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LETTER TO THE EDITORS

On a Localized Parseval Identity for the Finite Hilbert Transform

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Abstract
In this letter we present a dual weight version of a localized Parseval identity found by Coifman and Steinerberger for the finite Hilbert transform.

Keywords Finite Hilbert transform · Parseval identity · Chebychev polynomials

Mathematics Subject Classification Primary 44A15; Secondary 33C45

1 Introduction

All the functions considered in this letter are real-valued. Recall that the Hilbert transform H of a function f on $\mathbb{R}$ (in proper function spaces) is defined by

$$H(f)(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy,$$

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where “p.v." stands for the *principal value*. This singular integral operator has a local version, say on the interval $I = (-1, 1)$, and is given by

$$H_I(f)(x) = \text{p.v.}\frac{1}{\pi}\int_{-1}^{1}\frac{f(y)}{x-y}dy.$$

It is called the *finite Hilbert transform* and arises naturally in applied science. In particular, the resolution of the following airfoil equation from aerodynamics,

$$H_I(f)(x) = \phi(x) \quad (-1 < x < 1),$$

involves the inversion of H_I in proper function spaces. The airfoil equation is approached by Tricomi in [7] via establishing some convolution theorems for H_I; these convolution identities are motivated by his earlier study on mixed type equations. A byproduct from [7] is the following arcsine distribution uniqueness.

Theorem 1.1 (Tricomi 1951) *Let $f(x)(1-x^2)^{\frac{1}{4}} \in L_I^2 = L^2(-1, 1)$. If $H(f)$ vanishes identically on $I = (-1, 1)$, then for some real-valued constant c,*

$$f(x) = \frac{c}{\sqrt{1-x^2}}\chi_I(x).$$

Here, χ_I is the indicator function of I.

Remark 1.2 For an application of this to Erdös-Turán inequality, see [1].

Recently, this uniqueness result is revisited by Coifman and Steinerberger in [2]. They further observed the following localized Parseval identity for H_I.

Theorem 1.3 (Coifman and Steinerberger 2019) *Let $f(x)(1-x^2)^{\frac{1}{4}} \in L_I^2$. If the mean value of $f(x)(1-x^2)^{\frac{1}{2}}$ on I is 0, then*

$$\int_{-1}^{1} H_I(f)^2(x)\sqrt{1-x^2}dx = \int_{-1}^{1} f^2(x)\sqrt{1-x^2}dx. \tag{1.1}$$

This complements the global L^2-isometry for the standard Hilbert transform H:

$$\|H(f)\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Two proofs for the localized Parseval identity (1.1) are offered in [2]: one is by Chebychev orthogonal expansion (see e.g. [6]), another by working on the unit circle and using the formula of conjugate functions as carried out in [3]. In this letter we point out that the identity (1.1) admits the following dual weight version.

Theorem 1.4 *Let $f(x)(1-x^2)^{-\frac{1}{4}} \in L_I^2$. Then*

$$\int_{-1}^{1} H_I(f)^2(x)\frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{1} f^2(x)\frac{dx}{\sqrt{1-x^2}}. \tag{1.2}$$

Remark 1.5 Such a dual weight mechanism is indeed quite common in boundary value problems, see for example Rosén [5] on Euclidean upper half-space.

2 Proof of Theorem 1.4

Our proof of (1.2) adapts the Chebychev orthogonal expansion arguments in [2]. Consider f so that $\frac{f(x)}{\sqrt{1-x^2}}$ is a polynomial. For some $N \in \mathbb{N}$ we can write

$$\frac{f(x)}{\sqrt{1-x^2}} = \sum_{k=0}^{N} a_k U_k(x),$$

where $\{U_k\}$ denotes the family of Chebychev polynomials of the second kind. Thereby,

$$\begin{aligned}\int_{-1}^{1} f^2(x)\frac{dx}{\sqrt{1-x^2}} &= \int_{-1}^{1}\left(\frac{f(x)}{\sqrt{1-x^2}}\right)^2\sqrt{1-x^2}dx\\ &= \frac{\pi}{2}\sum_{k=0}^{N} a_k^2.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}H_I(f) &= \text{p.v.}\frac{1}{\pi}\int_{-1}^{1}\frac{f(y)/\sqrt{1-y^2}}{x-y}\sqrt{1-y^2}dy\\ &= \text{p.v.}\frac{1}{\pi}\int_{-1}^{1}\sum_{k=0}^{N}\frac{a_k U_k(y)}{x-y}\sqrt{1-y^2}dy.\end{aligned}$$

After using the crucial formulae (see for example [4, p. 187])

$$\text{p.v.}\frac{1}{\pi}\int_{-1}^{1}\frac{U_k(y)}{x-y}\sqrt{1-y^2}dy = -T_{k+1}(x),$$

where $\{T_k\}$ denotes the family of Chebychev polynomials of the first kind, we get

$$\int_{-1}^{1} H_I(f)^2(x)\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\sum_{k=0}^{N} a_k^2.$$

This proves the identity (1.2) for f such that $\frac{f(x)}{\sqrt{1-x^2}}$ is a polynomial. Note that the subspace of such functions is dense in $L^2\left(I, (1-x^2)^{-\frac{1}{2}}dx\right)$, so H_I extends to an isometry $\widetilde{H_I}$ on $L^2\left(I, (1-x^2)^{-\frac{1}{2}}dx\right)$. Moreover, this extension agrees with H_I since $L^2\left(I, (1-x^2)^{-\frac{1}{2}}dx\right)$ embeds into L_I^2. This finishes the proof of Theorem 1.4.

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