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On Planar Sampling with Gaussian Kernel in Spaces of Bandlimited Functions

Ilya Zlotnikov[1]

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Abstract
Let $I = (a, b) \times (c, d) \subset \mathbb{R}^2_+$ be an index set and let $\{G_\alpha(x)\}_{\alpha\in I}$ be a collection of Gaussian functions, i.e. $G_\alpha(x) = \exp(-\alpha_1 x_1^2 - \alpha_2 x_2^2)$, where $\alpha = (\alpha_1, \alpha_2) \in I$, $x = (x_1, x_2) \in \mathbb{R}^2$. We present a complete description of the uniformly discrete sets $\Lambda \subset \mathbb{R}^2$ such that every bandlimited signal f admits a stable reconstruction from the samples $\{f * G_\alpha(\lambda)\}_{\lambda\in\Lambda}$.

Keywords Multi-dimensional sampling · Dynamical sampling · Paley–Wiener spaces · Bernstein spaces · Gaussian kernel · Hermite polynomials · Delone set

1 Introduction

The sampling problem deals with recovery of bandlimited signals f from the collection of measurements $\{f(\lambda)\}_{\lambda\in\Lambda}$ taken at the points of some uniformly discrete set $\Lambda \subset \mathbb{R}^d$. The classical results deal with one dimensional signals that are elements of the Paley-Wiener or Bernstein spaces over a fixed interval $[-\sigma, \sigma]$. The sets Λ that provide the stable reconstruction, in this case, are completely described. For the Bernstein spaces, the answer is given in terms of a certain density of Λ and bandwidth parameter σ, see [5]. The result for Paley-Wiener spaces is more complicated, see [16, 18]. It cannot be expressed in terms of a density of Λ. We refer the reader to [5, 18] for the detailed exposition and the proofs.

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✉ Ilya Zlotnikov
zlotnikk@rambler.ru

1 St. Petersburg Department of V.A. Steklov, Mathematical Institute of the Russian Academy of Sciences, 27 Fontanka, St., Petersburg, Russia 191023

The complexity of the task significantly increases in the multi-dimensional setting. Landau [11] proved that the necessary conditions for stable sampling remain valid for the Paley-Wiener spaces over any domain (see [13] for a much simpler proof). A sufficient condition for a sampling of signals from the Bernstein space with spectrum in a ball was obtained by Beurling, see [6]. We also refer the reader to [15] for some extensions. However, there is a gap between the necessary and sufficient conditions. Moreover, even for the simplest spectra as balls or cubes, examples show that no description of sampling sets is possible in terms of a density of Λ, see Sect. 5.7 in [14].

Recently the so-called dynamical sampling problem (in what follows, we will more often use the term space-time sampling problem) attracted a lot of attention, see [2–4, 19], and references therein.

In this paper, we consider the following variant of dynamical sampling problem.

Main Problem
Let Λ be a uniformly discrete subset of $\mathbb{R}^n$ and let $G_\alpha(x)$ be a collection of functions parametrized by $\alpha \in I$. What assumptions should be imposed on the spatial set Λ, index set I, and functions G_α to enable the recovery of every band-limited signal f from its space-time samples $\{f * G_\alpha(\lambda)\}_{\lambda\in\Lambda,\alpha\in I}$?

For signals f from a Paley-Wiener space PW_σ (see the definition below) it means that the inequalities

$$D_1\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} \int_I |f * G_\alpha(\lambda)|^2\, d\alpha \leq D_2\|f\|_2^2 \quad \text{for every } f \in PW_\sigma \tag{1}$$

are true with some constants D_1 and D_2. Here, as usual, $\|\cdot\|$ denotes the L^2-norm.

Recall that a set $\Lambda = \{\lambda_k\} \subset \mathbb{R}^n$ is called uniformly discrete[1] (u.d.) if

$$\delta(\Lambda) := \inf_{\substack{\lambda\neq\lambda' \\ \lambda,\lambda'\in\Lambda}} |\lambda - \lambda'| > 0.$$

The constant $\delta(\Lambda)$ is called the separation constant of Λ.

In the one-dimensional setting, this problem appears in particular in connection with tasks of mathematical physics. Several examples are presented in [4]. One of them is the initial value problem for the heat equation

$$\frac{\partial}{\partial\alpha}u(x,\alpha) = \sigma^2\frac{\partial^2 u}{\partial x^2}(x,\alpha), \qquad \sigma \neq 0,\ x \in \mathbb{R},\ \alpha > 0, \tag{2}$$

with initial condition

$$u(x,0) = f(x). \tag{3}$$

[1] Sometimes, the term uniformly separated is used.

It is well-known that the solution is given by the formula

$$u(x, \alpha) = f * g_\alpha(x) = \int\limits_{\mathbb{R}^n} g_\alpha(x - y) f(y) dy, \tag{4}$$

where $g_\alpha(x) = \frac{1}{\sqrt{(4\pi\alpha\sigma)}} \exp\left(-\frac{x^2}{4\alpha\sigma}\right)$. Note that Main Problem applied to equation (2) provides the reconstruction of initial function f from the states $\{u(\lambda, \alpha)\}_{\lambda\in\Lambda,\alpha\in I}$.

A variant of Main Problem for the one-dimensional setting was considered by Aldroubi et al. in [4]. In particular, it was established that unlike the classical sampling setting, the assumptions that should be imposed on the set Λ to solve the Main Problem cannot be expressed in terms of some density of Λ, see Example 4.1 in [4]. More precisely, one may construct a set with an arbitrarily small density that provides stable reconstruction of the initial signal. Also in that paper, it was shown that for the solution of Main Problem we have to require Λ to be relatively dense.

In the one-dimensional setting, for a large collection of kernels, a solution of Main Problem was presented in [19]: It turns out the stable recovery from the samples on Λ is possible if and only if Λ is not (in a certain sense) "close" to an arithmetic progression.

It seems natural to extend the results of [4, 19] to the multi-dimensional situation. Below we focus on the two-dimensional variant of the problem for the case of Gaussian kernel

$$G_\alpha(x) = e^{-\alpha_1 x_1^2 - \alpha_2 x_2^2}, \qquad \alpha = (\alpha_1, \alpha_2) \in I,$$
$$I = (a, b) \times (c, d) \subset \mathbb{R}^2_+, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Our approach is similar to the one in [19]. However, this problem is considerably more involved than the one in the one-dimensional setting. One needs to apply some additional ideas. See Sect. 5 for some remarks on cases dimension higher than 2.

We pass to the description of the geometry of the sets Λ that solve the planar Main problem.

Definition 1 A curvilinear lattice in $\mathbb{R}^2$ defined by three vectors

$$t = (t_1, t_2) \in \mathbb{R}^2, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \quad \text{and} \quad r = (r_1, r_2) \in \mathbb{R}^2,\ r_1^2 + r_2^2 = 1,$$

is the set of all vectors $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ satisfying

$$l_{t,\xi,r} := \{\lambda \in \mathbb{R}^2 \mid r_1 \cos(\lambda_1\xi_1 + \lambda_2\xi_2 + t_1) = r_2 \cos(-\lambda_1\xi_1 + \lambda_2\xi_2 + t_2)\}.$$

The blue curves on Fig. 1 correspond to the curvilinear lattice $l_{t,\xi,r}$ with $t = (0, 0)$, $\xi = (1, 1)$, and $r = (1/\sqrt{10}, 3/\sqrt{10})$.

In what follows the notation $W(\Lambda)$ stands for the collection of all weak limits of translates of a uniformly discrete set Λ, see the definition in Sect. 2.

Condition (A): A uniformly discrete set $\Lambda = \{\lambda = (\lambda_1, \lambda_2)\} \subset \mathbb{R}^2$ satisfies condition (A) if every set $\Lambda^* \in W(\Lambda)$ is not empty and does not lie on any lattice $l_{t,\xi,r}$.

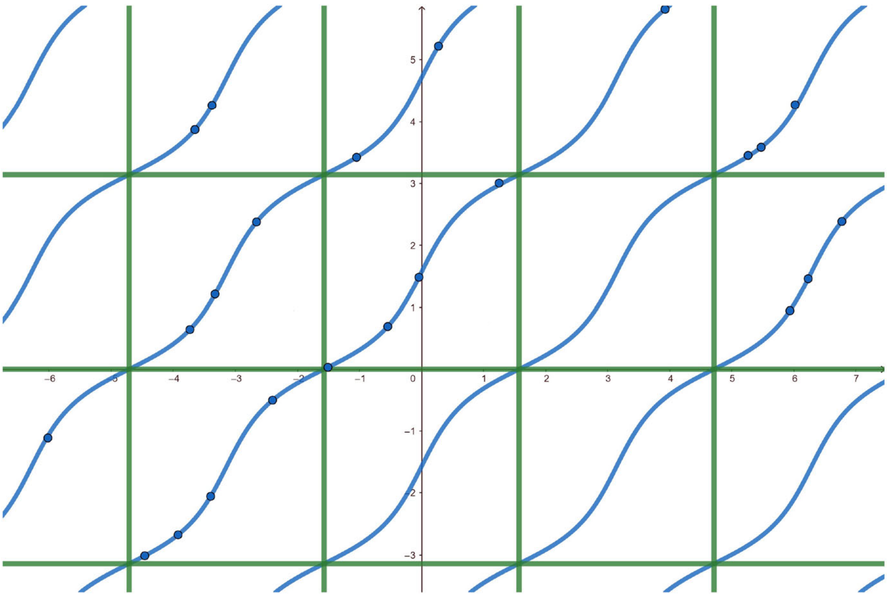

Fig. 1 Curvilinear lattice defined by $\cos(x+y) = 3\cos(y-x)$

Remark 1 A Delone set is a set that is both uniformly discrete and relatively dense. In particular, it is easy to check that every set that satisfies condition (A) is a Delone set.

We denote by PW_σ^2 the space of square integrable on $\mathbb{R}^2$ functions with spectrum supported in the square $[-\sigma, \sigma]^2$, i.e.

$$PW_\sigma^2 = \{f \in L^2(\mathbb{R}^2) \mid \operatorname{supp} \hat{f} \subset [-\sigma, \sigma]^2\},$$

where

$$\hat{f}(\xi_1, \xi_2) = \int\limits_{\mathbb{R}^2} e^{-i(\xi_1 x_1 + \xi_2 x_2)} f(x_1, x_2)\, dx_1 dx_2.$$

Now, we are ready to formulate the main result.

Theorem 1 *Given a u.d. set $\Lambda \subset \mathbb{R}^2$ and a rectangle $I = (a, b) \times (c, d)$ with $0 < a < b < \infty$, $0 < c < d < \infty$. The following statements are equivalent:*

(i) For every $\sigma > 0$ there are positive constants $D_1 = D_1(\sigma, I, \Lambda)$ and $D_2 = D_2(\sigma, I, \Lambda)$ such that (1) holds true.
(ii) Λ satisfies condition (A).

The paper is organized as follows. In Sect. 2 we give all necessary definitions and fix some notations. As it was mentioned above, we employ the approach from [19] and divide the solution into two parts. We start with solving Main Problem for the

Bernstein spaces B_σ and prove an analogue of Theorem 1 in Sect. 3. In Sect. 4 we investigate the connection between the sampling with Gaussian kernel in the Paley-Wiener and Bernstein spaces. We also prove the main result in Sect. 4. The remarks on multi-dimensional cases and some open problems that puzzle us are placed in Sect. 5.

2 Notations and Preliminaries

In the present paper, we deal with signals that belong to the Bernstein and Paley-Wiener spaces. Since we investigate Main Problem simultaneously for all bandwidth parameters, we may consider only the functions with the spectrum supported in squares. This leads us to

Definition 2 Given a positive number σ, we denote by B_σ the space of all entire functions f in $\mathbb{C}^2$ satisfying the estimate

$$|f(z)| \le Ce^{\sigma(|y_1|+|y_2|)}, \quad z=(z_1,z_2)\in\mathbb{C}^2, \quad z_j=x_j+iy_j\in\mathbb{C},\ j=1,2, \tag{5}$$

where the constant $C = C(f)$ depends only on f.

It is well-known that B_σ consists of the bounded continuous functions that are the inverse Fourier transforms of tempered distributions supported on the square $[-\sigma,\sigma]^2$. We refer the reader to [12] for more information about Bernstein spaces.

For $1 \le p < \infty$ we may define the Paley-Wiener spaces by the formula

$$PW_\sigma^p = B_\sigma \cap L^p(\mathbb{R}^2)$$

or equivalently

$$PW_\sigma^p = \{f \in L^p(\mathbb{R}^2) \mid \operatorname{supp} \hat{f} \subset [-\sigma,\sigma]^2\}.$$

Following [5] (see also Chapter 3.4 in [10, 14, 17]), we introduce auxiliary

Definition 3 Let $\{\Lambda_k\}$ and Λ be u.d. subsets of $\mathbb{R}^n$, satisfying $\delta(\Lambda_k) \ge \delta > 0,\ k \in \mathbb{N}$. We say that the sequence $\{\Lambda_k\}$ converges weakly to Λ if for every large $R > 0$ and small $\varepsilon > 0$ there exists such $N = N(R,\varepsilon)$ that

$$\Lambda_k \cap (-R,R)^n \subset \Lambda + (-\varepsilon,\varepsilon)^n,$$
$$\Lambda \cap (-R,R)^n \subset \Lambda_k + (-\varepsilon,\varepsilon)^n.$$

for all $k \ge N$.

Definition 4 By $W(\Lambda)$ we denote all weak limits of the translates $\Lambda_k := \Lambda - x_k$, where $\{x_k\} \subset \mathbb{R}^n$ is an arbitrarily bounded or unbounded sequence.

We supply these definitions with several examples concerning the condition (A).

Example 1 To construct the set that does not satisfy condition (A), one may consider the following perturbation of the rectangle lattice:

$$\Lambda = \left\{ \left(2\pi n + \frac{1}{2^{m^2+n^2}}, 2\pi m + \frac{1}{2^{|m|+|n|}} \right), \quad m, n \in \mathbb{Z} \right\}.$$

Taking any sequence $\{x_k\} \subset \mathbb{R}^2$ such that $|x_k| \to \infty$, one may check by the definition that the sequence $\Lambda - x_k$ weakly converges to the set $\Lambda = \{(2\pi n, 2\pi m), \quad m, n \in \mathbb{Z}\}$, which, clearly, lies in $l_{t,\xi,r}$ with $t = (0, 0)$, $\xi = (1, 1)$, and $r = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

The following example is inspired by the papers [10, 17], which considered a planar mobile sampling problems.

Example 2 Set

$$D_{\mathbb{Z}} = \left\{ (x, y) \subset \mathbb{R}^2 \quad | \quad x^2 + y^2 = 4\pi^2 k^2, \ k \in \mathbb{Z} \right\},$$

i.e. $D_{\mathbb{Z}}$ is a collection of the concentric equidistant circles with center $(0, 0)$. Now, one may consider Λ to be any u.d. set located on the circles $D_{\mathbb{Z}}$. One may check (see the proofs in [10, 17]) that any weak limit of translates for every unbounded sequence $\{x_n\}$ for $D_{\mathbb{Z}}$ lies on the parallel lines. The argument is based on the simple observation that the traces of translated circles in the rectangle $[-R, R]^2$ (for a fixed $R > 0$) are getting closer and closer to the parallel lines as the value $|x_n|$ increases. Moreover, the distance between these lines is $2\pi k$. For instance, one may take $x_n = (0, 2\pi n)$ and pass to a weak limit $\Lambda - x_n \to \Lambda^*$ to obtain that $\Lambda^* \subset l_{t,\xi,r}$ with $t = (0, 0)$, $\xi = (1, 1)$, and $r = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

The next example concerns the set that satisfy condition (A). In what follows, we skip some technical details.

Example 3 One may easily find a u.d. set Λ in $\mathbb{R}^2$ such that its projection P_Λ onto any rectangle $P = [0, a] \times [0, b]$ is dense in P, where

$$P_\Lambda = \{(\lambda_1 \bmod a, \quad \lambda_2 \bmod b) : \lambda = (\lambda_1, \lambda_2) \in \Lambda\}.$$

For instance, one may take a u.d. subset of $(\sqrt{2}\,\mathbb{Z} \cup \mathbb{Z}) \times (\sqrt{3}\,\mathbb{Z} \cup \mathbb{Z})$ (with a dense projection). Therefore, the set Λ and all its weak limits of translates do not lie on any curvilinear lattice, and Λ satisfy the condition (A).

Below we will use the simple fact that for every sequence x_k there is a subsequence x_{k_j} such that $\Lambda - x_{k_j}$ converges weakly.

Throughout this paper we will adopt the following notations:

- Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $n \in \mathbb{N}$. Define $|x| := \sqrt{x_1^2 + \cdots + x_n^2}$. Notation $x \cdot y$ stands for the scalar product of vectors x and y.
- Set $B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$, where $x \in \mathbb{R}^n$ and $r > 0$.

- Given $\lambda = (\lambda_1, \dots, \lambda_n)$ and $f \in L^\infty(\mathbb{R}^n)$, we set

$$f_\lambda(x) := f(x - \lambda) = f(x_1 - \lambda_1, \dots, x_n - \lambda_n).$$

- By $|A|$ we denote the n-th dimensional Lebesgue measure of a set $A \subset \mathbb{R}^n$.
- By C we denote different positive constants.

Basically, we will focus on two-dimensional case. It is convenient to fix the following notations.

- Given a point $x = (x_1, x_2) \in \mathbb{R}^2$, denote $\tilde{x} := (-x_1, x_2)$.
- A symmetrization operator S is defined by the formula

$$Sf(x) := f(x) + f(\tilde{x}) + f(-\tilde{x}) + f(-x), \quad f \in L^\infty(\mathbb{R}^2).$$

- Set $\mathbb{T} := \{|x| = 1,\ x \in \mathbb{R}^2\}$.

3 Sampling with Gaussian Kernel in Bernstein Spaces

An analogue of Theorem 1 for the Bernstein spaces is as follows:

Theorem 2 *Given a u.d. set $\Lambda \subset \mathbb{R}^2$ and $I = (a, b) \times (c, d)$ with $0 < a < b < \infty$, $0 < c < d < \infty$. The following statements are equivalent:*

(i) For every $\sigma > 0$ there is a constant $K = K(\sigma)$ such that

$$\|f\|_\infty \le K \sup_{\alpha \in I} \sup_{\lambda \in \Lambda} \|f * G_\alpha\|_\infty \quad \textit{for every } f \in B_\sigma.$$

(ii) Λ satisfies condition (A).

Above, as usual, $\|\cdot\|_\infty$ denotes the sup-norm

$$\|f\|_\infty := \sup_{x \in \mathbb{R}^2} |f(x)|.$$

3.1 Proof of Theorem 2, Part I

(ii) $\Rightarrow$ (i). In what follows we assume that (i) is not true. We have to show that (ii) fails, i.e. there is a set $\Lambda^* \in W(\Lambda)$ such that it lies on some curvilinear lattice. The proof is divided into 5 steps. For the convenience of the reader, we will briefly describe them here and then pass to the argument.

In Step 1, using the standard Beurling technique, we find $\Lambda^* \in W(\Lambda)$ and $g \in B_\sigma$ such that $g * G_\alpha$ vanishes on Λ^* for every $\alpha \in I$. Our next step is to show that $Sg_\lambda = 0$ for every $\lambda \in \Lambda^*$. In Step 3 we prove that Λ^* lies on some curvilinear lattice under the assumption that $g \in L^2(\mathbb{R}^2)$. In Steps 4 and 5, using some approximation technique, we show how to get rid of the requirement that g is square integrable.

1. Due to the assumption made one can find a sequence of Bernstein functions $f_n \in B_\sigma$ satisfying

$$\|f_n\|_\infty = 1, \quad \|f_n * G_\alpha|_\Lambda\|_\infty \le 1/n.$$

We may then introduce a sequence of functions

$$g_n(z) := f_n(z - x(n)), \quad z = (z_1, z_2), \quad x(n) = (x_1(n), x_2(n)),$$

where $x(n)$ are chosen so that $|f_n(x(n))| > 1 - \frac{1}{n}$, $n \in \mathbb{N}$. Then we have

$$\|g_n\|_\infty = 1 \quad \text{and} \quad \left\|g * G_\alpha|_{\Lambda + x(n)}\right\|_\infty \le 1/n, \; n \in \mathbb{N}.$$

Using the compactness property of Bernstein space (see, e.g., [14], Proposition 2.19), we may assume that sequence g_n converges (uniformly on compacts in $\mathbb{C}^2$) to some function $g \in B_\sigma$. Moreover, passing if necessary to a subsequence, we may assume that the translates $\Lambda + x(n)$ converge weakly to some u.d. set Λ^*. Of course, we may assume that Λ^* is non-empty. Otherwise, we have arrived at contradiction with condition (A). Clearly, g satisfies

$$\|g\|_\infty = 1, \text{ and for every } \alpha \in I: \quad g * G_\alpha|_{\Lambda^*} = 0, \quad \Lambda^* \in W(\Lambda). \tag{6}$$

For a point $z = (z_1, z_2) \in \mathbb{C}^2$ we consider its complex conjugate point $\bar{z} = (\bar{z_1}, \bar{z_2})$. Consider the decomposition $g(z) = \varphi(z) + i\psi(z)$, where

$$\varphi(z) := \frac{g(z) + \overline{g(\bar{z})}}{2}, \quad \psi(z) := \frac{g(z) - \overline{g(\bar{z})}}{2i}.$$

Then φ and ψ are real (on $\mathbb{R}^2$) entire functions satisfying (5). Thereby, functions φ and ψ belong to $\mathcal{B}_\sigma$, and since the kernel G_α takes only real values on $\mathbb{R}^2$, we have $(\varphi * G_\alpha)(\lambda) = 0$ and $(\psi * G_\alpha)(\lambda) = 0$ for every $\lambda \in \Lambda^*$. Thus, we can continue the argument assuming that g is a real-valued function.

2. Recall that the notations $\tilde{x}$ and Sf were introduced in Sect. 2.

Lemma 1 *Assume a function $g \in B_\sigma$ satisfies (6). Then for every $\lambda \in \Lambda^*$ the equality*

$$Sg_\lambda(x) = 0 \tag{7}$$

holds for a.e. $x \in \mathbb{R}^2$.

Proof Without loss of generality, we may assume that $\lambda = (0, 0)$ and $I = \left(\frac{1}{2}, 1\right)^2$. Observe that

$$(Sg * G_\alpha)(0, 0) = 4(g * G_\alpha)(0, 0) = 0 \quad \text{for every } \alpha \in I. \tag{8}$$

Set

$$h(x_1,x_2):=Sg(x_1,x_2)\exp\left\{-\frac{x_1^2+x_2^2}{4}\right\} \quad \text{and} \quad I_+:=\left(\frac{1}{2},\frac{3}{4}\right)^2.$$

Clearly, $h\in L^2(\mathbb{R}^2)$ and it is even in variables x_1 and x_2. Moreover, using (8), one can check that $(h*G_\alpha)(0,0)=0$ for any $\alpha\in I_+$.

For every multi-index $m=(m_1,m_2)\in\mathbb{N}^2$ and $u\in I_+$ we have

$$\frac{\partial^m}{\partial u_1^{m_1}\partial u_2^{m_2}}\int\limits_{\mathbb{R}^2} h(x_1,x_2)\exp\left\{-u_1x_1^2-u_2x_2^2\right\}dx_1dx_2=0.$$

In particular, h is orthogonal to every monomial $x_1^{\alpha_1}x_2^{\alpha_2}$ with even indexes α_1 and α_2 in the weighted space $L^2\left(\mathbb{R}^2,\exp\left\{-\frac{1}{2}(x_1^2+x_2^2)\right\}\right)$. Moreover, since h is even in any variable, from the symmetry, we see that h is orthogonal to every polynomial in this space. To finish the proof we use the completeness property of multi-dimensional analogues of Hermite polynomials. More precisely, we invoke Theorem 3.2.18 from [7] to deduce $h=0$. Consequently, $Sg(x)=0$ for every $x\in\mathbb{R}^2$, and the lemma follows. □

3. We will need a simple technical

Lemma 2 *Given a function $F\in L^2(\mathbb{R}^2)$ such that its inverse Fourier transform f is a real function. Then*

$$Sf_\lambda(x)=2\int\limits_{\mathbb{R}^2}\cos(x\cdot t)\,\mathrm{Re}\left(e^{i\lambda\cdot t}F(t)+e^{i\tilde{\lambda}\cdot t}F(\tilde{t})\right)dt.$$

Proof Indeed, we may write

$$f(x)=\mathrm{Re}\int\limits_{\mathbb{R}^2}e^{ix\cdot t}F(t)dt.$$

Therefore,

$$\begin{aligned} Sf_\lambda(x)&=\mathrm{Re}\int\limits_{\mathbb{R}^2}\left(e^{ix\cdot t}+e^{i\tilde{x}\cdot t}+e^{-ix\cdot t}+e^{-i\tilde{x}\cdot t}\right)e^{i\lambda\cdot t}F(t)dt\\ &=2\mathrm{Re}\int\limits_{\mathbb{R}^2}(\cos(x\cdot t)+\cos(\tilde{x}\cdot t))e^{i\lambda\cdot t}F(t)dt\\ &=2\mathrm{Re}\int\limits_{\mathbb{R}^2}\cos(x\cdot t)\left(e^{i\lambda\cdot t}F(t)+e^{i\tilde{\lambda}\cdot t}F(\tilde{t})\right)dt, \end{aligned}$$

which proves the lemma. □

3. If we additionally assume that $g \in L^2(\mathbb{R}^2)$, the result follows from the next statement.

Lemma 3 *Assume $g \in L^2(\mathbb{R}^2)$. Then Λ^* lies on some curvilinear lattice.*

Proof Denote by G the inverse Fourier transform of g. Recall that g is real, whence

$$G(t) = \overline{G(-t)} \quad \text{and} \quad G(\tilde{t}) = \overline{G(-\tilde{t})}.$$

Denote by $U(t) = \mathrm{Re}\left(e^{i\lambda\cdot t}G(t) + e^{i\tilde{\lambda}\cdot t}G(\tilde{t})\right)$. Since

$$\begin{aligned} 2U(t) &= 2\mathrm{Re}\left(e^{i\lambda\cdot t}G(t) + e^{i\tilde{\lambda}\cdot t}G(\tilde{t})\right) \\ &= \left(e^{i\lambda\cdot t}G(t) + e^{i\tilde{\lambda}\cdot t}G(\tilde{t})\right) + \overline{\left(e^{i\lambda\cdot t}G(t) + e^{i\tilde{\lambda}\cdot t}G(\tilde{t})\right)} \\ &= e^{i\lambda\cdot t}G(t) + e^{i\tilde{\lambda}\cdot t}G(\tilde{t}) + e^{-i\lambda\cdot t}G(-t) + e^{-i\tilde{\lambda}\cdot t}G(-\tilde{t}), \end{aligned}$$

we deduce that $U(t) = U(-t)$. Combining this observation with Lemmas 1 and 2, we see that equality

$$\mathrm{Re}\left(e^{i\lambda\cdot t}G(t) + e^{i\tilde{\lambda}\cdot t}G(\tilde{t})\right) = 0$$

holds for a.e. $t \in \mathbb{R}^2$ and for every $\lambda \in \Lambda^*$.

Recall that $G = 0$ a.e. outside $(-\sigma, \sigma)^2$. For every $\epsilon > 0$, find a real Schwartz function F_ϵ whose support lies on $[-\sigma, \sigma]^2$ satisfying $\|G - F_\epsilon\|_2 < \epsilon$. Then

$$\|F_\epsilon\|_2 \geq \|G\|_2 - \epsilon \tag{9}$$

and

$$\left(\int_{\mathbb{R}^2} \left|\mathrm{Re}\left(e^{i\lambda\cdot t}F_\epsilon(t) + e^{i\tilde{\lambda}\cdot t}F_\epsilon(\tilde{t})\right)\right|^2 dt\right)^{1/2} < 2\epsilon, \quad \lambda \in \Lambda^*. \tag{10}$$

Using these inequalities, one can check that there are a point t_ϵ, $t_\epsilon \in [-\sigma, \sigma]^2$ and constants C and c depending only on σ such that

$$|F_\epsilon(t_\epsilon)| > C \quad \text{and} \quad \left|\mathrm{Re}\left(e^{i\lambda\cdot t_\epsilon}F_\epsilon(t_\epsilon) + e^{i\tilde{\lambda}\cdot t_\epsilon}F_\epsilon(\tilde{t_\epsilon})\right)\right| < c\epsilon|F_\epsilon(t_\epsilon)|, \tag{11}$$

for every small enough ϵ. Indeed, by Plancherel theorem, we have $\|G\|_2 = \|g\|_2$. Since $\|g\|_\infty = 1$ and $g \in B_\sigma$, using Bernstein inequality, we deduce $\|G\|_2 = \|g\|_2 \geq C(\sigma)$. Now, assuming that for all t, the inequalities (11) do not hold true, by integration with respect to variable t and using the estimate (9), for sufficiently small ϵ we arrive at

$$\int_{\mathbb{R}^2} \left|\mathrm{Re}\left(e^{i\lambda\cdot t}F_\epsilon(t)+e^{i\tilde{\lambda}\cdot t}F_\epsilon(\tilde{t})\right)\right|^2 dt \geq c^2\epsilon^2\|F_\epsilon\|_2^2 \geq \frac{c^2}{2}\epsilon^2\|G\|_2^2 \geq \frac{c^2}{2}\epsilon^2 C^2(\sigma),$$

which contradicts to estimate (10) when $c > 2\sqrt{2}/C(\sigma)$.

Write

$$F_\epsilon(t_\epsilon) =: R_\epsilon e^{iu_\epsilon}, \quad F_\epsilon(\tilde{t}_\epsilon) =: r_\epsilon e^{iv_\epsilon}.$$

Then we get

$$|R_\epsilon| > C, \quad \left|R_\epsilon \cos(\lambda\cdot t_\epsilon + u_\epsilon) + r_\epsilon \cos(\tilde{\lambda}\cdot t_\epsilon + v_\epsilon)\right| < c\epsilon R_\epsilon,\ \lambda \in \Lambda^*.$$

Then normalizing we arrive at

$$\left|\frac{R_\epsilon}{\sqrt{R_\epsilon^2+r_\epsilon^2}}\cos(\lambda\cdot t_\epsilon + u_\epsilon) + \frac{r_\epsilon}{\sqrt{R_\epsilon^2+r_\epsilon^2}}\cos(\tilde{\lambda}\cdot t_\epsilon + v_\epsilon)\right| < c\epsilon.$$

Clearly, we may assume that $u_\epsilon \in [0, 2\pi]$ and $v_\varepsilon \in [0, 2\pi]$. Recall that $t_\epsilon \in [-\sigma, \sigma]^2$ and, of course,

$$\frac{R_\epsilon}{\sqrt{R_\epsilon^2+r_\epsilon^2}} \in [0, 1] \quad \text{and} \quad \frac{r_\epsilon}{\sqrt{R_\epsilon^2+r_\epsilon^2}} \in [0, 1].$$

Taking $\epsilon = \frac{1}{n}$ and passing if necessary to a subsequence, we deduce that Λ lies on some curvilinear lattice.

4. In what follows we assume that

$$g \in B_\sigma \setminus L^2(\mathbb{R}^2). \tag{12}$$

For $\epsilon > 0$ we set

$$h_\epsilon(\xi) := \frac{\sin(\epsilon\xi)}{\epsilon\xi}, \quad \Phi_\epsilon(x_1, x_2) := h_\epsilon(x_1)h_\epsilon(x_2), \quad \text{and} \quad \delta_\epsilon := \|g\Phi_\epsilon\|_2^{-1/2}.$$

□

The next statement easily follows from (12).

Lemma 4 *We have* $\delta_\epsilon \to 0$ *as* $\epsilon \to 0$.

We skip the simple proof.

Let us introduce auxiliary functions

$$\varphi_\epsilon(x) := \delta_\epsilon\Phi_\epsilon(x), \quad g_\epsilon(x) := g(x)\varphi_\epsilon(x), \quad x \in \mathbb{R}^2.$$

By Lemma 4,

$$\|g_\epsilon\|_2 = 1/\delta_\epsilon \to \infty, \quad \epsilon \to 0. \tag{13}$$

Lemma 5 *For every $\lambda \in \Lambda^*$ satisfying $|\lambda| < 1/\sqrt{\delta_\epsilon}$ we have*

$$\|S(g_\epsilon)_\lambda\|_2 \le C\sqrt{\delta_\epsilon}.$$

Proof By Lemma 1, $Sg_\lambda = 0$. Since the function φ_ϵ is even with respect to each variable, we have

$$\begin{aligned}&Sg_\lambda\varphi_\epsilon(x) = (Sg(\cdot-\lambda)\varphi_\epsilon(\cdot))(x)\\ &= g(x-\lambda)\varphi_\epsilon(x) + g(\tilde{x}-\lambda)\varphi_\epsilon(\tilde{x}) + g(-x-\lambda)\varphi_\epsilon(-x) + g(-\tilde{x}-\lambda)\varphi_\epsilon(-\tilde{x}) = 0.\end{aligned}$$

Hence,

$$\begin{aligned}&|S(g_\epsilon)_\lambda(x)| = |S(g\varphi_\epsilon)(\cdot-\lambda)(x)| = |S(g\varphi_\epsilon)(x-\lambda) - Sg(\cdot-\lambda)\varphi_\epsilon)(x)|\\ &\le |g(x-\lambda)(\varphi_\epsilon(x-\lambda)-\varphi_\epsilon(x))| + |g(\tilde{x}-\lambda)(\varphi_\epsilon(\tilde{x}-\lambda)-\varphi_\epsilon(\tilde{x}))|\\ &\quad + |g(-x-\lambda)(\varphi_\epsilon(-x-\lambda)-\varphi_\epsilon(-x))| + |g(-\tilde{x}-\lambda)(\varphi_\epsilon(-\tilde{x}-\lambda)-\varphi_\epsilon(-\tilde{x}))|.\end{aligned}$$

Below we focus on the estimate of the first term at the right hand-side of the inequality above. The remaining terms admit the same estimate.

Write $\lambda = (\lambda_1, \lambda_2)$. Observe that

$$\begin{aligned}|\varphi_\epsilon(x-\lambda) - \varphi_\epsilon(x)| \le \delta_\epsilon\Big(&|h_\epsilon(x_1-\lambda_1) - h_\epsilon(x_1)|\,|h_\epsilon(x_2-\lambda_2)|\\ &+ |h_\epsilon(x_2-\lambda_2) - h_\epsilon(x_2)|\,|h_\epsilon(x_1)|\Big).\end{aligned}$$

For $j = 1, 2$ using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\left(\int_{\mathbb{R}} |h_\epsilon(x_j-\lambda_j) - h_\epsilon(x_j)|^2 dx_j\right)^{1/2} &= \left(\int_{\mathbb{R}} \left|\int_0^{\lambda_j} h'_\epsilon(x_j-u)du\right|^2 dx_j\right)^{1/2}\\ &\le C|\lambda_j|\|h'_\varepsilon\|_2.\end{aligned}$$

One may check that $\|h_\epsilon\|_2 = C/\sqrt{\epsilon}$ and $\|h'_\epsilon\|_2 = C\sqrt{\epsilon}$. Since $\|g\|_\infty = 1$ and $|\lambda| \le 1/\sqrt{\delta_\epsilon}$, we arrive at

$$\|S(g_\epsilon)_\lambda\|_2 \le C\left(\int_{\mathbb{R}^2} |\varphi_\epsilon(x-\lambda) - \varphi_\epsilon(x)|^2\, dx\right)^{1/2} \le C\delta_\epsilon|\lambda|\|h_\epsilon\|_2\|h'_\epsilon\|_2 \le C\sqrt{\delta_\epsilon}.$$

That finishes the proof. □

5. Denote by $G_\epsilon := \widehat{g\varphi_\epsilon}$. Then $G_\epsilon \in L^2(\mathbb{R})$ vanishes a.e. outside some square $(-\sigma^*, \sigma^*)^2$ (it is easy to check that one may take $\sigma^* = \sigma + \epsilon$).

Using Lemma 5, for $|\lambda| \le 1/\sqrt{\delta_\epsilon}$ we get

$$\left(\int_{\mathbb{R}^2} \left|\mathrm{Re}\left(e^{i\lambda\cdot x}G_\epsilon(x) + e^{-i\lambda\cdot\tilde{x}}G_\epsilon(\tilde{x})\right)\right|^2 dx\right)^{1/2} \le C\sqrt{\delta_\epsilon}.$$

On the other hand, by (13), $\|G_\epsilon\|_2 \ge C$, for all small enough ϵ.

To finish the proof, we proceed as in the proof of Lemma 3.

3.2 Proof of Theorem 2, Part II

(i) $\Rightarrow$ (ii). We will argue by contradiction. Assume that for every $\sigma > 0$ there is a constant $K = K(\sigma)$ such that

$$\|f\|_\infty \le K \sup_{\alpha\in I}\sup_{\lambda\in\Lambda} \|f * G_\alpha\|_\infty, \quad f \in B_\sigma,$$

but condition (ii) is not satisfied, i.e. there exists some $\Lambda' \in W(\Lambda)$ such that Λ' lies on some curvilinear lattice. Clearly, to come to the contradiction it suffices to construct for every $\varepsilon > 0$ a function $f = f_\varepsilon$ such that

$$\|f\|_\infty \ge C, \quad \sup_{\alpha\in I}\sup_{\lambda\in\Lambda} |f * G_\alpha(\lambda)| \le C\varepsilon, \tag{14}$$

and $f \in \mathcal{B}_{\sigma^*}$ for some fixed σ^*.

Again, let us provide a brief description of the proof. We divide the proof into 4 steps. First, we build a function g such that $g * G_\alpha$ vanishes on Λ' for every $\alpha \in I$. A slight modification of g provides a function f, which satisfies (14). To verify the second estimate in (14) we split the set Λ into the sets $\Lambda_I = \Lambda \cap P$ and $\Lambda_O = \Lambda \cap (\mathbb{R}^2 \setminus P)$ for an appropriate rectangle P. In the steps 3 and 4, we show that f satisfies the relations (14) for $\lambda \in \Lambda_O$ and $\lambda \in \Lambda_I$ respectively.

Now we pass to the proof.

1. By our assumption, there exist $\Lambda' \in W(\Lambda)$, $\xi \in \mathbb{R}^2$, $(t_1, t_2) \in \mathbb{R}^2$, and $(r_1, r_2) \in \mathbb{T}$ such that for every $\lambda' \in \Lambda'$ the equality

$$r_1\cos(\lambda'\cdot\xi - t_1) - r_2\cos(\tilde{\lambda'}\cdot\xi - t_2) = 0 \tag{15}$$

holds. Set

$$g(x) = r_1\cos(\xi\cdot x + t_1) - r_2\cos(\tilde{\xi}\cdot x + t_2).$$

Clearly, $g \in B_\sigma$ for $\sigma = |\xi|$. Next, we will show that symmetrization of the function $g_{\lambda'}$ vanishes for every $\lambda' \in \Lambda'$.

Lemma 6 *The equality*

$$Sg_{\lambda'}(x) = 0$$

holds for every $x \in \mathbb{R}^2$ *and* $\lambda' \in \Lambda'$.

Proof After some simple calculations, we have

$$Sg_{\lambda'}(x) = r_1 \operatorname{Re}\left(e^{i(t_1-\xi\cdot\lambda')}S(e^{i\xi\cdot x})\right) - r_2 \operatorname{Re}\left(e^{i(t_2-\tilde{\xi}\cdot\lambda')}S(e^{i\tilde{\xi}\cdot x})\right),$$

where we, as usual, apply symmetrization operator S with respect to variable x. Clearly, $S(e^{i\tilde{\xi}\cdot x}) = S(e^{i\xi\cdot x}) = 2(\cos(\xi\cdot x) + \cos(\tilde{\xi}\cdot x))$. Thus, using (15), we have

$$Sg_{\lambda'}(x) = 2(\cos(\xi\cdot x) + \cos(\tilde{\xi}\cdot x))\left(r_1 \operatorname{Re} e^{i(t_1-\lambda'\cdot\xi)} + r_2 \operatorname{Re} e^{i(t_2-\tilde{\lambda}'\cdot\xi)}\right) = 0.$$

□

Consequently, for every $\lambda' \in \Lambda'$ and $\alpha \in I$ we have

$$g * G_\alpha(\lambda') = 0, \tag{16}$$

since G_α is even in every variable.

2. Fix small $\varepsilon > 0$ and take large $R = R(\varepsilon) > 0$ (we will specify its value later). Recall that $\Lambda' \in W(\Lambda)$. In particular, that means that one can find $v = (v_1, v_2) = v(R, \varepsilon) \in \mathbb{R}^2$ such that inside the square $[-R, R]^2$, the set $\Lambda - v$ is "close" to Λ':

$$\text{for every } \lambda \in \Lambda\cap(v+(-R,R)^2) \text{ there is } \lambda' \in \Lambda'\cap(-R,R)^2 : \quad \operatorname{dist}(\lambda - v, \lambda') \le \varepsilon. \tag{17}$$

Set $P = [v_1 - R, v_1 + R] \times [v_2 - R, v_2 + R]$ and consider the decomposition

$$\Lambda = \Lambda_I \cup \Lambda_O := (\Lambda \cap P) \cup \left(\Lambda \cap (\mathbb{R}^2 \setminus P)\right).$$

Consider

$$\Phi_\varepsilon(t) = \Phi_\varepsilon(t_1, t_2) = \frac{\sin(\varepsilon t_1)}{\varepsilon t_1}\frac{\sin(\varepsilon t_2)}{\varepsilon t_2}.$$

We define the function f by the formula

$$f(x) = \Phi_\varepsilon(x - v)g(x - v), \quad x \in \mathbb{R}^2,\ v \in \mathbb{R}^2.$$

Clearly, $\|f\|_\infty \ge C$, and it suffices to show that $|f * G_\alpha(\lambda)| \le C\varepsilon$ for every $\lambda \in \Lambda$. We will estimate the value $|f * G_\alpha(\lambda)|$ for $\lambda \in \Lambda_I$ and $\lambda \in \Lambda_O$ separately.

3. Assume that $\lambda \in \Lambda_O$. We may choose $R = R(\varepsilon) = \frac{1}{\varepsilon^2}$. Set $U = U_1 \times U_2 = [-\sqrt{R}, \sqrt{R}]^2$. For $s \in U$ we have

$$|f(\lambda - s)| \le \frac{\|g\|_\infty}{\varepsilon^2|\lambda_1 - s_1 - v_1||\lambda_2 - s_2 - v_2|} \le \frac{\|g\|_\infty}{\varepsilon^2|R - \sqrt{R}|^2} \le C\varepsilon^2\|g\|_\infty, \tag{18}$$

since $|\lambda_1 - v_1| \geq R$ and $|\lambda_2 - v_2| \geq R$. Next, it is easy to check that

$$J := \int_{\mathbb{R}} \int_{\mathbb{R}\setminus U_1} G_\alpha(s_1, s_2)\, ds_1 ds_2 + \int_{\mathbb{R}} \int_{\mathbb{R}\setminus U_2} G_\alpha(s_1, s_2)\, ds_2 ds_1 \leq C\varepsilon. \tag{19}$$

Now, to estimate $f * G_\alpha(\lambda)$ for $\lambda \in \Lambda_O$, we write

$$|f * G_\alpha(\lambda)| \leq \int_U |f(\lambda - s)| G_\alpha(s)\, ds + \int_{\mathbb{R}} \int_{\mathbb{R}\setminus U_1} |f(\lambda - s)| G_\alpha(s)\, ds$$
$$+ \int_{\mathbb{R}} \int_{\mathbb{R}\setminus U_2} |f(\lambda - s)| G_\alpha(s)\, ds.$$

Applying $\|f\|_\infty \leq 1$ and estimates (18) and (19), we arrive at

$$|f * G_\alpha(\lambda)| \leq C\varepsilon^2 \int_U G_\alpha(s)\, ds + J \leq C\varepsilon.$$

4. Now, assume that $\lambda \in \Lambda_I$. Take $\lambda' \in \Lambda'$, satisfying condition (17) corresponding to λ, i.e. $\operatorname{dist}(\lambda - v, \lambda') < \varepsilon$. Since $g * G_\alpha(\lambda') = 0$, we may write

$$f * G_\alpha(\lambda) = \int_{\mathbb{R}^2} f(\lambda - s) G_\alpha(s) ds + \Phi_\varepsilon(\lambda - v) \int_{\mathbb{R}^2} g(\lambda' - s) G_\alpha(s) ds$$
$$= \int_{\mathbb{R}^2} \big(\Phi_\varepsilon(\lambda - s - v)\, \big(g(\lambda - s - v) - g(\lambda' - s)\big)$$
$$+ g(\lambda' - s)\, (\Phi_\varepsilon(\lambda - v - s) - \Phi_\varepsilon(\lambda - v))\,\big) G_\alpha(s) ds.$$

Set

$$H_1 := |\Phi_\varepsilon(\lambda - s - v) - \Phi_\varepsilon(\lambda - v)|\,,$$
$$H_2 := \big|g(\lambda - s - v) - g(\lambda' - s)\big|\,.$$

Clearly,

$$|f * G_\alpha(\lambda)| \leq \int_{\mathbb{R}^2} \big(H_1 |g(\lambda' - s)| + H_2\, |h_\varepsilon(\lambda - v)|\big)\, G_\alpha(s) ds. \tag{20}$$

By Bernstein inequality and relation (17), we have

$$|H_1| \leq \varepsilon(|s_1| + |s_2|), \quad |H_2| \leq \varepsilon \|g'\|_\infty. \tag{21}$$

Combining estimates (20) and (21) together, we obtain

$$|f * G_\alpha(\lambda)| \leq C\varepsilon \int_{\mathbb{R}^2} (\|g'\|_\infty + |s_1| + |s_2|) G_\alpha(s_1, s_2) ds_1 ds_2 \leq C\varepsilon$$

that finishes the proof.

The following statement easily follows from Theorem 2.

Lemma 7 *Assume* Λ *and* I *satisfy the assumptions of Theorem 2 and condition* (i) *is fulfilled. Then for every* $\sigma > 0$ *there is a constant* C *such that*

$$\|f\|_\infty^2 \leq C \int_I \sup_{\lambda \in \Lambda} |f * G_\alpha(\lambda)|^2 \, d\alpha \tag{22}$$

for every $f \in PW_\sigma^2$.

4 Sampling with Gaussian Kernel in Paley–Wiener Spaces

4.1 Auxiliary Statements

Recall that our aim is to describe the geometry of sets $\Lambda \subset \mathbb{R}^2$ that for every $f \in PW_\sigma^2$ the estimates

$$D_1 \|f\|_2^2 \leq \sum_{\lambda \in \Lambda} \int_I |f * G_\alpha(\lambda)|^2 \, d\alpha \leq D_2 \|f\|_2^2, \tag{23}$$

hold with some constants D_1 and D_2 independent on f.

4.1.1 Bessel-Type Inequality

We start with showing that the right hand-side of (23) follows easily from classical sampling results for u.d. set Λ.

Proposition 1 *Assume* Λ *is a u.d. set,* $I = (a, b) \times (c, d)$*, where* $0 < a < b < \infty$, $0 < c < d < \infty$*. Then there is a constant* $D_2 = D_2(I, \Lambda)$ *such that*

$$\sum_{\lambda \in \Lambda} \int_I |f * G_\alpha(\lambda)|^2 \, d\alpha \leq D_2 \|f\|_2^2,$$

for every $f \in PW_\sigma^2$.

Proof Recall the *Bessel inequality* for Paley-Wiener spaces: if Λ is a u.d. subset of $\mathbb{R}^d$ then there is a constant $M = M(\Lambda, \sigma)$ such that

$$\sum_{\lambda \in \Lambda} |g(\lambda)|^2 \le M \|g\|_2^2 \tag{24}$$

for every $g \in PW_\sigma^2$, see [20], Chapter 2, Theorem 17.

Note that convolution with Gaussian Kernel G_α keeps the function in Paley-Wiener space. Using Young's convolution inequality and Bessel inequality, one can find a constant D_2 such that for every $f \in PW_\sigma^2$ the estimate

$$\sum_{\lambda \in \Lambda} \int_J |f * G_\alpha(\lambda)|^2 \, d\alpha \le C|J| \|f * G_\alpha\|_2^2 \le C \|f\|_2^2 \|G_\alpha\|_1^2 \le D_2 \|f\|_2^2$$

is true. That finishes the proof of proposition. □

4.1.2 Auxiliary Functions

In what follows we need some auxiliary functions with special properties. These functions should belong to Paley-Wiener spaces, have a large L^2-norm with a small L^2-norm of the gradient. Now, we specify these requirements.

Condition (B): Let ε be a small positive parameter. A family of functions $\{\Phi_\varepsilon\}$ satisfies condition (B) if

(β_1) $\Phi_\varepsilon(0, 0) = 1, \quad \|\Phi_\varepsilon\|_\infty = 1$;
(β_2) $\Phi_\varepsilon \in PW_\varepsilon^2$;
(β_3) $\|\Phi_\varepsilon\|_2 \to \infty$ as $\varepsilon \to 0$;
(β_4) $\|\nabla \Phi_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$.

Next, we provide a few examples to illustrate some additional difficulties that occur in the multi-dimensional setting. Then we present an example of functions Φ_ε that satisfy condition (B).

Example 4 Let us return to the one-dimensional case. Consider

$$\Phi_\varepsilon(x) = \frac{\sin(\varepsilon x)}{\varepsilon x}.$$

Observe that functions Φ_ε satisfy an analogue of condition (B) in the one-dimensional setting. Clearly, $\Phi_\varepsilon(0) = 1$, $\|\Phi_\varepsilon\|_\infty = 1$, and $\Phi_\varepsilon \in PW_\varepsilon^2$. One may easily check that

$$\|\Phi_\varepsilon\|_2 \le C\varepsilon^{-1/2} \quad \text{and} \quad \|\Phi'_\varepsilon\|_2 \le C\varepsilon^{1/2}.$$

These relations prove the one-dimensional analogues of (β_3) and (β_4).

The passage from Bernstein to Paley-Wiener spaces and back in [19] was based on the properties of the functions in Example 4. One may try to construct functions Φ_ε that satisfy condition (B) in the two-dimensional setting in the following natural way.

Example 5 Consider the function Φ_ε defined by the formula

$$\Phi_\varepsilon(x, y) = \frac{\sin(\varepsilon x)}{\varepsilon x}\frac{\sin(\varepsilon y)}{\varepsilon y}.$$

It is clear that conditions (β_1) and (β_2) are true. Property (β_3) follows from

$$\|\Phi_\varepsilon\|_2 \le C\varepsilon^{-1}.$$

However, one may easily check that $\|\nabla\Phi_\varepsilon\|_2$ does not converge to zero as $\varepsilon \to 0$.

However, in two-dimensional setting it is still possible to construct functions that satisfy condition (B). Now, we pass to the construction.

Lemma 8 *Assume* $\varepsilon > 0$. *There exist functions* Ψ_ε *such that*

$(P1)$ $\operatorname{supp}\Psi_\varepsilon \subset B_\varepsilon(0)$, $\Psi_\varepsilon \ge 0$,
$(P2)$ $C_1 \le \int\limits_{\mathbb{R}^2} \Psi_\varepsilon(x)\,dx \le C_2$, $0 < C_1 \le C_2 < \infty$,
$(P3)$ $\|\Psi_\varepsilon\|_2 \ge \frac{C}{\varepsilon^{3/4}}$,
$(P4)$ $\left(\int\limits_{\mathbb{R}^2} |\Psi_\varepsilon(x)|^2|x|^2\,dx\right)^{1/2} \le \frac{C}{\sqrt{\log\frac{1}{\varepsilon}}}$.

Proof Fix small $0 < \varepsilon < 1$ and denote the integer part of $\log\frac{1}{\varepsilon}$ by m. For integers n from $[m, 2m]$ we set $a_n = 2^{2n}/n$. Next, we define the function Ψ_ε layer by layer by the formula

$$\Psi_\varepsilon(x) = a_n, \quad x \in B_{2^{-n}}(0) \setminus B_{2^{-n-1}}(0).$$

For $|x| > \varepsilon$ and $|x| < \frac{\varepsilon^2}{2}$ we set $\Phi_\varepsilon(x) = 0$. Note that the area of the ring $B_{2^{-n}}(0) \setminus B_{2^{-n-1}}(0)$ is equal to $\frac{3\pi}{4}2^{-2n}$.

Clearly, Ψ_ε satisfy (P1). To verify (P2) we write

$$\int\limits_{\mathbb{R}^2} \Psi_\varepsilon(x)\,dx = \frac{3\pi}{4}\sum_{n=m}^{2m} 2^{-2n}a_n = C\sum_{n=m}^{2m}\frac{1}{n}.$$

Note that the right-hand side of this equation can be estimated with some fixed positive constants from above and below by

$$\int\limits_{\log\frac{1}{\varepsilon}}^{2\log\frac{1}{\varepsilon}} \frac{1}{t}\,dt = \log 2.$$

Thus, condition (P2) follows. Next, we have

$$\|\Psi_\varepsilon\|_2^2 = \frac{3\pi}{4}\sum_{n=m}^{2m} 2^{-2n}a_n^2 \geq C\sum_{n=m}^{2m}\frac{2^{2n}}{n^2} \geq C\int_{\log\frac{1}{\varepsilon}}^{2\log\frac{1}{\varepsilon}} 2^{2t}t^{-2}\,dt \geq \frac{C}{\varepsilon^2\log^2\frac{1}{\varepsilon}} \geq \frac{C}{\varepsilon^{3/2}},$$

and $(P3)$ follows. The estimate

$$\int_{\mathbb{R}^2} |\Psi_\varepsilon(x)|^2|x|^2\,dx \leq C\sum_{n=m}^{2m} 2^{-4n}a_n^2 \leq C\sum_{n=m}^{2m}\frac{1}{n^2} \leq \frac{C}{\log\frac{1}{\varepsilon}}$$

implies $(P4)$ that finishes the proof. □

Corollary 1 *There exist functions Φ_ε satisfying condition* (B).

Proof Denote by $c_\Psi = \int_{\mathbb{R}^2} \Psi_\varepsilon(x)\,dx$. By (P_2), c_Ψ is positive, finite, and separated from zero. Now, we may define Φ_ε as the Fourier transform of Ψ_ε with a proper normalization:

$$\Phi_\varepsilon(x) = \frac{1}{c_\Psi}\int_{\mathbb{R}^2} e^{-ix\cdot t}\Psi_\varepsilon(t)\,dt.$$

The property (β_2) follows from (P_1). Due to $\Psi_\varepsilon \geq 0$ and normalization condition (β_1) is fulfilled. Relations (P_3) and (P_4) imply estimates (β_3) and (β_4) respectively. □

4.2 From Bernstein to Paley–Wiener Spaces and Back

To prove Theorem 1, we will use the following statement, which describes the connection between sampling in Paley-Wiener and Bernstein spaces.

Theorem 3 *Let Λ be a u.d. set in $\mathbb{R}^2$, $I = (a,b)\times(c,d)$, $0 < a < b < \infty$, $0 < c < d < \infty$, and $\sigma' > \sigma > 0$.*

(i) Assume the inequality

$$\|f\|_\infty \leq K \sup_{\alpha\in I}\sup_{\lambda\in\Lambda}\|f * G_\alpha\|_\infty \quad \textit{for all } f \in B_{\sigma'} \tag{25}$$

holds with some constant $K = K(\sigma',\Lambda)$. Then there exists a constant $D_1 = D_1(\sigma,\Lambda)$ such that

$$D_1\|f\|_2^2 \leq \sum_{\lambda\in\Lambda}\int_I |f * G_\alpha(\lambda)|^2\,d\alpha \quad \textit{for every } f \in PW_\sigma^2 \tag{26}$$

is true.

(ii) *Assume that* (26) *holds with some constant* $D_1 = D_1(\sigma', \Lambda)$ *for all* $f \in PW_{\sigma'}$. *Then there is a constant* $K = K(\sigma', \Lambda)$ *such that* (25) *is true for every* $f \in B_\sigma$.

Remark 2 For a similar result for space sampling see [15].

Remark 3 In this theorem we do not need to require I to be a rectangle. One may take $I = (a, b) \subset \mathbb{R}$ with $0 < a < b < \infty$. In such a case by $G_\alpha(x)$ we mean $G_\alpha(x_1, x_2) = e^{-\alpha(x_1^2+x_2^2)}$ and $d\alpha$ is a standard one-dimensional Lebesgue measure.

The proof of Theorem 3 is similar to the proof of Theorem 3 in the paper [19]. We provide the argument for statement (i) and leave the proof of (ii) to the reader. The functions Φ_ε that satisfy condition (B) play a crucial role in our argument.

Proof of Theorem 3. Take $\varepsilon > 0$ such that $\sigma + \varepsilon < \sigma'$. By our assumption, for every $q \in B_\sigma$ the estimate

$$\|q\|_\infty \le C \sup_{\alpha \in I} \sup_{\lambda \in \Lambda} |q * G_\alpha(\lambda)|, \tag{27}$$

is true and our aim is to prove (26). Consider functions Φ_ε satisfying condition (B). Using (β_1), we get

$$\|f\|_2^2 = \int_{\mathbb{R}^2} |f(x)|^2 dx \le \int_{\mathbb{R}^2} \sup_{t \in \mathbb{R}^2} |\Phi_\varepsilon(x-t) f(t)|^2 \, dx. \tag{28}$$

Note that $q(t) := \Phi_\varepsilon(x-t) f(t) \in \mathcal{B}_{\sigma+\varepsilon}$, and we can apply Lemma 7 to obtain

$$|q(t)|^2 \le C \int_I \sup_{\lambda \in \Lambda} \left| \int_{\mathbb{R}^2} G_\alpha(\lambda - s) \Phi_\varepsilon(x-s) f(s) \, ds \right|^2 d\alpha, \tag{29}$$

where the constant C does not depend on t. To provide the estimate from above we may replace $\sup\limits_{\lambda \in \Lambda}$ by $\sum\limits_{\lambda \in \Lambda}$, and switch the order of integration and summation:

$$\|f\|_2^2 \le C \sum_{\lambda \in \Lambda} \int_I \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} G_\alpha(\lambda - s) \Phi_\varepsilon(x-s) f(s) \, ds \right|^2 dx \, d\alpha. \tag{30}$$

Denote by

$$Y_1 = \left| \Phi_\varepsilon(x-\lambda) \int_{\mathbb{R}^2} G_\alpha(\lambda - s) f(s) \, ds \right|^2, \tag{31}$$

$$Y_2 = \left| \int_{\mathbb{R}^2} G_\alpha(\lambda - s) \left(\Phi_\varepsilon(x-\lambda) - \Phi_\varepsilon(x-s) \right) f(s) \, ds \right|^2. \tag{32}$$

Using the inequality $|a+b|^2 \le C(|a|^2+|b|^2)$, we deduce from (30), (31), and (32) that

$$\|f\|_2^2 \le C \sum_{\lambda\in\Lambda} \int_I \int_{\mathbb{R}^2} (Y_1+Y_2)\, dx\, d\alpha. \tag{33}$$

Next, we estimate the terms with Y_1 and Y_2 separately. The value of $\sum_{\lambda\in\Lambda} \int_I \int_{\mathbb{R}^2} Y_1\, dx\, d\alpha$ is majorized by

$$\sum_{\lambda\in\Lambda} \int_I \left(\int_{\mathbb{R}^2} |\Phi_\varepsilon(x-\lambda)|^2\, dx \right) |(f*G_\alpha)(\lambda)|^2\, d\alpha \le \|\Phi_\varepsilon\|_2^2 \int_I \sum_{\lambda\in\Lambda} |(f*G_\alpha)(\lambda)|^2 d\alpha. \tag{34}$$

The inequalities for the second term are more complicated. Set

$$H(x;\lambda,s) = |\Phi_\varepsilon(x-\lambda) - \Phi_\varepsilon(x-s)|.$$

We start with the observation

$$H(x;\lambda,s) \le \left| \int_{s_1}^{\lambda_1} \frac{\partial\Phi_\varepsilon}{\partial x}(x-u_1, y-\lambda_2) du_1 \right| + \left| \int_{s_2}^{\lambda_2} \frac{\partial\Phi_\varepsilon}{\partial y}(x-s_1, y-u_2) du_2 \right|.$$

Using Cauchy-Schwarz inequality, we write

$$H^2(x;\lambda,s) \le C\Bigg((\lambda_1 - s_1) \int_{s_1}^{\lambda_1} \left| \frac{\partial\Phi_\varepsilon}{\partial x}(x-u_1, y-\lambda_2) \right|^2 du_1 + (\lambda_2 - s_2) \int_{s_2}^{\lambda_2} \left| \frac{\partial\Phi_\varepsilon}{\partial x}(x-s_1, y-u_2) \right|^2 du_2 \Bigg).$$

Thereby, for $\lambda = (\lambda_1, \lambda_2)$ and $s = (s_1, s_2)$ we get

$$\int_{\mathbb{R}^2} H^2(x;\lambda,s)\, dx \le C|s_1-\lambda_1||s_2-\lambda_2|\|\nabla\Phi_\varepsilon\|_2^2,$$

whence

$$\|H(\,\cdot\,;\lambda,s)\|_2^2 \le C|s-\lambda|^2\|\nabla\Phi_\varepsilon\|_2^2. \tag{35}$$

Now, we return to the estimation of the term with Y_2 in the formula (33). Applying Cauchy–Schwarz inequality, we arrive at

$$\sum_{\lambda\in\Lambda}\int_{\mathbb{R}^2} Y_2\,dx = \sum_{\lambda\in\Lambda}\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^2} f(s)G_\alpha(\lambda-s)H(x;\lambda,s)\,ds\right|^2 dx$$
$$\leq \sum_{\lambda\in\Lambda}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}|f(s)|^2 G_\alpha(\lambda-s)ds\int_{\mathbb{R}^2} G_\alpha(\lambda-s)H^2(x;\lambda,s)ds\right)dx.$$

With estimate (35) in hand, we continue

$$\sum_{\lambda\in\Lambda}\int_{\mathbb{R}^2} Y_2\,dx \leq \sum_{\lambda\in\Lambda}\left(\int_{\mathbb{R}^2}|f(s)|^2 G_\alpha(\lambda-s)ds\int_{\mathbb{R}^2} G_\alpha(\lambda-s)\|H^2(\cdot;\lambda,s)\|_2 ds\right)$$
$$\leq C\|\nabla\Phi_\varepsilon\|_2^2\sum_{\lambda\in\Lambda}\left(\int_{\mathbb{R}^2}|f(s)|^2 G_\alpha(\lambda-s)ds\int_{\mathbb{R}^2} G_\alpha(\lambda-s)|s-\lambda|^2 ds\right).$$

Clearly,

$$\int_{\mathbb{R}^2} G_\alpha(\lambda-s)|s-\lambda|^2 ds \leq C, \tag{36}$$

and since Λ is a u.d. set, we have

$$\sum_{\lambda\in\Lambda} G_\alpha(\lambda-s) \leq C. \tag{37}$$

Using relations (36) and (37), we finish the estimate of the term with Y_2:

$$\sum_{\lambda\in\Lambda}\int_I\int_{\mathbb{R}^2} Y_2\,dx\,d\alpha \leq C|I|\|\nabla\Phi_\varepsilon\|_2^2\|f\|_2^2. \tag{38}$$

Combining (33), (34), and (38) together, we get

$$\|f\|_2^2 \leq C_1|I|\|\nabla\Phi_\varepsilon\|_2^2\|f\|_2^2 + C_2\|\Phi_\varepsilon\|_2^2\int_I\sum_{\lambda\in\Lambda}|(f*G_\alpha)(\lambda)|^2 d\alpha. \tag{39}$$

To finish the proof we invoke properties (β_3) and (β_4). Indeed, taking sufficiently small $\varepsilon > 0$ we make the first summand less than $\frac{\|f\|_2^2}{2}$, and (26) follows. □

4.3 Proof of Theorem 1

Now, we are ready to prove the main result.

(i) $\Rightarrow$ (ii) Assume that for the set Λ condition (i) is satisfied. In particular, for any $\sigma > 0$ inequality (26) is true for every $f \in PW_\sigma^2$ with constant D_1 depending on σ. Then, Theorem 3 implies that for every $\sigma > 0$ inequality (25) holds true for every $f \in B_\sigma$ with constant K depending on σ. By Theorem 2, we deduce that Λ satisfy condition (A).

(ii) $\Rightarrow$ (i) Assume that condition (ii) is fulfilled. Recall that Proposition 1 ensures that the right hand-side estimate in (1) holds with some universal constant. Thus, it suffices to verify that inequality (26) is true for every $\sigma > 0$ and every $f \in PW_\sigma^2$ with some constant $D_1 = D_1(\sigma)$. By our assumption and Theorem 2, the inequality (25) is true for every $\sigma > 0$ and every $f \in B_\sigma$ with a constant K depending only on σ. Applying Theorem 3, we see that (26) holds true for every $\sigma > 0$ and $f \in PW_\sigma^2$ with a constant D_1 depending only on σ. Thus, condition (i) is true. That finishes the proof. □

5 Remarks

First, we would like to note that Theorems 1, 2, and 3 remain true for a wider collection of kernels that satisfy some additional assumptions similar to conditions $(\beta) - (\theta)$ in [19].

Second, one may check that our approach provides a complete solution to the Main Problem for the Bernstein spaces $B_{[-\sigma,\sigma]^n}$ for the Gaussian kernel in $\mathbb{R}^n$ with any index set $I = \prod_{i=1}^{n}[a_i, b_i]$. One may therefore formulate an analogue of Theorem 2 in multi-dimensional setting. However, the passage to Paley-Wiener spaces faces obstacles similar to those discussed in Example 5, Sect. 4.

Recall the frame inequalities for a continuous frame $\{e_x\}_{x\in X}$:

$$D_1\|f\|_p^p \le \int_X |\langle f, e_x\rangle|^p dx \le D_2\|f\|_p^p \tag{40}$$

Inequalities (1) correspond to the case $p = 2$, $X = I \times \Lambda$, and dx is product of n-dimensional Lebesgue measure on I and counting measure on Λ. As it was pointed to me by a reviewer, the inequalities (40) typically hold true for all range of Banach spaces $(X, \|\cdot\|_p)$, $1 \le p < \infty$ simultaneously provided the frame $\{e_x\}$ has a sufficiently good localization, see [1], [8], and [9]. However, in our setting we did not manage to prove the analogue of Theorem 1 for all $p \in [1, \infty)$ when the dimension $n > 2$.

On the other hand, using our approach, one may check that for every $n > 2$ there are a number $p(n)$ and functions Φ_ε such that for $p \ge p(n)$ we have

$$\|\Phi_\varepsilon\|_p \to \infty, \quad \|\nabla\Phi_\varepsilon\|_p \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus, a modification of the proof of Theorem 3 leads to a complete solution of the Main Problem for $PW^p_{[-\sigma,\sigma]^n}$ spaces with $p \geq p(n)$.

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