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Hardy's Theorem for Gabor Transform on Nilpotent Lie Groups

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Abstract
In this paper, we study the conjecture of Bansal, Kumar and Sharma, which is an analog of Hardy's theorem for Gabor transform in the setup of connected nilpotent Lie groups. To approach this conjecture, we use the orbit method and the Plancherel theory. When the Lie group G is simply connected, we show that the conjecture is true.

Keywords Hardy's theorem · Nilpotent Lie group · Gabor transform · Plancherel formula

Mathematics Subject Classification 22E25 · 43A25 · 43A32

1 Introduction

A classical version of the uncertainty principle states that an integrable function f defined on the real line and its Fourier transform $\hat{f}$ cannot be simultaneously and sharply localized unless $f = 0$ almost everywhere. An important result making this precise, is the Hardy Theorem (see [13]):

Theorem 1 *Let α, β, c be positive real numbers and f a measurable function on $\mathbb{R}$ such that:*

(i) $|f(t)| \leq ce^{-\alpha\pi t^2}, \quad t \in \mathbb{R}$,
(ii) $|\hat{f}(k)| \leq ce^{-\beta\pi k^2}, \quad k \in \mathbb{R}$.

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If $\alpha\beta > 1$, *then* $f = 0$ *a.e. If* $\alpha\beta = 1$ *then* $f(t) = be^{-\alpha\pi t^2}$, *for some constant b. If* $\alpha\beta < 1$, *then any finite linear combination of Hermite functions satisfies (i) and (ii).*

For the Fourier transformation we use the normalization

$$\hat{f}(k) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2i\pi tk}dt, \ k \in \mathbb{R}.$$

Note that Hardy's theorem is also valid on $\mathbb{R}^n$ (see [19, Theorem 4]). Much efforts have been deployed to prove Hardy-like theorems for various classes of non-Abelian connected Lie groups. Specifically, analogues and variants of Hardy's theorem have been shown for nilpotent Lie groups [1, 2, 14, 19, 21], some classes of solvable Lie groups [3], non-compact connected semisimple Lie groups G with finite center [9, 15, 17, 18] and motion groups [16, 20]. For more information and further references concerning the entire subject, we refer the readers to the excellent monograph by Thangavelu [22].

The continuous Gabor transform (also known as windowed Fourier transform) is a classical tool in mathematical signal processing. Roughly speaking, it is the Fourier transform of a signal f seen through a sliding window ψ. For $f \in L^2(\mathbb{R}^n)$ and a nonzero function $\psi \in L^2(\mathbb{R}^n)$ called a window function, the continuous Gabor transform with respect to ψ is defined on $\mathbb{R}^n \times \widehat{\mathbb{R}}^n$ by

$$\mathcal{G}_\psi f(x, w) := \int_{\mathbb{R}^n} f(y)\overline{\psi}(y - x)e^{-2i\pi\langle y,w\rangle}dy.$$

According to [11], we have for all $f_1, f_2, \psi_1, \psi_2 \in L^2(\mathbb{R}^n)$ the functions $\mathcal{G}_{\psi_1} f_1$ and $\mathcal{G}_{\psi_2} f_2$ belong to $L^2(\mathbb{R}^n \times \hat{\mathbb{R}}^n)$ and

$$\langle \mathcal{G}_{\psi_1} f_1, \mathcal{G}_{\psi_2} f_2\rangle_{L^2(\mathbb{R}^n\times\hat{\mathbb{R}}^n)} = \langle f_1, f_2\rangle_{L^2(\mathbb{R}^n)}\overline{\langle\psi_1, \psi_2\rangle}_{L^2(\mathbb{R}^n)}. \tag{1}$$

This transform plays an important role in time-frequency analysis namely by providing an interesting way to study the local frequency spectrum of signals. For a detailed discussion of time-frequency analysis, we refer the readers to [11]. It has been shown in the early 2000s that many uncertainty principles for the Fourier transform have a counterpart for the continuous Gabor transform (see e.g. [7, 12]). We specify that a Hardy-type theorem has been established in [12, Theorem 2.6.2].

Theorem 2 *Let* $f, \psi \in L^2(\mathbb{R}^n)$. *Assume that*

$$\left|\mathcal{G}_\psi f(x, w)\right| \leq Ce^{-\pi(\alpha x^2+\beta w^2)/2},$$

for some constants $\alpha, \beta, C > 0$. *Then three cases can occur.*

(i) *If* $\alpha\beta > 1$, *then either* $f = 0$ *a.e. or* $\psi = 0$ *a.e.*
(ii) *If* $\alpha\beta = 1$ *and* $\mathcal{G}_\psi f$ *is not zero almost everywhere, then both* f *and* ψ *are multiples of some time-frequency shift of the Gaussian* $e^{-\alpha\pi x^2}$.

(iii) *If $\alpha\beta < 1$, then the decay condition is satisfied whenever f and ψ are finite linear combinations of Hermite functions.*

On the other hand, the continuous Gabor transform has also been extended to separable locally compact unimodular group of type I (see [10]). One should notice that, in the Euclidean setting, the continuous Gabor transform has many symmetries which are lost in the Lie group setting (the dual of G does not identify with G) and this is then a serious obstacle for stating uncertainty principles for the continuous Gabor transform. However, some attempts to extend Theorem 2 on special classes of non-Abelian Lie groups have already been made. In particular, we cite here the work of Bansal, Kumar and Sharma [6], where the authors generalized Hardy's theorem for the Gabor transform on locally compact abelian groups having noncompact identity component and groups of the form $\mathbb{R}^n \times K$, where K is a compact group having irreducible representations of bounded dimension. When it comes to connected nilpotent Lie groups, only a conjecture are now available. In the same reference, the previous authors conjecture that if α, β and C are positive real numbers and f, ψ are square integrable functions on connected nilpotent Lie group $G = \exp \mathfrak{g}$ such that $\|\mathcal{G}_\psi f(g, \pi_l)\|_{HS} \leq C\mathrm{e}^{-\pi(\alpha\|g\|^2+\beta\|l\|^2)/2}$ for all $(g, l) \in G \times \mathcal{W}$, then $f = 0$ a.e. or $\psi = 0$ a.e. provided that $\alpha\beta > 1$. Here $\mathcal{W}$ is a suitable cross-section for the generic coadjoint orbits in $\mathfrak{g}^*$, the vector space dual of $\mathfrak{g}$, and $\|g\|$ and $\|l\|$ are substitutes (in terms of bases of $\mathfrak{g}$ and $\mathfrak{g}^*$) for the Euclidean norms on $\mathbb{R}^n$ and $\widehat{\mathbb{R}}^n = \mathbb{R}^n$. For details and unexplained notation see Sect. 2. They also proved that this conjecture fails for a connected nilpotent Lie group G having a square integrable irreducible representation. This paper is the first attempt to establish analog of Hardy's theorem for Gabor transform on nilpotent Lie groups. By exploiting Hardy's Theorem for $\mathbb{R}$ and a localized version of the Plancherel formula, we show in Sect. 4 that the above-mentioned conjecture holds. Our main result is the following:

Theorem 3 *Let G be connected, simply connected nilpotent Lie group. Let $f, \psi \in L^2(G)$ be such that*

$$\|\mathcal{G}_\psi f(g, \pi_l)\|_{HS}^2 \leq Ce^{-\frac{\pi}{2}(\alpha\|g\|^2+\beta\|l\|^2)}, \tag{2}$$

for all $(g, l) \in G \times \mathcal{W}$, where α, β and C are positive real numbers. If $\alpha\beta > 1$, then either $f = 0$ a.e. or $\psi = 0$ a.e.

2 Backgrounds

2.1 Continuous Gabor Transform

Let G be a separable locally compact unimodular group of type I, and let dg be its Haar measure. We endow the unitary dual of G with the Mackey Borel structure. We denote by $L^p(G)$ the space of L^p-functions on G for $p \geq 1$, and we define

$$\pi(f) = \int_G f(g)\pi(g)dg, \quad \pi \in \hat{G},\ f \in L^1(G).$$

Then by the abstract Plancherel theorem, there exists a unique Borel measure ρ on $\hat{G}$ such that for any function $f \in L^1(G) \cap L^2(G)$,

$$\int_G |f(g)|^2 dg = \int_{\hat{G}} \|\pi(f)\|_{HS}^2 d\rho(\pi),$$

where $\|\pi(f)\|_{HS} = \left(\mathrm{tr}\left(\pi(f)^* \pi(f)\right)\right)^{1/2}$ denotes the Hilbert-Schmidt norm of $\pi(f)$.

Let $f \in C_c(G)$, the set of all continuous complex-valued functions on G with compact supports, and ψ a fixed nonzero function in $L^2(G)$, usually called window function. For $(x, \pi) \in G \times \hat{G}$, the continuous Gabor transform of f with respect to the window function ψ is defined as a measurable field of operators on $G \times \hat{G}$ by

$$\mathcal{G}_\psi f(x, \pi) := \int_G f(g) \overline{\psi}(x^{-1} g) \pi(g) dg.$$

Let f_ψ^x be the function defined on G by

$$f_\psi^x(g) = f(g) \overline{\psi}(x^{-1} g), \qquad \forall g \in G.$$

Then, $f_\psi^x \in L^1(G) \cap L^2(G)$ and

$$\pi(f_\psi^x) = \int_G f_\psi^x(g) \pi(g) dg = \int_G f(g) \overline{\psi}(x^{-1} g) \pi(g) dg = \mathcal{G}_\psi f(x, \pi). \tag{3}$$

By the Plancherel theorem, $\mathcal{G}_\psi f(x, \pi)$ is a Hilbert-Schmidt operator for all $x \in G$ and for almost all $\pi \in \hat{G}$. Furthermore,

$$\int_G \int_{\hat{G}} \|\mathcal{G}_\psi f(x, \pi)\|_{HS}^2 d\rho(\pi)\, dx = \|\psi\|_2^2 \|f\|_2^2. \tag{4}$$

Thus, the continuous Gabor transform $\mathcal{G}_\psi : f \mapsto \mathcal{G}_\psi f$ $(f \in C_c(G))$ is a multiple of an isometry. So, we can extend $\mathcal{G}_\psi$ uniquely to a bounded linear operator on $L^2(G)$ which we still denote by $\mathcal{G}_\psi$ and this extension satisfies (4) for each $f \in L^2(G)$.

2.2 Nilpotent Lie Groups

We begin this subsection by reviewing some useful facts and notations for nilpotent Lie group. This material is quite standard, we refer the reader to [8] for details. We assume henceforth that $G = \exp \mathfrak{g}$ is a connected, simply connected nilpotent Lie group.

Let $\mathcal{B} = \{X_1, ..., X_n\}$ be a strong Malcev basis of $\mathfrak{g}$ passing through the center of $\mathfrak{g}$. We introduce a *norm function* on G by setting for $x = \exp(x_1 X_1 + \cdots + x_n X_n) \in G,\ x_j \in \mathbb{R}$,

$$\|x\| = \sqrt{(x_1^2 + \cdots + x_n^2)}.$$

The map:

$$\mathbb{R}^n \to G,\ (x_1, ..., x_n) \mapsto \exp\Big(\sum_{j=1}^{n} x_j X_j\Big)$$

is a diffeomorphism and maps the Lebesgue measure on $\mathbb{R}^n$ to the Haar measure on G. In this setup, we shall identify G as set with $\mathbb{R}^n$. We consider the Euclidean norm of $\mathfrak{g}^*$ with respect to the basis $\mathcal{B}^* = \{X_1^*, ..., X_n^*\}$, that is,

$$\Big\|\sum_{j=1}^{n} l_j X_j^*\Big\| = \sqrt{(l_1^2 + \cdots + l_n^2)} = \|l\|,\ \ l_j \in \mathbb{R}.$$

Let $\mathcal{U}$ denote the Zariski open subset of $\mathfrak{g}^*$ consisting of all elements in generic orbits with respect to the basis $\mathcal{B}^*$. Let S be the set of jump indices, and set $T = \{1, ..., n\}\backslash S$ and $V_T = \mathbb{R}\text{-span}\{X_i^* \ :\ i \in T\}$. Then, $\mathcal{W} = \mathcal{U} \cap V_T$ is a cross section of the generic orbits and $\mathcal{W}$ supports the Plancherel measure on $\hat{G}$. Let $Pf(l)$ denote the Pfaffian of the skew-symmetric matrix $M_S(l) = (l([X_i, X_j]))_{i,j\in S}$. Then, one has that:

$$|Pf(l)|^2 = \det M_S(l).$$

If dl is the Lebesgue measure on $\mathcal{W}$, then $d\tau = |Pf(l)|dl$ is a Plancherel measure for $\hat{G}$. Let dg be the Haar measure on G. For $\varphi \in L^1(G) \cap L^2(G)$, the Plancherel formula reads:

$$\|\varphi\|_2^2 = \int_G |\varphi(g)|^2 dg = \int_{\mathcal{W}} \|\pi_l(\varphi)\|_{HS}^2 d\tau(l). \tag{5}$$

3 Some Lemmas

In this section we prove three results, Lemmas 1, 2 and 3 which are required to prove Theorem 3.

For every $x, w \in \mathbb{R}^n$, we denote by $\mathcal{M}_w$ and $\mathcal{T}_x$ the modulation and the translation operators defined respectively on $L^2(\mathbb{R}^n)$ by

$$\forall z \in \mathbb{R}^n, \quad \mathcal{M}_w f(z) = e^{2i\pi\langle z,w\rangle} f(z),$$
$$\forall z \in \mathbb{R}^n, \quad \mathcal{T}_x f(z) = f(z - x).$$

Then we deduce that,

$$\forall z \in \mathbb{R}^n, \quad \mathcal{M}_w(\mathcal{T}_x f)(z) = e^{2i\pi\langle z,w\rangle} f(z - x), \tag{6}$$

and

$$\forall z \in \mathbb{R}^n, \quad \mathcal{T}_x(\mathcal{M}_w f)(z) = e^{-2i\pi\langle x,w\rangle} e^{2i\pi\langle z,w\rangle} f(z - x). \tag{7}$$

The results in the following lemma are quite standard.

Lemma 1 *Let $f, \psi \in L^2(\mathbb{R}^n)$ and $\xi, \lambda, y, z \in \mathbb{R}^n$. Then,*

(i) $\mathcal{G}_{(\mathcal{M}_\xi \mathcal{T}_z \psi)}(\mathcal{M}_\lambda \mathcal{T}_y f)(x, w) = e^{2i\pi\langle x,\xi\rangle} e^{-2i\pi\langle y, w-\lambda+\xi\rangle} \mathcal{G}_\psi f(x-y+z, w-\lambda+\xi)$. *In particular,* $\mathcal{G}_\psi(\mathcal{M}_\lambda \mathcal{T}_y f)(x, w) = e^{-2i\pi\langle y,w\rangle} e^{2i\pi\langle y,\lambda\rangle} \mathcal{G}_\psi f(x-y, w-\lambda)$.

(ii) $\mathcal{G}_\psi f(-x, -w) = e^{-2i\pi\langle x,w\rangle} \overline{\mathcal{G}_f \psi(x, w)}$.

(iii) *Let* $F(x, w) = \mathcal{G}_\psi f(x, w) \mathcal{G}_\psi f(-x, -w) e^{2i\pi\langle x,w\rangle}$. *Then,*

$$\hat{F}(\nu, \theta) = F(-\theta, \nu), \quad \nu, \theta \in \mathbb{R}^n.$$

Let's fix as above a strong Malcev basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ such that X_1 is in the center of $\mathfrak{g}$. For $a = (a_2, \ldots, a_n) \in \mathbb{R}^{n-1}$, let $(f_\psi^g)_a$ be the complex valued function defined on $\mathbb{R}$ by

$$(f_\psi^g)_a(t) = f_\psi^g(t, a) = f_\psi^g\Big(\exp\big(tX_1 + \sum_{j=2}^n a_j X_j\big)\Big).$$

For $k \in \mathbb{R}$ and $s = (s_2, \ldots, s_n) \in \mathbb{R}^{n-1}$, let $g = \exp\big(kX_1 + \sum_{j=2}^n s_j X_j\big)$ and $f_\psi^{k,s} = f_\psi^g$. It is easy to see that $f_\psi^g \in L^1(G)$ for all $g \in G$, it sufficient to use Cauchy–Schwarz inequality. Moreover by [5, Lemma 3.1], we have

$$\mathcal{G}_\psi f(g, \pi_l) = \pi_l(f_\psi^g), \tag{8}$$

for all $g \in G$. We should also mention that $f_\psi^g \in L^2(G)$, for almost all $g \in G$. In fact,

$$\int_G \int_G |f_\psi^g(x)|^2 dx\, dg = \int_G \int_G |f(x)|^2 |\psi(g^{-1}x)|^2 dx\, dg = \|f\|_2^2 \|\psi\|_2^2 < \infty.$$

Then obviously $\int_G |f_\psi^g(x)|^2 dx < \infty$, for almost all $g \in G$.

Lemma 2 *Let $f, \psi \in L^2(G)$ meet the condition (2) of Theorem 3. Then*

$$I(f_\psi^{k,s}) := \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \big|(f_\psi^{k,s})_a(t)\big| dt \right)^2 da < \infty,$$

for all $k \in \mathbb{R}$ and almost all $s = (s_2, \ldots, s_n) \in \mathbb{R}^{n-1}$.

Proof By using (2), we have

$$\begin{aligned} &\int_G \int_{\mathcal{W}} (1 + \|g\|^2) \|\mathcal{G}_\psi f(g, \pi_l)\|_{HS}^2 |Pf(l)| dl\, dg \\ &\quad \le \int_G \int_{\mathcal{W}} (1 + \|g\|^2) e^{-\pi(\alpha\|g\|^2 + \beta\|l\|^2)} |Pf(l)| dl\, dg. \end{aligned} \tag{9}$$

Assume that the degree of the polynomial function $Pf(l)$ is equal to δ. Then,

$$
\begin{aligned}
|Pf(l)| &\leq \text{cst } (1 + \|l\|^2)^{\frac{\delta}{2}}. \\
&\leq \text{cst } (1 + \|l\|^2)^{\delta}
\end{aligned}
$$

Therefore, the integral on the right hand side of (9) converges. Hence,

$$
\begin{aligned}
\infty &> \int_G (1 + \|g\|^2) \left(\int_{\mathcal{W}} \|\pi(f_\psi^g)\|_{HS}^2 |Pf(l)| dl \right) dg \\
&= \int_G \int_G (1 + \|g\|^2) |f(x)|^2 |\psi(g^{-1}x)|^2 dx\, dg
\end{aligned}
$$

(using Plancherel formula of G)

$$
\begin{aligned}
&\geq \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (1 + |k|^2) \Big| f\Big(\exp\Big(tX_1 + \sum_{j=2}^n a_j X_j \Big)\Big)\Big|^2 \\
&\times \Big| \psi\Big(\exp\Big(kX_1 + \sum_{j=2}^n s_j X_j \Big)^{-1} \exp\Big(tX_1 + \sum_{j=2}^n a_j X_j \Big)\Big)\Big|^2 dt\, da\, dk\, ds.
\end{aligned}
$$

Noting that,

$$
\begin{aligned}
&\exp\Big(kX_1 + \sum_{j=2}^n s_j X_j \Big)^{-1} \exp\Big(tX_1 + \sum_{j=2}^n a_j X_j \Big) \\
&= \exp\Big(\big(t - k + Q_1(a, s)\big) X_1 + \sum_{j=2}^n Q_j(a, s) X_j \Big), \qquad (10)
\end{aligned}
$$

where, for $1 \leq j \leq n$, Q_j is a polynomial function depending on $a = (a_2, ..., a_n)$ and $s = (s_2, ..., s_n)$. Furthermore, one can write

$$
Q_j(a, s) = a_j - s_j + Q'_j(a_{j+1}, ..., a_n, s_{j+1}, ..., s_n), \quad j = 2, ..., n. \qquad (11)
$$

It follows that,

$$
\begin{aligned}
\infty &> \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (1 + |k|^2) \Big| f\Big(\exp\Big(tX_1 + \sum_{j=2}^n a_j X_j \Big)\Big)\Big|^2 \\
&\times \Big| \psi\Big(\exp\Big(\big(t - k + Q_1(a, s)\big) X_1 + \sum_{j=2}^n Q_j(a, s) X_j \Big)\Big)\Big|^2 dt\, da\, dk\, ds \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \Big(1 + \big| t - r + Q_1(a, s) \big|^2 \Big) \Big| f\Big(\exp\Big(tX_1 + \sum_{j=2}^n a_j X_j \Big)\Big)\Big|^2
\end{aligned}
$$

$$\times \left|\psi\Big(\exp\Big(rX_1 + \sum_{j=2}^{n} Q_j(a,s)X_j\Big)\Big)\right|^2 dt\, da\, dr\, ds$$

(by substituting $r = t - k + Q_1(a,s)$ for k). Now let's use the change of variable $\sigma_j = Q_j(a,s)$, $j = 2, ..., n$, for fixed value of a. Note that, from (11) this change of variable has Jacobian 1. We then obtain,

$$\infty > \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \Big(1 + |t - r + R(a,\sigma)|^2\Big)\left|f\Big(\exp\Big(tX_1 + \sum_{j=2}^{n} a_jX_j\Big)\Big)\right|^2$$
$$\times \left|\psi\Big(\exp\Big(rX_1 + \sum_{j=2}^{n} \sigma_jX_j\Big)\Big)\right|^2 dt\, da\, dr\, d\sigma,$$

where $R(a,\sigma)$ is a polynomial function depending on $a = (a_2, ..., a_n)$ and $\sigma = (\sigma_2, ..., \sigma_n)$. Therefore,

$$\left|\psi\Big(\exp\Big(rX_1 + \sum_{j=2}^{n} \sigma_jX_j\Big)\Big)\right|^2 \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \Big(1 + |t - r + R(a,\sigma)|^2\Big)$$
$$\times \left|f\Big(\exp\Big(tX_1 + \sum_{j=2}^{n} a_jX_j\Big)\Big)\right|^2 dt\, da < \infty,$$

for almost all $r, \sigma_2, ..., \sigma_n \in \mathbb{R}$. As ψ is non identically zero, there exists $r_0, \sigma_0 = (\sigma_2^0, ..., \sigma_n^0)$ such that

$$\psi\Big(\exp\Big(r_0X_1 + \sum_{j=2}^{n} \sigma_j^0X_j\Big)\Big) \neq 0,$$

and

$$\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \Big(1 + |t - r_0 + R(a,\sigma_0)|^2\Big)\left|f\Big(\exp\Big(tX_1 + \sum_{j=2}^{n} a_jX_j\Big)\Big)\right|^2 dt\, da < \infty. \tag{12}$$

On the other hand, we have

$$I(f_\psi^{k,s}) = \int_{\mathbb{R}^{n-1}}\bigg(\int_{\mathbb{R}}\left|f\Big(\exp\Big(tX_1 + \sum_{j=2}^{n} a_jX_j\Big)\Big)\overline{\psi}\right.$$
$$\left.\Big(g^{-1}\exp\Big(tX_1 + \sum_{j=2}^{n} a_jX_j\Big)\Big)\right|dt\bigg)^2 da$$

$$= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \left| f\left(\exp\left(tX_1 + \sum_{j=2}^{n} a_j X_j \right) \right) \right| \right.$$
$$\left. \times \left| \psi\left(\exp\left((t - k + Q_1(a,s))X_1 + \sum_{j=2}^{n} Q_j(a,s)X_j \right) \right) \right| dt \right)^2 da$$

(using (10))

$$\leq \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\left| \psi\left(\exp\left((t - k + Q_1(a,s))X_1 + \sum_{j=2}^{n} Q_j(a,s)X_j \right) \right) \right|^2}{1 + \left| t - r_0 + R(a,\sigma_0) \right|^2} dt \right)$$
$$\times \left(\int_{\mathbb{R}} \left(1 + \left| t - r_0 + R(a,\sigma_0) \right|^2 \right) \left| f\left(\exp\left(tX_1 + \sum_{j=2}^{n} a_j X_j \right) \right) \right|^2 dt \right) da$$

(using Cauchy–Shwartz inequality),

$$\leq \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \left| \psi\left(\exp\left((t - k + Q_1(a,s))X_1 + \sum_{j=2}^{n} Q_j(a,s)X_j \right) \right) \right|^2 dt \right)$$
$$\times \left(\int_{\mathbb{R}} \left(1 + \left| t - r_0 + R(a,\sigma_0) \right|^2 \right) \left| f\left(\exp\left(tX_1 + \sum_{j=2}^{n} a_j X_j \right) \right) \right|^2 dt \right) da$$
$$= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \left| \psi\left(\exp\left(zX_1 + \sum_{j=2}^{n} Q_j(a,s)X_j \right) \right) \right|^2 dz \right)$$
$$\times \left(\int_{\mathbb{R}} \left(1 + \left| t - r_0 + R(a,\sigma_0) \right|^2 \right) \left| f\left(\exp\left(tX_1 + \sum_{j=2}^{n} a_j X_j \right) \right) \right|^2 dt \right) da$$

(using the change of variable $z = t - k + Q_1(a,s)$). By substituting $Q_j(a,s)$ for s_j, $j = 2, ..., n$, using Eq. (11), we have

$$\int_{\mathbb{R}^{n-1}} I(f_\psi^{k,s})\, ds \leq \|\psi\|_2^2 \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \left(1 + \left| t - r_0 + R(a,\sigma_0) \right|^2 \right)$$
$$\left| f\left(\exp\left(tX_1 + \sum_{j=2}^{n} a_j X_j \right) \right) \right|^2 dt\, da,$$

which is finite by (12). This implies that, $I(f_\psi^{k,s})$ is finite for all $k \in \mathbb{R}$ and almost all $s \in \mathbb{R}^{n-1}$. □

Before stating the next lemma, we need a localized version of the Plancherel measure (see [4]). Let $Z = \exp \mathfrak{z}$ be the center of G and fix a nonzero vector X_1 of $\mathfrak{z}$. Let $A = \exp \mathfrak{a} = \exp \mathbb{R}X_1$ be the closed connected subgroup of Z and $\chi = \chi_\zeta$,

$\zeta = l_1 X_1^* \in \mathfrak{a}^*$, be the unitary character of A, defined by

$$\chi_\zeta(\exp tX_1) = \mathrm{e}^{-2i\pi t l_1}.$$

Let $\hat{G}_\chi = \{\pi \in \hat{G} \ : \ \pi_{|A} = \chi \cdot I\}$. For $1 \leq p < +\infty$, let $L^p(G/A, \zeta)$ be the set of all measurable functions $\varphi : G \to \mathbb{C}$ such that $\varphi(xa) = \overline{\chi}(a)\varphi(x)$ for almost all $x \in G$ and $a \in A$ and such that

$$\|\varphi\|^p_{L^p(G/A)} = \int_{G/A} |\varphi(x)|^p d\dot{x} < +\infty.$$

Moreover, let $\mathfrak{g}^*_\zeta = \zeta + \mathfrak{a}^\perp$ and $\mathcal{W}_\zeta = \mathcal{W} \cap \mathfrak{g}^*_\zeta$. In this case, the Plancherel formula reads: if

$$\pi(\varphi) = \int_{G/A} \varphi(x)\pi(x)d\dot{x}, \ \pi \in \hat{G}_\chi,$$

then, for $\varphi \in L^1(G/A, \zeta) \cap L^2(G/A, \zeta)$ we have:

$$\|\varphi\|_2 = \Big(\int_{\mathcal{W}_\zeta} \|\pi_l(\varphi)\|^2_{HS} |Pf(l)| dl\Big)^{\frac{1}{2}}. \tag{13}$$

If d is the maximal dimension of coadjoint orbits in $\mathfrak{g}^*$, then T has $n-d$ elements and thus V_T can be identified with $\mathbb{R}^{n-d}$. We can identify V_T with $\mathbb{R}X_1^* \oplus \mathbb{R}^{n-d-1}$. We denote by

$$p_* : V_T \to \mathbb{R}X_1^*, \quad l \mapsto l_1 X_1^*$$

the canonical projection. As $\mathcal{W}$ is a Zariski open set of V_T, $p_*(\mathcal{W})$ is also a nonempty Zariski open set of $\mathbb{R}$. Then it will be convenient to write elements $l \in \mathcal{W}$, as (l_1, l') where $l_1 \in \ p_*(\mathcal{W})$ and $l' \in \mathcal{W}_{l_1} = \{l' \in \mathbb{R}^{n-d-1} \ : \ (l_1, l') \in \mathcal{W}\}$. It turns out that $\mathcal{W}_{l_1}$ is also a Zariski open set of $\mathbb{R}^{n-d-1}$ for each $l_1 \in p_*(\mathcal{W})$. The set $\mathcal{W}_{l_1}$ corresponds obviously to the cross-section $\mathcal{W}_\zeta$ used in the localized version of the Plancherel formula in (13). On the other hand, we obtain a decomposition of the Plancherel measure: for a function $F \in C_c(\mathcal{W})$, we have

$$\int_{\mathcal{W}} F(l)dl = \int_{\mathbb{R}} \int_{\mathcal{W}_{l_1}} F(l)dl' \, dl_1, \tag{14}$$

where the measure dl' is induced on $\mathcal{W}_{l_1}$ by the Lebesque measure on $\mathcal{W}$.

Lemma 3 *Let $f, \psi \in L^2(G)$ satisfying condition (2) of Theorem 3 and $\gamma \in]0, \beta[$. Then there exists $c > 0$, such that*

$$\int_{\mathbb{R}^{n-1}} \left|\widehat{(f_\psi^{k,s})_a}(l_1)\right|^2 da \leq c \exp(-\pi(\alpha k^2 + \alpha\|s\|^2 + \gamma l_1^2)),$$

for all $k, l_1 \in \mathbb{R}$ and almost all $s \in \mathbb{R}^{n-1}$.

Proof Let $h(k,s)$ be the function defined on $\mathbb{R}$ by

$$h(k,s)(\lambda) = \int_{\mathbb{R}^{n-1}} ((f_\psi^{k,s})_a * (f_\psi^{k,s})_a^\star)(\lambda) da,$$

where $\lambda \in \mathbb{R}$ and $(f_\psi^{k,s})_a^\star(\lambda) = \overline{(f_\psi^{k,s})_a(-\lambda)}$. Then, $h(k,s) \in L^1(\mathbb{R})$, for all $k \in \mathbb{R}$ and almost all $s \in \mathbb{R}^{n-1}$. In fact, from Lemma 2

$$\begin{aligned}\int_{\mathbb{R}} |h(k,s)(\lambda)| d\lambda &\le \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \left|(f_\psi^{k,s})_a(t)\right| \left|(f_\psi^{k,s})_a(t-\lambda)\right| dt\, da\, d\lambda \\ &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \left|(f_\psi^{k,s})_a(t)\right| dt \right)^2 da < \infty, \end{aligned} \tag{15}$$

for all $k \in \mathbb{R}$ and almost all $s \in \mathbb{R}^{n-1}$. Thus,

$$\widehat{h(k,s)}(l_1) = \int_{\mathbb{R}^{n-1}} \left|\widehat{(f_\psi^{k,s})_a}(l_1)\right|^2 da. \tag{16}$$

Identifying $A = \exp \mathbb{R} X_1$ with $\mathbb{R}$. Following the idea of Kaniuth and Kumar [14], for $u \in L^1(A) \cap L^2(A)$ define $u * f_\psi^{k,s}$ on G by

$$u * f_\psi^{k,s}(x) = \int_A u(z) f_\psi^{k,s}(z^{-1}x)\, dz$$

and then $h(k,s)_u : \mathbb{R} \to \mathbb{C}$ by

$$h(k,s)_u(t) = \int_{\mathbb{R}^{n-1}} \left((u * f_\psi^{k,s})_a * \left((u * f_\psi^{k,s})_a \right)^\star \right)(t)\, da.$$

It is not hard to see that

$$h(k,s)_u(t) = \int_{\mathbb{R}^{n-1}} \left((u * (f_\psi^{k,s})_a) * (u * (f_\psi^{k,s})_a)^\star \right)(t)\, da.$$

Therefore, for every $\eta_1 \in \mathbb{R}$

$$\begin{aligned}\widehat{h(k,s)_u}(\eta_1) &= \int_{\mathbb{R}^{n-1}} \left| \widehat{\left(u * (f_\psi^{k,s})_a\right)}(\eta_1) \right|^2 da \\ &= |\hat{u}(\eta_1)|^2 \int_{\mathbb{R}^{n-1}} \left| \widehat{(f_\psi^{k,s})_a}(\eta_1) \right|^2 da = |\hat{u}(\eta_1)|^2 \widehat{h(k,s)}(\eta_1) \end{aligned} \tag{17}$$

(using Eq. (16)). Hence,

$$\int_{\mathbb{R}} \left|\widehat{h(k,s)_u}(\eta_1)\right| d\eta_1 = \int_{\mathbb{R}} |\hat{u}(\eta_1)|^2 \left|\widehat{h(k,s)}(\eta_1)\right| d\eta_1$$

$$\leq \int_{\mathbb{R}} |\hat{u}(\eta_1)|^2 \|h(k,s)\|_1 d\eta_1 = \|u\|_2^2 \|h(k,s)\|_1,$$

which is finite by (15). By the inversion formula for $\mathbb{R}$, we have

$$\begin{aligned}\int_{\mathbb{R}} \widehat{h(k,s)_u}(\eta_1)\, d\eta_1 &= h(k,s)_u(0)\\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \left| u * (f_\psi^{k,s})_a(t)\right|^2 dt\, da = \|u * f_\psi^{k,s}\|_2^2. \end{aligned} \tag{18}$$

Now fix $l_1 \in \mathbb{R}$ and let $u_m \in L^1(A)$, $m \in \mathbb{N}^*$ such that $\widehat{u_m}(\eta_1) = 1$ for $\eta_1 \in V_m(l_1) = [l_1 - (1/2m), l_1 + (1/2m)]$ and $\widehat{u_m}(\eta_1) = 0$ on the complement of $V_m(l_1)$. Noting that, u_m is also in $L^2(A)$. Indeed, since $\widehat{u_m} \in L^1(A) \cap L^2(A)$,

$$\|u_m\|_2 = \|\check{u}_m\|_2 = \left\|\widehat{\widehat{u_m}}\right\|_2 = \|\widehat{u_m}\|_2 < \infty,$$

where $\check{u}_m(z) = u_m(-z)$.
As $\widehat{h(k,s)}$ is continuous and $V_m(l_1)$ has length $1/m$, we have: for all $k \in \mathbb{R}$ and almost all $s \in \mathbb{R}^{n-1}$,

$$\begin{aligned}\widehat{h(k,s)}(l_1) &= \lim_{m\to\infty} m \int_{V_m(l_1)} \widehat{h(k,s)}(\eta_1)\, d\eta_1 = \lim_{m\to\infty} m \int_{\mathbb{R}} \widehat{h(k,s)}(\eta_1) \mathbf{1}_{V_m(l_1)}\, d\eta_1\\ &= \lim_{m\to\infty} m \int_{\mathbb{R}} \widehat{h(k,s)_{u_m}}(\eta_1)\, d\eta_1 \qquad \text{(using Eq.(17))}\\ &= \lim_{m\to\infty} m \|u_m * f_\psi^{k,s}\|_2^2 \qquad \text{(using Eq.(18))}\\ &= \lim_{m\to\infty} m \int_{\mathbb{R}} \int_{\mathcal{W}_{\eta_1}} |Pf(\eta)| \|\pi_\eta(u_m * f_\psi^{k,s})\|_{HS}^2\, d\eta'\, d\eta_1\\ &\qquad\qquad \text{(using Eqs.(5) and (14))}\\ &= \lim_{m\to\infty} m \int_{\mathbb{R}} |\widehat{u_m}(\eta_1)|^2 \left(\int_{\mathcal{W}_{\eta_1}} |Pf(\eta)| \|\pi_\eta(f_\psi^{k,s})\|_{HS}^2\, d\eta'\right) d\eta_1\\ &= \lim_{m\to\infty} m \int_{V_m(l_1)} I\eta_1\, d\eta_1,\end{aligned}$$

where $\eta = (\eta_1, \eta')$ and $I\eta_1 = \displaystyle\int_{\mathcal{W}_{\eta_1}} |Pf(\eta)| \|\pi_\eta(f_\psi^{k,s})\|_{HS}^2\, d\eta'$.
Now by (2) and (8), we obtain

$$I\eta_1 \leq C \int_{\mathcal{W}_{\eta_1}} |Pf(\eta)| \exp(-\pi(\alpha k^2 + \alpha\|s\|^2 + \beta\|\eta\|^2)) d\eta'.$$

Since $Pf(\eta)$ is a polynominal function of η, there exists $R > 0$ such that

$$|Pf(\eta)| \exp(-\pi(\beta - \gamma)\|\eta\|^2) \leq 1$$

for all $\eta \in \mathfrak{g}^*$ with $\|\eta\| \geq R$. Let $K \geq 1$ such that $|Pf(\eta)| \leq K$ for all $\eta \in \mathfrak{g}^*$ with $\|\eta\| \leq R$. It follows that,

$$
\begin{aligned}
I\eta_1 &\leq CK\exp(-\pi(\alpha k^2 + \alpha\|s\|^2)) \int_{\mathcal{W}_{\eta_1}} \exp\left(-\pi\gamma\|\eta\|^2\right)d\eta' \\
&= CK\exp(-\pi(\alpha k^2 + \alpha\|s\|^2)) \int_{\mathbb{R}^{n-d-1}} \exp\left(-\pi\gamma\|(\eta_1, \eta')\|^2\right)d\eta' \\
&= CK\exp(-\pi(\alpha k^2 + \alpha\|s\|^2 + \gamma\eta_1^2)) \int_{\mathbb{R}^{n-d-1}} \exp\left(-\pi\gamma\|\eta'\|^2\right)d\eta' \\
&= c\exp(-\pi(\alpha k^2 + \alpha\|s\|^2 + \gamma\eta_1^2)),
\end{aligned}
$$

for some $c > 0$. Therefore,

$$
\begin{aligned}
\widehat{h(k,s)}(l_1) &\leq c \lim_{m\to\infty} m \int_{V_m(l_1)} \exp(-\pi(\alpha k^2 + \alpha\|s\|^2 + \gamma\eta_1^2))d\eta_1 \\
&= c\exp(-\pi(\alpha k^2 + \alpha\|s\|^2 + \gamma l_1^2)).
\end{aligned}
$$

Finally, Eq. (16) allows us to conclude. □

4 Proof of Theorem 3

For $a = (a_2, ..., a_n)$, $s = (s_2, ..., s_n) \in \mathbb{R}^{n-1}$, let $f_{a,s}$,$\psi_{a,s}$ be the complex-valued functions defined on $\mathbb{R}$ by

$$
\begin{aligned}
f_{a,s}(t) &= f\Big(\exp\Big(\big(t - Q_1(a,s)\big)X_1 + \sum_{j=2}^{n} a_j X_j\Big)\Big), \\
\text{and} \ \ \psi_{a,s}(t) &= \psi\Big(\exp\Big(tX_1 + \sum_{j=2}^{n} Q_j(a,s)X_j\Big)\Big),
\end{aligned}
$$

where the polynomial functions Q_j are defined as in (10). Then obviously $f_{a,s}$, $\psi_{a,s} \in L^2(\mathbb{R})$, for almost all $a \in \mathbb{R}^{n-1}$ and all $s \in \mathbb{R}^{n-1}$. For fixed λ, $y \in \mathbb{R}$, let $F_{\lambda,y}$ and $K_{\lambda,y}$ be the functions defined on $\mathbb{R} \times \mathbb{R}$ by

$$
F_{\lambda,y}(k, l_1) = \mathcal{G}_{\psi_{a,s}}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(k, l_1)\mathcal{G}_{\psi_{a,s}}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(-k, -l_1)e^{2i\pi k l_1},
$$

and

$$
K_{\lambda,y}(k, l_1) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} F_{\lambda,y}(k, l_1)\phi(a, s)da\, ds,
$$

where $\phi \in \mathcal{S}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, the Schwartz space of $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. Now for fixed $\mu \in \mathbb{R}$, let $R_{\lambda,y,\mu}$ be the function defined on $\mathbb{R}$ by

$$R_{\lambda,y,\mu}(k) = K_{\lambda,y}(k,.\widehat{)}(\mu), \tag{19}$$

where $K_{\lambda,y}(k,.\widehat{)}$ is the partial Fourier transform of $K_{\lambda,y}$ with respect the second variable l_1. It follows, using Lemma 1, that

$$\begin{aligned}\widehat{R}_{\lambda,y,\mu}(w) = \widehat{K}_{\lambda,y}(w,\mu) &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \hat{F}_{\lambda,y}(w,\mu)\phi(a,s)da\,ds\\ &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} F_{\lambda,y}(-\mu,w)\phi(a,s)da\,ds = K_{\lambda,y}(-\mu,w).\end{aligned} \tag{20}$$

Lemma 4 *There exists a positive constant C_1 such that*

$$\left|R_{\lambda,y,\mu}(k)\right| \leq C_1\, e^{-\alpha\pi k^2}.$$

Moreover, the constant C_1 does not depend on λ, μ and y.

Proof From Eq. (19) we have,

$$\begin{aligned}&|R_{\lambda,y,\mu}(k)|\\ &\quad = |K_{\lambda,y}(k,.\widehat{)}(\mu)|\\ &\quad \leq \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \left|\mathcal{G}_{\psi_a,s}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(k,l_1)\right|\\ &\qquad \left|\mathcal{G}_{\psi_a,s}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(-k,-l_1)\right|\left|\phi(a,s)\right|da\,ds\,dl_1\\ &\quad \leq \text{cst}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \left|\mathcal{G}_{\psi_a,s}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(k,l_1)\right|\\ &\qquad \left|\mathcal{G}_{\psi_a,s}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(-k,-l_1)\right|da\,ds\,dl_1.\end{aligned}$$

By using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}|R_{\lambda,y,\mu}(k)| \leq \text{cst}&\left(\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}^{n-1}} \left|\mathcal{G}_{\psi_a,s}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(k,l_1)\right|^2 da\right)ds\,dl_1\right)^{\frac{1}{2}}\\ &\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}^{n-1}} \left|\mathcal{G}_{\psi_a,s}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(-k,-l_1)\right|^2 da\right)ds\,dl_1\right)^{\frac{1}{2}}.\end{aligned}$$

Remark that,

$$\left|\mathcal{G}_{\psi_{a,s}}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(k,l_1)\right| = \left|\mathcal{G}_{\psi_{a,s}} f_{a,s}(k-y,l_1-\lambda)\right|$$

(using i) in Lemma 1)

$$= \left| \int_{\mathbb{R}} f_{a,s}(t)\overline{\psi_{a,s}}(t-k+y)\mathrm{e}^{-2i\pi t(l_1-\lambda)}dt \right|$$

$$= \left| \int_{\mathbb{R}} f\left(\exp\left((t-Q_1(a,s))\right)X_1 + \sum_{j=2}^{n} a_j X_j \right) \right.$$
$$\left. \times\, \overline{\psi}\left(\exp\left((t-k+y)X_1 + \sum_{j=2}^{n} Q_j(a,s)X_j \right)\right) \mathrm{e}^{-2i\pi t(l_1-\lambda)}dt \right|$$

$$= \left| \int_{\mathbb{R}} f\left(\exp\left(rX_1 + \sum_{j=2}^{n} a_j X_j \right)\right) \right.$$
$$\left. \times\overline{\psi}\left(\exp\left((r-(k-y)+Q_1(a,s))X_1 + \sum_{j=2}^{n} Q_j(a,s)X_j \right)\right) \mathrm{e}^{-2i\pi r(l_1-\lambda)}dr \right|$$

(by substituting $r = t - Q_1(a,s)$ for t)

$$= \left| \int_{\mathbb{R}} f\left(\exp\left(rX_1 + \sum_{j=2}^{n} a_j X_j \right)\right) \right.$$
$$\left. \times\overline{\psi}\left(\exp\left((k-y)X_1 + \sum_{j=1}^{n} s_j X_j \right)^{-1} \exp\left(rX_1 + \sum_{j=2}^{n} a_j X_j \right)\right) \mathrm{e}^{-2i\pi r(l_1-\lambda)}dr \right|$$

(using Eq. (10))

$$= \left| \int_{\mathbb{R}} f_{\psi}^{k-y,s}\left(\exp\left(rX_1 + \sum_{j=2}^{n} a_j X_j \right)\right) \mathrm{e}^{-2i\pi r(l_1-\lambda)}dr \right| = \left| (\widehat{f_{\psi}^{k-y,s}})_a(l_1-\lambda) \right|.$$

It results that,

$$\left|R_{\lambda,y,\mu}(k)\right| \le \mathrm{cst}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^{n-1}} \left|(\widehat{f_{\psi}^{k-y,s}})_a(l_1-\lambda)\right|^2 da \right) ds\, dl_1 \right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \left|(\widehat{f_{\psi}^{-k-y,s}})_a(-l_1-\lambda)\right|^2 da\, ds\, dl_1 \right)^{\frac{1}{2}}$$
$$\le \mathrm{cst}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{-\pi(\alpha|k-y|^2+\gamma(l_1-\lambda)^2+\alpha\|s\|^2)} ds\, dl_1 \right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{-\pi(\alpha|k+y|^2+\gamma(l_1+\lambda)^2+\alpha\|s\|^2)} ds\, dl_1 \right)^{\frac{1}{2}} \quad \text{(using Lemma 3)}$$

$$\leq \text{cst}\, e^{-\frac{\pi}{2}\alpha(|k-y|^2+|k+y|^2)} \left(\int_{\mathbb{R}^{n-1}} e^{-\pi\alpha\|s\|^2} ds\right)\left(\int_{\mathbb{R}} e^{-\pi\gamma l_1^2} dl_1\right)$$
$$\leq \text{cst}\, e^{-\pi\alpha k^2},$$

which is the desired result. □

Lemma 5 *There exists a positive constant C_2 such that*

$$\left|\hat{R}_{\lambda,y,\mu}(w)\right| \leq C_2\, e^{-\pi\gamma w^2}.$$

Moreover, the constant C_2 does not depend on λ, y and μ.

Proof By (20), we have

$$\left|\hat{R}_{\lambda,y,\mu}(w)\right| = \left|K_{\lambda,y}(-\mu, w)\right|$$
$$\leq \text{cst} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \left|\mathcal{G}_{\psi_{a,s}}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(-\mu, w)\right|\left|\mathcal{G}_{\psi_{a,s}}(\mathcal{M}_\lambda \mathcal{T}_y f_{a,s})(\mu, -w)\right| da ds.$$

As in the proof of the Lemma 4 we can show that,

$$\left|\hat{R}_{\lambda,y,\mu}(w)\right| \leq \text{cst}\, e^{-\frac{\pi\gamma}{2}(|w-\lambda|^2+|w+\lambda|^2)} e^{\frac{-\pi a}{2}(|\mu+y|^2+|-\mu+y|^2)} \times \left(\int_{\mathbb{R}^{n-1}} e^{-\pi\alpha\|s\|^2} ds\right)$$
$$\leq \text{cst}\, e^{-\pi\gamma w^2},$$

which is the desired result. □

We have shown finally that $R_{\lambda,y,\mu}$ verifies the decay conditions of Hardy theorem on $\mathbb{R}$. Since $\alpha\beta > 1$, we can choose $0 < \gamma < \beta$ such that $\alpha\gamma > 1$. We conclude that $R_{\lambda,y,\mu} = 0$ a.e. and $\hat{R}_{\lambda,y,\mu} = 0$ for all $\lambda, y, \mu \in \mathbb{R}$. In (20), allowing ϕ to vary through the space of Schwartz functions on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, we obtain $F_{\lambda,y}(-\mu, w) = 0$ for all λ, y, μ in $\mathbb{R}$ and almost all $w \in \mathbb{R}$. As $F_{-\lambda,-y}$ is continuous on $\mathbb{R} \times \mathbb{R}$,

$$|F_{-\lambda,-y}(0, 0)| = |G_{\psi_{a,s} f_{a,s}(y,\lambda)}|^2 = 0$$

(using i) in Lemma 1). Hence, $\mathcal{G}_{\psi_{a,s}} f_{a,s} = 0$ a.e. By using Eq. (4), we have

$$\|\psi_{a,s}\|_2^2 \|f_{a,s}\|_2^2 = 0,$$

which implies either $\psi_{a,s} = 0$ a.e. or $f_{a,s} = 0$ a.e. Observe that,

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \|f_{a,s}\|_2^2 \|\psi_{a,s}\|_2^2 da\, ds$$
$$= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \left|f\left(\exp\left((t - Q_1(a, s))\, X_1 + \sum_{j=2}^{n} a_j X_j\right)\right)\right|^2 dt\right)$$

$$\times \left(\int_{\mathbb{R}} \left| \psi \left(\exp \left(tX_1 + \sum_{j=2}^{n} Q_j(a,s) X_j \right) \right) \right|^2 dt \right) da\, ds = \|f\|_2^2 \|\psi\|_2^2$$

(by substituting $t - Q_1(a, s)$ for t and $Q_j(a, s)$ for s_j, $j = 2, ..., n$, using Eq. (11)). This allow us to achieve the proof.

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