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Uniform L^p Boundedness for Oscillatory Singular Integrals with C^∞ Phases

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Abstract

We establish the uniform boundedness of oscillatory singular integral operators on L^p spaces for C^∞ phases and Hölder class singular kernels. Our main result improves and unifies several existing L^p results for oscillatory singular integrals.

Keywords Oscillatory integrals · Singular integrals · Calderón–Zygmund kernels · L^p spaces · Hölder class

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1 Introduction

Both oscillatory and singular integrals have played very important roles in the history of harmonic analysis. Oscillatory singular integrals, as a hybrid between the two, have attracted a considerable amount of interest in the past few decades. In this paper we shall focus our attention on the L^p theory for oscillatory singular integral operators. The kernel of such an operator is given by the product of an oscillatory factor $e^{i\Phi(x,y)}$ and a Calderón-Zygmund type kernel function $K(x, y)$. More precisely, we define $T_{\Phi,K}$ by

$$T_{\Phi,K} f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i\Phi(x,y)} K(x, y) f(y) dy. \tag{1}$$

The phase function Φ is assumed to be real-valued. In [11], for any Calderón-Zygmund kernel $K(x, y)$ which is smooth away from $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$, Phong and Stein

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established the uniform L^p boundedness for all $T_{\Phi,K}$ with Φ being in the family of bilinear forms. Subsequently in [12], for any Calderón–Zygmund kernel $K(x, y)$ which is C^1 on $\mathbb{R}^n \times \mathbb{R}^n \backslash \Delta$, Ricci and Stein proved the L^p boundedness of $T_{\Phi,K}$ for all polynomial phase functions $\Phi(x, y) = P(x, y)$, with the bound on $\|T_{P,K}\|_{p,p}$ being uniform as long as a cap is placed on $\deg(P)$. Their result can be stated as follows.

Theorem 1.1 ([12]) *Let $n \in \mathbb{N}$ and $P(x, y)$ be a real-valued polynomial in $x, y \in \mathbb{R}^n$. Suppose that there is an $A > 0$ such that $K(x, y)$ satisfies*

$$|K(x, y)| \leq \frac{A}{|x-y|^n}; \tag{2}$$

$K(\cdot,\cdot) \in C^1(\mathbb{R}^n \times \mathbb{R}^n \backslash \Delta)$ and

$$|\nabla_x K(x, y)| + |\nabla_y K(x, y)| \leq \frac{A}{|x-y|^{n+1}} \tag{3}$$

for all $(x, y) \in (\mathbb{R}^n \times \mathbb{R}^n)\backslash\Delta$;

$$\|T_o\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq A \tag{4}$$

where

$$T_o f(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x, y) f(y) dy. \tag{5}$$

Then, for $1 < p < \infty$, there exists a $C_p > 0$ such that

$$\|T_{P,K} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{6}$$

for all $f \in L^p(\mathbb{R}^n)$. The constant C_p may depend on p, n, A and $\deg(P)$ but is independent of the coefficients of P.

Oscillatory singular integral operators with general C^∞ phase functions were studied in [9] where, among other things, the L^p boundedness was obtained under a "finite-type" phase function condition, both of which are described below.

Definition 1.1 Let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\Phi(x, y)$ be C^∞ in an open set containing (x_0, y_0). Φ is said to be of finite type at (x_0, y_0) if there exist two multi-indices $\alpha, \beta \in (\mathbb{N} \cup \{0\})^n$ such that $|\alpha|, |\beta| \geq 1$ and

$$\frac{\partial^{\alpha+\beta}\Phi}{\partial x^\alpha \partial y^\beta}(x_0, y_0) \neq 0.$$

Theorem 1.2 ([9]) *Let $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $\Phi_1(x, y), \ldots, \Phi_m(x, y)$ be C^∞ such that, for every $(u_1, \ldots, u_m) \in \mathbb{S}^{m-1}$, $\sum_{j=1}^m u_j \Phi_j(x, y)$ is of finite type at every point in*

$(\text{supp}(\varphi)) \cap \Delta$. *Let $K(x, y)$ satisfy (2), (3) and (4). Then, for $1 < p < \infty$, the operators $T_{\lambda\Phi,\varphi K}$ are uniformly bounded on $L^p(\mathbb{R}^n)$ for all $\Phi(x, y) = \sum_{j=1}^{m} u_j \Phi_j(x, y)$ where $\lambda \in \mathbb{R}$ and $(u_1, \ldots, u_m) \in \mathbb{S}^{m-1}$.*

For any polynomial phase function $P(x, y)$, if it has at least one nonzero term $a_{\alpha\beta}x^\alpha y^\beta$ with $\min\{|\alpha|, |\beta|\} \geq 1$, then the L^p boundedness of the corresponding oscillatory singular integral operators is covered by Theorem 1.2. Otherwise one has $P(x, y) = g(x)+h(y)$, in which case the L^p boundedness follows from $\|T_{P,K}\|_{p,p} = \|T_{0,K}\|_{p,p}$.

On the other hand, it has been well-known that Calderón-Zygmund singular integrals are bounded on L^p spaces even when the C^1 assumption and the bounds for ∇K in (3) are replaced by the following weaker Hölder type condition:

There exists a $\delta > 0$ such that

$$\begin{aligned} |K(x, y) - K(x', y)| &\leq \frac{A|x - x'|^\delta}{(|x - y| + |x' - y|)^{n+\delta}} \\ \text{whenever } |x - x'| &< (1/2)\max\{|x - y|, |x' - y|\}, \text{ and} \\ |K(x, y) - K(x, y')| &\leq \frac{A|y - y'|^\delta}{(|x - y| + |x - y'|)^{n+\delta}} \\ \text{whenever } |y - y'| &< (1/2)\max\{|x - y|, |x - y'|\}. \end{aligned} \tag{7}$$

In a recent paper [2], the results of Ricci and Stein in Theorem 1.1 were extended to allow $K(x, y)$ to be such a Hölder class kernel.

Theorem 1.3 ([2]) *Let $P(x, y)$ be a real-valued polynomial. Let $K(x, y)$ be a Hölder class Calderón-Zygmund kernel, i.e. there exist $\delta, A > 0$ such that $K(x, y)$ satisfies (2), (7) and (4). Then, for $1 < p < \infty$, there exists a $C_p > 0$ such that*

$$\|T_{P,K} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{8}$$

for all $f \in L^p(\mathbb{R}^n)$. The constant C_p may depend on p, n, δ, A and $\deg(P)$ but is independent of the coefficients of P.

See also [1, 6].

We now state the main result of this paper in which not only the kernels $K(x, y)$ are allowed to be in the Hölder class, but the phase functions can be fairly general.

Theorem 1.4 *Let U be an open set in $\mathbb{R}^m$ and G be a compact subset of U. Let $\Phi(x, y, u) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times U)$ and $\varphi(x, y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that, for every $u \in U$, $\Phi(\cdot, \cdot, u)$ is of finite type at every point in $(\text{supp}(\varphi)) \cap \Delta$. Let $K(x, y)$ be a Hölder class Calderón-Zygmund kernel, i.e. there exist $\delta, A > 0$ such that $K(x, y)$ satisfies (2), (7) and (4). Then, for $1 < p < \infty$, there exists a $C_p > 0$ such that*

$$\|T_{\lambda\Phi,\varphi K} f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{9}$$

for all $f \in L^p(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$ and $u \in G$. The constant C_p may depend on $p, n, m, \delta, A, \varphi$ and G but is independent of λ and u.

REMARKS.
(i) It is a well-known fact that the conclusion of Theorems 1.2 and 1.4 can fail if the "finite type" assumption for the phase functions is dropped [8, 9, 16].
(ii) The phase functions in Theorem 1.2 are subsumed in the family of phase functions in Theorem 1.4 as one can simply let $U = \mathbb{R}^m \backslash \{0\}$, $G = \mathbb{S}^{m-1}$ and

$$\Phi(x, y, u) = u \cdot (\Phi_1(x, y), \dots, \Phi_m(x, y)).$$

(iii) By (2), it is easy to see that Theorem 1.4 continues to hold if the smooth cut-off function $\varphi(x, y)$ is replaced by, say, $\chi_B(x - y)$, where B is the unit ball in $\mathbb{R}^n$.
(iv) The conclusion of Theorem 1.4 remains valid in the more general context of weighted spaces $L^p(\mathbb{R}^n, w(x)dx)$ with Muckenhoupt A_p weights. See Theorem 4.2.
(v) It follows from Theorem 1.4 that the operators $T_{\lambda\Phi,\varphi K}$ are uniformly bounded on L^p spaces for $\lambda \in \mathbb{R}$ and $u \in G$ if the phase function $\Phi(x, y, u)$ is real-analytic in $\mathbb{R}^n \times \mathbb{R}^n \times U$, where U is an open subset of $\mathbb{R}$ (i.e. m is taken to be 1) and G is a compact subset of U (see Theorem 5.1). It would be interesting to know whether the same holds for $m > 1$.

In the rest of the paper we shall use $A \lesssim B$ ($A \gtrsim B$) to mean that $A \le cB$ ($A \ge cB$) for a certain constant c whose actual value is not essential for the relevant arguments to work. We shall also use $A \approx B$ to mean "$A \lesssim B$ and $B \lesssim A$".

2 A van der Corput type lemma

A version of the classical van der Corput's lemma can be stated as follows.

Lemma 2.1 ([14]) *Let ϕ be a real-valued C^k function on $[a, b]$ satisfying $|\phi^{(k)}(x)| \ge 1$ for every $x \in [a, b]$. Suppose that $k \ge 2$, or that $k = 1$ and ϕ' is monotone on $[a, b]$. Then there exists a positive constant c_k such that, for every $\psi \in C^1([a, b])$,*

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \le c_k |\lambda|^{-1/k} \left(|\psi(b)| + \int_a^b |\psi'(x)| dx \right) \tag{10}$$

holds for all $\lambda \in \mathbb{R}$. The constant c_k is independent of λ, a, b, ϕ and ψ.

The following lemma, which is in the spirit of Lemma 2.1, is needed in our proof of Theorem 1.4.

Lemma 2.2 *Let $\phi \in C^\infty(\mathbb{R}^n)$ be real-valued and $\psi \in C_0^\infty(\mathbb{R}^n)$. Let $M > 0$, $k \in \mathbb{N}$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$ such that $|\alpha| = k$. Suppose that $|\partial^\beta \phi / \partial x^\beta(x)| \le M$ holds for all $|\beta| = k + 1$ and $x \in V_1$, where V_a is defined by*

$$V_a = \{x \in \mathbb{R}^n : dist(x, supp(\psi)) \le a \|\partial^\alpha \phi / \partial x^\alpha\|_{L^\infty(supp(\psi))}\}$$

for $a > 0$. *Let*

$$\|\psi\|_{0,1} = \|\psi\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n, v\in\mathbb{S}^{n-1}} \int_{\mathbb{R}} |\nabla\psi(x+tv)|dt.$$

Then there exists a $c > 0$ *such that*

$$\left|\int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\psi(x)dx\right| \le c(a^{-n}\|\psi\|_{0,1})|\lambda|^{-\varepsilon/k} \int_{V_a} \left|\frac{\partial^\alpha\phi(x)}{\partial x^\alpha}\right|^{-\varepsilon(1+1/k)} dx \qquad (11)$$

for all $a, \varepsilon \in (0, 1]$ *and* $\lambda \in \mathbb{R}$. *The constant c may depend on* M, α *(and thus* k*) but is otherwise independent of* $a, \varepsilon, \lambda, \psi$ *and* ϕ.

The above lemma is a refined version of Lemma 3.2 of [10]. We shall sketch its proof below where our focus will primarily be on providing the necessary details for the current incarnation.

Proof Without loss of generality we may assume that

$$|\{\partial^\alpha\phi/\partial x^\alpha = 0\} \cap \text{supp}(\psi)| = 0.$$

Let $A > 1$ be a suitably chosen constant which depends on M, n and α only, and let $r(x) = A^{-1}|\partial^\alpha\phi/\partial x^\alpha(x)|$ whenever it is nonzero. By applying the Vitali covering procedure, there exist $x_1, x_2, \ldots \in \{\partial^\alpha\phi/\partial x^\alpha \ne 0\} \cap \text{supp}(\psi)$ such that

$$\{\partial^\alpha\phi/\partial x^\alpha \ne 0\} \cap \text{supp}(\psi) \subseteq \bigcup_j B(x_j, r_j/2) \text{ where } r_j = r(x_j), \qquad (12)$$

$$\left\{B(x_j, r_j/10)\right\}_{j=1,2,\ldots} \text{ are pairwise disjoint.} \qquad (13)$$

It follows from our selection of A and a packing argument of Sogge and Stein in [13] (see also [14]) that, for each j, there exists a $v_j \in \mathbb{S}^{n-1}$ such that

$$|\partial^\alpha\phi/\partial x^\alpha(y)| \approx r_j; \qquad (14)$$

$$|(v_j \cdot \nabla)^k\phi(y)| \gtrsim r_j \qquad (15)$$

for all $y \in B(x_j, r_j)$ and

$$\sum_j \chi_{B(x_j, r_j)} \lesssim 1. \qquad (16)$$

Thus, there exists a partition of unity $\{\eta_j(x)\}_{j=1,2,\ldots}$ such that each η_j is supported in $B(x_j, r_j)$, $\sum_j \eta_j(x) = 1$ for $x \in \bigcup_j B(x_j, r_j/2)$, and

$$|\partial^\beta\eta_j/\partial x^\beta| \lesssim r_j^{-|\beta|} \qquad (17)$$

for all $\beta \in (\mathbb{N} \cup \{0\})^n$.

For $y = (y_1, y_2, \dots y_n) \in \mathbb{R}^n$, let $\tilde{y} = (y_2, \dots, y_n)$. For each j, let Γ_j denote an orthogonal linear transformation on $\mathbb{R}^n$ such that $\Gamma_j((1, 0, \dots, 0)) = v_j$. Then by (15), for $|y| \leq r_j$,

$$\left| \frac{\partial^k}{\partial y_1^k} \big(\phi(x_j + \Gamma_j(y)) \big) \right| \gtrsim r_j. \tag{18}$$

When $k \geq 2$, by using (18), Lemma 2.1 and (17), we have

$$\begin{aligned}
&\left| \int_{B(x_j, r_j)} e^{i\lambda\phi(x)} \psi(x) \eta_j(x) dx \right| \\
&\leq \int_{|\tilde{y}| \leq r_j} \left| \int_{-(r_j^2 - |\tilde{y}|^2)^{1/2}}^{(r_j^2 - |\tilde{y}|^2)^{1/2}} e^{i\lambda\phi(x_j + \Gamma_j(y))} \psi(x_j + \Gamma_j(y)) \eta_j(x_j + \Gamma_j(y)) dy_1 \right| d\tilde{y} \\
&\lesssim (\lambda r_j)^{-1/k} \int_{|\tilde{y}| \leq r_j} \Big(|\psi(x_j + \Gamma_j(((r_j^2 - |\tilde{y}|^2)^{1/2}, \tilde{y}))) \eta_j(x_j + \Gamma_j(((r_j^2 - |\tilde{y}|^2)^{1/2}, \tilde{y})))| \\
&\quad + \int_{-(r_j^2 - |\tilde{y}|^2)^{1/2}}^{(r_j^2 - |\tilde{y}|^2)^{1/2}} \left| \frac{\partial}{\partial y_1} \big(\psi(x_j + \Gamma_j(y)) \eta_j(x_j + \Gamma_j(y)) \big) \right| dy_1 \Big) d\tilde{y} \\
&\lesssim \|\psi\|_{0,1} (|\lambda| r_j)^{-1/k} r_j^{n-1}.
\end{aligned} \tag{19}$$

For $k = 1$, one cannot use Lemma 2.1 because the monotonicity of the first derivative of $\phi(x_j + \Gamma_j(y))$ in y_1 is not known. Fortunately we have the following upper bound for the corresponding second derivative:

$$\left| \frac{\partial^2}{\partial y_1^2} (\phi(x_j + \Gamma_j(y))) \right| = |(v_j \cdot \nabla)^2 \phi(x_j + \Gamma_j(y))| \leq M$$

for $|y| \leq r_j$, which allows us to use integration by parts and (15) to get

$$\begin{aligned}
&\left| \int_{B(x_j, r_j)} e^{i\lambda\phi(x)} \psi(x) \eta_j(x) dx \right| \\
&= \left| \int_{|\tilde{y}| \leq r_j} \int_{-(r_j^2 - |\tilde{y}|^2)^{1/2}}^{(r_j^2 - |\tilde{y}|^2)^{1/2}} \frac{\partial}{\partial y_1} \left(e^{i\lambda\phi(x_j + \Gamma_j(y))} \right) \frac{\psi(x_j + \Gamma_j(y)) \eta_j(x_j + \Gamma_j(y))}{(i\lambda) \partial / \partial y_1 (\phi(x_j + \Gamma_j(y)))} dy_1 d\tilde{y} \right| \\
&\lesssim \|\psi\|_{0,1} (|\lambda| r_j)^{-1} r_j^{n-1},
\end{aligned}$$

which is just (19) for the case $k = 1$.

Trivially we have

$$\left| \int_{B(x_j, r_j)} e^{i\lambda\phi(x)} \psi(x) \eta_j(x) dx \right| \lesssim \|\psi\|_{0,1} r_j^n. \tag{20}$$

By (19)–(20), for every j and every $\varepsilon \in (0, 1]$,

$$\left| \int_{B(x_j, r_j)} e^{i\lambda\phi(x)} \psi(x)\eta_j(x)dx \right| \lesssim \|\psi\|_{0,1}|\lambda|^{-\varepsilon/k} r_j^{-\varepsilon(1+1/k)} r_j^n. \tag{21}$$

By (21), (14) and (16), for every $a \in (0, 1]$,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} e^{i\lambda\phi(x)} \psi(x)dx \right| &\le \sum_j \left| \int_{B(x_j, r_j)} e^{i\lambda\phi(x)} \psi(x)\eta_j(x)dx \right| \\ &\lesssim (\|\psi\|_{0,1}|\lambda|^{-\varepsilon/k} a^{-n}) \sum_j r_j^{-\varepsilon(1+1/k)} (ar_j)^n \\ &\lesssim (\|\psi\|_{0,1}|\lambda|^{-\varepsilon/k} a^{-n}) \int_{\mathbb{R}^n} \left| \frac{\partial^\alpha \phi(x)}{\partial x^\alpha} \right|^{-\varepsilon(1+1/k)} \left(\sum_j \chi_{B(x_j, ar_j)}(x) \right) dx \\ &\lesssim (a^{-n}\|\psi\|_{0,1})|\lambda|^{-\varepsilon/k} \int_{V_a} \left| \frac{\partial^\alpha \phi(x)}{\partial x^\alpha} \right|^{-\varepsilon(1+1/k)} dx. \end{aligned}$$

□

3 Proof of Theorem 1.4

For $k \in \mathbb{N}$, $r > 0$ and $a \in \mathbb{R}^k$, let $B_k(a, r) = \{x \in \mathbb{R}^k : |x - a| < r\}$. For any function $F(x, y)$ defined on a product space $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$, and multi-indices $\alpha \in (\mathbb{N} \cup \{0\})^{n_1}$, $\beta \in (\mathbb{N} \cup \{0\})^{n_2}$, we let

$$D_1^\alpha F = \frac{\partial^\alpha F}{\partial x^\alpha}, \quad D_2^\beta F = \frac{\partial^\beta F}{\partial y^\beta}.$$

The same goes for functions defined on more general product spaces $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$.

Let $K(x, y)$ be a Hölder class Calderón-Zygmund kernel. Clearly, the three properties (2), (7) and (4) of $K(x, y)$ remain intact under the translation $(x, y) \to (x - \zeta, y - \zeta)$ for any $\zeta \in \mathbb{R}^n$. This observation, together with the compactness of supp(φ) and G, allows the proof of Theorem 1.4 to be reduced to the task of establishing the following:

Proposition 3.1 *Suppose that $\Phi(x, y, u)$ is C^∞ in an open neighborhood of the origin in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ and there are two nonzero multi-indices $\alpha_0, \beta_0 \in (\mathbb{N} \cup \{0\})^n$ such that*

$$D_1^{\alpha_0} D_2^{\beta_0} \Phi(0, 0, 0) \neq 0. \tag{22}$$

Then there exists an $r_0 > 0$ *such that for every* $p \in (1, \infty)$ *and every* $\varphi \in C_0^\infty(B_{2n}(0, r_0))$*, the operator*

$$T_{\lambda\Phi,\varphi K} : f \to p.v. \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y,u)} K(x, y)\varphi(x, y) f(y)dy$$

is uniformly bounded on $L^p(\mathbb{R}^n)$ *for* $\lambda > 2$ *and* $u \in B_m(0, r_0)$*.*

Proof Let $\lambda > 2$, $k_0 = |\alpha_0|$ and $l_0 = |\beta_0|$. Without loss of generality we may assume that

$$D_1^{\alpha_0} D_2^{\beta} \Phi(0, 0, 0) = 0 \tag{23}$$

for all $|\beta| < l_0$. By using a transformation $(x, y) \to (\Gamma(x), \Gamma(y))$ where Γ is an orthogonal transformation, if necessary, we may also assume that $\beta_0 = (l_0, 0, \dots, 0)$. Let

$$F(x, y, z, u) = D_1^{\alpha_0} \Phi(z, x, u) - D_1^{\alpha_0} \Phi(z, y, u).$$

Then

$$\frac{\partial^j F}{\partial y_1^j}(0, 0, 0, 0) = 0$$

for $0 \le j \le l_0 - 1$ and

$$\frac{\partial^{l_0} F}{\partial y_1^{l_0}}(0, 0, 0, 0) \neq 0.$$

By the Malgrange preparation theorem [4], there exist an $r_0 > 0$ and C^∞ functions $a_0(x, \tilde{y}, z, u), \dots, a_{l_0-1}(x, \tilde{y}, z, u)$ on $I^n \times I^{n-1} \times I^n \times I^m$ and $c(x, y, z, u)$ on $I^n \times I^n \times I^n \times I^m$, where $I = (-4r_0, 4r_0)$, such that

$$\begin{aligned} F(x, y, z, u) &= c(x, y, z, u) \\ &\times (y_1^{l_0} + a_{l_0-1}(x, \tilde{y}, z, u) y_1^{l_0-1} + \cdots + a_0(x, \tilde{y}, z, u)) \end{aligned} \tag{24}$$

and $|c(x, y, z, u)| \gtrsim 1$ for $(x, y, z, u) \in I^n \times I^n \times I^n \times I^m$.

Let $\eta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $0 \le \eta(x, y) \le 1$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$; $\eta(x, y) = 1$ for $|(x, y)| \le 1/2$; and $\eta(x, y) = 0$ for $|(x, y)| \ge 1$. For $t > 0$, let $\eta_t(x, y) = t^{-2n}\eta(x/t, y/t)$.

Also, let $\theta \in C^\infty(\mathbb{R}^n)$ be nonnegative such that $\theta(x) = 0$ for $|x| \le 4$ and $\theta(x) = 1$ for $|x| \ge 8$. Let $N_0 = 6(2n + 1)k_0 l_0$, $\rho = N_0^{-1}$ and

$$H_\lambda(x, y) = \frac{\varphi(x, y)}{J(\eta)} \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta_{\lambda^{-\rho}}(x - v, y - w) K(v, w)\theta(\lambda^\rho(v - w)) dv dw$$

where $J(\eta) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \eta(x, y)dxdy \gtrsim 1$.

When $H_\lambda(x, y) \neq 0$, there exists a $(v, w) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $|v - w| \geq 4\lambda^{-\rho}$ and $|(x, y) - (v, w)| < \lambda^{-\rho}$. Thus,

$$2\lambda^{-\rho} \leq |v - w|/2 \leq |x - y| \leq 3|v - w|/2.$$

By (2),

$$|H_\lambda(x, y)| \lesssim \frac{|\varphi(x, y)|}{|x - y|^n} \chi_{[2\lambda^{-\rho}, \infty)}(|x - y|). \tag{25}$$

Similarly, one can show that, for all $x, y \in \mathbb{R}^n$,

$$\|H_\lambda(x, \cdot)\|_{0,1} + \|H_\lambda(\cdot, y)\|_{0,1} \lesssim \lambda^{(n+1)\rho}. \tag{26}$$

We now decompose $T_{\lambda\Phi, \varphi K}$ as the sum of three operators:

$$T_{\lambda\Phi, \varphi K} f = T_1 f + T_2 f + T_3 f \tag{27}$$

where

$$T_1 f(x) = \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y,u)} H_\lambda(x, y) f(y)dy, \tag{28}$$

$$T_2 f(x) = \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y,u)} \big[K(x, y)\theta(\lambda^\rho(x - y))\varphi(x, y) - H_\lambda(x, y)\big] f(y)dy, \tag{29}$$

$$T_3 f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y,u)} K(x, y)(1 - \theta(\lambda^\rho(x - y)))\varphi(x, y) f(y)dy. \tag{30}$$

It follows from (25) that

$$\|T_1\|_{L^1(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} + \|T_1\|_{L^\infty(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)} \lesssim \ln(\lambda). \tag{31}$$

On the other hand, we have

$$T_1^* T_1 f(x) = \int_{\mathbb{R}^n} L(x, y) f(y)dy$$

where

$$L(x, y) = \int_{\mathbb{R}^n} e^{i\lambda(\Phi(z,x,u) - \Phi(z,y,u))} H_\lambda(z, x)\overline{H_\lambda(z, y)}dz.$$

By shrinking the support of φ if necessary, we may apply Lemma 2.2 with $\varepsilon = (3l_0)^{-1}$ to get

$$|L(x, y)| \lesssim \lambda^{-1/(3k_0 l_0)}(\|H_\lambda(\cdot, x)\overline{H_\lambda(\cdot, y)}\|_{0,1})\chi_{[0,2r_0]}(|x|)\chi_{[0,2r_0]}(|y|)$$

$$\times \int_{|z|\le 2r_0} |D_1^{\alpha_0}\Phi(z,x,u) - D_1^{\alpha_0}\Phi(z,y,u)|^{-(k_0+1)/(3k_0l_0)} dz. \tag{32}$$

By using (24), (26), (32), $(k_0+1)/(3k_0l_0) < 1$ and the lemma on page 182 of [12], for every $x \in \mathbb{R}^n$,

$$\begin{aligned}\int_{\mathbb{R}^n} |L(x,y)|dy &\lesssim \lambda^{-1/(3k_0l_0)}\lambda^{(2n+1)\rho} \int_{|z|\le 2r_0}\int_{|\tilde{y}|\le 2r_0}\Bigg(\int_{|y_1|\le 2r_0}\Big|y_1^{l_0} \\ &\quad + a_{l_0-1}(x,\tilde{y},z,u)y_1^{l_0-1} + \cdots + a_0(x,\tilde{y},z,u)\Big|^{-(k_0+1)/(3k_0l_0)} dy_1\Bigg)d\tilde{y}dz \\ &\lesssim \lambda^{-1/(6k_0l_0)}. \end{aligned} \tag{33}$$

Similary, we have

$$\int_{\mathbb{R}^n} |L(x,y)|dx \lesssim \lambda^{-1/(6k_0l_0)} \tag{34}$$

for all $y \in \mathbb{R}^n$. It follows from (33)–(34) that

$$\|T_1\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \lesssim \lambda^{-1/(12k_0l_0)}. \tag{35}$$

By interpolating between (31) and (35) we obtain

$$\|T_1\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim (\ln(\lambda))^{|1-2/p|}\lambda^{-(1-|1-2/p|)/(12k_0l_0)} \lesssim 1 \tag{36}$$

for $1 < p < \infty$.

To treat the term $T_2 f$, first we observe that

$$\begin{aligned} &|K(x,y)\theta(\lambda^\rho(x-y))\varphi(x,y) - H_\lambda(x,y)| \lesssim |\varphi(x,y)| \times \\ &\int_{\mathbb{R}^n\times\mathbb{R}^n} \eta_{\lambda^{-\rho}}(x-v,y-w)\big|K(x,y)\theta(\lambda^\rho(x-y)) - K(v,w)\theta(\lambda^\rho(v-w))\big|dvdw \\ &\lesssim |\varphi(x,y)|\Bigg[\int_{\mathbb{R}^n\times\mathbb{R}^n} \eta_{\lambda^{-\rho}}(x-v,y-w)\big|K(x,y)-K(v,w)\big|\theta(\lambda^\rho(x-y))dvdw \\ &\quad + \int_{\mathbb{R}^n\times\mathbb{R}^n} \eta_{\lambda^{-\rho}}(x-v,y-w)|K(v,w)|\big|\theta(\lambda^\rho(x-y)) - \theta(\lambda^\rho(v-w))\big|dvdw\Bigg]. \end{aligned}$$

Let the above two integrals be denoted by $I_1(x,y)$ and $I_2(x,y)$, respectively. For $I_1(x,y)$ to be nonzero, there must exist $v, w \in \mathbb{R}^n$ such that $|x-v| < \lambda^{-\rho}$, $|y-w| < \lambda^{-\rho}$, while $|x-y| \ge 4\lambda^{-\rho}$. Thus, $|v-w| \ge 2\lambda^{-\rho}$ and $|v-w| \approx |x-y|$. It follows from (7) that

$$\begin{aligned} \big|K(x,y)-K(v,w)\big| &\le \big|K(x,y)-K(v,y)\big| + \big|K(v,y)-K(v,w)\big| \\ &\lesssim \frac{|x-v|^\delta}{(|x-y|+|v-y|)^{n+\delta}} + \frac{|y-w|^\delta}{(|v-y|+|v-w|)^{n+\delta}} \end{aligned}$$

$$\lesssim \frac{\lambda^{-\rho\delta}\chi_{[4\lambda^{-\rho},\infty)}(|x-y|)}{|x-y|^{n+\delta}},$$

which implies that

$$|I_1(x,y)| \lesssim \frac{\lambda^{-\rho\delta}\chi_{[4\lambda^{-\rho},\infty)}(|x-y|)}{|x-y|^{n+\delta}}. \tag{37}$$

For $I_2(x,y)$ to be nonzero, there must exist $v, w \in \mathbb{R}^n$ such that $|x-v| < \lambda^{-\rho}$, $|y-w| < \lambda^{-\rho}$, while

$$\max\{|x-y|,\ |v-w|\} \geq 4\lambda^{-\rho}$$

and

$$\min\{|x-y|,\ |v-w|\} \leq 8\lambda^{-\rho}.$$

Thus, $|x-y| \approx |v-w|$ and

$$2\lambda^{-\rho} \leq |x-y| \leq 10\lambda^{-\rho},$$

which together imply that

$$|I_2(x,y)| \lesssim \frac{\chi_{[2\lambda^{-\rho},\,10\lambda^{-\rho}]}(|x-y|)}{|x-y|^n}. \tag{38}$$

By (37)–(38),

$$\|T_2\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \lambda^{-\rho\delta}\int_{|x|\geq 4\lambda^{-\rho}} \frac{dx}{|x|^{n+\delta}} + \int_{2\lambda^{-\rho}\leq|x|\leq 10\lambda^{-\rho}} \frac{dx}{|x|^n} \lesssim 1. \tag{39}$$

Now $T_3 f$ is the only term left to be treated. For any $h \in \mathbb{R}^n$, let $Q_h = h + (\lambda^{-\rho} I)^n$ and $Q_h^* = h + (9\lambda^{-\rho} I)^n$ where $I = (-1/2, 1/2]$. Let $\phi_\beta(x,u) = D_2^\beta \Phi(x,x,u)$ for $\beta \in (\mathbb{N}\cup\{0\})^n$ and define the polynomial $P_{h,u}(x,y)$ by

$$P_{h,u}(x,y) = \sum_{1\leq|\beta|\leq N_0-1}\left(\sum_{|\alpha|\leq N_0-|\beta|-1} \frac{1}{\alpha!\beta!} D_1^\alpha \phi_\beta(h,u)(x-h)^\alpha (y-x)^\beta\right).$$

Thus, for any $h \in \mathbb{R}^n$, $x \in Q_h^*$, $y \in Q_h$ and $|u| < r_0$,

$$|\Phi(x,y,u) - (\Phi(x,x,u) + P_{h,u}(x,y))||\varphi(x,y)| \lesssim \sum_{j=1}^{N_0} |x-y|^j |x-h|^{N_0-j}.$$

For any $f \in L^p(\mathbb{R}^n)$ and any $h \in \mathbb{R}^n$, we have $\mathrm{supp}(T_3(\chi_{Q_h} f)) \subseteq Q_h^*$ and thus,

$$\left| T_3(\chi_{Q_h} f)(x) - e^{i\lambda\Phi(x,x,u)} T_{\lambda P_{h,u},\, \tilde{K}}(\chi_{Q_h} f)(x) \right| \lesssim \sum_{j=1}^{N_0} \lambda^{1-(N_0-j)\rho} \int_{Q_h} \frac{|f(y)|dy}{|x-y|^{n-j}} \tag{40}$$

where $\tilde{K}(x, y) = K(x, y)(1 - \theta(\lambda^\rho(x - y)))\varphi(x, y)$. It is easy to verify that (2), (7) and (4) are all satisfied by $\tilde{K}(\cdot, \cdot)$ uniformly in λ. By (40) and Theorem 1.3,

$$\begin{aligned} \|T_3(\chi_{Q_h} f)\|_{L^p(\mathbb{R}^n)} &\lesssim \left(1 + \sum_{j=1}^{N_0} \lambda^{1-(N_0-j)\rho} \int_{|x|\leq 10\lambda^{-\rho}} \frac{dx}{|x|^{n-j}}\right) \times \|\chi_{Q_h} f\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|\chi_{Q_h} f\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{41}$$

By

$$\begin{aligned} |T_3 f|^p &= \left| \sum_{h\in(\lambda^{-\rho})\mathbb{Z}^n} \chi_{Q_h^*} T_3(\chi_{Q_h} f) \right|^p \\ &\leq \left| \sum_{h\in(\lambda^{-\rho})\mathbb{Z}^n} \chi_{Q_h^*} \right|^{p/p'} \left(\sum_{h\in(\lambda^{-\rho})\mathbb{Z}^n} |T_3(\chi_{Q_h} f)|^p \right) \lesssim \sum_{h\in(\lambda^{-\rho})\mathbb{Z}^n} |T_3(\chi_{Q_h} f)|^p \end{aligned}$$

and (41), we get

$$\|T_3\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim 1 \tag{42}$$

for $1 < p < \infty$. It follows from (27), (36), (39) and (42) that

$$\|T_{\lambda\Phi,\, \varphi K}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim 1$$

for $1 < p < \infty$. □

4 Extension to L^p spaces with A_p weights

As pointed earlier, the conclusions of Theorem 1.4 continue to hold when the spaces $L^p(\mathbb{R}^n, dx)$ is replaced by the weighted spaces $L^p(\mathbb{R}^n, w dx)$ as long as w is in the class A_p [7] whose definition is given below:

Definition 4.1 Let $p \in (1, \infty)$. A nonnegative, locally integrable function $w(\cdot)$ on $\mathbb{R}^n$ is said to be in the Muckenhoupt weight class $A_p(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|Q|}\int_Q w(y)dy\right)\left(\frac{1}{|Q|}\int_Q w(y)^{-1/(p-1)}dy\right)^{p-1} \leq C \tag{43}$$

holds for all cubes Q in $\mathbb{R}^n$. The smallest such constant C in (43) is the corresponding A_p constant of w.

Let

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p},$$

and

$$L^p(\mathbb{R}^n, wdx) = \{f : \|f\|_{p,w} < \infty\}.$$

We shall need the following result due to Coifman and Fefferman:

Theorem 4.1 ([3]) *For each $p \in (1, \infty)$ and each $w \in A_p(\mathbb{R}^n)$, there exists a $\nu \in (0, 1)$ such that $w^{1+\nu} \in A_p(\mathbb{R}^n)$. Both ν and the A_p constant of $w^{1+\nu}$ depend on n, p and the A_p constant of w only.*

We shall now state the weighted version of Theorem 1.4 and give a brief sketch of its proof while leaving out most of the technical details.

Theorem 4.2 *Let U be an open set in $\mathbb{R}^m$ and G be a compact subset of U. Let $\Phi(x, y, u) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times U)$ and $\varphi(x, y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that, for every $u \in U$, $\Phi(\cdot, \cdot, u)$ is of finite type at every point in $(\mathrm{supp}(\varphi)) \cap \Delta$. Let $K(x, y)$ be a Hölder class Calderón-Zygmund kernel, i.e. there exist $\delta, A > 0$ such that $K(x, y)$ satisfies (2), (7) and (4). Let $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$. Then there exists a positive constant $C_{p,w}$ such that*

$$\|T_{\lambda\Phi,\,\varphi K} f\|_{p,w} \le C_{p,w} \|f\|_{p,w} \tag{44}$$

for all $f \in L^p(\mathbb{R}^n, wdx)$, $\lambda \in \mathbb{R}$ and $u \in G$. The constant $C_{p,w}$ may depend on $p, n, m, \delta, A, \varphi$, G and A_p the constant of w, but is independent of λ and u.

Proof By (27), it suffices to prove $\|T_j f\|_{p,w} \lesssim \|f\|_{p,w}$ for $j = 1, 2, 3$ and $\lambda > 2$.

For T_1, by (25),

$$|T_1 f| \lesssim (\ln(\lambda))\mathcal{M} f,$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. By Theorem 4.1 and the weighted L^p boundedness of $\mathcal{M}$,

$$\|T_1 f\|_{p,w^{1+\nu}} \lesssim (\ln(\lambda)) \|f\|_{p,w^{1+\nu}} \tag{45}$$

for a certain $\nu > 0$ (see [5]). By (36) and (45) and a result of Stein and Weiss in [17], we obtain

$$\begin{aligned}\|T_1 f\|_{p,w} &\lesssim (\ln(\lambda))^{1/(1+\nu)+|1-2/p|} \lambda^{-(1-|1-2/p|)\nu/(12(1+\nu)k_0 l_0)} \|f\|_{p,w} \\ &\lesssim \|f\|_{p,w}\end{aligned}$$

For T_2, one can use (37)–(38) to get $|T_2 f| \lesssim \mathcal{M} f$ and thus

$$\|T_2 f\|_{p,w} \lesssim \|f\|_{p,w}.$$

Finally, for the treament of $T_3 f$, one uses Theorem 3.2 in [2] instead of Theorem 1.3 but otherwise follows the steps in the proof of Theorem 1.4 to arrive at

$$\|T_3 f\|_{p,w} \lesssim \|f\|_{p,w}. \qquad \square$$

5 Real analytic phases

In this section we will show how one can use Theorem 1.4 (and Theorem 4.2) to obtain the uniform L^p boundedness of oscillatory singular integral operators with Hölder class kernels and real-analytic phase functions $\lambda\Phi(x, y, u)$ when the parameter u is in a compact subset of $\mathbb{R}$.

Theorem 5.1 *Let U be an open set in $\mathbb{R}$ and G be a compact subset of U. Let $\varphi(x, y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $\Phi(x, y, u)$ be real-analytic for (x, y) in an open neighborhood of supp(φ) and $u \in U$. Let $K(x, y)$ be a Hölder class Calderón-Zygmund kernel, i.e. there exist $\delta, A > 0$ such that $K(x, y)$ satisfies (2), (7) and (4). Let $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$. Then there exists a positive constant $C_{p,w}$ such that*

$$\|T_{\lambda\Phi,\, \varphi K} f\|_{p,w} \leq C_{p,w}\|f\|_{p,w} \tag{46}$$

for all $f \in L^p(\mathbb{R}^n, wdx)$, $\lambda \in \mathbb{R}$ and $u \in G$. The constant $C_{p,w}$ may depend on $p, n, \delta, A, \varphi, G$ and the A_p constant of w, but is independent of λ and u.

Proof Without loss of generality we may assume that supp$(\varphi) = \overline{B(0, r_0)}$, $U = (-2r_0,\ 2r_0)$ and $G = [-r_0,\ r_0]$ for some $r_0 > 0$. Let

$$E = \{u \in [-r_0,\ r_0] :\ \Phi(\cdot\,, \cdot\,, u) \text{ fails to have finite type at some point}\}.$$

In the case where $E = \emptyset$, (46) follows from Theorem 4.2.

Suppose that $E \neq \emptyset$. For each $u_0 \in E$ and $1 \leq j, k \leq n$, there exists a (x_0, y_0) such that all partial derivatives

$$\left\{D_1^\alpha D_2^\beta \left(\frac{\partial^2 \Phi(x, y, u_0)}{\partial x_j \partial y_k}\right) :\ \alpha, \beta \in (\mathbb{N} \cup \{0\})^n\right\}$$

vanish at (x_0, y_0) which, by real-analyticity, implies that

$$\frac{\partial^2 \Phi(x, y, u_0)}{\partial x_j \partial y_k} = 0$$

for all $(x, y) \in \overline{B(0, r_0)}$ and $1 \leq j, k \leq n$.

If E has a limit point p, then there exists a sequence $\{u_l\}_{l=1}^{\infty}$ in $E\backslash\{p\}$ such that

$$\lim_{l\to\infty} u_l = p.$$

Thus,

$$\frac{\partial^2\Phi(x, y, u_l)}{\partial x_j \partial y_k} = 0$$

for all $(x, y) \in \overline{B(0, r_0)}$, $l \in \mathbb{N}$ and $1 \le j, k \le n$. Again by real-analyticity,

$$\frac{\partial^2\Phi(x, y, u)}{\partial x_j \partial y_k} = 0$$

for all $(x, y) \in \overline{B(0, r_0)}$, $u \in (-2r_0,\ 2r_0)$ and $1 \le j, k \le n$. Thus, $\Phi(x, y, u)$ can be written as $\phi(x, u) + \psi(y, u)$ and (46) follows trivially.

Thus we may now assume that E ($\neq \emptyset$) has no limit points. By using a translation and shrinking r_0 if necessary, we may further assume that $E = \{0\}$ and

$$\Phi(x, y, u) = \sum_{k=0}^{\infty} \left(\frac{u^k}{k!}\right) \frac{\partial^k\Phi(x, y, 0)}{\partial u^k}.$$

Since $\Phi(\cdot, \cdot, 0)$ fails to be of finite type at least at one point while for every $u \neq 0$, $\Phi(\cdot, \cdot, u)$ has finite type at every point, there exists a $k \in \mathbb{N}$ such that $\dfrac{\partial^k\Phi(x, y, 0)}{\partial u^k}$ has finite type at $(0, 0)$. Let k_0 be the smallest such k. Then each $\dfrac{1}{j!}\dfrac{\partial^j\Phi(x, y, 0)}{\partial u^j}$ can be written as $\phi_j(x) + \psi_j(y)$ for $0 \le j \le k_0 - 1$ and

$$\lambda\Phi(x, y, u) = \lambda \sum_{j=0}^{k_0-1} \big(\phi_j(x) + \psi_j(y)\big) + (\lambda u^{k_0})\Psi(x, y, u) \tag{47}$$

where

$$\Psi(x, y, u) = \frac{1}{k_0!}\frac{\partial^{k_0}\Phi(x, y, 0)}{\partial u^{k_0}} + \sum_{j=k_0+1}^{\infty} \left(\frac{u^{j-k_0}}{j!}\right) \frac{\partial^j\Phi(x, y, 0)}{\partial u^j}. \tag{48}$$

Since $\dfrac{\partial^{k_0}\Phi(\cdot, \cdot, 0)}{\partial u^{k_0}}$ has finite type at $(0, 0)$, by continuity, for $\tilde{r}_0 > 0$ sufficiently small and $|u| \le \tilde{r}_0$, $\Psi(\cdot, \cdot, u)$ also has finite type at every point of $\overline{B_{2n}(\tilde{r}_0)}$. Let

$$\tilde{f}(y) = e^{i\lambda\left(\sum_{j=0}^{k_0-1} \psi_j(y)\right)} f(y).$$

By Theorem 4.2 (after shrinking supp(φ) if necessary) and (47)–(48),

$$\|T_{\lambda\Phi,\,\varphi K} f\|_{p,w} = \|T_{(\lambda u^{k_0})\Psi,\,\varphi K} \tilde{f}\|_{p,w} \le C_p \|f\|_{p,w}.$$

□

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