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A Note on the Invertibility of the Gabor Frame Operator on Certain Modulation Spaces

Dae Gwan Lee[1] · **Friedrich Philipp**[2] · **Felix Voigtlaender**[3]

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Abstract

We consider Gabor frames generated by a general lattice and a window function that belongs to one of the following spaces: the Sobolev space $V_1 = H^1(\mathbb{R}^d)$, the weighted L^2-space $V_2 = L^2_{1+|x|}(\mathbb{R}^d)$, and the space $V_3 = \mathbb{H}^1(\mathbb{R}^d) = V_1 \cap V_2$ consisting of all functions with finite uncertainty product; all these spaces can be described as modulation spaces with respect to suitable weighted L^2 spaces. In all cases, we prove that the space of Bessel vectors in V_j is mapped bijectively onto itself by the Gabor frame operator. As a consequence, if the window function belongs to one of the three spaces, then the canonical dual window also belongs to the same space. In fact, the result not only applies to frames, but also to frame sequences.

Keywords Gabor frames · Sobolev space · Invariance · Dual frame · Regularity of dual window

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✉ Friedrich Philipp
friedrich.philipp@tu-ilmenau.de

Dae Gwan Lee
daegwans@gmail.com

Felix Voigtlaender
felix.voigtlaender@ku.de

[1] Mathematisch–Geographische Fakultät, Katholische Universität Eichstätt–Ingolstadt, 85071 Eichstätt, Germany

[2] Technische Universität Ilmenau, Institute for Mathematics, Weimarer Straße 25, 98693 Ilmenau, Germany

[3] Mathematical Institute for Machine Learning and Data Science (MIDS), Katholische Universität Eichstätt–Ingolstadt, Auf der Schanz 49, 85049 Ingolstadt, Germany

1 Introduction

Analyzing the time-frequency localization of functions is an important topic in harmonic analysis. Quantitative results on this localization are usually formulated in terms of function spaces such as Sobolev spaces, modulation spaces, or Wiener amalgam spaces. An especially important space is the *Feichtinger algebra* $S_0 = M^1$ [8, 16] which has numerous remarkable properties; see, e.g., [5, Sect. A.6] for a compact overview. Yet, in some cases it is preferable to work with more classical spaces like the Sobolev space $H^1(\mathbb{R}^d) = W^{1,2}(\mathbb{R}^d)$, the weighted L^2-space $L^2_{1+|x|}(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} : (1+|x|)f(x) \in L^2\}$, or the space $\mathbb{H}^1(\mathbb{R}^d) = H^1(\mathbb{R}^d) \cap L^2_{1+|x|}(\mathbb{R}^d)$ which consists of all functions $g \in L^2(\mathbb{R}^d)$ with finite uncertainty product

$$\left(\int_{\mathbb{R}^d} |x|^2 \cdot |g(x)|^2 \, dx\right)\left(\int_{\mathbb{R}^d} |\omega|^2 \cdot |\widehat{g}(\omega)|^2 \, d\omega\right) < \infty. \tag{1.1}$$

Certainly, one advantage of these classical spaces is that membership of a function in the space can be decided easily. We remark that all of these spaces fall into the scale of modulation spaces (see Sect. 3).

In Gabor analysis, it is known (see e.g., [12, Proposition 5.2.1] and [5, Theorem 12.3.2]) that for a Gabor frame generated by a lattice, the canonical dual frame is again a Gabor system (over the same lattice), generated by the so-called *dual window*. An important question is what kind of time-frequency localization conditions of the generating window are inherited by the dual window. Precisely, if $g \in L^2(\mathbb{R}^d)$ belongs to a certain "localization Banach space" V and if $\Lambda \subset \mathbb{R}^{2d}$ is such that (g, Λ) forms a Gabor frame for $L^2(\mathbb{R}^d)$, then does the canonical dual window belong to V as well? A celebrated result in time-frequency analysis states that this is true for the Feichtinger algebra $V = S_0(\mathbb{R}^d)$; see [14] for separable lattices Λ and [1, Theorem 7] for irregular sets Λ. In the case of separable lattices, the question has been answered affirmatively also for the Schwartz space $V = \mathcal{S}(\mathbb{R})$ [17, Proposition 5.5] and for the Wiener amalgam space $V = W(L^\infty, \ell^1_v)$ with a so-called *admissible weight* v; see [19]. Similarly, the setting of the spaces $V = W(C_\alpha, \ell^q_v)$ (with the Hölder spaces C_α) is studied in [26]—but except in the case $q = 1$, some additional assumptions on the window function g are imposed.

To the best of our knowledge, the question has not been answered for modulation spaces other than $V = M^1_v$, and in particular, not for any of the spaces $V = H^1(\mathbb{R}^d)$, $V = L^2_{1+|x|}(\mathbb{R}^d)$, and $V = \mathbb{H}^1(\mathbb{R}^d)$. In this note, we show that the answer is affirmative for all of these spaces:

Theorem 1.1 *Let $V \in \{H^1(\mathbb{R}^d), L^2_{1+|x|}(\mathbb{R}^d), \mathbb{H}^1(\mathbb{R}^d)\}$. Let $g \in V$ and let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice such that the Gabor system (g, Λ) is a frame for $L^2(\mathbb{R}^d)$ with frame operator S. Then the canonical dual window $S^{-1}g$ belongs to V. Furthermore, $(S^{-1/2}g, \Lambda)$ is a Parseval frame for $L^2(\mathbb{R}^d)$ with $S^{-1/2}g \in V$.*

We emphasize that we do *not* show that the inverse frame operator maps V into itself; in fact, it maps the smaller space $V_\Lambda = \{f \in V :$ the Gabor system (f, Λ) is a Bessel system$\}$ into itself; see Proposition 4.5 below.

To indicate the practical relevance of Theorem 1.1, recall that a Gabor frame (g, Λ) for $L^2(\mathbb{R}^d)$ allows for the frame expansion $f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)S^{-1}g\rangle \pi(\lambda)g$ for every $f \in L^2(\mathbb{R}^d)$, where $\pi(\lambda)$ denotes the time-frequency shift by λ (see (2.1) below). The sequence $(c_\lambda)_{\lambda\in\Lambda}$ of the frame coefficients $c_\lambda = \langle f, \pi(\lambda)S^{-1}g\rangle$, $\lambda \in \Lambda$, in general only belongs to $\ell^2(\Lambda)$, which means that if one truncates the sum $f = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g$ to N terms (as is necessary in practical applications), the L^2-approximation error may decay arbitrarily slowly as $N \to \infty$.

However, one can impose a faster decay of the coefficients— and therefore improve the decay rate of the L^2-approximation error— by restricting f and g to certain subspaces of $L^2(\mathbb{R}^d)$. For example, if $f, g \in V = S_0(\mathbb{R}^d)$, then it is well-known that $\big(\langle f, \pi(\lambda)S^{-1}g\rangle\big)_{\lambda\in\Lambda} \in \ell^1(\Lambda)$. Then, Stechkin's inequality (see e.g. [11, Propositions 2.3 and 2.11]) implies that if one truncates the sum $f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)S^{-1}g\rangle \pi(\lambda)g$ to the N terms with the largest frame coefficients, the resulting approximation error will be $\mathcal{O}(N^{-1/2})$. If $f, g \in V = \mathbb{H}^1(\mathbb{R}^d)$ and if (f, Λ) is a Bessel sequence, then Theorem 1.1 combined with the proof of Proposition 3.2 shows that $\big(\lambda_k \langle f, \pi(\lambda)S^{-1}g\rangle\big)_{\lambda\in\Lambda} \in \ell^2(\Lambda)$ for each $k = 1, \dots, 2d$. Since $\big((1 + |\lambda|)^{-1}\big)_{\lambda\in\Lambda} \in \ell^{2d,\infty}(\Lambda)$, this implies by Hölder's inequality for weak Lebesgue spaces (cf. [4, Theorem 5.23]) that $\big(\langle f, \pi(\lambda)S^{-1}g\rangle\big)_{\lambda\in\Lambda} \in \ell^{\frac{2d}{d+1},\infty}(\Lambda)$, so that Stechkin's inequality shows that the error of truncating the frame expansion to the N terms with the largest frame coefficients decays like $\mathcal{O}(N^{-1/(2d)})$. At least in dimension $d = 1$, this is just as good as for $V = S_0(\mathbb{R}^d)$.

As mentioned above, the corresponding statement of Theorem 1.1 for $V = S_0(\mathbb{R}^d)$ with separable lattices Λ was proved in [14]. In addition to several deeper insights, the proof given in [14] relies on a simple but essential argument showing that the frame operator $S = S_{\Lambda,g}$ maps V boundedly into itself, which is shown in [14] based on Janssen's representation of $S_{\Lambda,g}$. In our setting, this argument is not applicable, because—unlike in the case of $V = S_0(\mathbb{R}^d)$— there exist functions $g \in \mathbb{H}^1$ for which (g, Λ) is not an L^2-Bessel system. In addition, the series in Janssen's representation is not guaranteed to converge unconditionally in the strong sense for $\mathbb{H}^1$-functions, even if (g, Λ) is an L^2-Bessel system; see Proposition A.1. To bypass these obstacles, we introduce for each space $V \in \{H^1, L^2_{1+|x|}, \mathbb{H}^1\}$ the associated subspace V_Λ consisting of all those functions $g \in V$ that generate a Bessel system over the given lattice Λ.

We remark that most of the existing works concerning the regularity of the (canonical) dual window rely on deep results related to Wiener's $1/f$-lemma on absolutely convergent Fourier series. In contrast, our methods are based on elementary spectral theory (see Sect. 4) and on certain observations regarding the interaction of the Gabor frame operator with partial derivatives; see Proposition 3.2.

The paper is organized as follows: Sect. 2 discusses the concept of Gabor Bessel vectors and introduces some related notions. Then, in Sect. 3, we endow the space V_Λ (for each choice $V \in \{H^1, L^2_{1+|x|}, \mathbb{H}^1\}$) with a Banach space norm and show that the frame operator S maps V_Λ boundedly into itself, provided that the Gabor system (g, Λ) is an L^2-Bessel system and that the window function g belongs to V. Finally, we prove in Sect. 4 that for any $V \in \{H^1, L^2_{1+|x|}, \mathbb{H}^1\}$ the spectrum of S as an operator on V coincides with the spectrum of S as an operator on L^2. This easily implies our main result, Theorem 1.1.

2 Bessel Vectors

For $a, b \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$ we define the operators of translation by a and modulation by b as

$$T_a f(x) := f(x-a) \quad \text{and} \quad M_b f(x) := e^{2\pi i b\cdot x} \cdot f(x),$$

respectively. Both T_a and M_b are unitary operators on $L^2(\mathbb{R}^d)$ and hence so is the *time-frequency shift*

$$\pi(a,b) := T_a M_b = e^{-2\pi i a\cdot b} M_b T_a. \tag{2.1}$$

The Fourier transform $\mathcal{F}$ is defined on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by $\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\,dx$ and extended to a unitary operator on $L^2(\mathbb{R}^d)$. For $z = (z_1, z_2) \in \mathbb{R}^d \times \mathbb{R}^d \cong \mathbb{R}^{2d}$ and $f \in L^2(\mathbb{R}^d)$, a direct calculation shows that

$$\mathcal{F}[\pi(z)f] = e^{-2\pi i z_1\cdot z_2} \cdot \pi(Jz)\widehat{f}, \tag{2.2}$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

A (full rank) *lattice* in $\mathbb{R}^{2d}$ is a set of the form $\Lambda = A\mathbb{Z}^{2d}$, where $A \in \mathbb{R}^{2d\times 2d}$ is invertible. The volume of Λ is defined by $\mathrm{Vol}(\Lambda) := |\det A|$ and its density by $d(\Lambda) := \mathrm{Vol}(\Lambda)^{-1}$. The *adjoint lattice* of Λ is denoted and defined by $\Lambda^\circ := JA^{-\top}\mathbb{Z}^{2d}$.

The Gabor system generated by a window function $g \in L^2(\mathbb{R}^d)$ and a lattice $\Lambda \subset \mathbb{R}^{2d}$ is given by

$$(g, \Lambda) := \{\pi(\lambda)g : \lambda \in \Lambda\}.$$

We say that $g \in L^2(\mathbb{R}^d)$ is a *Bessel vector* with respect to Λ if the system (g, Λ) is a Bessel system in $L^2(\mathbb{R}^d)$, meaning that the associated *analysis operator* $C_{\Lambda,g}$ defined by

$$C_{\Lambda,g} f := \big(\langle f, \pi(\lambda)g\rangle\big)_{\lambda\in\Lambda}, \qquad f \in L^2(\mathbb{R}^d), \tag{2.3}$$

is a bounded operator from $L^2(\mathbb{R}^d)$ to $\ell^2(\Lambda)$. We define

$$\mathcal{B}_\Lambda := \{g \in L^2(\mathbb{R}^d) : (g, \Lambda) \text{ is a Bessel system}\},$$

which is a dense linear subspace of $L^2(\mathbb{R}^d)$ because each Schwartz function is a Bessel vector with respect to any lattice; see [9, Theorem 3.3.1]. It is well-known that $\mathcal{B}_\Lambda = \mathcal{B}_{\Lambda^\circ}$ (see, e.g., [9, Proposition 3.5.10]). In fact, we have for $g \in \mathcal{B}_\Lambda$ that

$$\|C_{\Lambda^\circ,g}\| = \mathrm{Vol}(\Lambda)^{1/2} \cdot \|C_{\Lambda,g}\|; \tag{2.4}$$

see [18, proof of Theorem 2.3.1]. The *cross frame operator* $S_{\Lambda,g,h}$ with respect to Λ and two functions $g, h \in \mathcal{B}_\Lambda$ is defined by

$$S_{\Lambda,g,h} := C^*_{\Lambda,h} C_{\Lambda,g}.$$

In particular, we write $S_{\Lambda,g} := S_{\Lambda,g,g}$ which is called the *frame operator* of (g, Λ). The system (g, Λ) is called a *frame* if $S_{\Lambda,g}$ is bounded and boundedly invertible on $L^2(\mathbb{R}^d)$, that is, if $A\, \mathrm{Id}_{L^2(\mathbb{R}^d)} \le S_{\Lambda,g} \le B\, \mathrm{Id}_{L^2(\mathbb{R}^d)}$ for some constants $0 < A \le B < \infty$ (called the frame bounds). In particular, a frame with frame bounds $A = B = 1$ is called a *Parseval frame*.

In our proofs, the so-called *fundamental identity of Gabor analysis* will play an essential role. This identity states that

$$\sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \langle \pi(\lambda)\gamma, h\rangle = d(\Lambda) \cdot \sum_{\mu\in\Lambda^\circ} \langle \gamma, \pi(\mu)g\rangle \langle \pi(\mu) f, h\rangle. \tag{2.5}$$

It holds, for example, if $f, h \in M^1(\mathbb{R}^d) = S_0(\mathbb{R}^d)$ (the Feichtinger algebra) and $g, \gamma \in L^2(\mathbb{R}^d)$; see [9, Theorem 3.5.11]. Here, we will use the following version of the fundamental identity:

Lemma 2.1 *The fundamental identity* (2.5) *holds if* $g, h \in \mathcal{B}_\Lambda$ *or* $f, \gamma \in \mathcal{B}_\Lambda$.

Proof The proof can be carried out similarly as [13, Theorem 4.3.2 (ii)] which shows (2.5) under the assumption that $g, \gamma \in \mathcal{B}_\Lambda$ and $\sum_{\mu\in\Lambda^\circ} |\langle \gamma, \pi(\mu)g\rangle| < \infty$. Note that the latter condition guarantees the absolute convergence of the right-hand side of (2.5). In our case, each of the conditions $g, h \in \mathcal{B}_\Lambda$ and $f, \gamma \in \mathcal{B}_\Lambda$ already implies the absolute convergence of both sides of (2.5) (by the Cauchy-Schwarz inequality) so that the proof in [13] is also valid here. □

3 Certain Subspaces of Modulation Spaces Invariant Under the Frame Operator

The L^2-Sobolev-space $H^1(\mathbb{R}^d) = W^{1,2}(\mathbb{R}^d)$ is the space of all functions $f \in L^2(\mathbb{R}^d)$ whose distributional derivatives $\partial_j f := \frac{\partial f}{\partial x_j}$, $j \in \{1, \dots, d\}$, all belong to $L^2(\mathbb{R}^d)$. We will frequently use the well-known characterization

$$H^1(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) : (1 + |\cdot|)\widehat{f}(\cdot) \in L^2(\mathbb{R}^d) \right\}$$

of $H^1(\mathbb{R}^d)$ in terms of the Fourier transform. With the weight function

$$w : \mathbb{R}^d \to \mathbb{R}, \qquad x \mapsto 1 + |x|,$$

we define the weighted L^2-space $L^2_w(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) : w(\cdot) f(\cdot) \in L^2(\mathbb{R}^d)\}$, equipped with the norm $\|f\|_{L^2_w} := \|w f\|_{L^2}$. It is then clear that $L^2_w(\mathbb{R}^d) = \mathcal{F}[H^1(\mathbb{R}^d)] = \mathcal{F}^{-1}[H^1(\mathbb{R}^d)]$. Finally, we define $\mathbb{H}^1(\mathbb{R}^d) = H^1(\mathbb{R}^d) \cap L^2_w(\mathbb{R}^d)$

which is the space of all functions $f \in H^1(\mathbb{R}^d)$ whose Fourier transform $\widehat{f}$ also belongs to $H^1(\mathbb{R}^d)$. Equivalently, $\mathbb{H}^1(\mathbb{R}^d)$ is the space of all functions $g \in L^2(\mathbb{R}^d)$ with finite uncertainty product (1.1).

It is worth to note that each of the spaces $H^1(\mathbb{R}^d)$, $L^2_w(\mathbb{R}^d)$, and $\mathbb{H}^1(\mathbb{R}^d)$ can be expressed as a modulation space $M^2_m(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^{2d}} |\langle f, \pi(z)\varphi\rangle|^2 \, |m(z)|^2 \, dz < \infty\}$ with a suitable weight function $m : \mathbb{R}^{2d} \to \mathbb{C}$, where $\varphi \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$ is any fixed function,[1] for instance a Gaussian. Indeed, we have

$$H^1(\mathbb{R}^d) = M^2_{m_1}(\mathbb{R}^d), \quad L^2_w(\mathbb{R}^d) = M^2_{m_2}(\mathbb{R}^d), \quad \text{and}$$
$$\mathbb{H}^1(\mathbb{R}^d) = H^1(\mathbb{R}^d) \cap L^2_w(\mathbb{R}^d) = M^2_{m_3}(\mathbb{R}^d),$$

with

$$m_1(x,\omega) = 1 + |\omega|, \quad m_2(x,\omega) = 1 + |x|, \quad \text{and} \quad m_3(x,\omega) = \sqrt{1 + |x|^2 + |\omega|^2},$$

see [12, Proposition 11.3.1] and [25, Corollary 2.3].

Our main goal in this paper is to prove for each of these spaces that if the window function g of a Gabor frame (g, Λ) belongs to the space, then so does the canonical dual window. In this section, we will mostly concentrate on the space $H^1(\mathbb{R}^d)$, since this will imply the desired result for the other spaces as well.

The corresponding result for the Feichtinger algebra $S_0(\mathbb{R}^d)$ was proved in [14] by showing the much stronger statement that the frame operator maps $S_0(\mathbb{R}^d)$ boundedly into itself and is in fact boundedly invertible on $S_0(\mathbb{R}^d)$. However, the methods used in [14] cannot be directly transferred to the case of a window function in $\mathbb{H}^1(\mathbb{R}^d)$ (or $H^1(\mathbb{R}^d)$), since the proof in [14] leverages two particular properties of the Feichtinger algebra which are not shared by $\mathbb{H}^1(\mathbb{R}^d)$:

(a) Every function from $S_0(\mathbb{R}^d)$ is a Bessel vector with respect to any given lattice;
(b) The series in Janssen's representation of the frame operator converges strongly (even absolutely in operator norm) to the frame operator when the window function belongs to $S_0(\mathbb{R}^d)$.

Indeed, it is well-known that $g \in L^2(\mathbb{R})$ is a Bessel vector with respect to $\mathbb{Z} \times \mathbb{Z}$ if and only if the Zak transform of g is essentially bounded (cf. [2, Theorem 3.1]), but [2, Example 3.4] provides an example of a function $g \in \mathbb{H}^1(\mathbb{R})$ whose Zak transform is not essentially bounded; this indicates that (a) does not hold for $\mathbb{H}^1(\mathbb{R}^d)$ instead of $S_0(\mathbb{R}^d)$. Concerning the statement (b) for $\mathbb{H}^1(\mathbb{R}^d)$, it is easy to see that if Janssen's representation converges strongly (with respect to some enumeration of $\mathbb{Z}^2$) to the frame operator of (g, Λ), then the frame operator must be bounded on $L^2(\mathbb{R})$ and thus the associated window function g is necessarily a Bessel vector. Therefore, the example above again serves as a counterexample: namely, the statement (b) fails for such a non-Bessel window function $g \in \mathbb{H}^1(\mathbb{R})$. Even more, we show in the Appendix that there exist Bessel vectors $g \in \mathbb{H}^1(\mathbb{R})$ for which Janssen's representation neither converges unconditionally in the strong sense nor conditionally in the operator norm.

[1] The definition of M^2_m is known to be independent of the choice of φ; see e.g., [12, Proposition 11.3.2].

We mention that in the case of the Wiener amalgam space $W(L^\infty, \ell^1_v)$ with an admissible weight v, the convergence issue was circumvented by employing Walnut's representation instead of Janssen's to prove the result for $W(L^\infty, \ell^1_v)$ in [19].

Fortunately, it turns out that establishing the corresponding result for $V = H^1(\mathbb{R}^d)$, $L^2_w(\mathbb{R}^d)$, and $\mathbb{H}^1(\mathbb{R}^d)$ only requires the invertibility of the frame operator on a particular subspace of V. Precisely, given a lattice $\Lambda \subset \mathbb{R}^{2d}$, we define

$$H^1_\Lambda(\mathbb{R}^d) := H^1(\mathbb{R}^d) \cap \mathcal{B}_\Lambda, \quad \mathbb{H}^1_\Lambda(\mathbb{R}^d) := \mathbb{H}^1(\mathbb{R}^d) \cap \mathcal{B}_\Lambda, \quad \text{and}$$
$$L^2_{w,\Lambda}(\mathbb{R}^d) := L^2_w(\mathbb{R}^d) \cap \mathcal{B}_\Lambda.$$

We equip the first two of these spaces with the norms

$$\|f\|_{H^1_\Lambda} := \|\nabla f\|_{L^2} + \|C_{\Lambda,f}\|_{L^2\to\ell^2} \qquad \text{and}$$
$$\|f\|_{\mathbb{H}^1_\Lambda} := \|\nabla f\|_{L^2} + \|\nabla \widehat{f}\|_{L^2} + \|C_{\Lambda,f}\|_{L^2\to\ell^2},$$

respectively, where

$$\|\nabla f\|_{L^2} := \sum_{j=1}^{d} \|\partial_j f\|_{L^2}$$

and $C_{\Lambda,f}$ is the analysis operator defined in (2.3). Finally, we equip the space $L^2_{w,\Lambda}(\mathbb{R}^d)$ with the norm

$$\|f\|_{L^2_{w,\Lambda}} := \|f\|_{L^2_w} + \|C_{\Lambda,f}\|_{L^2,\ell^2}, \qquad \text{where} \qquad \|f\|_{L^2_w} := \|w \cdot f\|_{L^2}.$$

We start by showing that these spaces are Banach spaces.

Lemma 3.1 *For a lattice $\Lambda \subset \mathbb{R}^{2d}$, the spaces $H^1_\Lambda(\mathbb{R}^d)$, $L^2_{w,\Lambda}(\mathbb{R}^d)$, and $\mathbb{H}^1_\Lambda(\mathbb{R}^d)$ are Banach spaces which are continuously embedded in $L^2(\mathbb{R}^d)$.*

Proof We naturally equip the space $\mathcal{B}_\Lambda \subset L^2(\mathbb{R}^d)$ with the norm $\|f\|_{\mathcal{B}_\Lambda} := \|C_{\Lambda,f}\|_{L^2\to\ell^2}$. Then $(\mathcal{B}_\Lambda, \|\cdot\|_{\mathcal{B}_\Lambda})$ is a Banach space by [15, Proposition 3.1]. Moreover, for $f \in \mathcal{B}_\Lambda$,

$$\|f\|_{L^2} = \left\|C^*_{\Lambda,f}\,\delta_{0,0}\right\|_{L^2} \le \|C^*_{\Lambda,f}\|_{\ell^2\to L^2} = \|f\|_{\mathcal{B}_\Lambda}, \tag{3.1}$$

which implies that $\mathcal{B}_\Lambda \hookrightarrow L^2(\mathbb{R}^d)$. Hence, if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1_\Lambda(\mathbb{R}^d)$, then it is a Cauchy sequence in both $H^1(\mathbb{R}^d)$ (equipped with the norm $\|f\|_{H^1} := \|f\|_{L^2} + \|\nabla f\|_{L^2}$) and in $\mathcal{B}_\Lambda$. Therefore, there exist $f \in H^1(\mathbb{R}^d)$ and $g \in \mathcal{B}_\Lambda$ such that $\|f_n - f\|_{H^1} \to 0$ and $\|f_n - g\|_{\mathcal{B}_\Lambda} \to 0$ as $n \to \infty$. But as $H^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ and $\mathcal{B}_\Lambda \hookrightarrow L^2(\mathbb{R}^d)$, we have $f_n \to f$ and $f_n \to g$ also in $L^2(\mathbb{R}^d)$, which implies $f = g$. Hence, $\|f_n - f\|_{H^1_\Lambda} \to 0$ as $n \to \infty$, which proves that $H^1_\Lambda(\mathbb{R}^d)$ is complete. The proof for $L^2_{w,\Lambda}(\mathbb{R}^d)$ and $\mathbb{H}^1_\Lambda(\mathbb{R}^d)$ is similar. □

Proposition 3.2 *Let* $\Lambda \subset \mathbb{R}^{2d}$ *be a lattice. If* $g, h \in H^1_\Lambda(\mathbb{R}^d)$, *then* $S_{\Lambda,g,h}$ *maps* $H^1_\Lambda(\mathbb{R}^d)$ *boundedly into itself with operator norm not exceeding* $\|g\|_{H^1_\Lambda}\|h\|_{H^1_\Lambda}$. *For* $f \in H^1_\Lambda(\mathbb{R}^d)$ *and* $j \in \{1, \ldots, d\}$ *we have*

$$\partial_j(S_{\Lambda,g,h} f) = S_{\Lambda,g,h}(\partial_j f) + d(\Lambda) \cdot C^*_{\Lambda^\circ, f}\, d_{j,\Lambda^\circ,g,h}, \tag{3.2}$$

where $d_{j,\Lambda^\circ,g,h} \in \ell^2(\Lambda^\circ)$ *is defined by*

$$(d_{j,\Lambda^\circ,g,h})_\mu := \big\langle \partial_j h, \pi(\mu)g\big\rangle + \big\langle h, \pi(\mu)(\partial_j g)\big\rangle, \quad \mu \in \Lambda^\circ. \tag{3.3}$$

Proof Let $f \in H^1_\Lambda(\mathbb{R}^d)$ and set $u := S_{\Lambda,g,h} f$. First of all, we have $u \in \mathcal{B}_\Lambda$. Indeed, a direct computation shows that $S_{\Lambda,g,h}$ commutes with $\pi(\lambda)$ for all $\lambda \in \Lambda$, and that $S^*_{\Lambda,g,h} = S_{\Lambda,h,g}$, which shows for $v \in L^2(\mathbb{R}^d)$ that

$$(C_{\Lambda,u} v)_\lambda = \langle v, \pi(\lambda)u\rangle = \langle v, \pi(\lambda) S_{\Lambda,g,h} f\rangle = \langle S_{\Lambda,h,g} v, \pi(\lambda) f\rangle = (C_{\Lambda,f}\, S_{\Lambda,h,g}\, v)_\lambda,$$

and therefore

$$\|C_{\Lambda,u}\| \le \|S_{\Lambda,h,g}\| \cdot \|C_{\Lambda,f}\| \le \|C_{\Lambda,g}\| \cdot \|C_{\Lambda,h}\| \cdot \|C_{\Lambda,f}\| < \infty, \tag{3.4}$$

since $S_{\Lambda,h,g} = C^*_{\Lambda,g} C_{\Lambda,h}$.

We now show that $u \in H^1(\mathbb{R}^d)$. To this end, note for $v \in H^1(\mathbb{R}^d)$, $a, b \in \mathbb{R}^d$, and $j \in \{1, \ldots, d\}$ that

$$\partial_j(M_b v) = 2\pi i \cdot b_j \cdot M_b v + M_b(\partial_j v) \qquad \text{and} \qquad \partial_j(T_a v) = T_a(\partial_j v)$$

and therefore

$$\partial_j(\pi(z)v) = 2\pi i \cdot z_{d+j} \cdot \pi(z)v + \pi(z)(\partial_j v).$$

Hence, setting $c_{\lambda,j} := 2\pi i \cdot \lambda_{d+j} \cdot \langle f, \pi(\lambda)g\rangle$ for $\lambda = (a, b) \in \Lambda$, we see that

$$c_{\lambda,j} = \langle \partial_j f, \pi(\lambda)g\rangle + \langle f, \pi(\lambda)(\partial_j g)\rangle. \tag{3.5}$$

In particular, $(c_{\lambda,j})_{\lambda\in\Lambda} \in \ell^2(\Lambda)$ for each $j \in \{1, \ldots, d\}$, because $f, g \in \mathcal{B}_\Lambda$ and $\partial_j f, \partial_j g \in L^2$.

In order to show that $\partial_j u$ exists and is in $L^2(\mathbb{R}^d)$, let $\phi \in C^\infty_c(\mathbb{R}^d)$ be a test function. Note that $C^\infty_c(\mathbb{R}^d) \subset \mathcal{B}_\Lambda$. Therefore, we obtain

$$\begin{aligned}
-\big\langle u, \partial_j \phi\big\rangle &= -\sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\big\rangle\big\langle \pi(\lambda)h, \partial_j \phi\big\rangle \\
&= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\big\rangle\big\langle 2\pi i \lambda_{d+j} \cdot \pi(\lambda)h + \pi(\lambda)(\partial_j h), \phi\big\rangle \\
&= \sum_{\lambda\in\Lambda} c_{\lambda,j} \cdot \big\langle \pi(\lambda)h, \phi\big\rangle + \sum_{\lambda\in\Lambda} \big\langle f, \pi(\lambda)g\big\rangle\big\langle \pi(\lambda)(\partial_j h), \phi\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&\overset{(3.5)}{=} \langle S_{\Lambda,g,h}(\partial_j f), \phi\rangle + \sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)(\partial_j g)\rangle\langle\pi(\lambda)h, \phi\rangle \\
&\quad + \sum_{\lambda\in\Lambda}\big\langle f, \pi(\lambda)g\big\rangle\big\langle\pi(\lambda)(\partial_j h), \phi\big\rangle \\
&\overset{(2.5)}{=} \langle S_{\Lambda,g,h}(\partial_j f), \phi\rangle + d(\Lambda)\sum_{\mu\in\Lambda^\circ}\Big[\big\langle h, \pi(\mu)(\partial_j g)\big\rangle + \big\langle\partial_j h, \pi(\mu)g\big\rangle\Big]\big\langle\pi(\mu)f, \phi\big\rangle \\
&= \Big\langle S_{\Lambda,g,h}(\partial_j f) + d(\Lambda)\sum_{\mu\in\Lambda^\circ}\Big[\big\langle h, \pi(\mu)(\partial_j g)\big\rangle + \big\langle\partial_j h, \pi(\mu)g\big\rangle\Big]\pi(\mu)f\,,\ \phi\Big\rangle \\
&= \Big\langle S_{\Lambda,g,h}(\partial_j f) + d(\Lambda)\cdot C^*_{\Lambda^\circ,f}\, d_j\,,\ \phi\Big\rangle,
\end{aligned}
$$

with $d_j = d_{j,\Lambda^\circ,g,h}$ as in (3.3). Note that $d_j \in \ell^2(\Lambda^\circ)$ because $g, h \in \mathcal{B}_\Lambda = \mathcal{B}_{\Lambda^\circ}$ and $\partial_j h, \partial_j g \in L^2$. Since $j \in \{1, \ldots, d\}$ is chosen arbitrarily, this proves that $u \in H^1(\mathbb{R}^d)$ with

$$
\partial_j u = S_{\Lambda,g,h}(\partial_j f) + d(\Lambda)\cdot C^*_{\Lambda^\circ,f}\, d_j \ \in\ L^2(\mathbb{R}^d)
$$

for $j \in \{1, \ldots, d\}$, which is (3.2). Next, recalling Eq. (2.4) we get

$$
\begin{aligned}
\|d_j\|_{\ell^2} &\le \|C_{\Lambda^\circ,h}\|\cdot\|\partial_j g\|_{L^2} + \|C_{\Lambda^\circ,g}\|\cdot\|\partial_j h\|_{L^2} \\
&= \mathrm{Vol}(\Lambda)^{1/2}\big(\|C_{\Lambda,h}\|\cdot\|\partial_j g\|_{L^2} + \|C_{\Lambda,g}\|\cdot\|\partial_j h\|_{L^2}\big),
\end{aligned}
$$

and $\|C^*_{\Lambda^\circ,f}\| = \mathrm{Vol}(\Lambda)^{1/2}\|C_{\Lambda,f}\|$. Therefore,

$$
\|\partial_j u\|_{L^2} \le \|S_{\Lambda,g,h}\|\cdot\|\partial_j f\|_{L^2} + \big(\|C_{\Lambda,h}\|\cdot\|\partial_j g\|_{L^2} + \|C_{\Lambda,g}\|\cdot\|\partial_j h\|_{L^2}\big)\,\|C_{\Lambda,f}\|.
$$

Hence, with (3.4), we see

$$
\begin{aligned}
\|S_{\Lambda,g,h}f\|_{H^1_\Lambda} &= \|\nabla u\|_{L^2} + \|C_{\Lambda,u}\| \le \sum_{j=1}^d \|\partial_j u\|_{L^2} + \|C_{\Lambda,g}\|\cdot\|C_{\Lambda,h}\|\cdot\|C_{\Lambda,f}\| \\
&\le \|S_{\Lambda,g,h}\|\cdot\|\nabla f\|_{L^2} + \big(\|C_{\Lambda,h}\|\cdot \\
&\quad \|\nabla g\|_{L^2} + \|C_{\Lambda,g}\|\cdot\|\nabla h\|_{L^2} + \|C_{\Lambda,g}\|\cdot\|C_{\Lambda,h}\|\big)\|C_{\Lambda,f}\| \\
&\le \|C_{\Lambda,g}\|\cdot\|C_{\Lambda,h}\|\cdot\|\nabla f\|_{L^2} + \big(\|\nabla g\|_{L^2} \\
&\quad + \|C_{\Lambda,g}\|\big)\big(\|\nabla h\|_{L^2} + \|C_{\Lambda,h}\|\big)\,\|C_{\Lambda,f}\| \\
&\le \|g\|_{H^1_\Lambda}\|h\|_{H^1_\Lambda}\cdot\|f\|_{H^1_\Lambda},
\end{aligned}
$$

and the proposition is proved. □

4 Spectrum and Dual Windows

Let X be a Banach space. As usual, we denote the set of bounded linear operators from X into itself by $\mathcal{B}(X)$. The *resolvent set* $\rho(T)$ of an operator $T \in \mathcal{B}(X)$ is the set of all $z \in \mathbb{C}$ for which $T - z := T - zI : X \to X$ is bijective. Note that $\rho(T)$ is always open in $\mathbb{C}$. The *spectrum* of T is the complement $\sigma(T) := \mathbb{C} \setminus \rho(T)$. The *approximate point spectrum* $\sigma_{ap}(T)$ is a subset of $\sigma(T)$ and is defined as the set of points $z \in \mathbb{C}$ for which there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset X$ such that $\|f_n\| = 1$ for all $n \in \mathbb{N}$ and $\|(T - z) f_n\| \to 0$ as $n \to \infty$. By [6, Proposition VII.6.7] we have

$$\partial\sigma(T) \subset \sigma_{ap}(T). \tag{4.1}$$

Lemma 4.1 *Let $(\mathcal{H}, \|\cdot\|)$ be a Hilbert space, let $S \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and let $X \subset \mathcal{H}$ be a dense linear subspace satisfying $SX \subset X$. If $\|\cdot\|_X$ is a norm on X such that $(X, \|\cdot\|_X)$ is complete and satisfies $X \hookrightarrow \mathcal{H}$, then $A := S|_X \in \mathcal{B}(X)$. If, in addition, $\sigma_{ap}(A) \subset \sigma(S)$, then $\sigma(A) = \sigma(S)$.*

Proof The fact that $A \in \mathcal{B}(X)$ easily follows from the closed graph theorem. Next, since $X \hookrightarrow \mathcal{H}$, there exists $C > 0$ with $\|f\| \leq C\|f\|_X$ for all $f \in X$. Assume now that additionally $\sigma_{ap}(A) \subset \sigma(S)$ holds. Note that $\sigma(S) \subset \mathbb{R}$, since S is self-adjoint. Since $\sigma(A) \subset \mathbb{C}$ is compact, the value $r := \max_{w\in\sigma(A)} |\Im w|$ exists. Choose $z \in \sigma(A)$ such that $|\Im z| = r$. Clearly, z cannot belong to the interior of $\sigma(A)$, and hence $z \in \partial\sigma(A)$. In view of Eq. (4.1), this implies $z \in \sigma_{ap}(A) \subset \sigma(S) \subset \mathbb{R}$, hence $r = 0$ and thus $\sigma(A) \subset \mathbb{R}$. Therefore, $\sigma(A)$ has empty interior in $\mathbb{C}$, meaning $\sigma(A) = \partial\sigma(A)$. Thanks to Eq. (4.1), this means $\sigma(A) \subset \sigma_{ap}(A)$, and hence $\sigma(A) \subset \sigma(S)$, since by assumption $\sigma_{ap}(A) \subset \sigma(S)$.

For the converse inclusion it suffices to show that $\rho(A) \cap \mathbb{R} \subset \rho(S)$. To see that this holds, let $z \in \rho(A) \cap \mathbb{R}$ and denote by E the spectral measure of the self-adjoint operator S. Since $\mathbb{R} \cap \rho(A) \subset \mathbb{R}$ is open, there are $a, b \in \mathbb{R}$ and $\delta_0 > 0$ such that $z \in (a, b)$ and $[a - \delta_0, b + \delta_0] \subset \rho(A)$. By Stone's formula (see, e.g., [21, Thm. VII.13]), the spectral projection of S with respect to $(a, b]$ can be expressed as

$$E((a,b])f = \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_{a+\delta}^{b+\delta} \left[(S - t - i\varepsilon)^{-1} f - (S - t + i\varepsilon)^{-1} f\right] dt, \quad f \in \mathcal{H},$$

where all limits are taken with respect to the norm of $\mathcal{H}$.

Note for $w \in \mathbb{C} \setminus \mathbb{R}$ that $w \in \rho(S) \subset \rho(A)$. Furthermore, $A - w = (S - w)|_X$, which easily implies $(S - w)^{-1}|_X = (A - w)^{-1}$. Hence, for $f \in X$,

$$\begin{aligned}
\|E((a,b])f\| &\leq \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi} \int_{a+\delta}^{b+\delta} \left\|(S - t - i\varepsilon)^{-1} f - (S - t + i\varepsilon)^{-1} f\right\| dt \\
&\leq C \cdot \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi} \int_{a+\delta}^{b+\delta} \left\|(A - t - i\varepsilon)^{-1} f - (A - t + i\varepsilon)^{-1} f\right\|_X dt
\end{aligned}$$

$$= C \cdot \lim_{\delta \downarrow 0} \frac{1}{2\pi} \int_{a+\delta}^{b+\delta} \lim_{\varepsilon \downarrow 0} \left\| (A - t - i\varepsilon)^{-1} f - (A - t + i\varepsilon)^{-1} f \right\|_X dt$$
$$= 0,$$

since the map $\rho(A) \to X, z \mapsto (A - z)^{-1} f$ is analytic and thus uniformly continuous on compact sets. This implies $E((a, b]) f = 0$ for all $f \in X$ and therefore $E((a, b]) = 0$ as X is dense in $\mathcal{H}$. But this means that $(a, b) \subset \rho(S)$ (see [21, Prop. on p. 236]) and thus $z \in \rho(S)$. □

For proving the invertibility of $S_{\Lambda,g}$ on H^1_Λ, $L^2_{w,\Lambda}$, and $\mathbb{H}^1_\Lambda$, we first focus on the space $H^1_\Lambda(\mathbb{R}^d)$. Note that if $g \in H^1_\Lambda(\mathbb{R}^d)$, then $S_{\Lambda,g}$ maps $H^1_\Lambda(\mathbb{R}^d)$ boundedly into itself by Proposition 3.2. For $g \in H^1_\Lambda(\mathbb{R}^d)$, we will denote the restriction of $S_{\Lambda,g}$ to $H^1_\Lambda(\mathbb{R}^d)$ by $A_{\Lambda,g}$; that is, $A_{\Lambda,g} := S_{\Lambda,g}|_{H^1_\Lambda(\mathbb{R}^d)} \in \mathcal{B}(H^1_\Lambda(\mathbb{R}^d))$.

Theorem 4.2 *Let $\Lambda \subset \mathbb{Z}^{2d}$ be a lattice and let $g \in H^1_\Lambda(\mathbb{R}^d)$. Then*

$$\sigma(A_{\Lambda,g}) = \sigma(S_{\Lambda,g}).$$

Proof For brevity, we set $A := A_{\Lambda,g}$ and $S := S_{\Lambda,g}$. Due to Lemma 4.1, we only have to prove that $\sigma_{ap}(A) \subset \sigma(S)$. For this, let $z \in \sigma_{ap}(A)$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset H^1_\Lambda(\mathbb{R}^d)$ such that $\|f_n\|_{H^1_\Lambda} = 1$ for all $n \in \mathbb{N}$ and $\|(A - z) f_n\|_{H^1_\Lambda} \to 0$ as $n \to \infty$. The latter means that, for each $j \in \{1, \ldots, d\}$,

$$\left\| \partial_j (S f_n) - z \cdot (\partial_j f_n) \right\|_{L^2} \to 0 \quad \text{and} \quad \left\| C_{\Lambda,(S-z) f_n} \right\| \to 0. \tag{4.2}$$

Suppose towards a contradiction that $z \notin \sigma(S)$. Since S is self-adjoint, this implies $\overline{z} \notin \sigma(S)$. Furthermore, because S is self-adjoint and commutes with $\pi(\lambda)$ for all $\lambda \in \Lambda$, we see for $f \in \mathcal{B}_\Lambda$ that $C_{\Lambda,(S-z)f} = C_{\Lambda,f} \circ (S - \overline{z})$ and hence $C_{\Lambda,f_n} = C_{\Lambda,(S-z)f_n} \circ (S - \overline{z})^{-1}$, which implies that $\|C_{\Lambda,f_n}\| \to 0$. Hence, also $\|C_{\Lambda^\circ,f_n}\| \to 0$ as $n \to \infty$ (see Eq. (2.4)). Now, by Eq. (3.2), we have

$$\partial_j (S f_n) - z \cdot (\partial_j f_n) = (S - z)(\partial_j f_n) + C^*_{\Lambda^\circ, f_n} d_j$$

with some $d_j \in \ell^2(\Lambda^\circ)$ which is *independent of* n. Hence, the first limit in (4.2) combined with $\|C_{\Lambda^\circ,f_n}\| \to 0$ implies that $\|(S - z)(\partial_j f_n)\|_{L^2} \to 0$ and thus $\|\partial_j f_n\|_{L^2} \to 0$ as $n \to \infty$ for all $j \in \{1, \ldots, d\}$, since $z \notin \sigma(S)$. Hence, $\|f_n\|_{H^1_\Lambda} = \sum_{j=1}^d \|\partial_j f_n\|_{L^2} + \|C_{\Lambda,f_n}\| \to 0$ as $n \to \infty$, in contradiction to $\|f_n\|_{H^1_\Lambda} = 1$ for all $n \in \mathbb{N}$. This proves that, indeed, $\sigma_{ap}(A) \subset \sigma(S)$. □

We now show analogous properties to Proposition 3.2 and Theorem 4.2 for $L^2_{w,\Lambda}(\mathbb{R}^d)$.

Corollary 4.3 *Let $\Lambda \subset \mathbb{Z}^{2d}$ be a lattice. If $g, h \in L^2_{w,\Lambda}(\mathbb{R}^d)$, then $S_{\Lambda,g,h}$ maps $L^2_{w,\Lambda}(\mathbb{R}^d)$ boundedly into itself.*

If $g = h$ and if $A^w_{\Lambda,g} := S_{\Lambda,g}|_{L^2_{w,\Lambda}(\mathbb{R}^d)} \in \mathcal{B}(L^2_{w,\Lambda}(\mathbb{R}^d))$ denotes the restriction of $S_{\Lambda,g}$ to $L^2_{w,\Lambda}(\mathbb{R}^d)$, then

$$\sigma(A^w_{\Lambda,g}) = \sigma(S_{\Lambda,g}).$$

Proof We equip the space $\mathcal{B}_\Lambda \subset L^2(\mathbb{R}^d)$ with the norm $\|f\|_{\mathcal{B}_\Lambda} := \|C_{\Lambda,f}\|_{L^2\to\ell^2}$, where we recall from Eq. (3.1) that $\|f\|_{L^2} \le \|f\|_{\mathcal{B}_\Lambda}$. Equation (2.2) shows that the Fourier transform is an isometric isomorphism from $\mathcal{B}_\Lambda$ to $\mathcal{B}_{\widehat{\Lambda}}$, where $\widehat{\Lambda} := J\Lambda$. Furthermore, it is well-known (see for instance [10, Sect. 9.3]) that the Fourier transform $\mathcal{F} : L^2 \to L^2$ restricts to an isomorphism of Banach spaces $\mathcal{F} : L^2_w(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$, where H^1 is equipped with the norm $\|f\|_{H^1} := \|f\|_{L^2} + \|\nabla f\|_{L^2}$. Taken together, we thus see that the Fourier transform restricts to an isomorphism $\mathcal{F} : L^2_{w,\Lambda}(\mathbb{R}^d) \to H^1_{\widehat{\Lambda}}(\mathbb{R}^d)$; here, we implicitly used that $\|f\|_{H^1_{\widehat{\Lambda}}} \asymp \|f\|_{H^1} + \|f\|_{\mathcal{B}_{\widehat{\Lambda}}}$, which follows from $\|\cdot\|_{L^2} \le \|\cdot\|_{\mathcal{B}_{\widehat{\Lambda}}}$.

Plancherel's theorem, in combination with Eq. (2.2) shows for $f \in L^2(\mathbb{R}^d)$ that

$$\begin{aligned}\mathcal{F}\big[S_{\Lambda,g,h}f\big] &= \sum_{\lambda\in\Lambda} \Big\langle \widehat{f}, \widehat{\pi(\lambda)g}\Big\rangle \widehat{\pi(\lambda)h} = \sum_{\lambda\in\Lambda} \langle \widehat{f}, \pi(J\lambda)\widehat{g}\rangle \pi(J\lambda)\widehat{h} \\ &= \sum_{\lambda\in\widehat{\Lambda}} \langle \widehat{f}, \pi(\lambda)\widehat{g}\rangle \pi(\lambda)\widehat{h} = S_{\widehat{\Lambda},\widehat{g},\widehat{h}}\widehat{f}.\end{aligned}$$

Since $A_{\widehat{\Lambda},\widehat{g},\widehat{h}} = S_{\widehat{\Lambda},\widehat{g},\widehat{h}}|_{H^1_{\widehat{\Lambda}}} : H^1_{\widehat{\Lambda}}(\mathbb{R}^d) \to H^1_{\widehat{\Lambda}}(\mathbb{R}^d)$ is well-defined and bounded by Proposition 3.2, the preceding calculation combined with the considerations from the previous paragraph shows that $A^w_{\Lambda,g,h} = S_{\Lambda,g,h}|_{L^2_{w,\Lambda}(\mathbb{R}^d)} : L^2_{w,\Lambda}(\mathbb{R}^d) \to L^2_{w,\Lambda}(\mathbb{R}^d)$ is well-defined and bounded, with

$$A^w_{\Lambda,g,h} = \mathcal{F}^{-1} \circ A_{\widehat{\Lambda},\widehat{g},\widehat{h}} \circ \mathcal{F}.$$

Finally, if $g = h$, we see $\sigma(A^w_{\Lambda,g,g}) = \sigma(A_{\widehat{\Lambda},\widehat{g},\widehat{g}}) = \sigma(S_{\widehat{\Lambda},\widehat{g},\widehat{g}}) = \sigma(S_{\Lambda,g,g})$, where the second step is due to Theorem 4.2, and the final step used the identity $S_{\Lambda,g,h} = \mathcal{F}^{-1} \circ S_{\widehat{\Lambda},\widehat{g},\widehat{h}} \circ \mathcal{F}$ from above. □

Finally, we establish the corresponding properties for $\mathbb{H}^1_\Lambda(\mathbb{R}^d) = H^1_\Lambda(\mathbb{R}^d) \cap L^2_{w,\Lambda}(\mathbb{R}^d)$.

Corollary 4.4 *Let $\Lambda \subset \mathbb{Z}^{2d}$ be a lattice. If $g, h \in \mathbb{H}^1_\Lambda(\mathbb{R}^d)$, then $S_{\Lambda,g,h}$ maps $\mathbb{H}^1_\Lambda(\mathbb{R}^d)$ boundedly into itself. If $g = h$ and $\mathbb{A}_{\Lambda,g} := S_{\Lambda,g}|_{\mathbb{H}^1_\Lambda(\mathbb{R}^d)} \in \mathcal{B}(\mathbb{H}^1_\Lambda(\mathbb{R}^d))$ denotes the restriction of $S_{\Lambda,g}$ to $\mathbb{H}^1_\Lambda(\mathbb{R}^d)$, then*

$$\sigma(\mathbb{A}_{\Lambda,g}) = \sigma(S_{\Lambda,g}). \tag{4.3}$$

Proof From the definition of $\mathbb{H}^1_\Lambda$ and the proof of Corollary 4.3 it is easy to see that $\mathbb{H}^1_\Lambda = H^1_\Lambda \cap L^2_{w,\Lambda}(\mathbb{R}^d)$, and $\|\cdot\|_{\mathbb{H}^1_\Lambda} \asymp \|\cdot\|_{H^1_\Lambda} + \|\cdot\|_{L^2_{w,\Lambda}}$. Therefore, Proposition 3.2 and Corollary 4.3 imply that $S_{\Lambda,g,h}$ maps $\mathbb{H}^1_\Lambda(\mathbb{R}^d)$ boundedly into itself.

Lemma 4.1 shows that to prove (4.3), it suffices to show $\sigma_{ap}(\mathbb{A}_{\Lambda,g}) \subset \sigma(S_{\Lambda,g})$. Thus, let $z \in \sigma_{ap}(\mathbb{A}_{\Lambda,g})$. Then there exists $(f_n)_{n\in\mathbb{N}} \subset \mathbb{H}^1_\Lambda(\mathbb{R}^d)$ with $\|f_n\|_{\mathbb{H}^1_\Lambda} = 1$ for all $n \in \mathbb{N}$ and $\|(\mathbb{A}_{\Lambda,g} - z)f_n\|_{\mathbb{H}^1_\Lambda} \to 0$ as $n \to \infty$. Thus, $\|(A_{\Lambda,g} - z)f_n\|_{H^1_\Lambda} \to 0$ and $\|(A^w_{\Lambda,g} - z)f_n\|_{L^2_{w,\Lambda}} \to 0$ as $n \to \infty$. Furthermore, there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \|f_{n_k}\|_{H^1_\Lambda} > 0$ or $\lim_{k\to\infty} \|f_{n_k}\|_{L^2_{w,\Lambda}} > 0$. Hence, $z \in \sigma(A_{\Lambda,g})$ or $z \in \sigma(A^w_{\Lambda,g})$. But Theorem 4.2 and Corollary 4.3 show $\sigma(A_{\Lambda,g}) = \sigma(A^w_{\Lambda,g}) = \sigma(S_{\Lambda,g})$. We have thus shown $\sigma_{ap}(\mathbb{A}_{\Lambda,g}) \subset \sigma(S_{\Lambda,g})$, so that Lemma 4.1 shows $\sigma(\mathbb{A}_{\Lambda,g}) = \sigma(S_{\Lambda,g})$. □

The next proposition shows that any operator obtained from $S_{\Lambda,g}$ through the holomorphic spectral calculus (see [22, Sects. 10.21–10.29] for a definition) maps each of the spaces $H^1_\Lambda(\mathbb{R}^d)$, $L^2_{w,\Lambda}(\mathbb{R}^d)$, and $\mathbb{H}^1_\Lambda(\mathbb{R}^d)$ into itself.

Proposition 4.5 *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice, let $V \in \{H^1_\Lambda(\mathbb{R}^d), L^2_{w,\Lambda}(\mathbb{R}^d), \mathbb{H}^1_\Lambda(\mathbb{R}^d)\}$, and $g \in V$. Then for any open set $\Omega \subset \mathbb{C}$ with $\sigma(S_{\Lambda,g}) \subset \Omega$, any analytic function $F : \Omega \to \mathbb{C}$, and any $f \in V$, we have $F(S_{\Lambda,g})f \in V$.*

Proof We only prove the claim for $V = H^1_\Lambda(\mathbb{R}^d)$; the proofs for the other cases are similar, using Corollaries 4.3 or 4.4 instead of Theorem 4.2. Thus, let $g \in H^1_\Lambda(\mathbb{R}^d)$ and set $S := S_{\Lambda,g}$ and $A := A_{\Lambda,g}$. Let $f \in H^1_\Lambda(\mathbb{R}^d)$ and define

$$h = -\frac{1}{2\pi i}\int_\Gamma F(z)\cdot(A - z)^{-1} f\, dz \in H^1_\Lambda(\mathbb{R}^d),$$

where $\Gamma \subset \Omega\backslash\sigma(S)$ is a finite set of closed rectifiable curves surrounding $\sigma(S) = \sigma(A)$ (existence of such curves is shown in [24, Theorem 13.5]). Note that the integral converges in $H^1_\Lambda(\mathbb{R}^d)$. Since $H^1_\Lambda(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$, it also converges (to the same limit) in $L^2(\mathbb{R}^d)$ and hence, by definition of the holomorphic spectral calculus,

$$F(S)f = -\frac{1}{2\pi i}\int_\Gamma F(z)\cdot(S - z)^{-1} f\, dz = h \in H^1_\Lambda(\mathbb{R}^d).$$

□

Our main result (Theorem 1.1) is now an easy consequence of Proposition 4.5.

Proof (Proof of Theorem 1.1) Using the fact that $S_{\Lambda,g}$ commutes with $\pi(\lambda)$ for all $\lambda \in \Lambda$, it is easily seen that $(S^{-1}_{\Lambda,g}\, g, \Lambda)$ is the canonical dual frame of (g, Λ) and that $(S^{-1/2}_{\Lambda,g}\, g, \Lambda)$ is a Parseval frame for $L^2(\mathbb{R}^d)$; see for instance, [5, Theorem 12.3.2]. Note that since (g, Λ) is a frame for $L^2(\mathbb{R}^d)$, we have $\sigma(S_{\Lambda,g}) \subset [A, B]$ where $0 < A \leq B < \infty$ are the optimal frame bounds for (g, Λ). Thus, we obtain $S^{-1}_{\Lambda,g}\, g \in V_\Lambda \subset V$ and $S^{-1/2}_{\Lambda,g}\, g \in V_\Lambda \subset V$ from Proposition 4.5 with $F(z) = z^{-1}$ and $F(z) = z^{-1/2}$ (with any suitable branch cut; for instance, the half-axis $(-\infty, 0]$), respectively, on $\Omega = \{x + iy : x \in (\frac{A}{2}, \infty), y \in \mathbb{R}\}$. □

Finally, we state and prove a version of Theorem 1.1 for Gabor frame *sequences*. For completeness, we briefly recall the necessary concepts. Generally, a (countable) family $(h_i)_{i\in I}$ in a Hilbert space $\mathcal{H}$ is called a *frame sequence*, if $(h_i)_{i\in I}$ is a frame for the subspace $\mathcal{H}' := \overline{\operatorname{span}}\{h_i : i \in I\} \subset \mathcal{H}$. In this case, the frame operator $S : \mathcal{H} \to \mathcal{H}, f \mapsto \sum_{i\in I}\langle f, h_i\rangle h_i$, is a bounded, self-adjoint operator on $\mathcal{H}$, and $S|_{\mathcal{H}'} : \mathcal{H}' \to \mathcal{H}'$ is boundedly invertible; in particular, ran $S = \mathcal{H}' \subset \mathcal{H}$ is closed, so that S has a well-defined *pseudo-inverse* $S^\dagger$, given by

$$S^\dagger = (S|_{\mathcal{H}'})^{-1} \circ P_{\mathcal{H}'} : \quad \mathcal{H} \to \mathcal{H}',$$

where $P_{\mathcal{H}'}$ denotes the orthogonal projection onto $\mathcal{H}'$. The *canonical dual system* of $(h_i)_{i\in I}$ is then given by $(h_i')_{i\in I} = (S^\dagger h_i)_{i\in I} \subset \mathcal{H}'$, and it satisfies $\sum_{i\in I}\langle f, h_i\rangle h_i' = \sum_{i\in I}\langle f, h_i'\rangle h_i = P_{\mathcal{H}'} f$ for all $f \in \mathcal{H}$.

Finally, in the case where $(h_i)_{i\in I} = (g, \Lambda)$ is a Gabor family with a lattice Λ, it is easy to see that $S \circ \pi(\lambda) = \pi(\lambda) \circ S$ and $\pi(\lambda)\mathcal{H}' \subset \mathcal{H}'$ for $\lambda \in \Lambda$, which implies $P_{\mathcal{H}'} \circ \pi(\lambda) = \pi(\lambda) \circ P_{\mathcal{H}'}$, and therefore $S^\dagger \circ \pi(\lambda) = \pi(\lambda) \circ S^\dagger$ for all $\lambda \in \Lambda$. Consequently, setting $\gamma := S^\dagger g$, we have $S^\dagger(\pi(\lambda)g) = \pi(\lambda)\gamma$, so that the canonical dual system of a Gabor frame sequence (g, Λ) is the Gabor system (γ, Λ), where $\gamma = S^\dagger g$ is called the *canonical dual window* of (g, Λ). Our next result shows that γ inherits the regularity of g, if one measures this regularity using one of the three spaces H^1, L^2_w, or $\mathbb{H}^1$.

Proposition 4.6 *Let $V \in \{H^1(\mathbb{R}^d), L^2_w(\mathbb{R}^d), \mathbb{H}^1(\mathbb{R}^d)\}$. Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice and let $g \in V$. If (g, Λ) is a frame sequence, then the associated canonical dual window γ satisfies $\gamma \in V$.*

Proof The frame operator $S : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ associated to (g, Λ) is non-negative and has closed range. Consequently, there exist $\varepsilon > 0$ and $R > 0$ such that $\sigma(S) \subset \{0\} \cup [\varepsilon, R]$; see for instance [3, Lemma A.2]. Now, with the open ball $B_\delta(0) := \{z \in \mathbb{C} : |z| < \delta\}$, define

$$\Omega := B_{\varepsilon/4}(0) \cup \left\{x + iy : x \in (\tfrac{\varepsilon}{2}, 2R), y \in (-\tfrac{\varepsilon}{4}, \tfrac{\varepsilon}{4})\right\} \subset \mathbb{C},$$

noting that $\Omega \subset \mathbb{C}$ is open, with $\sigma(S) \subset \Omega$. Furthermore, it is straightforward to see that

$$\varphi : \quad \Omega \to \mathbb{C}, \quad z \mapsto \begin{cases} 0, & \text{if } z \in B_{\varepsilon/4}(0), \\ z^{-1}, & \text{otherwise} \end{cases}$$

is holomorphic. Since the functional calculus for self-adjoint operators is an extension of the holomorphic functional calculus, [3, Lemma A.6] shows that $S^\dagger = \varphi(S)$. Finally, since $g \in V_\Lambda$, Proposition 4.5 now shows that $\gamma = S^\dagger g = \varphi(S)g \in V_\Lambda \subset V$ as well. □

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Appendix A: (Non)-convergence of Janssen's Representation for $\mathbb{H}^1$ Windows

In this appendix we provide a counterexample showing that Janssen's representation of the frame operator associated to a Bessel vector $g \in \mathbb{H}^1$ in general does not converge *unconditionally* with respect to the strong operator topology. We furthermore show that for convergence in *operator norm*, even conditional convergence fails in general.

For simplicity, we only consider the setting $d = 1$ and the lattice $\Lambda = \mathbb{Z} \times \mathbb{Z}$. Thus, given a function $g \in \mathbb{H}^1 = \mathbb{H}^1(\mathbb{R})$, we say that g is a *Bessel vector* if the Gabor system $(T_k M_\ell\, g)_{k,\ell\in\mathbb{Z}} \subset L^2(\mathbb{R})$ is a Bessel system. In this case, *Janssen's representation* of the frame operator $S := S_g := S_{\mathbb{Z}\times\mathbb{Z},g,g}$ is (formally) given by

$$S = \sum_{\ell,k\in\mathbb{Z}} \langle g, T_k M_\ell\, g\rangle\, T_k M_\ell. \tag{A.1}$$

We are interested in the question whether the series defining Janssen's representation is unconditionally convergent in the *strong operator topology* (*SOT*), as an operator on $L^2(\mathbb{R})$. We will construct a function $g \in \mathbb{H}^1$ for which this fails.

A.1. Properties of the Zak Transform

The construction of the counterexample is based on several properties of the *Zak transform* that we briefly recall. Given $f \in L^2(\mathbb{R})$, its Zak transform $Zf \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ is defined as

$$Zf(x,\omega) := \sum_{k\in\mathbb{Z}} f(x-k)e^{2\pi i k\omega},$$

where the series converges in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$; this is a consequence of the fact that

$$Z: \quad L^2(\mathbb{R}) \to L^2([0,1]^2) \text{ is unitary}, \tag{A.2}$$

as shown in [12, Theorem 8.2.3] and of the fact that the Zak transform Zf of a function $f \in L^2(\mathbb{R})$ is always *quasi-periodic*, meaning that

$$Zf(x+n,\omega) = e^{2\pi i n\omega} Zf(x,\omega) \qquad \text{and} \qquad Zf(x,\omega+n) = Zf(x,\omega) \quad \text{(A.3)}$$

for (almost) all $x, \omega \in \mathbb{R}$ and all $n \in \mathbb{Z}$; see [12, Equations (8.4) and (8.5)]. Another crucial property is the interplay between the Zak transform and the time-frequency shifts $T_k M_n$, as expressed by the following formula (found in [12, Eq. (8.7)]):

$$Z[T_k M_n f](x,\omega) = e^{2\pi i n x} e^{-2\pi i k\omega} Zf(x,\omega) = e_{n,-k}(x,\omega)\, Zf(x,\omega), \quad \text{(A.4)}$$

where we used the functions

$$e_{n,k}(x,\omega) := e^{2\pi i (nx+k\omega)} \quad \text{for } n,k \in \mathbb{Z} \text{ and } x,\omega \in \mathbb{R}.$$

Note that $(e_{n,k})_{n,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2([0,1]^2)$.

Finally, we note the following equivalence, taken from [2, Theorem 3.1]:

$$\forall\, g \in L^2(\mathbb{R}) : \quad g \text{ is a Bessel vector} \iff Zg \in L^\infty([0,1]^2). \quad \text{(A.5)}$$

A.2. Properties of $\mathbb{H}^1$

A further important property that we will use is the following characterization of the space $\mathbb{H}^1$ via the Zak transform, a proof of which is given in [3, Lemma 2.4].

$$\forall\, f \in L^2(\mathbb{R}) : \quad f \in \mathbb{H}^1 \iff Zf \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2). \quad \text{(A.6)}$$

It is crucial to observe that the Sobolev space $W^{1,2}(\mathbb{R}^2)$ belongs to the "borderline" case of the Sobolev embedding theorem, meaning $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2) \not\hookrightarrow L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$. In fact, it is easy to verify (see e.g. [7, Page 280]) for $x_0 := (\frac{1}{2}, \frac{1}{2})^T \in \mathbb{R}^2$ that the function

$$u_0 : \quad (0,1)^2 \to \mathbb{R}, \quad x \mapsto \ln\left(\ln\left(1 + \frac{1}{|x - x_0|}\right)\right)$$

belongs to $W^{1,2}((0,1)^2)$, but is not essentially bounded. Now, using the chain rule and the product rule for Sobolev functions (see e.g. [20, Exercise 11.51] and [7, Theorem 1 in Sect. 5.2.3]), we see that if $\varphi \in C_c^\infty((0,1)^2)$ is chosen such that $0 \le \varphi \le 1$ and such that $\varphi \equiv 1$ on a neighborhood of x_0, then the function

$$u : \quad \mathbb{R}^2 \to [0,\infty), \quad x \mapsto \varphi(x) \cdot \big(1 + \sin(u_0(x))\big) \quad \text{(A.7)}$$

satisfies $u \in W^{1,2}(\mathbb{R}^2)$, is continuous and bounded on $\mathbb{R}^2 \setminus \{x_0\}$, but $\lim_{x\to x_0} u(x)$ does not exist; this uses that $\lim_{x\to\infty} \sin(x)$ does not exist and that on each small ball $B_\varepsilon(x_0)$, the function u_0 attains all values from (M,∞), for a suitable $M = M(\varepsilon) > 0$.

A.3. A Connection to Fourier Series

In this subsection, we show that for any fixed window $g \in L^2(\mathbb{R})$ the unconditional convergence of Janssen's representation in the strong operator topology implies that the partial sums of a certain Fourier series are uniformly bounded in L^∞. This connection will be used in the next subsection to disprove the unconditional convergence of Janssen's representation in the strong operator topology.

Precisely, define $Q := [0, 1]^2$. For $H \in L^\infty(Q)$, define the associated multiplication operator as

$$M_H : \quad L^2(Q) \to L^2(Q), \quad F \mapsto F \cdot H.$$

It is well-known that $\|M_H\|_{L^2 \to L^2} = \|H\|_{L^\infty}$.

Let us fix any window $g \in L^2(\mathbb{R})$. Given a finite set $I \subset \mathbb{Z}^2$, we define

$$S_I : \quad L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad f \mapsto \sum_{(k,\ell)\in I} \langle g, T_k M_\ell g\rangle T_k M_\ell f.$$

Using Eq. A.4 and the isometry of the Zak transform, we then see

$$\begin{aligned}
Z(S_I f) &= \sum_{(k,\ell)\in I} \langle Zg, Z[T_k M_\ell g]\rangle_{L^2(Q)} Z[T_k M_\ell f] \\
&= Zf \cdot \sum_{(k,\ell)\in I} \langle Zg, Zg \cdot e_{\ell,-k}\rangle_{L^2(Q)} \cdot e_{\ell,-k} \\
&= Zf \cdot \sum_{(k,\ell)\in I} \langle Zg \cdot \overline{Zg}, e_{\ell,-k}\rangle_{L^2(Q)} \cdot e_{\ell,-k} \\
&= Zf \cdot \sum_{(k,\ell)\in I} \widehat{|Zg|^2}(\ell, -k) \cdot e_{\ell,-k} \\
&= M_{\mathcal{F}_{I'}[|Zg|^2]}[Zf],
\end{aligned}$$

where $I' := \{(\ell, -k) : (k, \ell) \in I\}$ and

$$\mathcal{F}_J H := \sum_{\alpha\in J} \widehat{H}(\alpha)\, e_\alpha \quad \text{with} \quad \widehat{H}(\alpha) = \langle H, e_\alpha\rangle_{L^2(Q)} \quad \text{for} \quad J \subset \mathbb{Z}^2.$$

In other words, we have

$$S_I = Z^{-1} \circ M_{\mathcal{F}_{I'}[|Zg|^2]} \circ Z. \tag{A.8}$$

Given $J \subset \mathbb{Z}^2$, define $J_* := \{(-\ell, k) : (k, \ell) \in J\}$ and note $(J_*)' = J$. Now, suppose that $(S_I)_I$ converges strongly to some (bounded) operator, as $I \to \mathbb{Z}^2$; this is always the case if Janssen's representation converges unconditionally (to S or some other operator) in the SOT. Then, given any sequence $(J_n)_{n\in\mathbb{N}}$ of finite subsets $J_n \subset \mathbb{Z}^2$ with $J_n \subset J_{n+1}$ and $\bigcup_{n=1}^\infty J_n = \mathbb{Z}^2$, we see $(J_n)_* \to \mathbb{Z}^2$ so that the sequence $(S_{(J_n)_*})_{n\in\mathbb{N}}$

converges strongly to some bounded operator. By the uniform boundedness principle, this shows $\|S_{(J_n)_*}\|_{L^2\to L^2} \le C$ for all $n \in \mathbb{N}$ and some $C > 0$. By Eqs. A.2 and A.8, and because of $((J_n)_*)' = J_n$, this implies

$$\big\|\mathcal{F}_{J_n}[|Zg|^2]\big\|_{L^\infty(Q)} = \big\|M_{\mathcal{F}_{J_n}[|Zg|^2]}\big\|_{L^2\to L^2} \le C \qquad \forall n \in \mathbb{N},$$

meaning that the partial Fourier sums $\mathcal{F}_{J_n}[|Zg|^2]$ of the function $|Zg|^2$ are uniformly bounded in $L^\infty(Q)$.

A.4. The Counterexample

In this subsection, we prove the following:

Proposition A.1 *There exists a Bessel vector $g \in \mathbb{H}^1(\mathbb{R})$ such that the series defining Janssen's representation of the frame operator $S = S_g = S_{\mathbb{Z}\times\mathbb{Z},g,g}$ associated to g is* not *unconditionally convergent in the strong operator topology.*

To prove the proposition, we consider the function $F := u : (0,1)^2 \to [0,\infty)$ introduced in Eq. A.7. The properties of F that we need are the following:

(1) F has compact support in $(0,1)^2$, say $\operatorname{supp} F \subset (\delta, 1-\delta)^2$ for some $\delta \in (0, \frac{1}{2})$.
(2) F is bounded, but discontinuous at $x_0 \in (0,1)^2$ (even after adjusting F on a set of measure zero).
(3) $F \in W^{1,2}\big((0,1)^2\big)$.

We now extend F by zero to $[0,1)^2$ and then extend 1-periodically in both coordinates to $\mathbb{R}^2$. Thanks to the compact support of F, it is easy to see that $F \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$.

Furthermore, we consider the function

$$G_0 : \quad \mathbb{R}^2 \to \mathbb{C}, \quad (x,\omega) \mapsto e^{2\pi i \lfloor x \rfloor \omega},$$

where for each $x \in \mathbb{R}$, $\lfloor x \rfloor \in \mathbb{Z}$ denotes the unique integer such that $x \in \lfloor x \rfloor + [0,1)$. It is then straightforward to verify that G_0 is quasi-periodic (see Eq. A.3), i.e., $G_0(x+m,\omega) = e^{2\pi i m\omega} G_0(x,\omega)$ and $G_0(x,\omega+m) = G_0(x,\omega)$ for $x,\omega \in \mathbb{R}$ and $m \in \mathbb{Z}$. Since F is 1-periodic in both coordinates, it is easy to see that $F \cdot G_0$ is quasi-periodic as well.

Finally, we choose a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ satisfying $\psi(x) = n$ for all $x \in n + [\delta, 1-\delta]$ with $n \in \mathbb{Z}$, and define

$$G : \quad \mathbb{R}^2 \to \mathbb{C}, \quad (x,\omega) \mapsto e^{2\pi i \psi(x)\omega}.$$

Using that $F(x,\omega) = 0$ for $n \in \mathbb{Z}$ and $x \in [n, n+1] \setminus (n+\delta, n+1-\delta)$, it is easy to check $F \cdot G_0 = F \cdot G$, so that $H := F \cdot G \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2) \subset L^2_{\mathrm{loc}}(\mathbb{R}^2)$ is quasi-periodic.

Since the Zak transform $Z : L^2(\mathbb{R}) \to L^2((0,1)^2)$ is unitary, there exists a unique function $g \in L^2(\mathbb{R})$ such that $(Zg)|_{(0,1)^2} = H|_{(0,1)^2}$. Since both Zg and H are quasi-periodic, this implies $Zg = H$ almost everywhere. Since $H \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ is bounded, Eqs. A.5,A.6 show that $g \in \mathbb{H}^1$ is a Bessel vector. Let us assume towards a contradiction that Janssen's representation of the frame operator associated to g converges unconditionally in the strong operator topology.

Note that $|Zg|^2 = |H|^2 = F^2$ is discontinuous at $x_0 \in (0,1)^2$ (since F is discontinuous there and also non-negative), even after possibly changing $|Zg|^2$ on a null-set. In particular, this implies that the Fourier coefficients $c_\alpha := \widehat{|Zg|^2}(\alpha)$ (for $\alpha \in \mathbb{Z}^2$) satisfy $c = (c_\alpha)_{\alpha\in\mathbb{Z}^2} \notin \ell^1(\mathbb{Z}^2)$, since otherwise the Fourier series of $|Zg|^2$ would be uniformly convergent. This implies $\sum_{\alpha\in\mathbb{Z}^2} |\Re c_\alpha| = \infty$ or $\sum_{\alpha\in\mathbb{Z}^2} |\Im c_\alpha| = \infty$. For simplicity, we assume the first case; the second case can be treated by similar arguments. This implies that there exists an enumeration $(\alpha_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}^2$ such that $|\sum_{n=1}^N \Re c_{\alpha_n}| \to \infty$ as $N \to \infty$. Indeed, if $\sum_{\alpha\in\mathbb{Z}^2} (\Re c_\alpha)_+ < \infty$ or $\sum_{\alpha\in\mathbb{Z}^2} (\Re c_\alpha)_- < \infty$, this is trivial (for every enumeration); otherwise, existence of the desired enumeration follows from the Riemann rearrangement theorem (see e.g., [23, Theorem 3.54]).

Now, define $J_n := \{\alpha_1, \ldots, \alpha_n\}$ for $n \in \mathbb{N}$. We have seen in Appendix 1 that the partial Fourier sums $\mathcal{F}_{J_n}[|Zg|^2]$ are uniformly bounded in L^∞, say $\|\mathcal{F}_{J_n}[|Zg|^2]\|_{L^\infty} \le C$ for all $n \in \mathbb{N}$. Since each $\mathcal{F}_{J_n}[|Zg|^2]$ is continuous (in fact, a trigonometric polynomial), this implies

$$
\begin{aligned}
C \ge \left|\mathcal{F}_{J_N}|Zg|^2(0)\right| &= \left|\sum_{\alpha\in J_N} \widehat{|Zg|^2}(\alpha)\cdot e^{2\pi i\langle\alpha,0\rangle}\right| \\
&= \left|\sum_{\alpha\in J_N} c_\alpha\right| \ge \left|\Re\sum_{n=1}^N c_{\alpha_n}\right| \to \infty \text{ as } N \to \infty,
\end{aligned}
$$

which is the desired contradiction.

A.5. Conditional Divergence of Janssen's Representation in the Operator Norm

We showed above that *unconditional* convergence of Janssen's representation (A.1) in the strong operator topology fails for some Bessel vector $g \in \mathbb{H}^1(\mathbb{R})$. A similar argument shows that convergence in the operator norm (with respect to *any* given enumeration) also fails in general: Using Eq. A.8 (or more generally the arguments in Appendix 1), it is relatively easy to see that if for some enumeration $\mathbb{Z}^2 = \{\alpha_n : n \in \mathbb{N}\}$ and $I_n := \{\alpha_1, \ldots, \alpha_n\}$, the sequence of partial sums $(S_{I_n})_{n\in\mathbb{N}}$ of Janssen's representation (A.1) converges in operator norm (not even necessarily to S), then the associated sequence $(\mathcal{F}_{I'_n}[|Zg|^2])_{n\in\mathbb{N}}$ of partial Fourier sums of $|Zg|^2$ is Cauchy in $L^\infty(Q)$ and thus converges uniformly on Q to a (necessarily continuous) function $\widetilde{H} : Q \to \mathbb{C}$. However, since $|Zg|^2 = |H|^2 = F^2 \in L^\infty(Q) \subset L^2(Q)$, we know that $\mathcal{F}_{I'_n}[|Zg|^2] \to F^2$ with convergence in $L^2(Q)$. Hence, $F^2 = \widetilde{H}$ almost everywhere on Q, where $\widetilde{H}$ is continuous. But we saw above that F^2 is discontinuous on Q, even after (possibly) changing it on a null-set. Thus, we have obtained the desired contradiction:

Proposition A.2 *There are Bessel vectors $g \in \mathbb{H}^1$ for which Janssen's representation fails to converge conditionally in the operator norm.*

However, we leave it as an open question whether Janssen's representation converges conditionally in the strong sense for Bessel vectors $g \in \mathbb{H}^1$.

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