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Sharp Estimates for Schrödinger Groups on Hardy Spaces for $0 < p \leq 1$

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Abstract

Let X be a space of homogeneous type with the doubling order n. Let L be a non-negative self-adjoint operator on $L^2(X)$ and suppose that the kernel of e^{-tL} satisfies a Gaussian upper bound. This paper shows that for $0 < p \leq 1$ and $s = n(1/p - 1/2)$,

$$\|(I + L)^{-s} e^{itL} f\|_{H^p_L(X)} \lesssim (1 + |t|)^s \|f\|_{H^p_L(X)}$$

for all $t \in \mathbb{R}$, where $H^p_L(X)$ is the Hardy space associated to L. This recovers the classical results in the particular case when $L = -\Delta$ and extends a number of known results.

Keywords Schrödinger group · Gaussian upper bound · Hardy space

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1 Introduction

Let (X, d, μ) be a metric space endowed with a nonnegative Borel measure μ. Denote by $B(x, r)$ the open ball of radius $r > 0$ and center $x \in X$, and by $V(x, r)$ its

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measure $\mu(B(x,r))$. In this paper we assume that the measure μ satisfies the doubling condition: there exists a constant $C > 0$ such that

$$V(x, 2r) \le CV(x, r) \tag{1}$$

for all $x \in X$, $r > 0$ and all balls $B(x, r)$.

We note that the doubling property (1) yields a constant $n > 0$ so that

$$V(x, \lambda r) \le C\lambda^n V(x, r), \tag{2}$$

for all $\lambda \ge 1$, $x \in X$ and $r > 0$; and that

$$V(x, r) \le C\Big(1 + \frac{d(x, y)}{r}\Big)^n V(y, r), \tag{3}$$

for all $x, y \in X$ and $r > 0$.

Suppose that L is a non-negative self-adjoint operator on $L^2(X)$. Suppose further that the operator L generates an analytic semigroup e^{-tL} whose kernels e^{-tL} satisfy the Gaussian estimate. That is, there exist constants $C, c > 0$ and $m > 1$ such that

$$|e^{-tL}(x, y)| \le \frac{C}{V(x, t^{1/m})} \exp\Big(-\frac{d(x, y)^{m/m-1}}{ct^{1/m-1}}\Big) \tag{G}$$

for all $x, y \in X$ and $t > 0$.

Through spectral theory we can define the Schrödinger group, for $t \in \mathbb{R}$,

$$e^{itL} = \int_0^\infty e^{it\lambda} dE_L(\lambda),$$

where $E_L(\lambda)$ is the spectral decomposition of L.

The mapping properties of the Schrödinger group e^{itL} has a wide range of applications spanning fields such as harmonic analysis and nonlinear dispersive equations. The Schrödinger group is bounded on $L^2(X)$ but not bounded in $L^p(X)$ for $p \ne 2$, even in the case when $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$. Despite this, $(1 + L)^{-s} e^{itL}$ is known to be L^p-bounded for s sufficiently large. It was shown in [7] that for every $1 < p < \infty$ and $t \in \mathbb{R}$,

$$\|(1 + L)^{-s} e^{itL} f\|_{L^p} \lesssim (1 + |t|)^s \|f\|_{L^p}, \qquad s > n\left|\frac{1}{2} - \frac{1}{p}\right|. \tag{4}$$

Similar results can be found in [2, 5, 7, 11, 20, 23] and the references therein.

In the classical case when $L = -\Delta$, we also have the following sharp estimate: for all $1 < p < \infty$ and $t > 0$ one has

$$\|(1 - \Delta)^{-s} e^{it\Delta} f\|_{L^p} \lesssim (1 + |t|)^s \|f\|_{L^p}, \qquad s = n\left|\frac{1}{2} - \frac{1}{p}\right|, \tag{5}$$

see [22]. Also for $p \leq 1$, it was proved by Miyachi [21] that for each $0 < p \leq 1$ and $t \in \mathbb{R}$ we have

$$\|(1-\Delta)^{-s} e^{it\Delta} f\|_{H^p(\mathbb{R}^n)} \lesssim (1+|t|)^s \|f\|_{H^p(\mathbb{R}^n)}, \qquad s = n\Big(\frac{1}{p} - \frac{1}{2}\Big), \tag{6}$$

where $H^p(\mathbb{R}^n)$ is the classical Hardy spaces. See [24].

Let us turn to some more recent results concerning (4)-(6), which also serves to motivate the results in our paper. The first concerns sharpness for $p > 1$. In comparison with (5), estimate (4) is not sharp. However this point has recently been addressed in [9]; more precisely, it was proved there that (4) also holds for $s = n\big|\frac{1}{2} - \frac{1}{p}\big|$.

Secondly, the following endpoint estimates for $p = 1$ were obtained in [8]:

$$\|(1+L)^{-n/2} e^{itL} f\|_{L^1} + \|(1+L)^{-n/2} e^{itL} f\|_{H^1_L} \lesssim (1+|t|)^{n/2} \|f\|_{H^1_L}, \tag{7}$$

under more general assumptions than G. Here $H^1_L(X)$ is the Hardy space associated to L (see Sect. 2 for the precise definition of $H^1_L(X)$). In this paper we address the sharp extension of (7) to $p < 1$ in the sense of (6). Our main result is the following.

Theorem 1.1 *Let L be a non-negative self-adjoint operator on $L^2(X)$ generating an analytic semigroup e^{-tL} whose kernels satisfy the Gaussian upper bound G. Then for each $0 < p \leq 1$ and $s = n(1/p - 1/2)$, we have*

$$\|(I+L)^{-s} e^{itL} f\|_{H^p_L(X)} \lesssim (1+|t|)^s \|f\|_{H^p_L(X)}, \qquad t \in \mathbb{R}, \tag{8}$$

where $H^p_L(X)$ is the Hardy space associated to L (defined in Sect. 2).

Some comments on Theorem 1.1 are in order.

(i) It is natural to speculate on the relationship between Theorem 1.1 and [8, Theorem 1.1]. While the endpoint $p = 1$ is implied by [8, Theorem 1.1], to the best of our knowledge, the result for $p < 1$ is new. It is also important to note that the approach in [8] is not immediately applicable to $p < 1$; indeed, the inequality (4.7) in [8], which plays a crucial role in the proof of [8, Theorem 1.1], is not true if the L^1-norm is replaced by the L^p-norm when $p < 1$. We believe therefore that any generalization of Theorem 1.1 under the less restrictive assumptions employed in [8] will require new ideas.

(ii) By using interpolation, estimate (8) implies the following sharp L^p estimate: for $1 < p < \infty$, we have

$$\|(1+L)^{-s} e^{itL} f\|_{L^p} \lesssim (1+|t|)^s \|f\|_{L^p}, \qquad s = n\left|\frac{1}{2} - \frac{1}{p}\right|.$$

See [8]. Thus, Theorem 1.1 completes the scale of sharp estimates for the Schrödinger group for all $0 < p < \infty$.

For $s > 0$, consider the operator defined by

$$I_{s,t}(L)f = st^{-s}\int_0^t (t-\lambda)^{s-1}e^{-i\lambda L} f\, d\lambda, \qquad t > 0,$$

and $I_{s,t}(L) = \bar{I}_{s,-t}(L)$ for $t < 0$. These operators are known as the 'Riesz means' associated to L. The Riesz means have close connections with the solution to the Schrödinger equation

$$\begin{cases} i\partial_t u + Lu = 0, \\ u(x,0) = f. \end{cases}$$

See for example [23].

By using Theorem 1.1, the spectral theorem in [14, Theorem 1.1], and a standard argument from [23], we can obtain the following result.

Corollary 1.2 *Assume that L satisfies the conditions of Theorem 1.1. Then for each $0 < p \le 1$ there exists a constant $C > 0$ independent of t such that*

$$\|I_{s,t}(L)f\|_{H^p_L(X)} \le C\|f\|_{H^p_L(X)}, \qquad s = \left(\frac{1}{p} - \frac{1}{2}\right)$$

for all $t \neq 0$.

The organization of this paper is as follows. In Sect. 2, we fix some notations that will be employed throughout the article and detail some properties of the Hardy spaces associated to the operator L. The proof of Theorem 1.1 will be given in Sect. 3. Finally, Sect. 4 will discuss some applications of the main result.

2 Preliminaries

2.1 Notations and Elementary Estimates on the Space of Homogeneous Type

As usual we use C and c to denote positive constants that are independent of the main parameters involved but may differ from line to line. The notation $A \lesssim B$ means $A \le CB$, and $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

The space of Schwarz functions on $\mathbb{R}^n$ is denoted by $\mathscr{S}(\mathbb{R}^n)$ and given $\psi \in \mathscr{S}(\mathbb{R})$, $\lambda \in \mathbb{R}$ and $j \in \mathbb{Z}$, we use the notation $\psi_j(\lambda) := \psi(2^{-j}\lambda)$. For $f \in \mathscr{S}(\mathbb{R}^n)$ we denote by $\mathcal{F}f$ the Fourier transform of f. That is,

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{n/2}}\int f(x)e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^n.$$

To simplify notation, we will often just use B for $B(x_B, r_B)$ and $V(E)$ for $\mu(E)$ for any measurable subset $E \subset X$. Also given $\lambda > 0$, we will write λB for the $B(x_B, \lambda r_B)$.

For each ball $B \subset X$ we set

$$S_0(B) = 0, \quad S_j(B) = 2^j B \backslash 2^{j-1} B \quad \text{for } j \in \mathbb{N}.$$

Let $w \in A_\infty$ and $0 < r < \infty$. The Hardy–Littlewood maximal function $\mathcal{M}_r$ is defined by

$$\mathcal{M}_r f(x) = \sup_{x \in B} \Big(\frac{1}{V(B)} \int_B |f(y)|^r d\mu(y) \Big)^{1/r}$$

where the sup is taken over all balls B containing x. We will drop the subscripts r when $r = 1$. It is well-known that for $0 < r < \infty$ one has

$$\|\mathcal{M}_r f\|_p \lesssim \|f\|_p, \tag{9}$$

whenever $p > r$.

The following elementary estimates will be used frequently. See for example [2].

Lemma 2.1 *Let* $\epsilon > 0$.

(a) For any $p \in [1, \infty]$ *we have*

$$\Big(\int_X \Big[\Big(1 + \frac{d(x,y)}{s}\Big)^{-n-\epsilon} \Big]^p d\mu(y) \Big)^{1/p} \lesssim V(x,s)^{1/p},$$

for all $x \in X$ *and* $s > 0$.

(b) For any $f \in L^1_{\mathrm{loc}}(X)$ *we have*

$$\int_X \frac{1}{V(x \wedge y, s)} \Big(1 + \frac{d(x,y)}{s}\Big)^{-n-\epsilon} |f(y)| d\mu(y) \lesssim \mathcal{M} f(x),$$

for all $x \in X$ *and* $s > 0$, *where* $V(x \wedge y, s) = \min\{V(x,s), V(y,s)\}$.

We also recall the Fefferman-Stein vector-valued maximal inequality in [17]. For $0 < p < \infty$, $0 < q \le \infty$ and $0 < r < \min\{p, q\}$, we have for any sequence of measurable functions $\{f_\nu\}$,

$$\Big\| \Big(\sum_\nu |\mathcal{M}_r f_\nu|^q \Big)^{1/q} \Big\|_p \lesssim \Big\| \Big(\sum_\nu |f_\nu|^q \Big)^{1/q} \Big\|_p. \tag{10}$$

2.2 Hardy Spaces Associated to the Operator *L*

We first recall from [16, 19] the definition of the Hardy spaces associated to an operator. Let L be a nonnegative self-adjoint operator on $L^2(X)$ satisfying the Gaussian upper bound G. Let $0 < p \le 1$. Then the Hardy space $H^p_L(X)$ is defined as the completion of

$$\{f \in L^2(X) : \mathcal{A}_L f \in L^p(X)\}$$

under the norm $\|f\|_{H_L^p(X)} = \|\mathcal{A}_L f\|_{L^p}$, where the square function $\mathcal{A}_L$ is defined as

$$\mathcal{A}_L f(x) = \Big(\int_0^\infty \int_{d(x,y)<t} |t^m L e^{-t^m L} f(y)|^2 \frac{d\mu(y)dt}{tV(x,t)}\Big)^{1/2}. \tag{11}$$

Next we have a notion of molecules from [16, 19].

Definition 2.2 *(Molecules for L)* Let $\epsilon > 0$, $0 < p \leq 1$ and $M \in \mathbb{N}$. A function $m(x)$ is called a $(p, 2, M, L, \epsilon)$-molecule associated to a ball $B \subset X$ of radius r_B if there exists a function $b \in D(L^M)$ such that

(i) $m = L^M b$;
(i) $\|L^k b\|_{L^2(S_j(B))} \leq 2^{-j\epsilon} r_B^{m(M-k)} V(2^j B)^{1/2-1/p}$ for all $k = 0, 1, \ldots, M$ and $j = 0, 1, 2 \ldots$.

The molecular property (ii) in particular can be thought of as a mild *locality* condition on the operator L.

Definition 2.3 *(Hardy spaces associated to L)* Given $\epsilon > 0$, $0 < p \leq 1$ and $M \in \mathbb{N}$, we say that $f = \sum \lambda_j m_j$ is a molecule $(p, 2, M, L, \epsilon)$-representation if $\{\lambda_j\}_{j=0}^\infty \in \ell^p$, each m_j is a $(p, 2, M, L, \epsilon)$-atom, and the sum converges in $L^2(X)$. The space $H^p_{L,\mathrm{mol},M,\epsilon}(X)$ is then defined as the completion of

$$\Big\{f \in L^2(X) : f \text{ has a molecule}(p, 2, M, L, \epsilon) - representation\Big\},$$

with the norm given by

$$\begin{aligned}\|f\|^p_{H^p_{L,\mathrm{mol},M,\epsilon}(X)} = \inf\Big\{\sum |\lambda_j|^p : f \\ = \sum \lambda_j m_j \text{ is a molecule}(p, 2, M, L, \epsilon) - representation\Big\}.\end{aligned}$$

The following gives a molecular characterization for the Hardy spaces $H_L^p(X)$.

Theorem 2.4 ([6, 16, 19]) *Let $\epsilon > 0$, $p \in (0, 1]$ and $M > \frac{n(2-p)}{2mp}$. Then the Hardy spaces $H^p_{L,\mathrm{mol},M,\epsilon}(X)$ and $H^p_L(X)$ coincide and have equivalent norms.*

We note that if $L = -\Delta$ then $H_L^p(\mathbb{R}^n)$ coincides with the standard Hardy space $H^p(\mathbb{R}^n)$ on $\mathbb{R}^n$ for $p \in (0, 1]$. In general, depending on the choice of the operator L, the space $H_L^p(\mathbb{R}^n)$ may be quite different to $H^p(\mathbb{R}^n)$. See for example [12].

2.3 Discrete Square Functions

In this section we obtain an inequality for certain square functions that will be important in the proof of Theorem 1.1.

In what follows, by a "partition of unity" we shall mean a function $\psi \in \mathscr{S}(\mathbb{R})$ such that supp $\psi \subset [1/2, 2]$, $\int \psi(\xi) \frac{d\xi}{\xi} \neq 0$ and

$$\sum_{j\in\mathbb{Z}} \psi_j(\lambda) = 1 \text{ on } (0, \infty).$$

where $\psi_j(\lambda) := \psi(2^{-j}\lambda)$ for each $j \in \mathbb{Z}$. Now let ψ be a partition of unity and define the discrete square function $S_{L,\psi}$ by

$$S_{L,\psi} f = \Big(\sum_{j\in\mathbb{Z}} |\psi_j(L) f|^2\Big)^{1/2},$$

which is bounded on $L^2(X)$ by Khintchine's inequality. We also have the following, which is the main result of this section.

Theorem 2.5 *Let ψ be a partition of unity. Then for each $0 < p \leq 1$, we have*

$$\|f\|_{H^p_L} \lesssim \|S_{L,\psi} f\|_p$$

for all $f \in H^p_L(X)$.

In order to prove the theorem we follow the ideas in [2]. Before presenting the proof we gather some technical elements which will play a core role in the proof of the theorem.

The first concerns certain kernel estimates.

Lemma 2.6 ([18]) *Let $\varphi, \psi \in \mathscr{S}(\mathbb{R})$ supported in $[1/2, 2]$. Then the kernel $K_{\varphi(tL)}$ of $\varphi(tL)$ satisfies the following: for any $N > 0$ there exists C such that*

$$|K_{\varphi(tL)}(x, y)| \leq \frac{C}{V(x \vee y, t^{1/m})}\Big(1 + \frac{d(x, y)}{t^{1/m}}\Big)^{-N}, \tag{12}$$

for all $t > 0$ and $x, y \in X$, where $V(x \vee y, t^{1/m}) = \max\{V(x, t^{1/m}), V(y, t^{1/m})\}$.

Next we introduce and give estimates for certain 'Peetre-type' maximal functions. For $\lambda > 0$, $j \in \mathbb{Z}$ and $\varphi \in \mathscr{S}(\mathbb{R})$ the Peetre-type function is defined, for $f \in \mathcal{L}^2(X)$, by

$$\varphi^*_{j,\lambda}(L) f(x) = \sup_{y\in X} \frac{|\varphi_j(L) f(y)|}{(1 + 2^{j/m} d(x, y))^\lambda}, \; x \in X. \tag{13}$$

Obviously, we have

$$\varphi^*_{j,\lambda}(L) f(x) \geq |\varphi_j(L) f(x)|, \quad x \in X.$$

Similarly, for $s, \lambda > 0$ we set

$$\varphi^*_\lambda(sL)f(x) = \sup_{y\in X} \frac{|\varphi(sL)f(y)|}{(1+d(x,y)/s^{1/m})^\lambda}, \quad f \in L^2(X). \tag{14}$$

Proposition 2.7 *Let $\psi \in \mathscr{S}(\mathbb{R})$ with $\operatorname{supp}\psi \subset [1/2, 2]$ and $\varphi \in \mathscr{S}(\mathbb{R})$ be a partition of unity. Then for any $\lambda > 0$ and $j \in \mathbb{Z}$ we have*

$$\sup_{s\in[2^{-j-1},2^{-j}]} \psi^*_\lambda(sL)f(x) \lesssim \sum_{k=j-2}^{j+3} \varphi^*_{k,\lambda}(L)f(x) \tag{15}$$

for all $f \in L^2(X)$ and $x \in X$.

Proof The proof can be done in the same way as [2, Proposition 2.16] with $s^{1/m}$ and $2^{j/m}$ in place of s and 2^j respectively. We omit the details. □

Proposition 2.8 *Let ψ be a partition of unity. Then for any $\lambda, s > 0$ and $r \in (0,1)$ we have:*

$$\psi^*_\lambda(sL)f(x) \lesssim \Big[\int_X \frac{1}{V(z,s^{1/m})} \frac{|\psi(sL)f(z)|^r}{(1+d(x,z)/s^{1/m})^{\lambda r}} d\mu(z)\Big]^{1/r}$$

for all $f \in L^2(X)$ and $x \in X$.

Proof The proof can be done in the same way as [2, Proposition 2.17] and we omit the details.

□

We next prove the following result.

Proposition 2.9 *Let ψ be a partition of unity. Then for $0 < p \le 1$ and $\lambda > n/p$ we have:*

$$\Big\|\Big[\sum_{j\in\mathbb{Z}} |\psi^*_{j,\lambda}(L)f|^2\Big]^{1/2}\Big\|_p \sim \|S_{L,\psi}f\|_p.$$

Proof Since $|\psi_j(\sqrt{L})f| \lesssim \psi^*_{j,\lambda}(L)f$, it suffices to prove that

$$\Big\|\Big[\sum_{j\in\mathbb{Z}} |\psi^*_{j,\lambda}(L)f|^2\Big]^{1/2}\Big\|_p \lesssim \|S_{L,\psi}f\|_p. \tag{16}$$

Choose $r < p$ so that $\lambda > n/r$. Then applying Proposition 2.8 and Lemma 2.1 we have

$$\psi^*_{j,\lambda}(L)f(x) \lesssim \Big[\int_X \frac{1}{V(z,2^{-j})} \frac{|\psi_j(L)f(z)|^r}{(1+2^jd(x,z))^{\lambda r}} d\mu(z)\Big]^{1/r} \lesssim \mathcal{M}_r\big(|\psi_j(L)f|\big)(x)$$

At this stage, we may apply the weighted Fefferman-Stein maximal inequality (10) to obtain (16) as desired. □

We now ready to prove Theorem 2.5.

Proof of Theorem 2.5: Setting $\varphi(\lambda) = \lambda e^{-\lambda}$. Observe that

$$|\varphi(tL)f(y)| \leq \varphi_\lambda^*(tL)f(x)$$

for all $\lambda > 0$ and $d(x,y) < t^{1/m}$. Therefore,

$$\begin{aligned}
&\Big(\int_0^\infty \int_{d(x,y)<t^{1/m}} |\varphi(tL)f(y)|^2 \frac{d\mu(y)dt}{tV(x,t^{1/m})}\Big)^{1/2} \\
&\leq \Big[\int_0^\infty \int_{d(x,y)<t^{1/m}} |\varphi_\lambda^*(tL)f(x)|^2 \frac{d\mu(y)dt}{tV(x,t^{1/m})}\Big]^{1/2} \\
&\lesssim \Big[\int_0^\infty |\varphi_\lambda^*(tL)f(x)|^2 \frac{dt}{t}\Big]^{1/2}.
\end{aligned}$$

Since

$$\|f\|_{H_L^p} = \Big\|\Big(\int_0^\infty \int_{d(x,y)<t^{1/m}} |\varphi(tL)f(y)|^2 \frac{d\mu(y)dt}{tV(x,t^{1/m})}\Big)^{1/2}\Big\|_p,$$

it suffices to prove that

$$\Big\|\Big[\int_0^\infty |\varphi_\lambda^*(tL)f(x)|^2 \frac{dt}{t}\Big]^{1/2}\Big\|_p \lesssim \|S_{L,\psi}f\|_p, \tag{17}$$

where ψ is a partition of unity.

By the spectral theory,

$$f = c_\psi \int_0^\infty \psi(sL)f\frac{ds}{s} \quad \text{in } L^2(X),$$

where $c_\psi = \Big[\int_0^\infty \psi(s)\frac{ds}{s}\Big]^{-1}$. Hence it follows that for every $t > 0$,

$$\varphi(tL)(f) = c_\psi \int_0^\infty \varphi(tL)\psi(sL)f\frac{ds}{s}. \tag{18}$$

Now let $\lambda > 0$, $t \in [2^{-\nu-1}, 2^{-\nu}]$ for some $\nu \in \mathbb{Z}$ and $M > \lambda$. For convenience we may assume $c_\psi = 1$. We then have

$$\varphi(tL)(f) = \int_0^\infty \psi(sL)\varphi(tL)f\frac{ds}{s}$$

$$
\begin{aligned}
&= \sum_{j\geq\nu}\int_{2^{-j-1}}^{2^{-j}} \psi(sL)\varphi(tL)f\frac{ds}{s} + \sum_{j<\nu}\int_{2^{-j-1}}^{2^{-j}} \psi(sL)\varphi(tL)f\frac{ds}{s} \\
&= \sum_{j\geq\nu}\int_{2^{-j-1}}^{2^{-j}} \left(\frac{s}{t}\right)^M (sL)^{-M}\psi(sL)(tL)^M\varphi(tL)f\frac{ds}{s} \\
&\quad + \sum_{j<\nu}\int_{2^{-j-1}}^{2^{-j}} \frac{t}{s}(sL)\psi(sL)e^{-tL}f\frac{ds}{s},
\end{aligned}
$$

where in the last line we used $\varphi(tL) = (tL)e^{-tL}$.

We now set $\psi_M(x) = x^{-M}\psi(x)$ and $\widetilde{\psi}(x) = x\psi(x)$. Then the above can be written as

$$
\begin{aligned}
\varphi(tL)(f) &= \sum_{j\geq\nu}\int_{2^{-j-1}}^{2^{-j}} \left(\frac{s}{t}\right)^M (tL)^M\varphi(tL)\psi_M(sL)f\frac{ds}{s} \\
&\quad + \sum_{j<\nu}\int_{2^{-j-1}}^{2^{-j}} \frac{t}{s}e^{-tL}\widetilde{\psi}(sL)f\frac{ds}{s}.
\end{aligned}
$$

Since $(tL)^M\varphi(tL) = (tL)^{M+1}e^{-tL}$ satisfies the Gaussian upper bound (see [13]), we have

$$
|(tL)^M\varphi(tL)\psi_M(sL)f(y)| \lesssim \int_X \frac{1}{V(y,t^{1/m})}\left(1+\frac{d(y,z)}{t^{1/m}}\right)^{-\lambda-N}|\psi_M(sL)f(z)|d\mu(z)
$$

where $N > n$.

It follows that

$$
\begin{aligned}
\frac{|(t^2L)^M\varphi(tL)\psi_M(tL)f(y)|}{(1+d(x,y)/t^{1/m})^\lambda} &\lesssim \int_X \frac{1}{V(y,t^{1/m})}\left(1+\frac{d(y,z)}{t^{1/m}}\right)^{-N} \\
\frac{|\psi_M(sL)f(z)|}{(1+d(x,z)/t^{1/m})^\lambda}d\mu(z)
\end{aligned}
$$

for $x, y \in X$.

Hence, for $j \geq \nu$, $t \in [2^{-\nu-1}, 2^{-\nu}]$ and $s \in [2^{-j-1}, 2^{-j}]$ we have

$$
\begin{aligned}
\frac{|(t^2L)^M\varphi(tL)\psi_M(sL)f(y)|}{(1+d(x,y)/t^{1/m})^\lambda} &\lesssim 2^{\lambda(j-\nu)}\psi^*_{M,\lambda}(sL)f(x) \\
&\qquad \int_X \frac{1}{V(y,t^{1/m})}\left(1+\frac{d(y,z)}{t^{1/m}}\right)^{-N}d\mu(y) \\
&\lesssim 2^{\lambda(j-\nu)}\psi^*_{M,\lambda}(sL)f(x).
\end{aligned}
$$

Since $\psi \in \mathscr{S}_m(\mathbb{R})$ and $\operatorname{supp}\psi \subset [1/2, 2]$, $x^{-2m}\psi(x) \in \mathscr{S}(\mathbb{R})$. Using Lemma 2.6 and an argument similar to the above, we obtain, for $j < \nu$, $t \in [2^{-\nu-1}, 2^{-\nu}]$ and

$s \in [2^{-j-1}, 2^{-j}]$,

$$\frac{|e^{-tL}\widetilde{\psi}(sL)f(y)|}{(1+d(x,y)/t^{1/m})^{\lambda}} \lesssim \widetilde{\psi}^*_{\lambda}(sL)f(x).$$

The above two estimates imply that

$$\begin{aligned}|\varphi^*_{\lambda}(tL)(f)| \le \sum_{j\ge\nu} 2^{-(j-\nu)(M-\lambda)} \sup_{s\in(2^{-j-1},2^{-j}]} \psi^*_{M,\lambda}(sL)f \\ + \sum_{j<\nu} 2^{-2m(\nu-j)} \sup_{s\in(2^{-j-1},2^{-j}]} \widetilde{\psi}^*_{\lambda}(sL)f.\end{aligned}$$

This, along with Proposition 2.7, implies that

$$\begin{aligned}|\varphi^*_{\lambda}(tL)(f)| &\lesssim \sum_{j\ge\nu-1} 2^{-(M-\lambda)(j-\nu)}\psi^*_{j,\lambda}(L)f + \sum_{j<\nu+3} 2^{-2m(\nu-j)}\psi^*_{j,\lambda}(L)f \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{-2m|\nu-j|}\psi^*_{j,\lambda}(L)f\end{aligned} \tag{19}$$

for all $t \in [2^{-\nu-1}, 2^{-\nu}]$ and $M > \lambda$.

By Young's inequality,

$$\begin{aligned}\Big(\int_0^\infty |\varphi^*_{\lambda}(tL)(f)|^2\frac{dt}{t}\Big)^{1/2} &\lesssim \Big(\sum_{\nu\in\mathbb{Z}}\Big[\sum_{j\in\mathbb{Z}} 2^{-(2m-\alpha)|\nu-j|}|\psi^*_{j,\lambda}(L)f|^2\Big)^{1/2} \\ &\lesssim \Big(\sum_{j\in\mathbb{Z}} |\psi^*_{j,\lambda}(L)f|^2\Big)^{1/2}.\end{aligned}$$

Hence, (17) follows from this and Proposition 2.9. The proof of Theorem 2.5 is thus complete. □

3 Estimates for the Schrödinger Group on Hardy Spaces

This section is devoted to the proof of Theorem 1.1. Before embarking on the proof, we need the following result from [8, Proposition 3.4]. Define

$$\|f\|_{\mathbf{B}^s} = \int_{-\infty}^{\infty} |\mathcal{F}f(\tau)|(1+|\tau|)^s d\tau,$$

where $\mathcal{F}f$ denotes the Fourier transform of f.

Lemma 3.1 ([8]) *Suppose that L is a non-negative self-adjoint operator on $L^2(X)$ and satisfies the Gaussian upper bound G. Then for every $s \ge 0$, there exists $C > 0$ such* *that for every* $j \in \mathbb{N} \cup \{0\}$,

$$\|1_B F(L) 1_{S_j(B)}\|_{2\to 2} \le C(\sqrt[m]{R} 2^j r_B)^{-s} \|\delta_R F\|_{\boldsymbol{B}^s}$$

for all balls B, *and all Borel functions* F *such that* $\operatorname{supp} F \subset [-R, R]$, *where* $\delta_R F(\cdot) = F(R\cdot)$.

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1: To prove the theorem, we will use Theorem 2.5 and the standard argument in, for example, [8, 14, 16, 19].

Set $F(\lambda) = (1+\lambda)^{-s} e^{it\lambda}$ with $t > 0$ and $s = n(1/p - 1/2)$. Let φ be a partition of unity. By Theorem 2.5 it suffices to show that there exists $C > 0$ such that

$$\|S_{L,\varphi} a\|_p \le C$$

for every $(p, 2, M, L, \epsilon)$ molecule a with $\epsilon > 0$ and $M > n(1/p - 1/2) + 1$.

Suppose a is a such a molecule that is associated with some ball B, and b be a function satisfying $a = L^M b$ from Definition 2.2. Using the following identity

$$\mathrm{Id} = (I - e^{-r_B^m L})^M + \sum_{k=1}^m (-1)^{k+1} C_k^M e^{-k r_B^m L} =: (I - e^{-r_B^m L})^M + P(r_B^m L)$$

we can write

$$\begin{aligned} S_{L,\varphi}(F(L)a) &= S_{L,\varphi}[(I - e^{-r_B^m L})^M F(L)a] + S_{L,\varphi}[(r_B^m L)^M P(r_B^m L) F(L) r_B^{-mM} b] \\ &\lesssim \sum_{k\ge 0} S_{L,\varphi}[(I - e^{-r_B^m L})^M F(L) a_k] \\ &\quad + \sum_{k\ge 0} S_{L,\varphi}[(r_B^m L)^M P(r_B^m L) F(L) r_B^{-mM} b_k] \\ &=: \sum_{k\ge 0} E_1^k + \sum_{k\ge 0} E_2^k, \end{aligned}$$

where $a_k = a.1_{S_k(B)}$ and $b_k = b.1_{S_k(B)}$.

Therefore, it suffices to prove that there exists $\epsilon' > 0$ such that

$$\|E_1^k\|_p + \|E_2^k\|_p \lesssim 2^{-k\epsilon'} (1+t)^{n(1/p-1/2)} \tag{20}$$

for each $k \in \mathbb{N} \cup \{0\}$.

Estimate for E_1^k : We now show that

$$\|E_1^k\|_p \lesssim 2^{-k\epsilon'} (1+t)^{n(1/p-1/2)}, \qquad k \in \mathbb{N} \cup \{0\} \tag{21}$$

for some $\epsilon' > 0$.

For each $k \geq 0$, setting $B_{t,k} = (1+t)2^k B$, we have

$$\begin{aligned}
\|E_1^k\|_p^p &= \|S_{L,\varphi}[(I - e^{-r_B^m L})^M F(L)a_k]\|_{L^p(4B_{t,k})}^p \\
&\quad + \|S_{L,\varphi}[(I - e^{-r_B^m L})^M F(L)a_k]\|_{L^p(X\setminus 4B_{t,k})}^p \\
&= E_{11}^k + E_{12}^k.
\end{aligned}$$

Using Hölder's inequality and the L^2-boundedness of $S_{L,\varphi}$ we obtain

$$\begin{aligned}
\|E_{11}^k\|_p^p &\lesssim V(2^k(1+t)B)^{\frac{2-p}{2}} \|S_{L,\varphi}[(I - e^{-r_B^m L})^M F(L)a_k]\|_{L^2(4B_{t,k})}^p \\
&\lesssim V(4(1+t)2^k B)^{1-p/2} \|a_k\|_2^p \\
&\lesssim 2^{-\epsilon kp} V(4(1+t)2^k B)^{1-p/2} V(2^k B)^{p/2-1} \\
&\lesssim 2^{-\epsilon kp}(1+t)^{n(1-p/2)},
\end{aligned}$$

where in the last inequality we used (2).

It remains to estimate the second term E_{12}^k. To do this, setting

$$F_{\ell,r_B}(\lambda) = \varphi_\ell(\lambda)(I - e^{-r_B^m \lambda})^M F(\lambda),$$

we then write

$$\begin{aligned}
\|E_{12}^k\|_p^p &= \Big\| \Big(\sum_{\ell\in\mathbb{Z}} |F_{\ell,r_B}(L)a_k|^2 \Big)^{1/2} \Big\|_{L^p(X\setminus 4B_{t,k})}^p \\
&\lesssim \Big\| \sum_{\ell\in\mathbb{Z}} |F_{\ell,r_B}(L)a_k| \Big\|_{L^p(X\setminus 4B_{t,k})}^p \\
&\lesssim \sum_{\ell\in\mathbb{Z}} \|F_{\ell,r_B}(L)a_k\|_{L^p(X\setminus 4B_{t,k})}^p \\
&= \sum_{\ell\geq\ell_0} \ldots + \sum_{\ell<\ell_0} \ldots =: F_1^k + F_2^k,
\end{aligned} \tag{22}$$

where ℓ_0 is the largest integer such that $2^{\ell_0(m-1)/m} \leq 2^k r_B$.

We estimate F_1^k first. To do this, we write

$$\begin{aligned}
F_1^k &= \sum_{\ell\geq\ell_0} \|F_{\ell,r_B}(L)a_k\|_{L^p(X\setminus 4B_{t,k})}^p \\
&\leq \sum_{\ell\geq\ell_0} \sum_{j\geq\frac{\ell-\ell_0}{2}} \|F_{\ell,r_B}(L)a_k\|_{L^p(S_j(B_{t,k}))}^p + \sum_{\ell\geq\ell_0} \|F_{\ell,r_B}(L)a_k\|_{L^p(B(x_B,2^{\ell(m-1)/m}(1+t)))}^p \\
&=: F_{11}^k + F_{12}^k.
\end{aligned}$$

By Hölder's inequality and property (ii) of Definition 2.2 we obtain

$$\begin{aligned} F_{12}^k &\lesssim \sum_{\ell \geq \ell_0} V(B(x_B, 2^{\ell(m-1)/m}(1+t)))^{1-p/2} \|F_{\ell,r_B}(L)a\|^p_{L^2(B(x_B,2^{\ell(m-1)/m}(1+t)))} \\ &\lesssim \sum_{\ell \geq \ell_0} V(B(x_B, 2^{\ell(m-1)/m}(1+t)))^{1-p/2} \|F_{\ell,r_B}\|^p_\infty \|a_k\|^p_2. \end{aligned}$$

This, along with the fact that $\|F_{\ell,r_B}\|_\infty \lesssim \min\{1, (2^\ell r_B^m)^M\} 2^{-\ell n(1/p-1/2)}$, implies that

$$\begin{aligned} F_{12}^k \lesssim \sum_{\ell \geq \ell_0} & 2^{-kp\epsilon} V(B(x_B, 2^{\ell(m-1)/m}(1+t)))^{1-p/2} \\ & \min\{1, (2^\ell r_B^m)^{pM}\} 2^{-\ell n(1-p/2)} V(2^k B)^{1-p/2}. \end{aligned}$$

On the other hand, since $2^{\ell_0(m-1)/m} \sim 2^k r_B$, we have, for $\ell \geq \ell_0$,

$$\begin{aligned} \frac{V(x_B, 2^{\ell(m-1)/m}(1+t))}{V(2^k B)} &\sim \frac{V(x_B, 2^{(\ell-\ell_0)(m-1)/m}(1+t)2^k r_B)}{V(2^k B)} \\ &\lesssim [2^{(\ell-\ell_0)(m-1)/m}(1+t)]^n \\ &\sim (1+t)^n [2^{\ell(m-1)/m}(2^k r_B)^{-1}]^n \\ &\lesssim (1+t)^n [2^{\ell(m-1)/m} r_B^{-1}]^n \end{aligned}$$

We thus deduce that

$$\begin{aligned} F_{12}^k &\lesssim 2^{-kp\epsilon}(1+t)^{n(1-p/2)} \sum_{\ell \geq \ell_0} \min\{1, (2^\ell r_B^m)^{pM}\} 2^{-\ell n(1-p/2)} [2^{\ell(m-1)/m} r_B^{-1}]^{n(1-p/2)} \\ &\lesssim 2^{-kp\epsilon}(1+t)^{n(1-p/2)} \sum_{\ell \geq \ell_0} \min\{1, (2^\ell r_B^m)^{pM}\} [2^{\ell/m} r_B]^{-n(1-p/2)} \\ &\lesssim 2^{-kp\epsilon}(1+t)^{n(1-p/2)}. \end{aligned}$$

We now take care of F_{11}^k. For $\ell \geq \ell_0$ and $j \geq \frac{(\ell-\ell_0)(m-1)}{m}$ we have

$$\begin{aligned} F_{11}^k &\leq \sum_{\ell \geq \ell_0} \sum_{j \geq \frac{\ell-\ell_0}{2}} \|F_{\ell,r_B} a_k\|^p_{L^2(S_j(B_{t,k}))} V(2^j B_{t,k})^{1-p/2} \\ &\lesssim \sum_{\ell \in \mathbb{N}} \sum_{j \geq \frac{(\ell-\ell_0)(m-1)}{m}} \|F_{r_B,\ell}(L)\|^p_{L^2(S_k(B)) \to L^2(S_j(B_{t,k}))} \|a_k\|^p_2 V(2^j B_{t,k})^{1-p/2} \\ &\lesssim \sum_{\ell \in \mathbb{N}} \sum_{j \geq \frac{(\ell-\ell_0)(m-1)}{m}} 2^{-kp\epsilon} \|F_{r_B,\ell}(L)\|^p_{L^2(S_k(B)) \to L^2(S_j(B_{t,k}))} V(2^k B)^{p/2-1} V(2^j B_{t,k})^{1-p/2}. \end{aligned}$$

This, in combination with the doubling property (2), yields that

$$F_{11}^k \lesssim \sum_{\ell \ge \ell_0} \sum_{j \ge \frac{(\ell-\ell_0)(m-1)}{m}} 2^{-kp\epsilon}[2^j(1+t)]^{n(1-p/2)} \|F_{r_B,\ell}(L)\|^p_{L^2(S_k(B)) \to L^2(S_j(B_{t,k}))}. \tag{23}$$

By Lemma 3.1, for $\alpha = n(1/p - 1/2) + \theta$ with $\theta \in (0, \epsilon)$, we have

$$\|F_{r_B,\ell}(L)\|_{L^2(S_k(B)) \to L^2(S_j(B_{t,k}))} \lesssim (2^{\ell/m} 2^j (1+t) 2^k r_B)^{-\alpha} \|\delta_{2^\ell} F_{r_B,\ell}\|_{\mathbf{B}^\alpha}. \tag{24}$$

We claim that for $\alpha > 0$,

$$\|\delta_{2^\ell} F_{r_B,\ell}\|_{\mathbf{B}^\alpha} \lesssim \max\{1, 2^{(\alpha - n(1/p-1/2))\ell}\}(1+t)^\alpha \min\{1, (2^\ell r_B^m)^M\}. \tag{25}$$

To show this, as in [8], we write

$$\begin{aligned}\|\delta_{2^\ell} F_{r_B,\ell}\|_{\mathbf{B}^\alpha} &= \|\varphi(\lambda)(I - e^{-2^\ell r_B^m \lambda})^M F(2^\ell \lambda)\|_{\mathbf{B}^\alpha} \\ &\lesssim \|\varphi(\lambda)(I - e^{-2^\ell r_B^m \lambda})^M\|_{\mathbf{B}^\alpha} \|\varphi(\lambda) F(2^\ell \lambda)\|_{\mathbf{B}^\alpha}.\end{aligned}$$

It is easy to see that

$$\|\varphi(\lambda)(I - e^{-2^\ell r_B^m \lambda})^M\|_{\mathbf{B}^\alpha} \lesssim \min\{1, (2^\ell r_B^m)^M\}.$$

On the other hand,

$$\mathcal{F}(\varphi(\lambda) F(2^\ell \lambda))(\tau) = \int_{-\infty}^{\infty} \varphi(\lambda) \frac{e^{i(2^\ell t - \tau)\lambda}}{(1 + 2^\ell \lambda)^s} d\lambda,$$

where $s = n(1/p - 1/2)$. Next, from integration by parts, we have, for each $N \in \mathbb{N}$,

$$\mathcal{F}(\varphi(\lambda) F(2^\ell \lambda))(\tau) \le C_N \min\{1, 2^{-\ell s}\}(1 + |2^\ell t - \tau|)^{-N}.$$

As a consequence,

$$\begin{aligned}\|\varphi(\lambda) F(2^\ell \lambda)\|_{\mathbf{B}^\alpha} &\lesssim \min\{1, 2^{-\ell s}\} \int_{\mathbb{R}} (1 + |2^\ell t - \tau|)^{-N} (1 + |\tau|)^\alpha d\tau \\ &\lesssim \max\{1, 2^{(\alpha - s)\ell}\}(1+t)^\alpha,\end{aligned}$$

which proves (25).

Substituting (25) into (24) we then obtain

$$\begin{aligned}&\|F_{r_B,\ell}(L)\|_{L^2(S_k(B)) \to L^2(S_j(B_{t,k}))} \lesssim (2^{\ell/m} 2^j 2^k r_B)^{-\alpha} \\ &\quad \max\{1, 2^{(\alpha - n(1/p-1/2))\ell}\} \min\{1, (2^\ell r_B^m)^M\}.\end{aligned}$$

This, in combination with (23), implies that for $\alpha = n(1/p - 1/2) + \theta$ with $\theta \in (0, \epsilon)$,

$$\begin{aligned} F_{11}^k &\lesssim \sum_{\ell \geq \ell_0} \sum_{j \geq \frac{(\ell-\ell_0)(m-1)}{m}} 2^{-kp\epsilon}(1+t)^{n(1-p/2)} 2^{-jp\theta} (2^{\ell/m} 2^k r_B)^{-\alpha p} \\ &\quad \max\{1, 2^{\ell\theta p}\} \min\{1, (2^\ell r_B^m)^{pM}\} \\ &=: \sum_{\ell \geq \ell_0 \colon \ell < 0} \dots + \sum_{\ell \geq \ell_0 \colon \ell \geq 0} \dots . \end{aligned}$$

For the first sum, we have

$$\begin{aligned} \sum_{\ell \geq \ell_0 \colon \ell < 0} \dots &\lesssim \sum_{\ell \geq \ell_0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)} (2^{\ell/m} 2^k r_B)^{-\alpha p} \min\{1, (2^\ell r_B^m)^{pM}\} \\ &\lesssim \sum_{\ell \geq \ell_0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)} (2^{\ell/m} r_B)^{-\alpha p} \min\{1, (2^\ell r_B^m)^{pM}\} \\ &\lesssim 2^{-kp\epsilon}(1+t)^{n(1-p/2)}, \end{aligned}$$

as long as $M > \alpha$.

For the contribution of the second sum we have

$$\begin{aligned} \sum_{\ell \geq \ell_0 \colon \ell \geq 0} \dots &\lesssim \sum_{\ell \geq \ell_0 \colon \ell \geq 0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)} 2^{-\frac{p\theta(\ell-\ell_0)(m-1)}{m}} (2^{\ell/m} 2^k r_B)^{-\alpha p} 2^{\ell\theta p} \\ &\quad \min\{1, (2^\ell r_B^m)^{pM}\} \\ &\lesssim \sum_{\ell \geq \ell_0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)} [2^{\ell(m-1)/m} (2^k r_B)^{-1}]^{-\theta p} (2^{\ell/m} 2^k r_B)^{-\alpha p} 2^{\theta p \ell} \\ &\quad \min\{1, (2^\ell r_B^m)^{pM}\} \\ &\lesssim \sum_{\ell \geq \ell_0 \vee 0} 2^{-kp(\epsilon-\theta)}(1+t)^{n(1-p/2)} (2^{\ell/m} r_B)^{-p(\alpha-\theta)} \\ &\quad \min\{1, (2^\ell r_B^m)^{pM}\} \\ &\lesssim 2^{-kp(\epsilon-\theta)}(1+t)^{n(1-p/2)}, \end{aligned}$$

where we used the fact that $2^{\ell_0(m-1)/m} \sim 2^k r_B$ in the second inequality. Therefore, it holds that

$$F_{11}^k \lesssim 2^{-k\epsilon'}(1+t)^{n(1-p/2)}$$

for some $\epsilon' > 0$.

Collecting the estimates of F_{11}^k and F_{12}^k, we arrive at

$$F_1^k \lesssim 2^{-k\epsilon'}(1+t)^{n(1-p/2)}$$

for some $\epsilon' > 0$.

It remains to handle the term F_2^k. Indeed, we have

$$F_2^k = \sum_{\ell<\ell_0} \|F_{\ell,r_B}(L)a_k\|^p_{L^p(X\setminus 4B_{t,k})} = \sum_{\ell<\ell_0}\sum_{j\geq 3} \|F_{\ell,r_B}(L)a_k\|^p_{L^p(S_j(B_{t,k}))}.$$

Arguing similarly to the estimate of F_{11}, we have

$$\begin{aligned}
F_2^k &\lesssim \sum_{\ell<\ell_0}\sum_{j\geq 3} 2^{-kp\epsilon}(1+t)^{n(1-p/2)}2^{-jp\theta}(2^{\ell/m}2^k r_B)^{-\alpha p}\max\{1, 2^{\theta\ell p}\}\min\{1, (2^\ell r_B^m)^{pM}\}\\
&\lesssim \sum_{\ell<\ell_0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)}(2^{\ell/m}2^k r_B)^{-\alpha p}\max\{1, 2^{\theta\ell p}\}\min\{1, (2^\ell r_B^m)^{pM}\}\\
&\lesssim \sum_{\ell<0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)}(2^{\ell/m} r_B)^{-\alpha p}\min\{1, (2^\ell r_B^m)^{pM}\}\\
&\quad + \sum_{0<\ell<\ell_0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)}(2^{\ell/m}2^k r_B)^{-\alpha p}2^{\theta\ell p}\min\{1, (2^\ell r_B^m)^{pM}\}.
\end{aligned}$$

It is clear that

$$\sum_{\ell<0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)}(2^{\ell/m} r_B)^{-\alpha p}\min\{1, (2^\ell r_B^m)^{pM}\} \lesssim 2^{-kp\epsilon}(1+t)^{n(1-p/2)},$$

as long as $M > \alpha$.

For the second sum, we have

$$\begin{aligned}
&\sum_{0<\ell<\ell_0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)}(2^{\ell/m}2^k r_B)^{-\alpha p}2^{\theta\ell p}\min\{1, (2^\ell r_B^m)^{pM}\}\\
&\qquad \lesssim \sum_{0<\ell<\ell_0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)}(2^{\ell/m}2^k r_B)^{-\alpha p}2^{\theta\ell p}\min\{1, (2^\ell r_B^m)^{pM}\}\\
&\qquad \lesssim \sum_{0<\ell<\ell_0} 2^{-kp\epsilon}(1+t)^{n(1-p/2)}[2^{\ell(m-1)/m}(2^k r_B)^{-1}]^{-\theta p}(2^{\ell/m}2^k r_B)^{-\alpha p}2^{\theta p\ell}\\
&\qquad\quad \min\{1, (2^\ell r_B^m)^{pM}\}\\
&\qquad \lesssim \sum_{\ell>0} 2^{-kp(\epsilon-\theta)}(1+t)^{n(1-p/2)}(2^{\ell/m} r_B)^{-p(\alpha-\theta)}\min\{1, (2^\ell r_B^m)^{pM}\}\\
&\qquad \lesssim 2^{-kp(\epsilon-\theta)}(1+t)^{n(1-p/2)}
\end{aligned}$$

where in the second inequality we used the fact that

$$2^{\ell(m-1)/m}(2^k r_B)^{-1} \leq 2^{\ell_0(m-1)/m}(2^k r_B)^{-1} \leq 1,$$

along with $2^{-\ell/2}r_B \geq 2^{-\ell_0/2}r_B \geq 1$. Therefore we may conclude

$$F_2^k \lesssim 2^{-kp\epsilon'}(1+t)^{n(1-p/2)}$$

for some $\epsilon' > 0$. and this, along with the estimate of F_1^k and (22), implies that

$$E_{12}^k \lesssim 2^{-kp\epsilon'}(1+t)^{n(1-p/2)},$$

completing the proof of (21).

Estimate for E_2^k : We now show that

$$\|E_2^k\|_p \lesssim 2^{-k\epsilon'}(1+t)^{n(1/p-1/2)}, \qquad k = 0, 1, 2, \dots \tag{26}$$

for some $\epsilon' > 0$.

Set $G_{\ell,r_B}(\lambda) = \varphi_\ell(\lambda)(r_B^m\lambda)^M P(r_B^m\lambda)F(\lambda)$. Then we have

$$\|G_{\ell,r_B}\|_\infty \lesssim \min\{(2^\ell r_B^m)^{-M}, (2^\ell r_B^m)^M\}2^{-\ell n(1/p-1/2)}.$$

Arguing similarly to (25), we see that

$$\|\delta_{2^\ell} G_{r_B,\ell}\|_{\mathbf{B}^\alpha} \lesssim \max\{1, 2^{(\alpha-n(1/p-1/2))\ell}\}(1+t)^\alpha \min\{(2^\ell r_B^m)^{-M}, (2^\ell r_B^m)^M\},$$

as along as $M > n(1/p-1/2)+1 > \alpha = n(1/p-1/2)+\theta$ with $\theta \in (0, \epsilon)$.

At this stage, proceed along the same lines as in the proof of (21) to obtain (26). This completes the proof of (20), and thus of Theorem 1.1. □

4 Some Applications

Our framework is sufficiently general to include a large variety of applications; in this section we survey a few of the more interesting cases.

4.1 Laplacian-Like Operators

Let us here consider two additional conditions on the operator L: *Hölder regularity:* there exists $\delta_0 \in (0, 1]$ so that whenever $d(x, \bar{x}) < t^{1/m}$ we have

$$|e^{-tL}(x,y) - e^{-tL}(\bar{x},y)| \lesssim \left(\frac{d(x,\bar{x})}{t^{1/m}}\right)^{\delta_0} \frac{1}{V\left(x, \sqrt{t}\right)} \exp\left(-\frac{d(x,y)^{m/(m-1)}}{ct^{1/(m-1)}}\right), \tag{H}$$

Conservation: for all $y \in X$ and $t > 0$ we have

$$\int_X e^{-tL}(x,y)\, d\mu(x) = 1. \tag{C}$$

Examples of typical operators satisfying G, H and C include the $2k$-higher order elliptic operator in divergence form with smooth coefficients, the homogeneous sub-Laplacian on a homogeneous group and the Laplace-Beltrami operator on a doubling manifold admits the Poincaré's inequality as in [1].

We recall the definition of the Hardy spaces $H^p(X)$ for $\frac{n}{n+1} < p \leq 1$ from [10]. For $0 < p \leq 1$, we say that a function a is a $(2, p)$ atom if there exists a ball B such that

(i) $\operatorname{supp} a \subset B$;
(ii) $\|a\|_{L^2} \leq V(B)^{1/2-1/p}$;
(iii) $\int a(x)\mu(x) = 0$.

For $p = 1$ the atomic Hardy space H^1 is defined as follows. We say that a function $f \in H^1(X)$, if $f \in L^1$ and there exist a sequence $(\lambda_j)_{j\in\mathbb{N}} \in l^1$ and a sequence of $(2, 1)$-atoms $(a_j)_{j\in\mathbb{N}}$ such that $f = \sum_j \lambda_j a_j$. We set

$$\|f\|_{H^1} = \inf\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\}.$$

For $0 < p < 1$, as in [10], we need to introduce the Lipschitz space $\mathfrak{L}_\alpha$. We say that the function $f \in \mathfrak{L}_\alpha$ if there exists a constant $c > 0$, such that

$$|f(x) - f(y)| \leq c|B|^\alpha$$

for all ball B and $x, y \in B$. The best constant c above can be taken to be the norm of f and is denoted by $\|f\|_{\mathfrak{L}_\alpha}$.

Now let $0 < p < 1$ and $\alpha = 1/p - 1$. We say that a function $f \in H^p(X)$, if $f \in (\mathfrak{L}_\alpha)^*$ and there is a sequence $(\lambda_j)_{j\in\mathbb{N}} \in l^p$ and a sequence of $(2, p)$-atoms $(a_j)_{j\in\mathbb{N}}$ such that $f = \sum_j \lambda_j a_j$. Furthermore, we set

$$\|f\|_{H^p} = \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{1/p} : f = \sum_j \lambda_j a_j\right\}.$$

Note that when $0 < p < 1$, the quantity $\|\cdot\|_{H^p}$ is not the norm but $d(f, g) := \|f - g\|_{H^p}$ forms a metric.

Lemma 4.1 *Let L be a nonnegative self-adjoint operator satisfying G, H and C. Then $H^p_L(X) \equiv H^p(X)$ for $\frac{n}{n+\delta_0} < p \leq 1$.*

Proof The proof of this lemma is fairly standard but we could not find in the existing literature. Thus for the reader's benefit, we will provide a sketch of its proof. Firstly, arguing similarly to Lemma 9.1 in [16], we have that every $(p, 2, M, L, \epsilon)$ molecule m satisfies

$$\int_X m(x)d\mu(x) = 0.$$

Therefore, by the argument as in the proof of [3, Proposition 4.16] we can show that $\|m\|_{H^p(X)} \lesssim 1$ uniformly for every $(p, 2, M, L, \epsilon)$ molecule m with $M > \frac{n(2-p)}{2mp}$, $\epsilon > n$ and $\frac{n}{n+1} < p \leq 1$. It follows immediately that $H^p_L(X) \subset H^p(X)$.

Conversely, if a is a $(2, p)$ atom with $\frac{n}{n+\delta_0} < p \le 1$, then by a standard argument we can show that $\|\mathcal{A}a\|_p \lesssim 1$, where $\mathcal{A}$ is the square function defined by (11). It follows $\|a\|_{H^p_L} \lesssim 1$ and this gives $H^p(X) \subset H^p_L(X)$. The proof is thus complete. □

From Lemma 4.1 and Theorem 1.1 we deduce the following.

Theorem 4.2 *Let L be a nonnegative self-adjoint operator satisfying G, H and C. Then for each $\frac{n}{n+\delta_0} < p \le 1$ and $s = n(1/p - 1/2)$ we have*

$$\|(I+L)^{-s}e^{itL}f\|_{H^p(X)} \lesssim (1+|t|)^s\|f\|_{H^p(X)}, \qquad t \in \mathbb{R}.$$

4.2 Hermite Operators

Let $\mathcal{H} = -\Delta + |x|^2$ be the Hermite operator on $\mathbb{R}^n$ with $n \ge 1$. Let $p_t(x, y)$ denote the kernel of the semigroup $e^{-t\mathcal{H}}$. It is clear that $p_t(x, y)$ enjoys the Gaussian upper bound G. Moreover we have an explicit representation for the kernel $p_t(x, y)$:

$$p_t(x,y) = \frac{1}{\pi^{n/2}}\Big(\frac{e^{-2t}}{1-e^{-4t}}\Big)^{n/2}\exp\Big(-\frac{1}{4}\frac{1+e^{-2t}}{1-e^{-2t}}|x-y|^2 - \frac{1}{4}\frac{1-e^{-2t}}{1+e^{-2t}}|x+y|^2\Big)$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$. This representation is well known – see for example [26].

Let $\rho(x) = \min\{1, |x|^{-1}\}$ for $x \in \mathbb{R}^n$. Let $p \in (0, 1]$. A function a is called a (p, ∞, ρ)-*atom* associated to the ball $B(x_0, r)$ if

(i) $\operatorname{supp} a \subset B(x_0, r)$;
(ii) $\|a\|_{L^\infty} \le |B(x_0, r)|^{-1/p}$;
(iii) $\int x^\alpha a(x)dx = 0$ for all $|\alpha| \le \lfloor n(1/p - 1)\rfloor$ if $r < \rho(x_0)/4$.

The Hardy space $H^p_{at,\rho}(\mathbb{R}^n)$ is then defined to be the set of all functions f which can be expressed in the form $f = \sum_j \lambda_j a_j$ where $(\lambda_j)_j \in \ell^p$ and a_j are (p, ∞, ρ)-atoms. Its norm is given by

$$\|f\|_{H^p_{at,\rho}(\mathbb{R}^n)} := \inf\Big\{\Big(\sum_j |\lambda_j|^p\Big)^{1/p} : f = \sum_j \lambda_j a_j\Big\},$$

where the infimum is taken over all possible atomic decompositions of f. From the definition, it is obvious that $H^p(\mathbb{R}^n) \subsetneq H^p_{at,\rho}(\mathbb{R}^n)$ for all $p \in (0, 1]$; more importantly, we have $H^p_{at,\rho}(\mathbb{R}^n) \equiv H^p_{\mathcal{H}}(\mathbb{R}^n)$ for all $0 < p \le 1$ (see for instance [4, 15]), thus the Hardy space associated to the Hermite operator *contains* the standard Hardy spaces $H^p(\mathbb{R}^n)$.

Theorem 4.3 *Let $\mathcal{H} = -\Delta + |x|^2$ be the Hermite operator on $\mathbb{R}^n$ with $n \ge 1$. Then for each $0 < p \le 1$ and $s = n(1/p - 1/2)$ we have*

$$\|\mathcal{H}^{-s}e^{it\mathcal{H}}f\|_{H^p_{at,\rho}(\mathbb{R}^n)} \lesssim \|f\|_{H^p_{at,\rho}(\mathbb{R}^n)}, \qquad t \in \mathbb{R}.$$

Proof Since $\mathcal{H}$ is a nonnegative self-adjoint operator and satisfies the Gaussian upper bound G with $m = 2$, then by Theorem 1.1 and the coincidence $H^p_{at,\rho}(\mathbb{R}^n) \equiv H^p_{\mathcal{H}}(\mathbb{R}^n)$ for every $0 < p \le 1$, we have

$$\|(I + \mathcal{H})^{-s} e^{it\mathcal{H}} f\|_{H^p_{at,\rho}(\mathbb{R}^n)} \lesssim (1 + |t|)^s \|f\|_{H^p_{at,\rho}(\mathbb{R}^n)}.$$

On the other hand, it is well-known that the spectrum of $\mathcal{H}$ is contained in $[1, \infty)$ (see [26]). It follows that

$$\|\mathcal{H}^{-s} e^{it\mathcal{H}} f\|_{H^p_{at,\rho}(\mathbb{R}^n)} \lesssim (1 + |t|)^s \|f\|_{H^p_{at,\rho}(\mathbb{R}^n)}.$$

It is also well-known that

$$e^{it\mathcal{H}} f = \int_{\mathbb{R}^n} \exp\left(2\frac{(|x|^2 + |y|^2)\cos 2t - 2\langle x, y\rangle}{i \sin 2t}\right) f(y)dy,$$

which implies that the flow $e^{it\mathcal{H}}$ is time-periodic with the period $T = 2\pi$. Therefore,

$$\|\mathcal{H}^{-s} e^{it\mathcal{H}} f\|_{H^p_{at,\rho}(\mathbb{R}^n)} \lesssim \|f\|_{H^p_{at,\rho}(\mathbb{R}^n)},$$

which completes our proof. □

Note that in [25], Thangavelu proved that for each $t > 0$,

$$\|\mathcal{H}^{-s} e^{it\mathcal{H}} f\|_{L^1(\mathbb{R}^n)} \le C_t \|f\|_{H^1(\mathbb{R}^n)},$$

where C_t is a constant dependent on t. In comparison, our result in Theorem 4.3 improves upon the result in [25] significantly, even in the case $p = 1$ since $H^1_{at,\rho}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. Moreover, the constant in Theorem 4.3 is independent of t.

We now consider an application of Theorem 4.3 to the Schödinger equation

$$\begin{cases} i\partial_t u + \mathcal{H}\sqcap = 0, \\ u(x, 0) = f. \end{cases} \tag{27}$$

For each $0 < p \le 1$ and $s > 0$ we define the Hardy-Sobolev space $\dot{H}^{p,s}_{\mathcal{H}}(\mathbb{R}^n)$ associated to $\mathcal{H}$ by

$$\|f\|_{\dot{H}^{p,s}_{\mathcal{H}}(\mathbb{R}^n)} = \|\mathcal{H}^{s/2} f\|_{H^p_{at,\rho}(\mathbb{R}^n)}.$$

It is well-known (see [2]) that

$$\|f\|_{\dot{H}^{p,s}_{\mathcal{H}}(\mathbb{R}^n)} = \left\| \Big[\sum_j (2^{-js} |\psi_j(\sqrt{\mathcal{H}}))^2 f|^2 \Big]^{1/2} \right\|_p.$$

This means that similar to the classical setting, the Hardy-Sobolev space $\dot{H}^{p,s}_{\mathcal{H}}(\mathbb{R}^n)$ can be viewed as a Triebel–Lizorkin type space $\dot{F}^{s}_{p,2}(\mathbb{R}^n)$ that is associated to $\mathcal{H}$.

Returning to the equation (27), we note that its solution can be formally written as $u = e^{it\mathcal{H}} f$. From Theorem 4.3 we can then deduce the following result.

Corollary 4.4 *Suppose u is a solution to (27) and let $0 < p \le 1$. If the initial data $f \in \dot{H}^{p,s}_{\mathcal{H}}(\mathbb{R}^n)$ with $s = n(1/p - 1/2)$, then we have*

$$\|u\|_{H^p_{at,\rho}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p,2s}_{\mathcal{H}}(\mathbb{R}^n)}.$$

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