omega_0, \omega_1\}$, where $\omega_0 \neq \omega_1$. In order to do this, we follow a three-step procedure.

(1) We note that $|\mathcal{G}f(\cdot, \omega)|^2$ is bandlimited, for all $\omega \in \mathbb{R}$, and use the Nyquist–Shannon sampling theorem to recover $|\mathcal{G}f|^2$ on $\mathbb{R} \times \{\omega_0, \omega_1\}$.
(2) Relating this to the Bargmann transform of f shows that it suffices to analyse the recovery of a second-order entire function from magnitude measurements on two parallel lines. In this direction, we show that a second-order entire function is either uniquely determined (up to global phase) by its magnitude on two parallel lines or it has infinitely many evenly spaced zeroes.
(3) Finally, we make use of the bandlimitedness of f again and show that the Bargmann transform of f can only have infinitely many evenly spaced zeroes if $f = 0$.

Let us start with the realisation of item 1. We note that the following lemma already follows from the considerations in [4]. A considerably simpler proof based on a different convention for the Paley–Wiener spaces is given here.

Lemma 1 *Let $B > 0$, $\omega \in \mathbb{R}$ and $f \in \mathrm{PW}_B^2$. Then, $x \mapsto |\mathcal{G}f(x, \omega)|^2 \in \mathrm{PW}_{2B}^2$.*

Proof We have $x \mapsto e^{2\pi i x \omega} \mathcal{G}f(x, \omega) \in \mathrm{PW}_B^2$ since it is the (inverse) Fourier transform of $\mathcal{F}f \cdot \mathrm{T}_\omega \phi \in L^2(\mathbb{R})$ which satisfies $\mathrm{supp}(\mathcal{F}f \cdot \mathrm{T}_\omega \phi) \subset [-B, B]$ [8, Eq. (3.5) on p. 39]. Therefore, $x \mapsto |\mathcal{G}f(x, \omega)|^2 \in L^1(\mathbb{R})$ and applying the Fourier convolution theorem to

$$x \mapsto |\mathcal{G}f(x, \omega)|^2 = \mathcal{G}f(x, \omega)\overline{\mathcal{G}f(x, \omega)}$$

shows that $x \mapsto |\mathcal{G}f(x, \omega)|^2 \in \mathrm{PW}_{2B}^1 \subset \mathrm{PW}_{2B}^2$. □

Next, we move on to item 2 of our three-step procedure: the analysis of the recovery of a second-order entire function from magnitude measurements on two parallel lines. Interestingly, it can be shown that it is impossible to reconstruct a finite order entire function (up to global phase) from magnitude information on two parallel lines [9]. It is therefore also impossible to recover a square-integrable signal from Gabor phase retrieval measurements on two parallel lines; a fact which has been used to construct the counterexamples to Gabor phase retrieval in [3]. We are not considering general

square-integrable signals here however and the bandlimitedness assumption turns out
to be sufficient to exclude counterexamples.

Let us start by realising that fixing the magnitude of an entire function on two
parallel lines enforces a periodicity in its zeroes. More precisely, we show that, if two
entire functions F and G have magnitudes that agree on $\mathbb{R} \cup (\mathbb{R} + i\tau)$, then $m_F - m_G$
is $(2i\tau)$-periodic.

Lemma 2 *Let $\tau > 0$ and let $F, G \in \mathbb{C} \to \mathbb{C}$ be two entire functions such that
$|F| = |G|$ on $\mathbb{R} \cup (\mathbb{R} + i\tau)$. Then,*

$$m_F(z + 2i\tau) - m_G(z + 2i\tau) = m_F(z) - m_G(z), \quad z \in \mathbb{C}.$$

Proof Let $z \in \mathbb{C}$ denote an arbitrary complex number. According to [10, Proposition
1 on p. 261], $|F| = |G|$ on $\mathbb{R}$ implies that

$$F(z)\overline{F(\overline{z})} = G(z)\overline{G(\overline{z})}.$$

Therefore, after looking at the zeroes of the above equation exclusively, we have

$$m_F(z) + m_F(\overline{z}) = m_G(z) + m_G(\overline{z}).$$

The same argument applied to $F_\tau(z) := F(z + i\tau)$ and $G_\tau(z) := G(z + i\tau)$ yields

$$m_F(z + i\tau) + m_F(\overline{z} + i\tau) = m_G(z + i\tau) + m_G(\overline{z} + i\tau)$$

such that we can conclude that

$$m_F(z + 2i\tau) - m_G(z + 2i\tau) = m_G(\overline{z}) - m_F(\overline{z}) = m_F(z) - m_G(z).$$

$\square$

The periodicity in $m_F - m_G$ directly implies that the zeroes (with multiplicities)
of F and G agree everywhere if they agree on the strip $\mathbb{R} + i(-\tau, \tau]$. Combining
this insight with the Hadamard factorisation theorem yields that a second-order entire
function is either uniquely determined (up to global phase) by its magnitude on the
two parallel lines $\mathbb{R} \cup (\mathbb{R} + i\tau)$ or that it has at least one zero in the strip $\mathbb{R} + i(-\tau, \tau]$.

Corollary 3 *Let $\tau > 0$ and let $F, G \in \mathbb{C} \to \mathbb{C}$ be two entire functions of second-order
such that $|F| = |G|$ on $\mathbb{R} \cup (\mathbb{R} + i\tau)$. If $m_F - m_G = 0$ on $\mathbb{R} + i(-\tau, \tau]$, then $F \sim G$.*

Proof If $m_F - m_G = 0$ on $\mathbb{R} + i(-\tau, \tau]$, then Lemma 2 implies $m_F - m_G = 0$
such that the zeroes (with multiplicity) of F and G agree. It therefore follows from
Hadamard's factorisation theorem (cf. [11, Sect. 8.24 on p. 250]) that

$$F(z) = e^{Q(z)} G(z), \quad z \in \mathbb{C},$$

where $Q \in \mathbb{C}_2[z]$ is a quadratic polynomial. As $|F| = |G|$ on $\mathbb{R}$, we furthermore have

$$|F(x)| = e^{\Re Q(x)} |G(x)| = e^{\Re Q(x)} |F(x)|, \quad x \in \mathbb{R},$$

which implies that $\exp(\Re Q) = 1$ holds on $\mathbb{R} \setminus Z(F)$. As F is entire, $Z(F)$ has no accumulation point in $\mathbb{R}$ and is thus of measure zero. It follows that $\Re Q = 0$ almost everywhere (and thus everywhere) on $\mathbb{R}$. We can now write

$$Q(z) = i \left(\alpha + \lambda_1 z + \lambda_2 z^2 \right), \quad z \in \mathbb{C},$$

for some $\alpha, \lambda_1, \lambda_2 \in \mathbb{R}$. The same argument as above shows that $|F| = |G|$ on $\mathbb{R} + i\tau$ implies $\Re Q = 0$ on $\mathbb{R} + i\tau$. Therefore, we find

$$\Re Q(x + i\tau) = \lambda_1 \tau + 2\lambda_2 \tau x = 0, \quad x \in \mathbb{R},$$

which proves that $\lambda_1 = \lambda_2 = 0$ and thus $F = e^i G$. $\square$

Remark 2 Corollary 3 actually holds for general entire functions of finite order $F, G \in \mathbb{C} \to \mathbb{C}$. The proof remains mostly the same: only the polynomial $Q \in \mathbb{C}[z]$ is of arbitrary order instead of quadratic.

We can finally turn to item 3 of our three step procedure: using bandlimitedness to show that $f = 0$ if $\mathcal{B}f$ has infinitely many evenly spaced zeroes. The inspiration for this step actually comes from [2, 4], where a Müntz–Szász type result from [12] is used in order to recover bandlimited f from $|\mathcal{G}f|$ on $\mathbb{R} \times \mathbb{N}$. We rely on a slight generalisation of that same Müntz–Szász type result for this paper.

Theorem 4 (Zalik's theorem; cf. Theorem 4 in [12]) *Let $p \in [1, \infty)$, let $a, b \in \mathbb{R}$ be such that $a < b$, let $r > 0$ and let $(c_n)_{n \in \mathbb{N}} \in \mathbb{C}$ be a sequence of distinct complex numbers such that there exists a $\delta > 0$ and an $N_0 \in \mathbb{N}$ with*

$$\left| \Re \left[c_n - \tfrac{1}{2} \right] \right| \geq \delta \left| c_n - \tfrac{1}{2} \right|, \quad n \geq N_0.$$

Then,

$$\left\{ t \mapsto e^{-r^2(t - c_n)^2} \, ; \, n \in \mathbb{N} \right\}$$

is complete in $L^p([a, b])$ if and only if

$$\sum_{n \in \mathbb{N}, \, c_n \neq 0} |c_n|^{-1}$$

diverges.

Proof The theorem follows from the original proof in [12] with some small modifications. $\square$

Remark 3 Zalik's original result is stated for $p = 2$ and *real numbers* $(c_n)_{n \in \mathbb{N}}$. We do not make use of the added generality in the integrability parameter p here but the proof of our main result does require the sequence $(c_n)_{n \in \mathbb{N}}$ to be complex-valued.

3 The Main Result

We are now in a position to state and prove our main result: general bandlimited signals can be recovered from their Gabor transform magnitudes sampled at twice the Nyquist rate in two frequency bins.

Theorem 5 (Main result) *Let $B > 0$, let $\omega_0, \omega_1 \in \mathbb{R}$ be such that $\omega_0 < \omega_1$ and let $f, g \in \mathrm{PW}_B^2$. Then, $f \sim g$ if (and only if) $|\mathcal{G}f| = |\mathcal{G}g|$ on $\frac{1}{4B}\mathbb{Z} \times \{\omega_0, \omega_1\}$.*

Proof Let us suppose that $|\mathcal{G}f| = |\mathcal{G}g|$ on $\frac{1}{4B}\mathbb{Z} \times \{\omega_0, \omega_1\}$ and fix $j \in \{0, 1\}$. According to Lemma 1, the functions

$$x \mapsto |\mathcal{G}f(x, \omega_j)|^2, \quad x \mapsto |\mathcal{G}g(x, \omega_j)|^2$$

are in PW_{2B}^2. By the Nyquist–Shannon sampling theorem, we therefore find that

$$\left|\mathcal{G}f(x, \omega_j)\right|^2 = \left|\mathcal{G}g(x, \omega_j)\right|^2, \quad x \in \mathbb{R}. \tag{2}$$

Next, we define the second-order entire functions $F(z) := \mathcal{B}f(z + i\omega_0)$ and $G(z) := \mathcal{B}g(z + i\omega_0)$, for $z \in \mathbb{C}$, which satisfy

$$|F(x)| = |G(x)|, \quad |F(x + i\tau)| = |G(x + i\tau)|, \quad x \in \mathbb{R},$$

for $\tau := \omega_1 - \omega_0 > 0$, according to Eq. (2). In the rest of this proof, we will distinguish between two cases: $m_F - m_G = 0$ on $\mathbb{R} \times i(-\tau, \tau]$ and $m_F - m_G \neq 0$ on $\mathbb{R} \times i(-\tau, \tau]$. In the first case, i.e. when $m_F - m_G = 0$ on $\mathbb{R} \times i(-\tau, \tau]$, Corollary 3 implies that $F \sim G$. Since the Bargmann transform is injective[2], it follows that $f \sim g$.

In the second case, we may without loss of generality assume that there exists a complex number $z_0 \in \mathbb{R} \times (-\tau, \tau]$ such that $m_F(z_0) - m_G(z_0) > 0$: indeed, there exists a complex number at which $m_F - m_G$ is non-zero and we can exchange F and G if we have $m_F(z_0) - m_G(z_0) < 0$. By Lemma 2, we therefore find that

$$m_F(z_0 + 2ik\tau) - m_G(z_0 + 2ik\tau) = m_F(z_0) - m_G(z_0) > 0,$$

for $k \in \mathbb{Z}$. It follows that $(z_0 + 2ik\tau)_{k \in \mathbb{Z}} \in \mathbb{C}$ forms a sequence of zeroes of F, i.e.

$$\mathcal{B}f(z_0 + i(\omega_0 + 2k\tau)) = 0, \quad k \in \mathbb{Z}.$$

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[2] Actually, the range of the Bargmann transform can be equipped with an inner product and thereby turned into a Hilbert space which known as the Fock space $\mathcal{F}^2(\mathbb{C})$. The Bargmann transform then turns out to be a unitary operator mapping $L^2(\mathbb{R})$ onto $\mathcal{F}^2(\mathbb{C})$ [8, Theorem 3.4.3 on p. 56].

The Bargmann transform satisfies the nice symmetry $\mathcal{B} f(-\mathrm{i}z) = \mathcal{B}\mathcal{F} f(z)$, for $z \in \mathbb{C}$ (c.f. [13, Eq. (3.10a) on p. 207]). Therefore,

$$0 = \mathcal{B} f(z_0 + \mathrm{i}(\omega_0 + 2k\tau)) = \mathcal{B}\mathcal{F} f(\mathrm{i}z_0 - \omega_0 - 2k\tau)$$

$$= \mathrm{e}^{\frac{\pi}{2}(\mathrm{i}z_0 - \omega_0 - 2k\tau)^2} \int_{-B}^{B} \mathcal{F} f(\xi) \mathrm{e}^{-\pi(\xi - \mathrm{i}z_0 + \omega_0 + 2k\tau)^2} \, \mathrm{d}\xi$$

which implies that $\mathcal{F} f$ is orthogonal to $\xi \mapsto \mathrm{e}^{-\pi(\xi - \mathrm{i}z_0 + \omega_0 + 2k\tau)^2}$ in $L^2([-B, B])$, for all $k \in \mathbb{Z}$. Theorem 4 now implies that $\mathcal{F} f = 0$ and thus $f = 0$. Hence, $F = 0$ and as $|F| = |G|$ the identity theorem of complex analysis can be used to show that $G = 0$ such that $g = 0 = f$. $\qquad\square$

Our main result may alternatively be stated without reference to the Gabor transform. Indeed, it is equivalent to the following theorem.

Theorem 6 *Let $B > 0$ and let $\omega_0, \omega_1 \in \mathbb{R}$ be such that $\omega_0 \neq \omega_1$. If $f, g \in \mathrm{PW}_B^2$ satisfy*

$$\left| \phi * (\mathrm{M}_{-\omega_j} f) \right| = \left| \phi * (\mathrm{M}_{-\omega_j} g) \right|, \quad j = 0, 1,$$

then $f \sim g$.

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