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Concentration of Eigenfunctions of Schrödinger Operators

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Abstract

We consider the limit measures induced by the rescaled eigenfunctions of Schrödinger operators with even confining potentials. We show that the limit measure is supported on $[-1, 1]$ and with the density proportional to $(1-|x|^\beta)^{-1/2}$ when the non-perturbed potential resembles $|x|^\beta$, $\beta > 0$, for large x, and with the uniform density for super-polynomially growing potentials. We compare these results to analogous results in orthogonal polynomials and semiclassical defect measures.

Keywords Schrödinger operators · Eigenfunctions · Limit measure

Mathematics Subject Classification 35P15 · 35L05

1 Introduction

Let A be a Schrödinger operator acting in $L^2(\mathbb{R})$

$$A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Q(x), \tag{1.1}$$

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where Q is a real-valued, even potential which tends to $+\infty$ as $|x| \to \infty$. More precisely, we suppose that $Q = V + W$, where V is sufficiently regular (see Assumptions I) and W is its possibly irregular perturbation (satisfying Assumption II that guarantees that W is small in a suitable sense). Our main condition on the potential is that V satisfies

$$\exists \beta \in (0, \infty], \quad \forall x \in (-1, 1), \quad \lim_{t \to +\infty} \frac{V(xt)}{V(t)} = \omega_\beta(x), \tag{1.2}$$

where

$$\omega_\beta(x) := \begin{cases} |x|^\beta, & \beta \in (0, \infty), \\ 0, & \beta = \infty. \end{cases} \tag{1.3}$$

As explained in [17, Sec. 1.3], the existence of the limiting function in (1.2) already implies that ω_β is a power of $|x|$ or zero; functions V satisfying (1.2) with $\beta < \infty$ are called *regularly varying*.

It is well-known (also under much weaker assumptions on Q) that the operator A, defined via its quadratic form, is self-adjoint with compact resolvent, hence its spectrum is real and discrete. In fact, all eigenvalues $\{\lambda_k\}$ of A are simple, thus they can be ordered increasingly and the corresponding eigenspaces are one-dimensional. Since the potential Q is real, eigenfunctions $\{\psi_k\}$ related to $\{\lambda_k\}$ can be selected as real functions satisfying

$$A\psi_k = \lambda_k \psi_k, \quad \|\psi_k\| = \|\psi_k\|_{L^2(\mathbb{R})} = 1, \quad k \in \mathbb{N}. \tag{1.4}$$

These conditions do not determine ψ_k uniquely, since $-\psi_k$ satisfies the same conditions; nonetheless, the squares $\{\psi_k^2\}$ are already uniquely determined.

Let x_{λ_k} be positive turning points of V corresponding to eigenvalues $\{\lambda_k\}$, *i.e.*

$$V(x_{\lambda_k}) = \lambda_k, \quad x_{\lambda_k} > 0, \quad k \in \mathbb{N}, \quad k > k_0; \tag{1.5}$$

here $k_0 \in \mathbb{N}$ is sufficiently large so that x_{λ_k} are well-defined by (1.5), see also Assumption I. We define non-negative normalized measures on $\mathbb{R}$ induced by the eigenfunctions $\{\psi_k\}$ by

$$d\mu_k := x_{\lambda_k} \, \psi_k(x_{\lambda_k} x)^2 \, dx, \quad x \in \mathbb{R}, \quad k \in \mathbb{N}, \quad k > k_0. \tag{1.6}$$

This rescaling transforms the classically forbidden region $\{x : V(x) > \lambda_k\}$ with (super)-exponential decay of ψ_k to $\mathbb{R} \setminus [-1, 1]$ while the rescaled functions $\psi_k(x_{\lambda_k} \cdot)$ oscillate in $[-1, 1]$. Notice that W enters the definition of $\{x_{\lambda_k}\}$, and thus the rescaling of eigenfunctions, since $\{\lambda_k\}$ are eigenvalues of the operator with the potential $Q = V + W$; the assumptions on the size of W comparing to V, see Assumption II and Proposition 2.2, allow for treating W perturbatively.

In this paper, we prove that measures (1.6) converges (as $k \to \infty$) to a limiting concentration measure supported on $[-1, 1]$

$$d\mu_* := \frac{\Gamma(\frac{1}{2} + \frac{1}{\beta})}{2\pi^{\frac{1}{2}}\Gamma(1 + \frac{1}{\beta})} \frac{\mathbb{1}_{[-1,1]}(x)}{(1 - \omega_\beta(x))^{\frac{1}{2}}}\, dx, \tag{1.7}$$

see Theorem 2.3. This generalizes the classical result for the harmonic oscillator, *i.e.* $Q(x) = x^2$, namely *the arcsine law* for the concentration measure

$$\frac{1}{\pi} \frac{\mathbb{1}_{[-1,1]}(x)}{\sqrt{1 - x^2}}\, dx \tag{1.8}$$

of the Hermite functions. Limiting measures of the type (1.7) were found for rescaled eigenfunctions with a different normalization for polynomial, possibly complex, potentials in [3, Thm. 2]. The concentration of eigenfunctions is in particular used in estimates of norms of the spectral projections of non-self-adjoint Schrödinger operators obtained through conjugation, see [14], in particular, Sect. 3.

Notice that the condition (1.2) does not require V to be a polynomial. For instance, the potentials below satisfy both technical Assumption I and the condition (1.2):

$$V(x) = |x|^\alpha \log(1 + x^2), \quad \alpha > 0, \tag{1.9}$$

lead to the limit

$$\omega_\alpha(x) = |x|^\alpha, \quad x \in (-1, 1), \tag{1.10}$$

while for the fast-growing potentials

$$V(x) = \exp(|x|^\gamma), \quad \gamma > 0, \tag{1.11}$$

the limit reads

$$\omega_\infty(x) = 0, \quad x \in (-1, 1); \tag{1.12}$$

the latter is not a special case, see Proposition 2.1.i). Moreover, one can include further, possibly irregular and unbounded perturbations W, see Proposition 2.2 for examples of admissible W.

We emphasize that while the limiting function, if exists, is always homogeneous, this not required for V; see examples (1.9) and (1.11) above. Thus rescaling leads to a semi-classical operator only in very special cases; a relation of our result and so called semi-classical defect measures in these special cases can be found in Sect. 5.2 below.

This paper is organized as follows. Our results with precise assumptions are formulated in Sect. 2 and they are proved in Sect. 3 relying on asymptotic formulas for the eigenfunctions $\{\psi_k\}$ summarized in Sect. 3.1. In Sect. 4 we prove the asymptotic formulas following and slightly extending the ideas and results in the book [18, §22.27] and in [7]. Finally, in Sect. 5 our results are compared to the existing literature in more detail.

1.1 Notation

Throughout the paper, we employ notations and results summarized in Sect. 3.1. In particular, to avoid many appearing constants, for $a, b \geq 0$, we write $a \lesssim b$ if there exists a constant $C > 0$, independent of any relevant variable or parameter, such that $a \leq Cb$; the relation $a \gtrsim b$ is introduced analogously. By $a \approx b$ it is meant that $a \lesssim b$ and $a \gtrsim b$. The natural numbers are denoted by $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2 Assumptions and Results

Our results are obtained under the following assumptions on the potential $Q = V + W$. The conditions on V, similar to those used in [7, 18], guarantee that V is an even confining potential with sufficient regularity to obtain convenient asymptotic formulas for eigenfunctions (associated with large eigenvalues) of the corresponding Schrödinger operator, see Sects. 3.1 and 4 for details. The conditions on W ensure that it is indeed a small perturbation which does not essentially affect the shape of the eigenfunctions.

Assumption I Let $V : \mathbb{R} \to \mathbb{R}$ satisfy the following conditions.

(i) $V \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ is even,

$$\lim_{|x| \to +\infty} V(x) = +\infty, \tag{2.1}$$

(ii) there exists $\xi_0 > 0$ such that $V \in C^3(\mathbb{R} \setminus [-\xi_0, \xi_0])$,

$$V(x) > 0, \;\; V'(x) > 0, \qquad x \geq \xi_0, \tag{2.2}$$

and

$$\frac{V'^2}{V^{\frac{5}{2}}} \in L^1((\xi_0, \infty)), \qquad \frac{V''}{V^{\frac{3}{2}}} \in L^1((\xi_0, \infty)), \tag{2.3}$$

(iii) there exists $\nu \geq -1$ such that for all $x \geq \xi_0$

$$\begin{aligned} &V'(x) \approx V(x) x^{\nu}, \\ &|V''(x)| \lesssim V'(x) x^{\nu}, \quad |V'''(x)| \lesssim V'(x) x^{2\nu}. \end{aligned} \tag{2.4}$$

□

Assumption I is an extension of conditions in [18, §22.27] where the case $\nu = -1$, *i.e.* polynomial-like potentials, is analyzed; conditions analogous to Assumption I are used also in [1, 9] where the resolvent estimates of non-self-adjoint Schrödinger operators are given. The assumptions of [7] allow for fast growing potentials and are based on suitable restrictions of V''', see [7, Condition 2].

The first assumption (2.4) implies there are two constants $0 < c_1 \le c_2 < \infty$ such that for all $x \ge \xi_0$

$$\begin{aligned} x^{c_1} \lesssim V(x) \lesssim x^{c_2}, \qquad & \nu = -1, \\ \exp(c_1 x^{\nu+1}) \lesssim V(x) \lesssim \exp(c_2 x^{\nu+1}), \quad & \nu > -1. \end{aligned} \tag{2.5}$$

This can be seen from (with $\xi_0 \le x_1 \le x_2$)

$$\log \frac{V(x_2)}{V(x_1)} = \int_{x_1}^{x_2} \frac{V'(s)}{V(s)}\,\mathrm{d}s \approx \int_{x_1}^{x_2} s^{\nu}\,\mathrm{d}s = \begin{cases} \dfrac{x_2^{\nu+1} - x_1^{\nu+1}}{\nu+1}, & \nu > -1, \\ \log \dfrac{x_2}{x_1}, & \nu = -1. \end{cases} \tag{2.6}$$

The crucial technical observation used frequently in the proofs is that (2.4) imply that, for any $\varepsilon \in (0,1)$ and all sufficiently large $x > 0$, we have

$$V^{(j)}(x+\Delta) \approx V^{(j)}(x), \qquad |\Delta| \le \varepsilon x^{-\nu}, \quad j = 0, 1, \tag{2.7}$$

i.e. we have a control of how much V and V' varies over the intervals of size $x^{-\nu}$, see Lemma 4.1. Assumptions (2.3) and (2.4) also imply that

$$\frac{V'(x)}{V(x)^{\frac{3}{2}}} = o(1), \qquad x \to +\infty, \tag{2.8}$$

see Lemma 3.2, which is almost optimal condition for the separation property of the domain of the self-adjoint Schrödinger operator $B = -\mathrm{d}^2/\mathrm{d}x^2 + V(x)$, namely,

$$\mathrm{Dom}(B) = W^{2,2}(\mathbb{R}) \cap \{f \in L^2(\mathbb{R}) \,:\, Vf \in L^2(\mathbb{R})\}, \tag{2.9}$$

see [4, 5, 8]; note that the separation property might be lost for A due to the possibly irregular W.

The following proposition relates the parameter ν and the condition (1.2).

Proposition 2.1 *Let V satisfy Assumption I.*

(i) *If $\nu > -1$, then V satisfies the condition (1.2) with $\beta = \infty$.*
(ii) *If $\nu = -1$ and V satisfies the condition (1.2), then $\beta \in (0, \infty)$.*

Proof Let $x \in (0,1)$ be fixed. From (2.6), we have that for all $t \ge \xi_0/x$

$$\log \frac{V(t)}{V(xt)} \approx \begin{cases} \dfrac{t^{\nu+1}}{\nu+1}(1 - x^{\nu+1}), & \nu > -1, \\ -\log x, & \nu = -1. \end{cases} \tag{2.10}$$

Thus, if $\nu > -1$, we get that for every $x \in (0,1)$

$$\lim_{t \to +\infty} \frac{V(xt)}{V(t)} = 0. \tag{2.11}$$

If $\nu = -1$ and the condition (1.2) holds, then for every $x \in (0, 1)$

$$x^{\beta_1} \le \lim_{t \to +\infty} \frac{V(xt)}{V(t)} \le x^{\beta_2} \tag{2.12}$$

where $\beta_1, \beta_2 \in (0, \infty)$ are independent of x. □

In the next step, we formulate a condition on the perturbation W that guarantees that it is small in a suitable sense (arising in the proof of Theorem 3.3). The appearing weight w_1^{-2} is naturally related with the main part of the potential V, although, the precise formula (3.24) might seem more complicated to grasp. It includes the turning point x_λ of V, the quantity a_λ (the value of V' at the turning point) and a "natural small region" around the turning point (characterized by δ and δ_1), see Sect. 3.1 for details. Examples of perturbations satisfying Assumption II are given in Proposition 2.2 below.

Assumption II Let w_1 be as in (3.24) below. Let $W : \mathbb{R} \to \mathbb{R}$ be even, locally integrable and satisfy

$$\mathcal{J}_W(\lambda) := \int_0^\infty \frac{W(s)}{w_1(s)^2}\, \mathrm{d}s = o(1), \quad \lambda \to +\infty. \tag{2.13}$$

□

Proposition 2.2 *Let $V(x) = |x|^\beta$, $\beta > 0$, and let $W = W_1 + W_2$ where* $\operatorname{supp} W_1$ *is compact, $W_1 \in L^1(\mathbb{R})$, $W_2 \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and let $|W_2(x)| \lesssim |x|^\gamma$, $x \in \mathbb{R}$, for some $\gamma \in \mathbb{R}$. Then (2.13) is satisfied if $\beta > 2\gamma + 2$. Moreover, if $\beta > 1$, already $W_1 \in L^1(\mathbb{R})$ suffices (one can omit the condition on the compactness of support of W_1).*

Proof For all large $\lambda > 0$, we get (let $\operatorname{supp} W_1 \subset [-x_1, x_1]$)

$$\int_0^\infty \frac{W_1(s)}{w_1(s)^2}\, \mathrm{d}s = \int_0^{x_1} \frac{W_1(s)}{(\lambda - s^\beta)^{\frac{1}{2}}}\, \mathrm{d}s \lesssim \frac{\|W_1\|_{L^1}}{\lambda^{\frac{1}{2}}}. \tag{2.14}$$

For $\beta > 1$ and $W_1 \in L^1(\mathbb{R})$ without the condition on $\operatorname{supp} W_1$, one can use (3.20) and (3.19) to obtain

$$\int_0^\infty \frac{W_1(s)}{w_1(s)^2}\, \mathrm{d}s \lesssim \frac{1}{a_\lambda^{\frac{1}{3}}} \int_0^\infty W_1(s)\, \mathrm{d}s \lesssim \lambda^{\frac{1-\beta}{3\beta}} \|W_1\|_{L^1}. \tag{2.15}$$

Next, changing the integration variable $s = x_\lambda t$ and using (3.19), we get (with the assumption $\beta > 2\gamma + 2$)

$$\begin{aligned} \int_0^\infty \frac{W_2(s)}{w_1(s)^2}\, \mathrm{d}s &\lesssim \int_0^1 \frac{W_2(s)}{w_1(s)^2}\, \mathrm{d}s + \frac{x_\lambda^{1+\gamma}}{\lambda^{\frac{1}{2}}} \int_0^\infty \frac{t^\gamma\, \mathrm{d}t}{|1 - t^\beta|^{\frac{1}{2}}} + \frac{x_\lambda^\gamma (\delta + \delta_1)}{a_\lambda^{\frac{1}{3}}} \\ &\lesssim \lambda^{-\frac{1}{2}} + \lambda^{\frac{2\gamma+2-\beta}{2\beta}} + \lambda^{\frac{3\gamma+2-2\beta}{3\beta}} = o(1), \quad \lambda \to +\infty. \end{aligned} \tag{2.16}$$

□

Conditions on W in Proposition 2.2, in particular $\beta > 2\gamma + 2$ or $W \in L^1(\mathbb{R})$ when $\beta > 1$, arise also in [9, 13], where the Riesz basis property of eigenfunctions, eigenvalue asymptotics and resolvent estimates are analyzed for *complex* W.

Our main result reads as follows.

Theorem 2.3 *Let $Q = V + W$ where V and W satisfy Assumptions I and II, respectively. Let V satisfy in addition the condition (1.2) and let $\{\mu_k\}$, μ_* be as in (1.6), (1.7), respectively. Let*

$$\mathcal{F}_V := \{f \in L^\infty_{\mathrm{loc}}(\mathbb{R}) \,:\, \exists M \geq 0,\; f \exp(-M|V|^{\frac{1}{2}}) \in L^\infty(\mathbb{R})\}. \tag{2.17}$$

Then, for every $f \in \mathcal{F}_V$, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}} f(x)\,\mathrm{d}\mu_k(x) = \frac{\Gamma(\frac{1}{2}+\frac{1}{\beta})}{2\pi^{\frac{1}{2}}\Gamma(1+\frac{1}{\beta})} \int_{-1}^{1} \frac{f(x)}{(1-\omega_\beta(x))^{\frac{1}{2}}}\,\mathrm{d}x. \tag{2.18}$$

Hence, in particular, the measures $\{\mu_k\}$ converge weakly to the limit measure μ_* as $k \to \infty$.

2.1 Distribution of Zeros

We remark that the related result on the number of zeros of the eigenfunction ψ_k in $[-\varepsilon x_{\lambda_k}, \varepsilon x_{\lambda_k}]$, $\varepsilon \in (0, 1]$, denoted by $N_k(\varepsilon x_{\lambda_k})$, is

$$\lim_{k\to\infty} \frac{N_k(\varepsilon x_{\lambda_k})}{k} = \frac{\Gamma(\frac{3}{2}+\frac{1}{\beta})}{\pi^{\frac{1}{2}}\Gamma(1+\frac{1}{\beta})} \int_{-\varepsilon}^{\varepsilon} (1-\omega_\beta(x))^{\frac{1}{2}}\,\mathrm{d}x, \qquad \varepsilon \in (0, 1]. \tag{2.19}$$

This generalizes the classical results for the harmonic oscillator, *i.e.* $Q(x) = x^2$, namely the semi-circle law for the limiting distribution of the number of zeros of Hermite functions,

$$\lim_{k\to\infty} \frac{N_k(\varepsilon\sqrt{2k+1})}{k} = \frac{2}{\pi} \int_{-\varepsilon}^{\varepsilon} \sqrt{1-x^2}\,\mathrm{d}x, \qquad \varepsilon \in (0, 1], \tag{2.20}$$

see *e.g.* [6, 11, 16]. A generalization of (2.19) for polynomial, possibly complex, potentials has been given in [3].

The distribution of zeros of eigenfunctions ψ_k, see (2.19), is closely related to the distribution of eigenvalues of A and it is essentially proved in [19, Sect. 7]. Indeed, without the perturbation W, *i.e.* $W = 0$, the eigenvalues of A satisfy

$$\frac{\pi^{\frac{1}{2}}\Gamma(1+\frac{1}{\beta})}{\Gamma(\frac{3}{2}+\frac{1}{\beta})} x_{\lambda_k} \lambda_k^{\frac{1}{2}} = \pi k(1+o(1)), \qquad k \to \infty, \tag{2.21}$$

see [19, Sect. 7], [7, Theorem. 2], so (2.19) follows from [19, Lemma. 7.3, Theorem. 7.4]. To include W, one could check that (2.21) remains valid for $V + W$, *e.g.* like in [13, Theorem. 6.6], and adjust the arguments in [19, Sect. 7]. Alternatively, one can use the asymptotic formulas for $\{\psi_k\}$ and $\{\psi_k'\}$ in Sect. 3.1; the latter can be derived by differentiating (4.43). The zeros of ψ_k for $|x| < x_{\lambda_k}$ are in a neighborhood of the zeros of

$$J_{\frac{1}{3}}(\zeta(x)) + J_{-\frac{1}{3}}(\zeta(x)), \quad \zeta(x) = \int_x^{x_{\lambda_k}} (\lambda - V(s))^{\frac{1}{2}}\, \mathrm{d}s, \tag{2.22}$$

and, for large ζ, using asymptotic formulas for Bessel functions, see [15, §10.17], these are in a neighborhood of zeros of

$$\sin\left(\zeta(x) + \frac{\pi}{4}\right), \qquad |x| < x_{\lambda_k}. \tag{2.23}$$

3 The Proofs

We start with an implication of the condition (1.2) for integrals frequently appearing in our analysis and proceed with the proof of Theorem 2.3.

Lemma 3.1 *Let V satisfy Assumption I and the condition (1.2). Then, for every $g \in L^\infty((-1, 1))$,*

$$\begin{aligned} \lim_{t\to+\infty} \int_{-1}^{1} \left(1 - \frac{V(xt)}{V(t)}\right)^{\frac{1}{2}} g(x)\,\mathrm{d}x &= \int_{-1}^{1} \big(1 - \omega_\beta(x)\big)^{\frac{1}{2}} g(x)\,\mathrm{d}x, \\ \lim_{t\to+\infty} \int_{-1}^{1} \left(1 - \frac{V(xt)}{V(t)}\right)^{-\frac{1}{2}} g(x)\,\mathrm{d}x &= \int_{-1}^{1} \big(1 - \omega_\beta(x)\big)^{-\frac{1}{2}} g(x)\,\mathrm{d}x. \end{aligned} \tag{3.1}$$

Proof Both statements follow by (1.2) and the dominated convergence theorem. Since V is even, it suffices to consider the integrals on $(0, 1)$ only.

First let $x \in [0, 1/2]$ and let $\xi_0 > 0$ be as in Assumption I. Since $V \in C(\mathbb{R})$ and $V(y)$ is positive and increasing for $y \geq \xi_0$, see (2.2), we get that

$$\begin{aligned} \frac{|V(xt)|}{V(t)} &\leq \frac{\max_{0\leq y\leq \xi_0} |V(y)| + \max_{\xi_0 \leq y \leq \frac{t}{2}} V(y)}{V(t)} \\ &\leq \frac{V(\frac{t}{2})}{V(t)} \left(1 + \frac{\max_{0\leq y\leq\xi_0} |V(y)|}{V(\frac{t}{2})}\right), \qquad t \geq 2\xi_0. \end{aligned} \tag{3.2}$$

Thus (2.1) and (1.2) imply that there exists $\varepsilon_0 > 0$ such that for all $x \in [0, 1/2]$ and all $t > t_0$ with $t_0 \geq 2\xi_0$ (independent of x) we have

$$\frac{|V(xt)|}{V(t)} \leq 1 - \varepsilon_0. \tag{3.3}$$

Combining (3.3) and the assumption that V is eventually increasing on $\mathbb{R}_+$, see (2.2), we have that $V(xt) \leq V(t)$ for all $x \in [0, 1]$ and all $t > t_0$. Thus the existence of an integrable bound in the first limit follows.

For the second limit, we use inequalities (2.10). These imply in particular that there is a constant $\varsigma > 0$ (depending only on ν) such that for all $x \in [1/2, 1)$ and all $t \geq 2\xi_0$

$$\frac{V(xt)}{V(t)} \leq \frac{U(xt)}{U(t)}, \quad \text{where} \quad U(x) := \begin{cases} x^{\varsigma}, & \nu = -1, \\ \exp\left(\varsigma x^{\nu+1}\right), & \nu > -1. \end{cases} \tag{3.4}$$

For $\nu = -1$, combining (3.3) and (3.4) for $x \in [\frac{1}{2}, 1)$, we arrive at the integrable bound

$$\left(1 - \frac{V(xt)}{V(t)}\right)^{-\frac{1}{2}} |g(x)| \leq \begin{cases} \varepsilon_0^{-\frac{1}{2}} |g(x)|, & x \in [0, \frac{1}{2}), \\ (1 - x^{\varsigma})^{-\frac{1}{2}} |g(x)|, & x \in [\frac{1}{2}, 1). \end{cases} \tag{3.5}$$

For $\nu > -1$, we show that for all $x \in [\frac{1}{2}, 1]$ and all sufficiently large $t \geq 2\xi_0$ (independently of x)

$$1 - \frac{U(xt)}{U(t)} \geq 1 - x^{\nu+1}. \tag{3.6}$$

To see this, we introduce $y = 1 - x^{\nu+1} \in [0, y_0]$ with $y_0 = 1 - (1/2)^{\nu+1} < 1$ and $s = \varsigma t^{\nu+1}$. Then (3.6) holds if

$$e^{sy}(1 - y) - 1 \geq 0 \tag{3.7}$$

for all $y \in [0, y_0]$ and all large $s > 0$ (independently of y). Since $e^{sy} \geq 1 + sy$, we get

$$e^{sy}(1 - y) - 1 \geq y(s(1 - y) - 1), \tag{3.8}$$

thus (3.7) holds if

$$s \geq \frac{1}{1 - y_0}. \tag{3.9}$$

Hence the sought integrable bound reads

$$\left(1 - \frac{V(xt)}{V(t)}\right)^{-\frac{1}{2}} |g(x)| \leq \begin{cases} \varepsilon_0^{-\frac{1}{2}} |g(x)|, & x \in [0, \frac{1}{2}), \\ (1 - x^{\nu+1})^{-\frac{1}{2}} |g(x)|, & x \in [\frac{1}{2}, 1). \end{cases} \tag{3.10}$$

□

3.1 Summary of Properties of Eigenfunctions of Schrödinger Operators

We summarize properties eigenfunctions of Schrödinger operators with even confining potentials $Q = V + W$ satisfying Assumptions I and II. The details and proofs are given in Sect. 4; this slightly extends the reasoning in [18, §22.27] and [7].

Since Q is an even function by assumption, we can restrict ourselves to $(0, +\infty)$. Following the notations of [7], we introduce (for large enough $\lambda > 0$)

$$\begin{aligned}
V(x_\lambda) &= \lambda, \quad (x_\lambda > 0)\\
a_\lambda &= V'(x_\lambda),\\
\zeta = \zeta(x,\lambda) &= \begin{cases} \displaystyle\int_x^{x_\lambda} (\lambda - V(s))^{\frac{1}{2}}\,\mathrm{d}s, & 0 < x < x_\lambda,\\ \displaystyle\mathrm{i}\int_{x_\lambda}^{x} (V(s) - \lambda)^{\frac{1}{2}}\,\mathrm{d}s, & x > x_\lambda, \end{cases}\\
b = b(x,\lambda) &= \left(\frac{\zeta}{\zeta'}\right)^{\frac{1}{2}}, \quad \text{where} \quad \arg b = \begin{cases} 0, & x > x_\lambda,\\ \frac{\pi}{2}, & x < x_\lambda, \end{cases}\\
u = u(x,\lambda) &= bK_{\frac{1}{3}}(-\mathrm{i}\zeta),\\
v = v(x,\lambda) &= bI_{\frac{1}{3}}(-\mathrm{i}\zeta);
\end{aligned} \tag{3.11}$$

here $K_{1/3}$, $I_{1/3}$ are modified Bessel functions of order $1/3$. Furthermore, we define

$$\kappa_\lambda := \int_{x_\lambda}^{\infty} \left(\frac{|V''(t)|}{V(t)^{\frac{3}{2}}} + \frac{V'(t)^2}{V(t)^{\frac{5}{2}}}\right)\mathrm{d}t. \tag{3.12}$$

The functions u and v are known to be two linearly independent solutions of the differential equation

$$-f'' + (V - \lambda) f = Kf, \tag{3.13}$$

where

$$K = K(x,\lambda) = -\left(\frac{b''}{b} + \frac{1}{9b^4}\right) = \frac{1}{4}\left(\frac{5}{9}\frac{\lambda - V}{\zeta^2} - \frac{V''}{\lambda - V} - \frac{5}{4}\frac{V'^2}{(\lambda - V)^2}\right); \tag{3.14}$$

moreover, the Wronskian of u and v satisfies

$$W[u,v](x) = u(x)v'(x) - v(x)u'(x) = 1. \tag{3.15}$$

The L^2-solution of Schrödinger equation $-y'' + Qy = \lambda y$ is then found by solving the integral equation (obtained by variation of constants)

$$y(x) = u(x) + \int_x^{\infty} G(x,s)(K(s) + W(s))y(s)\,\mathrm{d}s, \tag{3.16}$$

where $G(x,s) = u(x)v(s) - v(x)u(s)$, see Theorem 3.3 and its proof in Sect. 4.

Next, for $0 \le x < x_\lambda$, one gets

$$u(x) = \frac{\pi}{\sqrt{3}}|b|\left(J_{\frac{1}{3}}(\zeta) + J_{-\frac{1}{3}}(\zeta)\right), \qquad v(x) = -|b|J_{\frac{1}{3}}(\zeta). \tag{3.17}$$

The positive numbers δ and δ_1 are defined by

$$\zeta(x_\lambda - \delta) = -\mathrm{i}\zeta(x_\lambda + \delta_1) = 1 \tag{3.18}$$

and they satisfy

$$\delta + \delta_1 = o(x_\lambda^{-\nu}), \qquad \delta \approx \delta_1 \approx a_\lambda^{-\frac{1}{3}}, \quad \lambda \to +\infty, \tag{3.19}$$

see Lemma 4.1 and its proof for details. As $\lambda \to +\infty$, we have

$$V(x_\lambda) - V(x_\lambda - \delta) \approx a_\lambda \delta \approx a_\lambda^{\frac{2}{3}}, \quad V(x_\lambda + \delta_1) - V(x_\lambda) \approx a_\lambda \delta_1 \approx a_\lambda^{\frac{2}{3}}, \tag{3.20}$$

see Lemma 4.1 below.

If $|x| < x_\lambda$ stays away from turning points, ζ is large and so it is useful to employ asymptotic formulas for Bessel functions with large argument, see [15, §10.17]. In particular, one obtains

$$u^2(x) = \frac{\pi}{(\lambda - V(x))^{\frac{1}{2}}}(1 + \sin 2\zeta + R_1(\zeta)), \quad |x| < x_\lambda, \tag{3.21}$$

where (see also [7, Sec. 7])

$$|R_1(\zeta)| = \mathcal{O}(\zeta^{-1}), \qquad \zeta \to +\infty. \tag{3.22}$$

For the absolute values of u and v, we have that, for all large enough $\lambda > 0$,

$$|u(x)| \lesssim \big(w_1(x)w_2(x)\big)^{-1}, \qquad |v(x)| \lesssim w_1(x)^{-1}w_2(x), \qquad x > 0, \tag{3.23}$$

with the weights

$$\begin{aligned} w_1(x) &= \begin{cases} |\lambda - V(x)|^{\frac{1}{4}}, & x \in (0, x_\lambda - \delta) \cup (x_\lambda + \delta_1, \infty), \\ a_\lambda^{\frac{1}{6}}, & x \in [x_\lambda - \delta, x_\lambda + \delta_1], \end{cases} \\ w_2(x) &= \begin{cases} 1, & x \in (0, x_\lambda + \delta_1], \\ e^{-\mathrm{i}\zeta}, & x \in (x_\lambda + \delta_1, \infty), \end{cases} \end{aligned} \tag{3.24}$$

see Lemma 4.2 below. Notice that $\arg \zeta(x) = \pi/2$ for $x > x_\lambda$ thus $|u(x)|$ is exponentially decreasing while $|v(x)|$ is allowed to be exponentially increasing as $x \to +\infty$.

Next, from Assumption I we obtain the following estimates, frequently occurring in our statements and proofs.

Lemma 3.2 *Let V satisfy Assumption I and let x_λ and a_λ be as in (3.11). Then, as $\lambda \to +\infty$,*

$$\left(\frac{x_\lambda^{2\nu}}{\lambda}\right)^{\frac{1}{2}} \approx \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{2}} \approx \frac{V'(x_\lambda)}{V(x_\lambda)^{\frac{3}{2}}} \lesssim \kappa_\lambda = \int_{x_\lambda}^{\infty} \left(\frac{|V''(t)|}{V(t)^{\frac{3}{2}}} + \frac{V'(t)^2}{V(t)^{\frac{5}{2}}}\right) \mathrm{d}t = o(1), \qquad \lambda \to +\infty. \tag{3.25}$$

Proof The claims follow from $V'(x) \approx V(x)x^\nu$ for x sufficiently large, see (2.4), and

$$\frac{V'(x_\lambda)}{V(x_\lambda)^{\frac{3}{2}}} = -\int_{x_\lambda}^{\infty} \left(\frac{V'(t)}{V(t)^{\frac{3}{2}}}\right)' \mathrm{d}t \tag{3.26}$$

together with (2.3). □

Finally, we have that

$$\int_0^\infty u(x)^2 \,\mathrm{d}x = \left(\int_0^{x_\lambda} \frac{\pi \,\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}}\right)\left(1 + \mathcal{O}\left(\frac{1}{x_\lambda} + \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{6}} \frac{\log \frac{a_\lambda}{x_\lambda^{3\nu}}}{x_\lambda^{1+\nu}}\right)\right) = \left(\int_0^{x_\lambda} \frac{\pi \,\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}}\right)(1 + o(1)), \quad \lambda \to +\infty, \tag{3.27}$$

see Lemma 4.3 below.

The following theorem shows that the function u is the main term in the asymptotic formula for eigenfunctions of the operator A from (1.1). The proof is given at the end of Sect. 4. One can check that the eigenvalues of A are simple and eigenfunctions are even or odd functions (since Q is assumed to be even). Thus the eigenvalues and eigenfunctions of A can be found by determining $\lambda > 0$ for which solutions y in (3.29) of the differential equation (3.28) satisfy a Dirichlet ($y(0) = 0$) or a Neumann ($y'(0) = 0$) boundary condition at 0.

Theorem 3.3 *Let $Q = V + W$ where V and W satisfy Assumptions I and II, respectively. Let x_λ and u be as in (3.11), let w_1, w_2 be as in (3.24), let κ_λ as in (3.12) and let $\mathcal{J}_W$ be as in (2.13). Then, for every sufficiently large $\lambda > 0$, there is a solution of*

$$-y'' + (Q - \lambda)y = 0 \tag{3.28}$$

on $(0, +\infty)$ such that

$$y = u + r, \tag{3.29}$$

where

$$|r(x)| \le \frac{C(\lambda)}{w_1(x)w_2(x)}, \qquad x > 0, \tag{3.30}$$

and

$$C(\lambda) = \mathcal{O}(\lambda^{-\frac{1}{2}} + \kappa_\lambda + \mathcal{J}_W(\lambda)) = o(1), \quad \lambda \to +\infty. \tag{3.31}$$

Moreover

$$\begin{aligned} &\int_0^\infty y^2(x)\,\mathrm{d}x \\ &= \left(\int_0^{x_\lambda} \frac{\pi\,\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}}\right)\left(1 + C(\lambda) + \mathcal{O}\left(\frac{1}{x_\lambda} + \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{6}} \frac{\log \frac{a_\lambda}{x_\lambda^{3\nu}}}{x_\lambda^{1+\nu}}\right)\right) \\ &= \left(\int_0^{x_\lambda} \frac{\pi\,\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}}\right)(1 + o(1)), \qquad \lambda \to +\infty. \end{aligned} \tag{3.32}$$

3.2 Proof of Theorem 2.3

Since the eigenfunctions $\{\psi_k\}$ are even or odd, we consider only $x \in (0, \infty)$. We select the eigenfunctions $\{\psi_k\}$ such that

$$\psi_k(x) = \frac{y_k(x)}{\|y_k\|} = \frac{u_k(x) + r_k(x)}{\|y_k\|}, \quad x > 0, \tag{3.33}$$

where $y_k = y(\cdot, \lambda_k)$, $u_k = u(\cdot, \lambda_k)$ and $r_k = y_k - u_k$, see Sect. 3.1 and in particular Theorem 3.3. Hence, the densities $\{\phi_k\}$ of the measures $\{\mu_k\}$, see (1.6), satisfy

$$\begin{aligned} \phi_k(x) &= x_{\lambda_k}\, \psi_k(x_{\lambda_k} x)^2 \\ &= x_{\lambda_k} \frac{u_k(x_{\lambda_k} x)^2 + 2r_k(x_{\lambda_k} x) u_k(x_{\lambda_k} x) + r_k(x_{\lambda_k} x)^2}{\|y_k\|^2}. \end{aligned} \tag{3.34}$$

In the sequel, notations and results summarized in Sect. 3.1 are used, moreover, we introduce the constant (for $\beta \in (0, \infty]$)

$$\Omega'_\beta := \int_{-1}^{1} (1 - |t|^\beta)^{-\frac{1}{2}}\,\mathrm{d}t = \frac{2\pi^{\frac{1}{2}}\Gamma(1 + \frac{1}{\beta})}{\Gamma(\frac{1}{2} + \frac{1}{\beta})}. \tag{3.35}$$

We also drop the subscript k and work with quantities like $y = y(\cdot, \lambda)$ as $\lambda \to +\infty$.

First, Lemma 3.1, (3.32) and the change of integration variables $x = x_\lambda t$ imply

$$\|y\|^2 = 2\left(\int_0^{x_\lambda} \frac{\pi\,\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}}\right)(1 + o(1)) = \frac{\pi\,\Omega'_\beta x_\lambda}{\lambda^{\frac{1}{2}}}(1 + o(1)), \quad \lambda \to +\infty. \tag{3.36}$$

Thus with $f \in \mathcal{F}_V$, see (2.17), and the change of integration variables, we get

$$\int_0^\infty \phi(x) f(x)\,\mathrm{d}x = \frac{1}{\pi \Omega'_\beta} \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \left(\int_0^\infty y(x)^2 f\left(\frac{x}{x_\lambda}\right)\,\mathrm{d}x \right) (1 + o(1)), \quad \lambda \to +\infty; \tag{3.37}$$

the integral indeed converges for $f \in \mathcal{F}_V$ as can be seen from (3.42), (3.43) below and the behavior of y at infinity, see (3.29), (3.30), (3.23) and (3.24).

First we show that the contribution from the region around the turning point is negligible. It follows from (3.19) and (3.25) that

$$\frac{\delta_1}{x_\lambda} \approx \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{3}} \frac{1}{x_\lambda^{\nu+1}} = o(1), \quad \lambda \to +\infty, \tag{3.38}$$

hence, since $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$,

$$\operatorname{ess\,sup}_{0 \le x \le x_\lambda + \delta_1} \left| f\left(\frac{x}{x_\lambda}\right) \right| = \mathcal{O}(1), \quad \lambda \to +\infty. \tag{3.39}$$

Employing estimates (3.23), (3.30), (3.39) and (3.19) in the last step, we obtain

$$\mathcal{I}_1 := \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_{x_\lambda - \delta}^{x_\lambda + \delta_1} y(x)^2 \left| f\left(\frac{x}{x_\lambda}\right) \right| \,\mathrm{d}x \lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \frac{(1 + C(\lambda)^2)(\delta + \delta_1)}{a_\lambda^{\frac{1}{3}}} \lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda a_\lambda^{\frac{2}{3}}}. \tag{3.40}$$

Similarly, since $x_\lambda^{-\nu} \le x_\lambda$ and $\delta_1 = o(x_\lambda^{-\nu})$ as $\lambda \to +\infty$, see (3.19), we get (using (3.23), (3.20), changing the integration variables $-\mathrm{i}\zeta(x) = |\zeta(x)| = t$ and observing that $|\zeta(x)|' = (V(x) - \lambda)^{1/2}$)

$$\begin{aligned}
\mathcal{I}_2 &:= \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_{x_\lambda + \delta_1}^{x_\lambda + \frac{x_\lambda^{-\nu}}{2}} y(x)^2 \left| f\left(\frac{x}{x_\lambda}\right) \right| \,\mathrm{d}x \\
&\lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_{x_\lambda + \delta_1}^{x_\lambda + \frac{x_\lambda^{-\nu}}{2}} \frac{(1 + C(\lambda)^2) e^{-2|\zeta(x)|}}{(V(x) - \lambda)^{\frac{1}{2}}} \,\mathrm{d}x \\
&\lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_{x_\lambda + \delta_1}^{x_\lambda + \frac{x_\lambda^{-\nu}}{2}} \frac{(V(x) - \lambda)^{\frac{1}{2}} e^{-2|\zeta(x)|}}{V(x) - \lambda} \,\mathrm{d}x \lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda a_\lambda^{\frac{2}{3}}} \int_1^\infty e^{-2t}\,\mathrm{d}t \lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda a_\lambda^{\frac{2}{3}}}.
\end{aligned} \tag{3.41}$$

We investigate the region $(x_\lambda + x_\lambda^{-\nu}/2, \infty)$ and also explain the convergence of the integral in (3.37). To this end, we recall that by assumption $f \in \mathcal{F}_V$, see (2.17), thus with some $M > 0$

$$\left|f\left(\frac{x}{x_\lambda}\right)\right| \exp(-|\zeta(x)|) \\ \leq \|f \exp(-M|V|^{\frac{1}{2}})\|_{L^\infty} \exp\left(-|\zeta(x)|\Big(1 - M\frac{|V(\frac{x}{x_\lambda})|^{\frac{1}{2}}}{|\zeta(x)|}\Big)\right) \tag{3.42}$$

and we show below that

$$\sup_{x > x_\lambda + \frac{1}{2}x_\lambda^{-\nu}} \frac{\left|V\left(\frac{x}{x_\lambda}\right)\right|^{\frac{1}{2}}}{|\zeta(x)|} = o(1), \qquad \lambda \to +\infty. \tag{3.43}$$

To prove (3.43), notice that for $x > x_\lambda$ and assuming that λ is sufficiently large that $x_\lambda > \xi_0$

$$\left(\frac{V(x) - \lambda}{V(x)}\right)' = \frac{\lambda V'(x)}{V(x)^2} > 0 \tag{3.44}$$

and, using (2.4) and (2.7),

$$\frac{V(x_\lambda + \frac{1}{2}x_\lambda^{-\nu}) - V(x_\lambda)}{V(x_\lambda + \frac{1}{2}x_\lambda^{-\nu})} \approx \frac{V'(x_\lambda)x_\lambda^{-\nu}}{V(x_\lambda)} \approx 1. \tag{3.45}$$

Thus, for $x > x_\lambda + x_\lambda^{-\nu}/2$,

$$|\zeta(x)| = \int_{x_\lambda}^{x} (V(t) - \lambda)^{\frac{1}{2}}\, \mathrm{d}t = \int_{x_\lambda}^{x} \frac{V'(t)}{V'(t)}(V(t) - \lambda)^{\frac{1}{2}}\, \mathrm{d}t \gtrsim \frac{(V(x) - \lambda)^{\frac{3}{2}}}{\max_{x_\lambda \leq t \leq x} V'(t)} \\ = \frac{(V(x) - \lambda)^{\frac{3}{2}}}{V(x)^{\frac{3}{2}}} \frac{V(x)^{\frac{3}{2}}}{\max_{x_\lambda \leq t \leq x} V'(t)} \gtrsim \min\{x_\lambda^{-\nu}, x^{-\nu}\} V(x)^{\frac{1}{2}}. \tag{3.46}$$

Hence for $\nu < 0$ we immediately arrive at

$$\frac{\left|V\left(\frac{x}{x_\lambda}\right)\right|^{\frac{1}{2}}}{|\zeta(x)|} \lesssim \frac{\left|V\left(\frac{x}{x_\lambda}\right)\right|^{\frac{1}{2}}}{V(x)^{\frac{1}{2}} x_\lambda^{|\nu|}} \leq \frac{1}{x_\lambda^{|\nu|}}. \tag{3.47}$$

For $\nu \geq 0$, we use (2.6) to get (with $\xi_0 > 0$ from Assumption I and some $c > 0$)

$$\frac{\left|V\left(\frac{x}{x_\lambda}\right)\right|^{\frac{1}{2}}}{|\zeta(x)|} \lesssim \frac{x^\nu \left|V\left(\frac{x}{x_\lambda}\right)\right|^{\frac{1}{2}}}{V(x)^{\frac{1}{2}}} \\ \lesssim \max_{x_\lambda \leq x \leq \xi_0 x_\lambda} \left(\frac{x^{2\nu}}{V(x)}\right)^{\frac{1}{2}} + x^\nu \exp\left(-c x^{\nu+1}(1 + \mathcal{O}(x_\lambda^{-\nu-1}))\right), \tag{3.48}$$

thus (3.43) follows also in this case (recall (3.25)).

As a consequence of (3.42) and (3.43) we obtain in particular that

$$\operatorname{ess\,sup}_{x \geq x_\lambda + \frac{1}{2}x_\lambda^{-\nu}} \left| f\left(\frac{x}{x_\lambda}\right)\right| \exp(-|\zeta(x)|) = \mathcal{O}(1), \quad \lambda \to +\infty \tag{3.49}$$

which we use in the estimate of integral

$$\mathcal{I}_3 := \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_{x_\lambda + \frac{1}{2}x_\lambda^{-\nu}}^{\infty} y(x)^2 \left| f\left(\frac{x}{x_\lambda}\right)\right| \mathrm{d}x. \tag{3.50}$$

In detail, employing (3.49), (3.23), (3.30), changing the integration variables $-\mathrm{i}\zeta(x) = |\zeta(x)| = t$ and using (2.7) and (2.4) in the last steps, we get

$$\begin{aligned} \mathcal{I}_3 &\lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_{x_\lambda + \frac{1}{2}x_\lambda^{-\nu}}^{\infty} \frac{(1 + C(\lambda)^2) e^{-|\zeta(x)|}}{(V(x) - \lambda)^{\frac{1}{2}}} \mathrm{d}x \\ &\lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \frac{1}{V(x_\lambda + \frac{1}{2}x_\lambda^{-\nu}) - V(x_\lambda)} \int_0^\infty e^{-t}\, \mathrm{d}t \lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \frac{1}{V'(x_\lambda) x_\lambda^{-\nu}} \lesssim \frac{1}{x_\lambda \lambda^{\frac{1}{2}}}. \end{aligned} \tag{3.51}$$

Thus in summary, using (2.4), (3.25) and $\nu \geq -1$, we get

$$\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \lesssim \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{6}} \frac{1}{x_\lambda^{1+\nu}} + \frac{1}{x_\lambda \lambda^{\frac{1}{2}}} = o(1), \quad \lambda \to +\infty. \tag{3.52}$$

We continue with the integral over $(0, x_\lambda - \delta)$, see (3.37), where we use the representation of u^2 from (3.21), *i.e.*

$$y^2 = \frac{\pi}{(\lambda - V)^{\frac{1}{2}}}(1 + \sin 2\zeta + R_1(\zeta)) + 2ur + r^2. \tag{3.53}$$

The main contribution in (3.37) reads (employing Lemma 3.1)

$$\begin{aligned} \mathcal{I}_4 &:= \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_0^{x_\lambda - \delta} \frac{\pi f\left(\frac{x}{x_\lambda}\right)}{(\lambda - V(x))^{\frac{1}{2}}} \mathrm{d}x = \pi \int_0^{1 - \frac{\delta}{x_\lambda}} \left(1 - \frac{V(x_\lambda x)}{V(x_\lambda)}\right)^{-\frac{1}{2}} f(x)\, \mathrm{d}x \\ &= \int_0^1 \frac{\pi f(x)\, \mathrm{d}x}{(1 - \omega_\beta(x))^{\frac{1}{2}}} + o(1), \quad \lambda \to +\infty. \end{aligned} \tag{3.54}$$

Thus, to prove (2.18), we need to show that the remaining terms are negligible.

Employing the estimates on $|u|$, $|r|$, see (3.23), (3.30), we get by changing the integration variables $x = x_\lambda t$ and applying Lemma 3.1 that (recall that $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$)

$$
\begin{aligned}
\mathcal{I}_5 :=& \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_0^{x_\lambda-\delta} \left(|u(x)||r(x)| + r(x)^2\right) \left| f\left(\frac{x}{x_\lambda}\right)\right| \, \mathrm{d}x \\
\lesssim& \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_0^{x_\lambda-\delta} \frac{C(\lambda)+C(\lambda)^2}{(\lambda - V(x))^{\frac{1}{2}}} \, \mathrm{d}x \lesssim \Omega'_\beta C(\lambda) = o(1), \quad \lambda \to +\infty.
\end{aligned} \tag{3.55}
$$

Thus the contribution from the integrals with $2ur + r^2$ is indeed negligible.

Using (3.22), (4.7), (4.4), (2.4) and (3.25), we obtain (recall that $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, $-\zeta' = (\lambda - V)^{\frac{1}{2}}$ and see also (4.20))

$$
\begin{aligned}
\mathcal{I}_6 :=& \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_0^{x_\lambda-\delta} \frac{|R_1(\zeta)|}{(\lambda - V(x))^{\frac{1}{2}}} \left| f\left(\frac{x}{x_\lambda}\right)\right| \, \mathrm{d}x \\
\lesssim& \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \left(\frac{1}{\zeta(x_\lambda - \frac{1}{2}x_\lambda^{-\nu})} \int_0^{x_\lambda - \frac{1}{2}x_\lambda^{-\nu}} \frac{\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}} \right. \\
& \left. + \int_{x_\lambda - \frac{1}{2}x_\lambda^{-\nu}}^{x_\lambda-\delta} \frac{\mathrm{d}x}{\zeta(x)(\lambda - V(x))^{\frac{1}{2}}} \right) \\
\lesssim& \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \left(\left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{2}} \int_0^{x_\lambda - \frac{1}{2}x_\lambda^{-\nu}} \frac{\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}} + \frac{\log \zeta(x_\lambda - \frac{1}{2}x_\lambda^{-\nu})}{\lambda - V(x_\lambda - \delta)} \right) \\
\lesssim& \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{2}} \Omega'_\beta + \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{6}} \frac{\log \frac{a_\lambda}{x_\lambda^{3\nu}}}{x_\lambda^{1+\nu}} = o(1), \quad \lambda \to +\infty.
\end{aligned} \tag{3.56}
$$

Finally, we analyze the term with $\sin 2\zeta$, see (3.53). For every $\varepsilon > 0$ there is $g \in C_0^\infty((0,1))$ such that $\|f - g\|_{L^1((0,1))} < \varepsilon$. With this $\varepsilon > 0$, we define $\delta_\varepsilon := \varepsilon x_\lambda^{-\nu}$; notice that $\delta = o(\delta_\varepsilon)$ as $\lambda \to +\infty$, see (3.19). Then

$$
\begin{aligned}
& \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \left| \int_0^{x_\lambda-\delta} \frac{\sin 2\zeta(x)}{(\lambda - V(x))^{\frac{1}{2}}} f\left(\frac{x}{x_\lambda}\right) \mathrm{d}x \right| \\
\le& \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \int_{x_\lambda - \delta_\varepsilon}^{x_\lambda-\delta} \frac{1}{(\lambda - V(x))^{\frac{1}{2}}} \left| f\left(\frac{x}{x_\lambda}\right)\right| \mathrm{d}x \\
& + \lambda^{\frac{1}{2}} \int_0^{1-\frac{\delta_\varepsilon}{x_\lambda}} \frac{|f(t) - g(t)|}{(\lambda - V(x_\lambda t))^{\frac{1}{2}}} \, \mathrm{d}t \\
& + \frac{\lambda^{\frac{1}{2}}}{x_\lambda} \left| \int_0^{x_\lambda - \delta_\varepsilon} \frac{\sin 2\zeta(x)}{(\lambda - V(x))^{\frac{1}{2}}} g\left(\frac{x}{x_\lambda}\right) \mathrm{d}x \right| \\
=:& \mathcal{I}_8 + \mathcal{I}_9 + \mathcal{I}_{10}.
\end{aligned} \tag{3.57}
$$

Using that $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, (2.7), (2.4) and the mean value theorem (with $\eta_\lambda \in (x_\lambda - \delta_\varepsilon, x_\lambda)$),

$$\begin{aligned}\mathcal{I}_8 &\lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda}\frac{(V(x_\lambda)-V(x_\lambda-\delta_\varepsilon))^{\frac{1}{2}}}{V'(x_\lambda)} = \frac{\lambda^{\frac{1}{2}}}{x_\lambda}\frac{(V'(\eta_\lambda)\varepsilon x_\lambda^{-\nu})^{\frac{1}{2}}}{V'(x_\lambda)}\\ &\lesssim \varepsilon^{\frac{1}{2}}\frac{\lambda^{\frac{1}{2}}}{x_\lambda}\frac{(V'(x_\lambda)x_\lambda^{-\nu})^{\frac{1}{2}}}{V'(x_\lambda)} \lesssim \frac{\varepsilon^{\frac{1}{2}}}{x_\lambda^{1+\nu}}.\end{aligned} \tag{3.58}$$

From $\|f-g\|_{L^1((0,1))} < \varepsilon$, (2.7) and (2.4), we get

$$\mathcal{I}_9 \lesssim \varepsilon\frac{\lambda^{\frac{1}{2}}}{(V(x_\lambda)-V(x_\lambda-\delta_\varepsilon))^{\frac{1}{2}}} \lesssim \varepsilon\frac{\lambda^{\frac{1}{2}}}{(V'(x_\lambda)\varepsilon x_\lambda^{-\nu})^{\frac{1}{2}}} \lesssim \varepsilon^{\frac{1}{2}}. \tag{3.59}$$

By integration by parts and (3.20),

$$\begin{aligned}\mathcal{I}_{10} &\lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda}\left(\left|\left[g\left(\frac{x}{x_\lambda}\right)\frac{\cos 2\zeta(x)}{\lambda - V(x)}\right]_0^{x_\lambda-\delta_\varepsilon}\right| + \int_0^{x_\lambda-\delta_\varepsilon}\left|\left(\frac{g\left(\frac{x}{x_\lambda}\right)}{\lambda-V(x)}\right)'\right|\,\mathrm{d}x\right)\\ &\lesssim \frac{\lambda^{\frac{1}{2}}}{x_\lambda}\left(\frac{\|g\|_{L^\infty}}{\varepsilon\lambda} + \int_0^{x_\lambda-\delta_\varepsilon}\frac{\|g'\|_{L^\infty}}{x_\lambda(\lambda-V(x))} + \frac{\|g\|_{L^\infty}V'(x)}{(\lambda-V(x))^2}\,\mathrm{d}x\right)\\ &\lesssim \frac{\|g\|_{L^\infty}+\|g'\|_{L^\infty}}{\varepsilon x_\lambda \lambda^{\frac{1}{2}}}.\end{aligned} \tag{3.60}$$

Putting the estimates from above together, we finally obtain

$$\limsup_{\lambda\to+\infty}\left|\int_0^\infty \phi(x)f(x)\,\mathrm{d}x - \frac{1}{\Omega'_\beta}\int_0^1 \frac{f(x)\,\mathrm{d}x}{(1-\omega_\beta(x))^{\frac{1}{2}}}\right| \lesssim \varepsilon^{\frac{1}{2}}, \tag{3.61}$$

thus the claim (2.18) follows since $\varepsilon > 0$ was arbitrary. □

4 Eigenfunctions of Schrödinger Operators with Even Confining Potentials

In this section, we collect technical lemmas and proofs of results summarized in Sect. 3.1; these are used in the proof of the main Theorem 2.3. Notice that in this section we *do not assume* that (1.2) holds. The proofs follow mostly the reasoning in [18, §22.27] and [7].

Lemma 4.1 *Let V satisfy Assumption I, let ξ_0 be as in (2.2), let x_λ, a_λ, ζ be as in (3.11) and δ, δ_1 as in (3.18). Let $\varepsilon \in (0,1)$. Then, for all sufficiently large $\lambda > 0$ and* *all sufficiently large x, the following hold.*

$$V^{(j)}(x+\Delta) \approx V^{(j)}(x), \qquad |\Delta| \le \varepsilon x^{-\nu}, \quad j=0,1, \tag{4.1}$$

$$|\zeta(x_\lambda \pm \varepsilon x_\lambda^{-\nu})| \quad \approx \left(\frac{a_\lambda}{x_\lambda^{3\nu}}\right)^{\frac{1}{2}}, \tag{4.2}$$

$$\delta \approx \delta_1 \approx a_\lambda^{-\frac{1}{3}}, \tag{4.3}$$

$$V(x_\lambda) - V(x_\lambda - \delta) \approx a_\lambda \delta \approx a_\lambda^{\frac{2}{3}}, \quad V(x_\lambda + \delta_1) - V(x_\lambda) \approx a_\lambda \delta_1 \approx a_\lambda^{\frac{2}{3}}. \tag{4.4}$$

Proof Using Assumption I, for $\nu > -1$, we have

$$\begin{aligned}\left|\log \frac{V(x+\Delta)}{V(x)}\right| = \left|\int_x^{x+\Delta} \frac{V'(t)}{V(t)}\,\mathrm{d}t\right| &\lesssim \left| |x+\Delta|^{\nu+1} - |x|^{\nu+1} \right| \\ &\lesssim x^\nu |\Delta| + \mathcal{O}(|\Delta|^2 x^{\nu-1}),\end{aligned} \tag{4.5}$$

for $\nu = -1$,

$$\begin{aligned}\left|\log \frac{V(x+\Delta)}{V(x)}\right| = \left|\int_x^{x+\Delta} \frac{V'(t)}{V(t)}\,\mathrm{d}t\right| &\lesssim \left|\log\left(1+\frac{\Delta}{x}\right)\right| \\ &\le \max\{|\log(1-\varepsilon)|, |\log(1+\varepsilon)|\};\end{aligned} \tag{4.6}$$

the case with $j=1$ is similar.

Using (4.1) for V' and the mean value theorem in the last step, we get

$$\begin{aligned}\zeta(x_\lambda - \varepsilon x_\lambda^{-\nu}) &= \int_{x_\lambda - \varepsilon x_\lambda^{-\nu}}^{x_\lambda} \frac{V'(t)}{V'(t)} (\lambda - V(t))^{\frac{1}{2}}\,\mathrm{d}t \\ &\approx \frac{1}{a_\lambda}(V(x_\lambda) - V(x_\lambda - \varepsilon x_\lambda^{-\nu}))^{\frac{3}{2}} \approx \left(\frac{a_\lambda}{x_\lambda^{3\nu}}\right)^{\frac{1}{2}};\end{aligned} \tag{4.7}$$

the case with $x_\lambda + \varepsilon x_\lambda^{-\nu}$ is analogous.

The number δ must satisfy

$$\delta = o(x_\lambda^{-\nu}), \qquad \lambda \to +\infty \tag{4.8}$$

for otherwise $\zeta(x_\lambda - \delta) \to +\infty$ by (4.2) and (3.25). Then, using the definition of δ, see (3.18), we get similarly as in (4.7),

$$1 = \zeta(x_\lambda - \delta) = \int_{x_\lambda - \delta}^{x_\lambda} \frac{V'(t)}{V'(t)} (\lambda - V(t))^{\frac{1}{2}}\,\mathrm{d}t \approx \frac{1}{a_\lambda}(a_\lambda \delta)^{\frac{3}{2}} \tag{4.9}$$

and thus (4.3) follows. The reasoning for δ_1 is analogous.

Relations (4.4) follow by the mean value theorem, (4.8), (4.1) and (4.3). □

Lemma 4.2 *Let V satisfy Assumption I, let u, v be as in (3.11) and let w_1, w_2 be as in (3.24). Then, for all sufficiently large $\lambda > 0$, we have*

$$|u(x)| \lesssim \big(w_1(x)w_2(x)\big)^{-1}, \qquad |v(x)| \lesssim w_1(x)^{-1}w_2(x), \qquad x > 0. \tag{4.10}$$

Proof For $x \in (0, x_\lambda - \delta) \cup (x_\lambda + \delta_1, \infty)$, where $|\zeta| > 1$, the inequalities (4.10) follow from the definitions of u and v and asymptotic expansions of the corresponding Bessel functions for a large argument, see *e.g.* [15, Chap. 10]; we omit details.

In the region around the turning point x_λ, one has $|\zeta| \le 1$ and so expansions of Bessel functions for a small argument are used, see *e.g.* [15, Chap. 10]. More precisely, for u and $x_\lambda - \delta \le x \le x_\lambda$, one has, see (3.17),

$$|u(x)| = \frac{\pi}{\sqrt{3}}\,|b|\,\Big|J_{\frac{1}{3}}(\zeta) + J_{-\frac{1}{3}}(\zeta)\Big| \lesssim \left(\frac{\zeta}{|\zeta'|^3}\right)^{\frac{1}{6}}. \tag{4.11}$$

Similarly as in (4.7), we obtain

$$\zeta(x) \approx \frac{(\lambda - V(x))^{\frac{3}{2}}}{a_\lambda} = \frac{|\zeta'(x)|^3}{a_\lambda}, \qquad x_\lambda - \delta \le x \le x_\lambda, \tag{4.12}$$

thus $|u(x)| \approx a_\lambda^{-\frac{1}{6}}$. The case $x_\lambda < x < x_\lambda + \delta_1$ is similar.
The estimates for v are obtained analogously. □

Lemma 4.3 *Let V satisfy Assumption I and u, x_λ and a_λ be as in (3.11). Then*

$$\begin{aligned}\int_0^\infty u(x)^2\,\mathrm{d}x &= \left(\int_0^{x_\lambda} \frac{\pi\,\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}}\right)\left(1 + \mathcal{O}\left(\frac{1}{x_\lambda} + \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{6}} \frac{\log \frac{a_\lambda}{x_\lambda^{3\nu}}}{x_\lambda^{1+\nu}}\right)\right)\\ &= \left(\int_0^{x_\lambda} \frac{\pi\,\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}}\right)(1 + o(1)), \quad \lambda \to +\infty.\end{aligned} \tag{4.13}$$

Proof Using (3.21), we obtain

$$\begin{aligned}\int_0^\infty u(x)^2\,\mathrm{d}x = &\int_0^{x_\lambda} \frac{\pi}{(\lambda - V(x))^{\frac{1}{2}}}\,\mathrm{d}x + \pi\int_0^{x_\lambda - \delta} \frac{\sin 2\zeta(x) + R_1(\zeta(x))}{(\lambda - V(x))^{\frac{1}{2}}}\,\mathrm{d}x\\ &+ \int_{x_\lambda - \delta}^{x_\lambda + \delta_1} u(x)^2\,\mathrm{d}x + \int_{x_\lambda + \delta_1}^\infty u^2(x)\,\mathrm{d}x\\ &- \int_{x_\lambda - \delta}^{x_\lambda} \frac{\pi}{(\lambda - V(x))^{\frac{1}{2}}}\,\mathrm{d}x.\end{aligned} \tag{4.14}$$

First we notice that

$$\int_0^{x_\lambda} \frac{\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}} = \frac{1}{\lambda^{\frac{1}{2}}}\int_0^{x_\lambda} \frac{\mathrm{d}x}{(1 - \frac{V(x)}{\lambda})^{\frac{1}{2}}} \gtrsim \frac{x_\lambda}{\lambda^{\frac{1}{2}}}. \tag{4.15}$$

Using (4.10) and (4.3), we get

$$\int_{x_\lambda-\delta}^{x_\lambda+\delta_1} u(x)^2\,\mathrm{d}x \lesssim a_\lambda^{-\frac{2}{3}}. \tag{4.16}$$

Since $\delta \approx a_\lambda^{-\frac{1}{3}} = o(x_\lambda^{-\nu})$ as $\lambda \to +\infty$, see (4.3) and (4.8), using (4.1), we get

$$\int_{x_\lambda-\delta}^{x_\lambda} \frac{\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}} = \int_{x_\lambda-\delta}^{x_\lambda} \frac{V'(x)\,\mathrm{d}x}{V'(x)(\lambda - V(x))^{\frac{1}{2}}} \lesssim \frac{(a_\lambda\delta)^{\frac{1}{2}}}{a_\lambda} \approx a_\lambda^{-\frac{2}{3}}. \tag{4.17}$$

Using (4.10), the definition (3.18) of δ_1 and (4.4), we have

$$\begin{aligned} \int_{x_\lambda+\delta_1}^{\infty} u(x)^2\,\mathrm{d}x &\lesssim \int_{x_\lambda+\delta_1}^{\infty} \frac{e^{-2\int_{x_\lambda}^{x}(V(s)-\lambda)^{\frac{1}{2}}\,\mathrm{d}s}}{(V(x)-\lambda)^{\frac{1}{2}}}\,\mathrm{d}x \\ &= \int_{x_\lambda+\delta_1}^{\infty} \frac{(V(x)-\lambda)^{\frac{1}{2}} e^{-2\int_{x_\lambda}^{x}(V(s)-\lambda)^{\frac{1}{2}}\,\mathrm{d}s}}{V(x)-\lambda}\,\mathrm{d}x \\ &\lesssim \frac{1}{V(x_\lambda+\delta_1)-\lambda}\int_1^{\infty} e^{-2t}\,\mathrm{d}t \lesssim \frac{1}{a_\lambda\delta_1} \approx a_\lambda^{-\frac{2}{3}}. \end{aligned} \tag{4.18}$$

The second mean value theorem for integrals (from which the point $\xi_1 = \xi_1(\lambda)$ arises below), the fact that V is increasing for $x > \xi_0$ (see (2.2)) and (4.4) yield (recall that by (3.11) $-\zeta' = (\lambda - V)^{\frac{1}{2}}$)

$$\begin{aligned} \left|\int_0^{x_\lambda-\delta} \frac{\sin 2\zeta(x)\,\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}}\right| &\lesssim \lambda^{-\frac{1}{2}} + \frac{1}{\lambda - V(\xi_0)}\left|\int_{\xi_0}^{\xi_1}(-\zeta'(x))\sin 2\zeta(x)\,\mathrm{d}x\right| \\ &\quad + \frac{1}{\lambda - V(x_\lambda-\delta)}\left|\int_{\xi_1}^{x_\lambda-\delta}(-\zeta'(x))\sin 2\zeta(x)\,\mathrm{d}x\right| \\ &\lesssim \lambda^{-\frac{1}{2}} + \frac{1}{a_\lambda^{\frac{2}{3}}}\left|\int_1^{\zeta(\xi_1)}\sin 2t\,\mathrm{d}t\right| \lesssim \lambda^{-\frac{1}{2}} + a_\lambda^{-\frac{2}{3}}. \end{aligned} \tag{4.19}$$

Using (3.22), (4.7) and (4.4), we have

$$
\begin{aligned}
&\int_0^{x_\lambda-\delta} \frac{|R_1(\zeta(x))|}{(\lambda - V(x))^{\frac{1}{2}}} \,\mathrm{d}x \\
&\lesssim \int_0^{x_\lambda-\delta} \frac{\mathrm{d}x}{\zeta(x)(\lambda - V(x))^{\frac{1}{2}}} \\
&\lesssim \frac{1}{\zeta(x_\lambda - \frac{1}{2}x_\lambda^{-\nu})} \int_0^{x_\lambda-\delta} \frac{\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}} + \int_{x_\lambda-\frac{1}{2}x_\lambda^{-\nu}}^{x_\lambda-\delta} \frac{\mathrm{d}x}{\zeta(x)(\lambda - V(x))^{\frac{1}{2}}} \\
&\lesssim \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{2}} \int_0^{x_\lambda-\delta} \frac{\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}} + \frac{\log \zeta(x_\lambda - \frac{1}{2}x_\lambda^{-\nu})}{V(x_\lambda) - V(x_\lambda - \delta)} \\
&\lesssim \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{2}} \int_0^{x_\lambda-\delta} \frac{\mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}} + \frac{\log \frac{a_\lambda}{x_\lambda^{3\nu}}}{a_\lambda^{\frac{2}{3}}}.
\end{aligned}
\tag{4.20}
$$

From (2.4) we have

$$
\frac{\lambda^{\frac{1}{2}}}{x_\lambda a_\lambda^{\frac{2}{3}}} \approx \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{6}} \frac{1}{x_\lambda^{1+\nu}}, \tag{4.21}
$$

thus the claim (4.13) follows by putting together all estimates from above (and (3.25)). □

Lemma 4.4 *Let V satisfy Assumption I, let K be as in (3.14), let w_1 be as in (3.24) and let κ_λ be as in (3.12). Then*

$$
\mathcal{J}_K(\lambda) := \int_0^\infty \frac{K(s)}{w_1(s)^2} \,\mathrm{d}s = \mathcal{O}(\lambda^{-\frac{1}{2}} + \kappa_\lambda) = o(1), \qquad \lambda \to +\infty. \tag{4.22}
$$

Proof We follow and extend the strategy in [18, §22.27]. We split the integral into several regions; we define $\delta_\lambda' := \varepsilon_1 x_\lambda^{-\nu}$ and $\delta_\lambda'' := \varepsilon_2 x_\lambda^{-\nu}$, where $\varepsilon_1, \varepsilon_2 \in (0, 1)$ will be determined below.

• $0 \le s \le \xi_0$: Notice that $\zeta(s) \gtrsim \lambda^{\frac{1}{2}}$, hence (recall that $-\zeta' = (\lambda - V)^{\frac{1}{2}}$)

$$
\int_0^{\xi_0} \frac{|K(s)|}{w_1(s)^2} \,\mathrm{d}s \lesssim \int_0^{\xi_0} \frac{-\zeta'(s)}{\zeta(s)^2} \,\mathrm{d}s + \frac{1}{\lambda^{\frac{1}{2}}} \lesssim \frac{1}{\lambda^{\frac{1}{2}}}. \tag{4.23}
$$

• $\xi_0 \le s \le x_\lambda - \delta_\lambda'$: We give the estimate for any value of $\varepsilon_1 \in (0, 1)$; ε_1 will be specified below, see (4.39),

$$
\begin{aligned}
\int_{\xi_0}^{x_\lambda-\delta_\lambda'} \frac{|K(s)|}{w_1(s)^2} \,\mathrm{d}s &\lesssim \int_{\xi_0}^{x_\lambda-\delta_\lambda'} \frac{-\zeta'(s)}{\zeta(s)^2} \,\mathrm{d}s + \left| \int_{\xi_0}^{x_\lambda-\delta_\lambda'} \frac{V''(s) \,\mathrm{d}s}{(\lambda - V(s))^{\frac{3}{2}}} \right| \\
&\quad + \int_{\xi_0}^{x_\lambda-\delta_\lambda'} \frac{V'(s)^2 \,\mathrm{d}s}{(\lambda - V(s))^{\frac{5}{2}}}.
\end{aligned}
\tag{4.24}
$$

The first integral on the r.h.s. is estimated using (4.7)

$$\int_{\xi_0}^{x_\lambda-\delta'_\lambda} \frac{-\zeta'(s)}{\zeta(s)^2}\,\mathrm{d}s \le \frac{1}{\zeta(x_\lambda-\delta'_\lambda)} \lesssim \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{2}}. \tag{4.25}$$

Since by (2.4)

$$\lambda - V(x_\lambda-\delta'_\lambda) \approx a_\lambda\delta'_\lambda \approx \lambda, \tag{4.26}$$

we have for the third integral on the r.h.s. in (4.24) that (we use (2.4) and (3.25))

$$\begin{aligned}\int_{\xi_0}^{x_\lambda-\delta'_\lambda} \frac{V'(s)^2\,\mathrm{d}s}{(\lambda-V(s))^{\frac{5}{2}}} &\lesssim \frac{\lambda\max\{1,x_\lambda^\nu\}}{\lambda^{\frac{5}{2}}}\int_{\xi_0}^{x_\lambda-\delta'_\lambda} V'(s)\,\mathrm{d}s\\ &\lesssim \max\left\{\frac{1}{\lambda^{\frac{1}{2}}},\left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{2}}\right\}.\end{aligned} \tag{4.27}$$

Integration by parts in the second integral on the r.h.s. in (4.24), the choice of δ'_λ, (4.1) and (4.27) lead to (with $\xi_0 > 0$ as in (2.2))

$$\begin{aligned}\left|\int_{\xi_0}^{x_\lambda-\delta'_\lambda} \frac{V''(s)\,\mathrm{d}s}{(\lambda-V(s))^{\frac{3}{2}}}\right| &\lesssim \frac{V'(x_\lambda-\delta'_\lambda)}{(\lambda-V(x_\lambda-\delta'_\lambda))^{\frac{3}{2}}} + \frac{V'(\xi_0)}{(\lambda-V(\xi_0))^{\frac{3}{2}}}\\ &\quad + \int_{\xi_0}^{x_\lambda-\delta'_\lambda} \frac{V'(s)^2\,\mathrm{d}s}{(\lambda-V(s))^{\frac{5}{2}}}\\ &\lesssim \max\left\{\frac{1}{\lambda^{\frac{1}{2}}},\left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{2}}\right\}.\end{aligned} \tag{4.28}$$

Putting together the estimates above, we arrive at

$$\int_0^{x_\lambda-\delta'_\lambda} \frac{|K(s)|}{w_1(s)^2}\,\mathrm{d}s \lesssim \frac{1}{\lambda^{\frac{1}{2}}} + \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{2}}. \tag{4.29}$$

• $x_\lambda + \delta''_\lambda \le s$: The estimates are again obtained for any value of $\varepsilon_2 \in (0,1)$ which will be specified later. The important observations are (based on the choice of δ''_λ and (2.4))

$$\begin{aligned}V(x_\lambda+\delta''_\lambda) - V(x_\lambda) &\approx a_\lambda x_\lambda^{-\nu} \approx \lambda,\\ |\zeta(x_\lambda+\delta''_\lambda)| &\gtrsim \left(\frac{a_\lambda}{x_\lambda^{3\nu}}\right)^{\frac{1}{2}}.\end{aligned} \tag{4.30}$$

Moreover, since $V'(x) > 0$ for all sufficiently large $x > 0$,

$$\left(\frac{V(x)}{V(x)-\lambda}\right)' = -\frac{\lambda V'(x)}{(V(x)-\lambda)^2} < 0, \tag{4.31}$$

and (see (2.4))

$$\frac{V(x_\lambda + \delta''_\lambda)}{V(x_\lambda+\delta''_\lambda) - V(x_\lambda)} \approx \frac{\lambda}{a_\lambda x_\lambda^{-\nu}} \approx 1, \tag{4.32}$$

we obtain (recall (3.25))

$$\begin{aligned}\int_{x_\lambda+\delta''_\lambda}^{\infty} \frac{|K(s)|}{w_1(s)^2}\,\mathrm{d}s &\lesssim \int_{x_\lambda+\delta''_\lambda}^{\infty} \frac{|\zeta(s)|'}{|\zeta(s)|^2}\,\mathrm{d}s + \int_{x_\lambda+\delta''_\lambda}^{\infty} \frac{|V''(s)|}{V(s)^{\frac{3}{2}}} + \frac{V'(s)^2}{V(s)^{\frac{5}{2}}}\,\mathrm{d}s \\ &\lesssim \left(\frac{x_\lambda^{3\nu}}{a_\lambda}\right)^{\frac{1}{2}} + \kappa_\lambda \lesssim \kappa_\lambda.\end{aligned} \tag{4.33}$$

• $x_\lambda - \delta'_\lambda \le s \le x_\lambda$: We integrate by parts twice in the formula for ζ and obtain

$$\zeta = \frac{2}{3}\frac{(\lambda - V)^{\frac{3}{2}}}{V'}\left(1 - \frac{2}{5}\frac{(\lambda - V)V''}{V'^2} - T\right), \tag{4.34}$$

where

$$T(s) = \frac{2}{5}\frac{V'(s)}{(\lambda - V(s))^{\frac{3}{2}}} \int_s^{x_\lambda} (\lambda - V(t))^{\frac{5}{2}} \left(\frac{V''(t)}{V'(t)^3}\right)' \mathrm{d}t. \tag{4.35}$$

Using (2.4) and (4.1), we obtain

$$\frac{(\lambda - V(s))V''(s)}{V'(s)^2} \lesssim \frac{a_\lambda \delta'_\lambda x_\lambda^{\nu}}{a_\lambda} \lesssim \varepsilon_1. \tag{4.36}$$

To estimate T, we first notice that by (2.4), (4.1) and (3.25)

$$\left|\left(\frac{V''(t)}{V'(t)^3}\right)'\right| \lesssim \frac{|V'''(t)|}{V'(t)^3} + \frac{V''(t)^2}{V'(t)^4} \lesssim \left(\frac{x_\lambda^{\nu}}{a_\lambda}\right)^2. \tag{4.37}$$

Thus, inserting $V'(t)/V'(t)$ and using (4.1),

$$\begin{aligned}|T(s)| &\lesssim \frac{x_\lambda^{2\nu}}{a_\lambda^2 (\lambda - V(s))^{\frac{3}{2}}} \int_s^{x_\lambda} V'(t)(\lambda - V(t))^{\frac{5}{2}}\,\mathrm{d}t \lesssim \frac{x_\lambda^{2\nu}}{a_\lambda^2}(\lambda - V(s))^2 \\ &\lesssim \frac{x_\lambda^{2\nu}}{a_\lambda^2}(\lambda - V(x_\lambda - \delta'_\lambda))^2 \lesssim \varepsilon_1^2.\end{aligned} \tag{4.38}$$

Hence it is possible to select $\varepsilon_1 \in (0, 1)$ so small that

$$\left| \frac{2}{5} \frac{(\lambda - V)V''}{V'^2} - T \right| \leq \frac{1}{4} \tag{4.39}$$

Using Taylor's theorem for ζ^{-2} and cancellations in K, one arrives at (employing (4.38), (2.4) and (3.24))

$$\begin{aligned} \frac{|K(s)|}{w_1(s)^2} &\lesssim \frac{|K(s)|}{(\lambda - V(s))^{\frac{1}{2}}} \lesssim \frac{V'(s)^2}{(\lambda - V(s))^{\frac{5}{2}}} \left[\left(\frac{(\lambda - V(s))V''(s)}{V'(s)^2} \right)^2 + |T(s)| \right] \\ &\lesssim \frac{x_\lambda^{2\nu}}{(\lambda - V(s))^{\frac{1}{2}}}; \end{aligned} \tag{4.40}$$

in the first step we use in addition (see (3.24) and (3.20))

$$\begin{aligned} &(\lambda - V(s))^{\frac{1}{2}} = w_1(s)^2, \quad s \in [x_\lambda - \delta_\lambda', x_\lambda - \delta), \\ &(\lambda - V(s))^{\frac{1}{2}} \leq (V(x_\lambda) - V(x_\lambda - \delta))^{\frac{1}{2}} \approx a_\lambda^{\frac{1}{3}} = w_1(s)^2, \quad s \in [x_\lambda - \delta, x_\lambda]. \end{aligned} \tag{4.41}$$

Hence,

$$\int_{x_\lambda - \delta_\lambda'}^{x_\lambda} \frac{|K(s)|}{w_1(s)^2} \, \mathrm{d}s \lesssim \frac{x_\lambda^{2\nu}}{a_\lambda} (\lambda - V(x_\lambda - \delta_\lambda'))^{\frac{1}{2}} \lesssim \left(\frac{x_\lambda^{3\nu}}{a_\lambda} \right)^{\frac{1}{2}}. \tag{4.42}$$

• $x_\lambda \leq s \leq x_\lambda + \delta_\lambda''$: The estimate and the choice of ε_2 in this region is analogous to the previous case. We omit the details.

In summary, putting all estimates together and using (3.25), we obtain the claim (4.22). □

Proof of Theorem 3.3 We follow the steps in [7]; the main differences are the additional perturbation W and new estimate of $\mathcal{J}_K(\lambda)$ from Lemma 4.4.

Using (3.15) and variation of constants, we can find a solution (distributional, since $W \in L^1_{\mathrm{loc}}(\mathbb{R})$ only) of (3.28) by solving the integral equation

$$y(x) = u(x) + \int_x^\infty G(x, s)(K(s) + W(s))y(s) \, \mathrm{d}s, \tag{4.43}$$

where $G(x, s) = u(x)v(s) - v(x)u(s)$. Using the notation $\hat{f}$ for a function f multiplied by $w_1 w_2$, we rewrite the integral equation (4.43) as

$$\hat{y}(x) = \hat{u}(x) + \int_x^\infty H(x, s) \frac{K(s) + W(s)}{w_1(s)^2} \hat{y}(s) \, \mathrm{d}s; \tag{4.44}$$

here

$$H(x, s) = \big(\hat{u}(x)\hat{v}(s) - \hat{v}(x)\hat{u}(s)\big) \, w_2(s)^{-2} \tag{4.45}$$

and $|H(x,s)| \lesssim 1$ in $0 \le x \le s$, see (3.23).

Let

$$\mathcal{J}_{K+W}(\lambda) := \int_0^\infty \frac{K(s) + W(S)}{w_1(s)^2}\, \mathrm{d}s = \mathcal{J}_K + \mathcal{J}_W. \tag{4.46}$$

If $\mathcal{J}_{K+W}(\lambda) = o(1)$ as $\lambda \to +\infty$, then the norm of the integral operator in (4.44) in $L^\infty(\mathbb{R}_+)$ decays as $\lambda \to +\infty$. Thus we can solve the equation (4.44), moreover, the solution can be expressed as

$$\hat{y} = \hat{u} + \hat{r}, \qquad \|\hat{r}\|_{L^\infty(\mathbb{R}_+)} \lesssim \frac{\mathcal{J}_{K+W}(\lambda)}{1 - \mathcal{J}_{K+W}(\lambda)} =: C(\lambda). \tag{4.47}$$

Returning back to y, we obtain (3.29) and (3.30).

The estimate on $\mathcal{J}_K$ is the main technical step of the proof, see Lemma 4.4 above, the decay of $\mathcal{J}_W$ is guaranteed by Assumption II.

Finally, the formula (3.32) for the L^2-norm of y follows from (3.27) as in [7, Thm. 1]. Namely,

$$y^2 = u^2 + \frac{\hat{r}(2\hat{u} + \hat{r})}{w_1^2 w_2^2} \tag{4.48}$$

and

$$\begin{aligned} \int_0^\infty \frac{\mathrm{d}x}{w_1(x)^2 w_2(x)^2} &= \int_0^{x_\lambda - \delta} \frac{\pi\, \mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}} + \mathcal{O}\left(\frac{\delta + \delta_1}{a_\lambda^{\frac{1}{3}}}\right) \\ &\quad + \int_{x_\lambda + \delta_1}^\infty \frac{e^{2\mathrm{i}\zeta(x)}\, \mathrm{d}x}{(V(x) - \lambda)^{\frac{1}{2}}} + \int_{x_\lambda - \delta}^{x_\lambda} \frac{\pi}{(\lambda - V(x))^{\frac{1}{2}}}\, \mathrm{d}x \\ &= \int_0^{x_\lambda} \frac{\pi\, \mathrm{d}x}{(\lambda - V(x))^{\frac{1}{2}}} + \mathcal{O}(a_\lambda^{-\frac{2}{3}}), \qquad \lambda \to +\infty, \end{aligned} \tag{4.49}$$

see the proof of Lemma 4.3 for more details on the estimates. The claim (3.32) then follows from (3.27), (4.49) and $\|\hat{r}(2\hat{u} + \hat{r})\|_{L^\infty} \lesssim C(\lambda)$, see (4.47) and (3.23).

5 Comparison with Existing Results

5.1 Concentration Measures for Orthogonal Polynomials

It is interesting to compare the concentration phenomenon (2.18) of measures (1.6) with its analogue in the case of orthogonal polynomials $\{p_n(x)\}$ for the weights $\exp(-|x|^\alpha)$, $\alpha > 0$, or even more general non-even weights $w(x) = \exp(-\tilde{w}(x))$ with properly chosen $\tilde{w}$. Following [10, 12], let

$$\kappa_\alpha := \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)}, \qquad w_\alpha(x) := \exp(-\kappa_\alpha |x|^\alpha), \qquad \alpha > 0; \tag{5.1}$$

the corresponding system of orthogonal polynomials $\{p_n(x)\}$

$$\int_{\mathbb{R}} p_n(x)p_m(x)w_\alpha(x)\,\mathrm{d}x = \delta_{mn}, \qquad m, n \in \mathbb{Z}, \tag{5.2}$$

has the property that, for sufficiently small $\delta > 0$ and for every $x \in [\delta, 1-\delta]$, as $n \to \infty$,

$$p_n(n^{\frac{1}{\alpha}}x)\sqrt{w_\alpha(n^{\frac{1}{\alpha}}x)} = \sqrt{\frac{2}{\pi n^{\frac{1}{\alpha}}}}(1-x^2)^{-\frac{1}{4}}\left[\cos\left(n\pi\int_1^x \psi_\alpha(y)\,\mathrm{d}y + \frac{1}{2}\arcsin x\right) + \mathcal{O}(n^{-1})\right], \tag{5.3}$$

where

$$\psi_\alpha(y) = \frac{\alpha}{\pi}y^{\alpha-1}\int_1^{\frac{1}{y}} \frac{u^{\alpha-1}}{\sqrt{u^2-1}}\,\mathrm{d}u \tag{5.4}$$

and where the implicit constant in $\mathcal{O}(n^{-1})$ is allowed to depend on δ. Formula (5.3) and elementary trigonometry imply that, as $n \to \infty$,

$$n^{\frac{1}{\alpha}}p_n^2(n^{\frac{1}{\alpha}}x)w_\alpha(n^{\frac{1}{\alpha}}x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\left[1 + \cos\left(2n\pi\int_1^x \psi_\alpha(y)\,\mathrm{d}y + \arcsin x\right) + \mathcal{O}(n^{-1})\right]. \tag{5.5}$$

Thus, for any $f \in C([-1, 1])$, Riemann-Lebesgue lemma gives

$$\lim_{n\to\infty}\int_\delta^{1-\delta} f(x)n^{\frac{1}{\alpha}}p_n^2(n^{\frac{1}{\alpha}}x)w_\alpha(n^{\frac{1}{\alpha}}x)\,\mathrm{d}x = \frac{1}{\pi}\int_\delta^{1-\delta}\frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x. \tag{5.6}$$

The analogous limit holds on the interval $[-1+\delta, -\delta]$ because the polynomials p_n are either even or odd. Moreover, by [10, Thm.1.16],

$$\sup_{n\geq 1}\sup_{x\in\mathbb{R}} n^{\frac{1}{\alpha}}p_n^2(n^{\frac{1}{\alpha}}x)w_\alpha(n^{\frac{1}{\alpha}}x)\sqrt{|1-x^2|} < \infty, \tag{5.7}$$

so

$$\lim_{n\to\infty}\int_{-1}^{1} f(x)n^{\frac{1}{\alpha}}p_n^2(n^{\frac{1}{\alpha}}x)w_\alpha(n^{\frac{1}{\alpha}}x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x. \tag{5.8}$$

On the whole real line, one can use the following inequalities, see [12, Thm.19, p.16, Eq.(1.66)]. Let $a > 1$ and P be a polynomial of degree smaller than or equal to n. Then

$$\int_{|x|\geq a} P^2(n^{\frac{1}{\alpha}}x)\omega_\alpha(n^{\frac{1}{\alpha}}x)\,\mathrm{d}x \leq C_1\exp(-C_2 n)\int_{-1}^{1} P^2(n^{\frac{1}{\alpha}}x)\omega_\alpha(n^{\frac{1}{\alpha}}x)\,\mathrm{d}x \tag{5.9}$$

for all $n \geq 1$; the constants C_1, C_2 depend on a, but not on n or P. These inequalities imply

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) n^{\frac{1}{\alpha}} p_n^2(n^{\frac{1}{\alpha}} x) w_\alpha(n^{\frac{1}{\alpha}} x)\, \mathrm{d}x = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, \mathrm{d}x \tag{5.10}$$

for any bounded continuous function on $\mathbb{R}$.

A striking difference between (5.10) and (2.18) is that in the case of orthogonal polynomials the concentration measure *does not depend on* α, or $\tilde{w}$ in a more general case of weights $\exp(-\tilde{w}(x))$.

5.2 Semi-classical Defect Measures

In classical mechanics, *cf.* [2], a particle with position $x(t)$ subject to the differential equation

$$\begin{cases} \ddot{x}(t) + V(x(t)) = 0, \\ (x(0), \dot{x}(0)) = (x_0, \xi_0) \end{cases} \tag{5.11}$$

remains for all times on the energy surface

$$(x(t), \dot{x}(t)) \in \{(x, \xi)\ :\ \xi^2 + V(x) = \xi_0^2 + V(x_0)\}$$

and travels along the trajectory $(\dot{x}(t), \dot{\xi}(t))$ obeying

$$(\dot{x}(t), \dot{\xi}(t)) = (2\xi(t), -V'(x(t))).$$

The classical-quantum correspondence suggests that, in the high-energy limit, the L^2-mass of an eigenfunction should be distributed in the same way as the average position of a classical particle: since a classical particle passes through an interval $[x_*, x_* + \mathrm{d}x]$ in physical space with velocity near $\eta(x_*)$ or $-\eta(x_*)$, where

$$\eta(x_*) = \sqrt{\lambda - V(x_*)}, \tag{5.12}$$

we obtain the heuristic (for a normalization constant c_0)

$$|u(x)|^2\, \mathrm{d}x \text{ “}=\text{''}\ \frac{c_0}{\eta(x)}\, \mathrm{d}x = \frac{c_0}{\sqrt{\lambda - V(x)}}\, \mathrm{d}x, \tag{5.13}$$

which agrees with Theorem 2.3 after the corresponding scaling.

To make this correspondence precise, one can use the notion of semiclassical defect measures (see, for instance, [20, Ch. 5]). The following discussion will be under weaker hypotheses than Theorem 2.3, because our goal is only to show that the precise asymptotics obtained agree with the semiclassical prediction.

Let $V : \mathbb{R} \to \mathbb{R}$ be even, smooth and suppose that there exists some $\beta > 0$ such that

$$\left| V^{(k)}(x) \right| \lesssim (1 + |x|)^{\beta - k}, \qquad k \in \mathbb{N}_0, \quad x \in \mathbb{R}. \tag{5.14}$$

Suppose also that

$$V'(x) > 0, \quad x > 0. \tag{5.15}$$

and that there exists $x_0 > 0$ such that

$$V'(x) \gtrsim (1+x)^{\beta-1}, \quad x > x_0; \tag{5.16}$$

the latter implies that, for $|x|$ sufficiently large,

$$V(x) \approx (1+|x|)^{\beta}.$$

We consider the semiclassical Schrödinger operator

$$A_\hbar = -\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$$

in the limit $\hbar \to 0^+$.

For example, if $V(x) = |x|^\beta$ for $\beta \in 2\mathbb{N}$ (so that $V(x)$ is infinitely differentiable), scaling gives a unitary equivalence

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\beta \sim \hbar^{-\frac{2\beta}{2+\beta}} \left(-\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\beta \right).$$

Other potentials can be treated by rescaling and controlling the error, but this analysis is outside the aim of this work. We emphasize that the assumptions on Q in Theorem 2.3 are significantly weaker than the hypotheses on V here, *cf.* (1.2), Assumption I and II and comments in Introduction.

Suppose that for $\lambda_0 > \inf V(x)$, there exists a sequence $\{\hbar_k\}_{k\in\mathbb{N}}$ of positive numbers tending to zero and eigenfunctions $\{u_k\}_{k\in\mathbb{N}}$ obeying $\|u_k\| = 1$ and

$$A_{\hbar_k} u_k = \lambda_0 u_k.$$

For each u_k, one can define the functional

$$\varphi_k(b) = \int_{\mathbb{R}} \overline{u_k(x)} b^w_{\hbar_k}(x, \hbar_k D_x) u_k(x)\,\mathrm{d}x, \quad b \in C_c^\infty(\mathbb{R}).$$

Here, $D_x = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ and $b^w_\hbar(x, \hbar D_x)$ is the Weyl quantization (see e.g. [20, Ch. 4]); when $b \in C_c^\infty(\mathbb{R}^2)$, the Weyl quantization of b is a compact operator on $L^2(\mathbb{R})$ which takes $\mathscr{S}'(\mathbb{R})$ to $\mathscr{S}(\mathbb{R})$.

Following [20, Thm. 5.2] there is a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ with $\hbar_{k_j} \to 0^+$ for which the functionals φ_k converge to a non-negative Radon measure μ in the sense that, for each $b \in C_c^\infty(\mathbb{R}^2)$,

$$\lim_{j\to\infty} \varphi_{k_j}(b) = \int_{\mathbb{R}^2} b(x,\xi)\,\mathrm{d}\mu(x,\xi). \tag{5.17}$$

We will show that this μ is unique and that therefore $\varphi_k \to \mu$ in the same sense since every subsequence admits a further subsequence tending to μ.

By [20, Thm. 5.3 or Thm. 6.4],

$$\operatorname{supp}\mu \subseteq \{\xi^2 + V(x) = \lambda_0\}, \tag{5.18}$$

so let us define, in analogy with (5.12),

$$\eta(x) = \sqrt{\lambda_0 - V(x)} \tag{5.19}$$

for those x such that $V(x) < \lambda_0$. There exists a measure ν_+ such that, when $\operatorname{supp} b \subset \{\xi > 0\}$, then

$$\int_{\mathbb{R}^2} b(x,\xi)\,\mathrm{d}\mu(x,\xi) = \int_{\{V(x)<\lambda_0\}} b(x,\eta(x))\,\mathrm{d}\nu_+(x). \tag{5.20}$$

By [20, Thm. 5.4], for any $b \in C_c^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \{a,b\}(x,\xi)\,\mathrm{d}\mu(x,\xi) = 0, \tag{5.21}$$

where the Poisson bracket $\{a,b\}$ of the symbol $a(x,\xi) = \xi^2 + V(x)$ of $A_\hbar$ with b is

$$\{a,b\} = a_\xi b_x - a_x b_\xi = 2\xi b_x - V'(x) b_\xi.$$

This corresponds to invariance of μ under the classical Hamilton flow associated to $a(x,\xi)$, which in the case of a Schrödinger operator corresponds to (5.11).

Finally, since in our situation the support of μ is compact, we show that

$$\int_{\mathbb{R}^2} \mathrm{d}\mu(x,\xi) = 1 \tag{5.22}$$

as follows. For any $b(x,\xi) \in C_c^\infty(\mathbb{R}^2)$ such that $b \equiv 1$ on $\{\xi^2 + V(x) = \lambda_0\}$, we use that the Weyl quantization of the constant 1 function is the identity operator to write

$$1 = \int_{\mathbb{R}} |u_{k_j}(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}} \overline{u_{k_j}(x)}\left(b^w(x,\hbar_k k_j D_x) + (1-b)^w(x,\hbar_{k_j} D_x)\right) u_{k_j}(x)\,\mathrm{d}x. \tag{5.23}$$

By [20, Thm. 6.4],

$$(1-b)^w(x,\hbar_{k_j} D_x) u_k(x) = \mathcal{O}(\hbar_{k_j}^\infty), \tag{5.24}$$

meaning that its $L^2(\mathbb{R})$ norm is smaller than any power of $\hbar_{k_j}$ as $\hbar_{k_j} \to 0^+$, and by the definition (5.17) of $\mu(x,\xi)$ and the fact that $b \equiv 1$ on $\operatorname{supp}\mu$,

$$\lim_{k\to\infty} \int_{\mathbb{R}} \overline{u_k(x)} b^w(x,\hbar_k D_x) u_k(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} b(x,\xi)\,\mathrm{d}\mu(x,\xi) = \int_{\mathbb{R}^2} \mathrm{d}\mu(x,\xi). \tag{5.25}$$

Taking (5.23), (5.24), and (5.25) together proves (5.22).

We now prove that a measure μ satisfying the properties of a semiclassical defect measure must have the form matching the classical heuristic (5.13) generalized in Theorem 2.3.

Proposition 5.1 *Let $V(x) \in C^\infty(\mathbb{R};\mathbb{R})$ satisfy (5.14), (5.16), and (5.15). Let $\lambda_0 > V(0) = \inf V(x)$, and let μ be a measure satisfying (5.18), (5.21), and (5.22) and let η be as in (5.19). Then the measure μ obeys for all $b \in C_c^\infty(\mathbb{R}^2)$*

$$\int b(x,\xi)\,\mathrm{d}\mu = c_0 \int_{-x_{\lambda_0}}^{x_{\lambda_0}} (b(x,\eta(x)) + b(x,-\eta(x)))\,\frac{\mathrm{d}x}{\eta(x)},$$

where the normalization constant c_0 is such that $\int \mathrm{d}\mu = 1$.

Proof We observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}b(x,\eta(x)) &= b_x(x,\eta(x)) + \eta'(x)b_\xi(x,\eta(x))\\ &= b_x(x,\eta(x)) - \frac{V'(x)}{2\eta}b_\xi(x,\eta(x))\\ &= \frac{1}{2\eta(x)}\left(2\eta(x)b_x(x,\eta(x)) - V'(x)b_\xi(x,\eta(x))\right)\\ &= \frac{1}{2\eta(x)}\{a,b\}(x,\eta(x)).\end{aligned} \tag{5.26}$$

Letting $b \in C_c^\infty(\mathbb{R}^2)$ be such that $\operatorname{supp} b \subset \{\xi > \delta\}$ for some $\delta > 0$, we obtain from (5.20), (5.21), and (5.26) that

$$\int \left(\frac{\mathrm{d}}{\mathrm{d}x}b(x,\eta(x))\right) 2\eta(x)\,\mathrm{d}\nu_+(x)$$

vanishes. Taking $b(x,\xi) = f(x)\chi_{[\delta,\delta^{-1}]}(\xi)$ for $f \in C_c^\infty(\mathbb{R})$ arbitrary and for χ a cutoff function, letting $\delta \to 0^+$ allows us to conclude that

$$\int f'(x)\,\eta(x)\,\mathrm{d}\nu_+(x) = 0$$

for all $f \in C_c^\infty(\mathbb{R})$. Therefore along $\{\xi^2 + V(x) = \lambda_0\}$,

$$\mathrm{d}\nu_+(x) = \frac{c_+}{\eta}\mathrm{d}x$$

for some c_+ which is positive because μ is a positive measure.

When $\operatorname{supp} b(x,\xi) \subset \{\xi < 0\}$, the same argument shows that there is some $c_- > 0$ such that

$$\int b(x,\xi)\,\mathrm{d}\mu(x,\xi) = \int b(x,-\eta(x))\,\frac{c_-}{\eta}(x)\mathrm{d}x.$$

One can show then that $c_+ = c_-$ by projecting onto the ξ variable instead of the x variable: let

$$\tilde{x}(\xi) = V^{-1}(\lambda_0 - \xi^2)$$

where the inverse image is chosen nonnegative. Note that because $V(x)$ is strictly increasing and unbounded on $[0, \infty)$, one may define $\tilde{x}(\xi)$ if and only if $\lambda_0 - \xi^2 \geq V(0)$. Let $\mathrm{d}\rho_+(\xi)$ be such that when $\operatorname{supp} b \subset \{x > 0\}$,

$$\int b(x, \xi)\, \mathrm{d}\mu(x, \xi) = \int_{\{|\xi| \leq \sqrt{\lambda_0 - V(0)}\}} b(\tilde{x}(\xi), \xi)\, \mathrm{d}\rho_+(\xi).$$

Then $\tilde{x}'(\xi) = -\frac{2\xi}{V'(\tilde{x}(\xi))}$,

$$\frac{\mathrm{d}}{\mathrm{d}x} b(\tilde{x}(\xi), \xi) = V'(\tilde{x}(\xi))\{a, b\}(\tilde{x}(\xi), \xi).$$

The earlier argument (along with the fact that $V'(x) > 0$ for $x > 0$) shows that there is some $d_+ > 0$ such that

$$\mathrm{d}\rho_+(\xi) = \frac{d_+}{V'(\tilde{x}(\xi))}\, \mathrm{d}\xi.$$

On $\{\xi^2 + V(x) = \lambda_0\}$, note that

$$\left|\frac{\mathrm{d}\xi}{\mathrm{d}x}\right| = \frac{|V'(x)|}{2|\tilde{\xi}(x)|}.$$

Since the pull-backs of $\mathrm{d}\nu_\pm(x) = \frac{c_\pm}{|\xi|}\mathrm{d}x$ and $\mathrm{d}\rho_+$ agree on $a^{-1}(\{\lambda_0\}) \cap \{x > 0, \xi > 0\}$ and since $\mathrm{d}\rho_+$ and $\mathrm{d}\nu_-$ agree on $\{x > 0, \xi < 0\}$ we can conclude that $c_+ = d_+/2 = c_-$. We remark that this argument is not available in the case $\omega_\beta = 0$ corresponding to a very rapidly-growing potential.

Finally, we conclude that $c_0 = c_+$ is such that $\int \mathrm{d}\mu = 1$ by the hypothesis (5.22). ☐

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