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Revisiting Yano Extrapolation Theory

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Abstract

We prove a pointwise estimate for the decreasing rearrangement of Tf, where T is any sublinear operator satisfying the weak-type boundedness

$$T : L^{p,1}(\mu) \to L^{p,\infty}(\nu), \quad \forall p : 1 < p_0 < p \le p_1 < \infty,$$

with norm controlled by $C\varphi\left(\left[p_0^{-1} - p^{-1}\right]^{-1}\right)$ and φ satisfies some admissibility conditions. The pointwise estimate is:

$$(Tf)^*_\nu(t) \lesssim \frac{1}{p_0 - 1}\left(\frac{1}{t^{\frac{1}{p_0}}}\int_0^t \varphi\left(1 - \log\frac{r}{t}\right) f^*_\mu(r)\frac{dr}{r^{1-\frac{1}{p_0}}} + \frac{1}{t^{\frac{1}{p_1}}}\int_t^\infty f^*_\mu(r)\frac{dr}{r^{1-\frac{1}{p_1}}}\right).$$

In particular, this estimate allows to obtain extensions of Yano's extrapolation results.

Keywords Yano's extrapolation theory · Zygmund's extrapolation theory · Calderón type operators · Decreasing rearrangement estimates

Mathematics Subject Classification 46E30 · 28A10 · 47A30

Dedicated to the 80th anniversary of Professor Stefan Samko.

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1 Introduction

In 1951, Yano (see [10, 11]) using the ideas of Titchmarsh in [9], proved that for every sublinear operator T satisfying that, for some $\alpha > 0$ and for every $1 < p \leq p_1$,

$$T : L^p(\mu) \longrightarrow L^p(\nu), \qquad \frac{C}{(p-1)^\alpha},$$

where $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ are two finite measure spaces, it holds that

$$T : L(\log L)^\alpha(\mu) \longrightarrow L^1(\nu).$$

Here and all over the paper, given two function spaces E and F,

$$T : E \longrightarrow F, \qquad C,$$

means that, for every function $f \in E$,

$$||Tf||_F \leq C||f||_E.$$

Also, let us recall that the space $L(\log L)^\alpha(\mu)$ is defined as the set of μ-measurable functions such that

$$||f||_{L(\log L)^\alpha(\mu)} = \int_0^\infty f_\mu^*(t) \left(1 + \log^+ \frac{1}{t}\right)^\alpha dt < \infty,$$

where $a^+ = \max(a, 0)$, for every $a \in \mathbb{R}$, f_μ^* is the decreasing rearrangement of f with respect to the measure μ defined as

$$f_\mu^*(t) = \inf\left\{y > 0 : \lambda_f^\mu(y) \leq t\right\}, \qquad t > 0,$$

and

$$\lambda_f^\mu(y) = \mu\left(\left\{x \in \mathbb{R}^n : |f(x)| > y\right\}\right), \qquad y > 0,$$

is the distribution function of f with respect to μ. (Here we are using the standard notation $\mu(E) = \int_E d\mu(x)$ for every μ-measurable set $E \subseteq X$. If $d\mu = dx$, we shall write f^*, $\lambda_f(y)$ and $|E|$. See [4] for more details about this topic).

If the measures involved are not finite, but they are σ-finite, it was proved in [5] and [6] that under a weaker condition on the operator T, namely

$$\left(\int_{\mathcal{N}} |T\chi_A(x)|^p \, d\nu(x)\right)^{1/p} \leq \frac{C}{p-1}\mu(A)^{1/p},$$

for every μ-measurable set $A \subset \mathcal{M}$ and every $1 < p \leq p_0$, with C independent of A and p, then

$$T : L(\log L)^{\alpha}(\mu) \longrightarrow M(\phi),$$

where $M(\phi)$ is the maximal Lorentz space associated to the function $\phi(t) = \frac{t}{1+\log^+ t}$, $t > 0$; that is,

$$\|f\|_{M(\phi)} = \sup_{t>0} \phi(t) f_{\nu}^{**}(t) = \sup_{t>1} \frac{t f_{\nu}^{**}(t)}{1+\log t},$$

where $f_{\nu}^{**}(t) = \frac{1}{t}\int_0^t f_{\nu}^{*}(s)\, ds$, $t > 0$. These results belong to what is known as Yano's extrapolation theory.

On the other hand, in [11,p. 119] it was seen that if T is a sublinear operator satisfying

$$\|Tf\|_{L^p(\nu)} \leq Cp\|f\|_{L^p(\mu)}, \tag{1.1}$$

for every p near ∞ and for μ and ν being finite measures, then

$$T : L^{\infty}(\mu) \longrightarrow L_{\exp}(\nu),$$

where $L_{\exp}(\nu)$ is the set of ν-measurable functions satisfying that

$$||f||_{L_{\exp}(\nu)} = \sup_{0<t<1} \frac{f_{\nu}^{**}(t)}{1+\log\frac{1}{t}},$$

and this result was also extended to the case of general measures (see [6]) proving that, if T is a sublinear operator satisfying (1.1), then

$$\sup_{0<t<1} \frac{(Tf)_{\nu}^{**}(t)}{1+\log\frac{1}{t}} \lesssim ||f||_{L^{\infty}(\mu)} + \int_1^{\infty} f_{\mu}^{**}(s)\frac{ds}{s}, \tag{1.2}$$

where, as usual, we write $A \lesssim B$ if there exists a positive constant $C > 0$, independent of A and B, such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$. These results belong to what is known as Zygmund's extrapolation theory.

For the purpose of this work, it is also interesting to mention that results analogous to the ones mentioned above are known in the case that, for some $1 < p_0 < p_1 < \infty$ and every $p_0 < p < p_1$,

$$T : L^p(\mu) \longrightarrow L^p(\nu)$$

with

$$||T||_{L^p(\mu)\longrightarrow L^p(\nu)} \lesssim \frac{1}{(p-p_0)^{\alpha}} \quad \text{or} \quad ||T||_{L^p(\mu)\longrightarrow L^p(\nu)} \lesssim \frac{1}{(p_1-p)^{\alpha}}, \qquad \alpha > 0.$$

The exact statements are the following:

Theorem 1.1 [7] *Let T be a sublinear operator satisfying that for every $p_0 < p \le p_1$,*

$$T : L^p(\mu) \longrightarrow L^p(\nu), \qquad \frac{C}{p - p_0}.$$

Then,

$$\sup_{t>1} \frac{\left(\int_0^t (Tf)_\nu^*(s)^{p_0}\, ds\right)^{1/p_0}}{1 + \log t} \lesssim \|f\|_{L^{p_0}(\mu)} + \int_0^1 \frac{\left(\int_0^r f_\mu^*(s)^{p_0}\, ds\right)^{1/p_0}}{r}\, dr.$$

Theorem 1.2 [7] *Let T be a sublinear operator satisfying that for every $p_0 \le p < p_1$,*

$$T : L^p(\mu) \longrightarrow L^p(\nu), \qquad \frac{C}{p_1 - p}.$$

Then,

$$\sup_{0<t<1} \frac{\left(\int_t^\infty (Tf)_\nu^*(s)^{p_1}\, ds\right)^{1/p_1}}{1 + \log \frac{1}{t}} \lesssim \|f\|_{L^{p_1}(\mu)} + \int_1^\infty \frac{\left(\int_r^\infty f_\mu^*(s)^{p_1}\, ds\right)^{1/p_1}}{r}\, dr.$$

There is also a Yano's extrapolation theorem concerning weak-type spaces. In 1996, Antonov [2] proved that there is almost everywhere convergence for the Fourier series of every function in $L \log L \log_3 L(\mathbb{T})$, where $\mathbb{T}$ represents the unit circle and, for an arbitrary σ-finite measure μ,

$$||f||_{L \log L \log_3 L(\mu)} = \int_0^\infty f_\mu^*(t) \log_1 \frac{1}{t} \log_3 \frac{1}{t}\, dt < \infty,$$

with

$$\log_1 t = 1 + \log^+ t \quad \text{and} \quad \log_k t = \log_1 \log_{k-1} t \ \text{ for } k > 1, \qquad t > 0. \tag{1.3}$$

Indeed, even though he did not write it explicitly, behind his ideas there is an extrapolation argument (see [3, 5] for more details). Before we make its statement precise, let us recall that, given $1 \le p < \infty$ and $0 < q < \infty$, the Lorentz spaces $L^{p,q}(\mu)$ are defined as the set of μ-measurable functions f such that

$$\|f\|_{L^{p,q}(\mu)} = \left(\int_0^\infty t^{\frac{q}{p}-1} f_\mu^*(t)^q\, dt\right)^{1/q} = \left(p \int_0^\infty t^{q-1} \lambda_f^\mu(y)^{\frac{q}{p}}\, dy\right)^{1/q} < \infty,$$

and

$$\|f\|_{L^{p,\infty}(\mu)} = \sup_{t>0} t^{\frac{1}{p}} f_\mu^*(t) = \sup_{y>0} y\lambda_f^\mu(y)^{\frac{1}{p}} < \infty.$$

Theorem 1.3 *If T is a sublinear operator such that for some $\alpha > 0$ and for every $1 < p \le p_0$,*

$$T : L^{p,1}(\mu) \to L^{p,\infty}(\nu), \qquad \frac{C}{(p-1)^\alpha},$$

then

$$T : L(\log L)^\alpha \log_3 L(\mu) \to L^{1,\infty}(\nu).$$

Since all the spaces mentioned above are rearrangement invariant, all the results could be also obtained if we find a good estimate for the function $(Tf)^*$ and, moreover, in this case, we can deduce boundedness of T in other rearrangement invariant spaces as well. At this point, we should recall the following well-known pointwise estimate:

Theorem 1.4 [4,Ch. 4.4 Theorem 4.11] *Let $1 \le p_0 < p_1 < \infty$. A sublinear operator T satisfies that*

$$T : L^{p_0,1}(\mu) \to L^{p_0,\infty}(\nu) \qquad \text{and} \qquad T : L^{p_1,1}(\mu) \to L^{p_1,\infty}(\nu),$$

if and only if, for every $t > 0$ and every μ-measurable function f,

$$(Tf)_\nu^*(t) \lesssim \frac{1}{t^{\frac{1}{p_0}}} \int_0^t f_\mu^*(s) \frac{ds}{s^{1-\frac{1}{p_0}}} + \frac{1}{t^{\frac{1}{p_1}}} \int_t^\infty f_\mu^*(s) \frac{ds}{s^{1-\frac{1}{p_1}}}.$$

Moreover, with the goal of finding interesting pointwise estimates for the function $(Tf)^*$ under weaker conditions on T, the following results have been recently proved in [1] (see Definition 2.1 for the notion of admissible function):

Theorem 1.5 *Let T be a sublinear operator and φ some admissible function. For every $1 \le p < \infty$,*

$$T : L^{p,1}(\mu) \to L^{p,\infty}(\nu), \qquad C\varphi(p),$$

if and only if, for every $t > 0$ and every μ-measurable function f,

$$(Tf)_\nu^*(t) \lesssim \frac{1}{t} \int_0^t f_\mu^*(s)\, ds + \int_t^\infty \left(1 + \log \frac{s}{t}\right)^{-1} \varphi\left(1 + \log \frac{s}{t}\right) f_\mu^*(s) \frac{ds}{s}.$$

If the boundedness information is only for $p \ge p_0 > 1$, we also have the following result (see [1]).

Theorem 1.6 *Take $1 \le p_0 < p_1 \le \infty$ and let T be a sublinear operator and φ some admissible function. Assume that for every $p_0 \le p < p_1$,*

$$T : L^{p,1}(\mu) \to L^{p,\infty}(\nu), \qquad C\varphi\left(\left[\frac{1}{p} - \frac{1}{p_1}\right]^{-1}\right).$$

Then, for every $t > 0$ and every μ-measurable function f:

(i) *If $p_1 < \infty$,*

$$(Tf)^*_\nu(t) \lesssim \frac{1}{t^{\frac{1}{p_0}}} \int_0^t f^*_\mu(s) \frac{ds}{s^{1-\frac{1}{p_0}}} + \frac{1}{t^{\frac{1}{p_1}}} \int_t^\infty \varphi\left(1 + \log\frac{s}{t}\right) f^*_\mu(s) \frac{ds}{s^{1-\frac{1}{p_1}}}. \tag{1.4}$$

(ii) *If $p_1 = \infty$,*

$$(Tf)^*_\nu(t) \lesssim \frac{1}{t^{\frac{1}{p_0}}} \int_0^t f^*_\mu(s) \frac{ds}{s^{1-\frac{1}{p_0}}} + \int_t^\infty \left(1 + \log\frac{s}{t}\right)^{-1} \varphi\left(1 + \log\frac{s}{t}\right) f^*_\mu(s) \frac{ds}{s}. \tag{1.5}$$

Conversely, if (1.4) holds then, for every $p_0 \le p < p_1$,

$$||T||_{L^{p,1}(\mu)\to L^{p,\infty}(\nu)} \lesssim \left[\frac{1}{p} - \frac{1}{p_1}\right]^{-1} \varphi\left(\left[\frac{1}{p} - \frac{1}{p_1}\right]^{-1}\right),$$

while if (1.5) holds, $||T||_{L^{p,1}(\mu)\to L^{p,\infty}(\nu)} \lesssim \varphi(p)$.

We observe that, if $\varphi(p) = p$ (which is an admissible function) and $p_1 = \infty$, then

$$(Tf)^{**}_\nu(t) \lesssim \frac{1}{t^{\frac{1}{p_0}}} \int_0^t f^{**}_\mu(s) \frac{ds}{s^{1-\frac{1}{p_0}}} + \int_t^\infty f^*_\mu(s) \frac{ds}{s},$$

and hence

$$\begin{aligned} \sup_{0<t<1} \frac{(Tf)^{**}_\nu(t)}{1 + \log\frac{1}{t}} &\lesssim \|f\|_\infty + \int_1^\infty f^*_\mu(s)\frac{ds}{s} + \sup_{0<t<1} \frac{1}{1+\log\frac{1}{t}} \int_t^1 f^*_\mu(s)\frac{ds}{s} \\ &\lesssim \|f\|_\infty + \int_1^\infty f^*_\mu(s)\frac{ds}{s} \simeq \|f\|_\infty + \int_1^\infty f^{**}_\mu(s)\frac{ds}{s}, \end{aligned}$$

and we recover (1.2).

Our goal in this note is to prove results similar to those in Theorems 1.5 and 1.6, to obtain extensions of Yano's extrapolation results. Moreover, we want to emphasize here that we obtain stronger results with a simpler proof because contrary to what happens in the proof of the above results of Yano and Zygmund, where the function

f is decomposed in an infinite sum of functions f_n, our proof follows the idea of Theorem 1.4 where the function f is decomposed as the sum of just two functions.

The paper is organized as follows. In Sect. 2, we present previous results, the necessary definitions and some technical lemmas which shall be used later on, and Sect. 3 contains our main results.

2 Definitions, Previous Results and Lemmas

2.1 Admissible Functions

Definition 2.1 [1] A function $\varphi : [1, \infty] \to [1, \infty]$ is said to be *admissible* if it satisfies the following conditions:

(a) $\varphi(1) = 1$ and φ is log-concave, that is

$$\theta \log \varphi(x) + (1 - \theta) \log \varphi(y) \le \log \varphi(\theta x + (1 - \theta) y), \quad \forall x, y \ge 1, \ 0 \le \theta \le 1.$$

(b) There exist $\gamma, \beta > 0$ such that for every $x \ge 1$,

$$\frac{\gamma}{x} \le \frac{\varphi'(x)}{\varphi(x)} \le \frac{\beta}{x}. \tag{2.1}$$

Observe that (2.1) implies that φ is increasing, as well as that

$$x^{\gamma} \le \varphi(x) \le x^{\beta}.$$

Besides, since for every $x, y \ge 1$,

$$\log \varphi(xy) = \int_1^y (\log \varphi)'(s)\, ds + \int_y^{xy} (\log \varphi)'(s) ds \le \log \varphi(y) + \beta \log x,$$

it also holds that

$$\varphi(xy) \le x^{\beta} \varphi(y). \tag{2.2}$$

Example 2.2 Given $m \in \mathbb{N}$ and using the notation in (1.3), if $\gamma > 0$ and $\beta_1, \ldots, \beta_m \ge 0$, the function

$$\varphi(x) = x^{\gamma} \prod_{k=1}^{m} \left(\log_k x\right)^{\beta_k}, \qquad x \ge 1,$$

is admissible.

The next lemma is a simple computation for admissible functions which shall be fundamental in the proof of our main results.

Lemma 2.3 *Let φ be an admissible function. For $x \in \mathbb{R}$ and $1 \leq q_0 < \infty$,*

$$\inf_{q \in [q_0, \infty)} \varphi(q) e^{-\frac{x}{q}} \leq \begin{cases} \varphi(q_0) e^{-\frac{x}{q_0}}, & \text{if } x \geq 0, \\ q_0^{\beta} e^{\frac{1}{q_0}} \varphi(1-x), & \text{if } x < 0. \end{cases}$$

Proof If $x \geq 0$, the infimum is attained at $q = q_0$, and if $x < 0$, we take $q = q_0(1-x)$ and make use of (2.2). □

2.2 Calderón Type Operators

Definition 2.4 Let $1 \leq p_0, p_1 \leq \infty$ and let φ be an admissible function. Then, for every positive and real valued measurable function f and $t > 0$, we define

$$P_{p_0,\varphi} f(t) := \frac{1}{t^{\frac{1}{p_0}}} \int_0^t \varphi\left(1 - \log \frac{s}{t}\right) f(s) \frac{ds}{s^{1-\frac{1}{p_0}}},$$
$$Q_{p_1} f(t) := \frac{1}{t^{\frac{1}{p_1}}} \int_t^\infty f(s) \frac{ds}{s^{1-\frac{1}{p_1}}},$$

and

$$R_{p_0,p_1,\varphi} f(t) := P_{p_0,\varphi} f(t) + Q_{p_1} f(t).$$

In particular, if $p_0 = 1$, $p_1 = \infty$, and $\varphi(x) = 1$, we recover the Calderón operator [4]

$$Rf(t) := Pf(t) + Qf(t), \qquad t > 0,$$

where P and Q are respectively the Hardy operator and its conjugate

$$Pf(t) = \frac{1}{t} \int_0^t f(s)\, ds, \qquad Qf(t) = \int_t^\infty f(s) \frac{ds}{s}, \qquad t > 0.$$

We observe that, in general,

$$R_{p_0,p_1,\varphi} f(t) = \int_0^1 \varphi(1 - \log s) f(st) \frac{ds}{s^{1-\frac{1}{p_0}}} + \int_1^\infty f(st) \frac{ds}{s^{1-\frac{1}{p_1}}}, \qquad t > 0. \tag{2.3}$$

Lemma 2.5 *Let $1 \leq p_0, p_1 \leq \infty$. For an arbitrary measure μ and every μ-measurable function f,*

$$R_{p_0,p_1,\varphi}(f_\mu^*)^{**}(t) = R_{p_0,p_1,\varphi}(f_\mu^{**})(t), \qquad t > 0.$$

Proof By (2.3), clearly, $R_{p_0,p_1,\varphi}(f_\mu^*)$ is a decreasing function. Then, it holds that

$$R_{p_0,p_1,\varphi}(f_\mu^*)^{**}(t) = P\big(R_{p_0,p_1,\varphi}(f_\mu^*)\big)(t), \qquad t > 0,$$

and the result follows immediately by Fubini's theorem. □

3 Main Results

Throughout this section, if not specified, $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ are two arbitrary measure spaces.

Theorem 3.1 *Take $1 < p_0 < p_1 < \infty$ and let T be a sublinear operator and φ some admissible function. If for every $p_0 < p \le p_1$,*

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu), \qquad C\varphi\left(\left[\frac{1}{p_0} - \frac{1}{p}\right]^{-1}\right), \tag{3.1}$$

then, for every $t > 0$ and every μ-measurable function f,

$$\begin{aligned}(Tf)_\nu^*(t) &\lesssim \frac{1}{p_0 - 1}\left(\frac{1}{t^{\frac{1}{p_0}}}\int_0^t \varphi\left(1 - \log\frac{r}{t}\right) f_\mu^*(r)\frac{dr}{r^{1-\frac{1}{p_0}}} + \frac{1}{t^{\frac{1}{p_1}}}\int_t^\infty f_\mu^*(r)\frac{dr}{r^{1-\frac{1}{p_1}}}\right)\\ &= \frac{1}{p_0 - 1} R_{p_0,p_1,\varphi}(f_\mu^*)(t).\end{aligned} \tag{3.2}$$

Conversely, if $(Tf)_\nu^*(t) \lesssim R_{p_0,p_1,\varphi}(f_\mu^*)(t)$ for every $t > 0$, then, for each $p_0 < p \le p_1$, (3.1) holds.

Proof First assume that (3.1) applies for every $p_0 < p \le p_1$. Then, if $f = \chi_E$ for some μ-measurable set $E \subseteq \mathbb{R}^n$ such that $\mu(E) < \infty$, for every $t > 0$,

$$\begin{aligned}R_{p_0,p_1,\varphi}(f_\mu^*)(t) &= \frac{1}{t^{\frac{1}{p_0}}}\int_0^t \varphi\left(1 - \log\frac{r}{t}\right)\chi_{(0,\mu(E))}(r)\frac{dr}{r^{1-\frac{1}{p_0}}} + \frac{1}{t^{\frac{1}{p_1}}}\int_t^\infty \chi_{(0,\mu(E))}(r)\frac{dr}{r^{1-\frac{1}{p_1}}}\\ &= \left(\frac{1}{t^{\frac{1}{p_0}}}\int_0^t \varphi\left(1 - \log\frac{r}{t}\right)\frac{dr}{r^{1-\frac{1}{p_0}}} + \frac{1}{t^{\frac{1}{p_1}}}\int_t^{\mu(E)} \frac{dr}{r^{1-\frac{1}{p_1}}}\right)\chi_{(0,\mu(E))}(t)\\ &\quad + \left(\frac{1}{t^{\frac{1}{p_0}}}\int_0^{\mu(E)} \varphi\left(1 - \log\frac{r}{t}\right)\frac{dr}{r^{1-\frac{1}{p_0}}}\right)\chi_{(\mu(E),\infty)}(t)\\ &\ge p_0\left[\left(\frac{\mu(E)}{t}\right)^{\frac{1}{p_1}}\chi_{(0,\mu(E))}(t) + \varphi\left(1 - \log\frac{\mu(E)}{t}\right)\left(\frac{\mu(E)}{t}\right)^{\frac{1}{p_0}}\chi_{(\mu(E),\infty)}(t)\right],\end{aligned}$$

where in the last estimate we have used that $p_1 > p_0$, $\varphi(1) = 1$ and that $\varphi\left(1 - \log\frac{s}{t}\right)$ is a decreasing function on $s \in (0, t)$.

Hence, since by hypothesis, for every $p_0 < p \le p_1$ we have that

$$(T\chi_E)^*_\nu(t) \le C\varphi\left(\left[\frac{1}{p_0}-\frac{1}{p}\right]^{-1}\right)\left(\frac{\mu(E)}{t}\right)^{\frac{1}{p}} = C\left[\varphi\left(q\right)\left(\frac{\mu(E)}{t}\right)^{-\frac{1}{q}}\right]\left(\frac{\mu(E)}{t}\right)^{\frac{1}{p_0}},\ t>0,$$

with $\frac{1}{q}=\frac{1}{p_0}-\frac{1}{p}$, the result for characteristic functions plainly follows by taking the infimum for $q \in \left[\frac{p_1p_0}{p_1-p_0},\infty\right)$ (see Lemma 2.3) since then, for every $t>0$,

$$\begin{aligned}(T\chi_E)^*_\nu(t) &\lesssim \left(\frac{\mu(E)}{t}\right)^{\frac{1}{p_1}}\chi_{(0,\mu(E))}(t)+\varphi\left(1-\log\frac{\mu(E)}{t}\right)\left(\frac{\mu(E)}{t}\right)^{\frac{1}{p_0}}\chi_{(\mu(E),\infty)}(t)\\ &\lesssim R_{p_0,p_1,\varphi}((\chi_E)^*_\mu)(t).\end{aligned}$$

The extension to simple functions with sets of finite measure with respect to μ follows the same lines as the proof of Theorem III.4.7 of [4]. We include the computations adapted to our case for the convenience of the reader. First of all, consider a positive simple function

$$f=\sum_{j=1}^{n}a_j\chi_{F_j},$$

where $F_1\subseteq F_2\subseteq\cdots\subseteq F_n$ have finite measure with respect to μ. Then

$$f^*_\mu=\sum_{j=1}^{n}a_j\chi_{[0,\mu(F_j))}.$$

Using what we have already proved for characteristic functions we get that for every $t>0$,

$$\begin{aligned}(Tf)^{**}_\nu(t) &\lesssim \sum_{j=1}^{n}a_j\left(T(\chi_{F_j})\right)^{**}_\nu(t) \lesssim \sum_{j=1}^{n}a_j\left(R_{p_0,p_1,\varphi}(\chi_{[0,\mu(F_j))})\right)^{**}(t)\\ &= \left(R_{p_0,p_1,\varphi}\left(\sum_{j=1}^{n}a_j\chi_{[0,\mu(F_j))}\right)\right)^{**}(t) = R_{p_0,p_1,\varphi}(f^*_\mu)^{**}(t).\end{aligned}$$

Further, since $R_{p_0,p_1,\varphi}(f^*_\mu)^{**}=R_{p_0,p_1,\varphi}(f^{**}_\mu)$ (see Lemma 2.5) we obtain that

$$(Tf)^{**}_\nu(t)\lesssim R_{p_0,p_1,\varphi}(f^{**}_\mu)(t),\qquad t>0. \tag{3.3}$$

Now fix $t>0$ and consider the set $E=\{x: f(x)>f^*_\mu(t)\}$. Using this set define

$$g=(f-f^*_\mu(t))\chi_E \quad\text{and}\quad h=f^*_\mu(t)\chi_E+f\chi_{E^c},$$

so that $f = g + h$ and

$$g^*_\mu(s) = (f^*_\mu(s) - f^*_\mu(t))\chi_{(0,t)}(s) \quad \text{and} \quad h^*_\mu(s) = \min\{f^*_\mu(s), f^*_\mu(t)\}, \qquad s > 0.$$

Since (3.1) holds with $p = p_1$, the corresponding weak inequality leads to

$$\begin{aligned}(Th)^*_\nu(t/2) &\lesssim \frac{1}{t^{\frac{1}{p_1}}}\int_0^\infty h^*_\mu(s)\,\frac{ds}{s^{1-\frac{1}{p_1}}} \lesssim f^*_\mu(t) + \frac{1}{t^{\frac{1}{p_1}}}\int_t^\infty f^*_\mu(s)\,\frac{ds}{s^{1-\frac{1}{p_1}}}\\ &\le P_{p_0,\varphi}(f^*_\mu)(t) + Q_{p_1}(f^*_\mu)(t),\end{aligned}$$

where we have used that $f^*_\mu(t) \le P_{p_0,\varphi}(f^*_\mu)(t)$.

On the other hand, on account of (3.3) we get

$$(Tg)^{**}_\nu(t) \lesssim R_{p_0,p_1,\varphi}(g^{**}_\mu)(t) = P_{p_0,\varphi}(g^{**}_\mu)(t) + Q_{p_1}(g^{**}_\mu)(t), \tag{3.4}$$

and for the first term of the right hand side of (3.4) we deduce that

$$\begin{aligned}P_{p_0,\varphi}(g^{**}_\mu)(t) &= \frac{1}{t^{\frac{1}{p_0}}}\int_0^t \varphi\left(1-\log\frac{s}{t}\right)\frac{1}{s}\int_0^s g^*_\mu(r)\,dr\,\frac{ds}{s^{1-\frac{1}{p_0}}}\\ &\le \frac{1}{t^{\frac{1}{p_0}}}\int_0^t \varphi\left(1-\log\frac{s}{t}\right)\int_0^s f^*_\mu(r)\,dr\,\frac{ds}{s^{2-\frac{1}{p_0}}}\\ &= \frac{1}{t^{\frac{1}{p_0}}}\int_0^t f^*_\mu(r)\int_r^t \varphi\left(1-\log\frac{s}{t}\right)\frac{ds}{s^{2-\frac{1}{p_0}}}\,dr\\ &\le \frac{p_0}{p_0-1}\left(\frac{1}{t^{\frac{1}{p_0}}}\int_0^t \varphi\left(1-\log\frac{r}{t}\right) f^*_\mu(r)\frac{dr}{r^{1-\frac{1}{p_0}}}\right)\\ &= \frac{p_0}{p_0-1}P_{p_0,\varphi}(f^*_\mu)(t),\end{aligned} \tag{3.5}$$

while for the second term

$$Q_{p_1}(g^{**}_\mu)(t) = \frac{1}{t^{\frac{1}{p_1}}}\int_t^\infty \frac{1}{s}\int_0^s g^*_\mu(r)\,dr\,\frac{ds}{s^{1-\frac{1}{p_1}}} \le \frac{p_1}{p_1-1}f^{**}_\mu(t) \le \frac{p_0}{p_0-1}P_{p_0,\varphi}(f^*_\mu)(t).$$

Thus,

$$(Tf)^*_\nu(t) \le 2(Tg)^{**}_\nu(t) + (Th)^*_\nu(t/2) \lesssim \frac{1}{p_0-1}R_{p_0,p_1,\varphi}(f^*_\mu)(t),$$

and the general case follows from the density of the simple functions in the μ-measurable ones and dividing a μ-measurable function in its positive and negative parts.

Conversely, assume that $(Tf)^*_\nu(t) \lesssim R_{p_0,p_1,\varphi}(f^*_\mu)(t)$ for every $t > 0$ and fix some $p \in (p_0, p_1]$. The operator $R_{p_0,p_1,\varphi}$ is a kernel operator; that is

$$R_{p_0,p_1,\varphi} f(t) = \int_0^\infty k(t,r) f(r) dr, \qquad t > 0,$$

where the kernel is

$$k(t,r) = \varphi\left(1 - \log \frac{r}{t}\right)\left(\frac{r}{t}\right)^{\frac{1}{p_0}} \chi_{[0,t)}(r)\frac{1}{r} + \left(\frac{r}{t}\right)^{\frac{1}{p_1}} \chi_{[t,\infty)}(r)\frac{1}{r}. \tag{3.6}$$

By virtue of [8,Theorem 3.3], the norm $\left\| R_{p_0,p_1,\varphi} \right\|_{L^{p,1}(\mu)\to L^{p,\infty}}$ can be estimated by

$$A_k := \sup_{t>0}\left(\sup_{s>0}\left(\frac{t}{s}\right)^{\frac{1}{p}} \int_0^s k(t,r)dr\right).$$

Now observe that for $\beta_0 = \max(1,\beta)$ and for every $0 < \alpha \le 1$, by means of (2.2),

$$\varphi(1 - \log x) \le \varphi\left(\frac{\beta_0}{\alpha} x^{-\frac{\alpha}{\beta_0}}\right) \le \varphi\left(\frac{\beta_0}{\alpha}\right) x^{-\frac{\alpha\beta}{\beta_0}} \le \beta_0^{\beta_0}\varphi\left(\frac{1}{\alpha}\right) x^{-\alpha}, \qquad 0 < x \le 1.$$

Take $\alpha = \frac{1}{p_0} - \frac{1}{p} \in (0,1)$. Hence, if $0 < s \le t$,

$$\int_0^s k(t,r)\,dr = \frac{1}{t^{\frac{1}{p_0}}}\int_0^s \varphi\left(1 - \log\frac{r}{t}\right)\frac{dr}{r^{1-\frac{1}{p_0}}} \le p_1\beta_0^{\beta_0}\varphi\left(\left[\frac{1}{p_0} - \frac{1}{p}\right]^{-1}\right)\left(\frac{s}{t}\right)^{\frac{1}{p}},$$

while if $s > t$, we obtain

$$\begin{aligned}\int_0^s k(t,r)\,dr &= \frac{1}{t^{\frac{1}{p_0}}}\int_0^t \varphi\left(1 - \log\frac{r}{t}\right)\frac{dr}{r^{1-\frac{1}{p_0}}} + \frac{1}{t^{\frac{1}{p_1}}}\int_t^s \frac{dr}{r^{1-\frac{1}{p_1}}}\\ &\le p_1\beta_0^{\beta_0}\varphi\left(\left[\frac{1}{p_0} - \frac{1}{p}\right]^{-1}\right) + p_1\left(\frac{s}{t}\right)^{\frac{1}{p}}.\end{aligned}$$

In consequence, we have that

$$A_k \le p_1\beta_0^{\beta_0}\varphi\left(\left[\frac{1}{p_0} - \frac{1}{p}\right]^{-1}\right)\sup_{t>0}\left(\sup_{s>t}\left(\frac{t}{s}\right)^{\frac{1}{p}}\left[1 + \left(\frac{s}{t}\right)^{\frac{1}{p}}\right]\right) = 2p_1\beta_0^{\beta_0}\varphi\left(\left[\frac{1}{p_0} - \frac{1}{p}\right]^{-1}\right).$$

□

Remark 3.2 It is worth mentioning that in order to prove that (3.1) implies (3.2), the only properties that we have used of φ are that φ is a nondecreasing function such that $\varphi(1) = 1$ and that for every constant $C \ge 1$, $\varphi(Cx) \approx \varphi(x)$.

As we mentioned in the introduction, one application of these pointwise estimates is to deduce extensions of Yano's extrapolation results as the following corollary shows. First, for an arbitrary measure μ, some exponent $1 \leq p < \infty$ and some admissible function φ, we define the function space $L^{p,1}\varphi(\log L)(\mu)$ as the set of μ-measurable functions f satisfying

$$\|f\|_{L^{p,1}\varphi(\log L)(\mu)} = \int_0^\infty \varphi\left(1+\log^+\frac{1}{r}\right) f_\mu^*(r)\frac{dr}{r^{1-\frac{1}{p}}} < \infty.$$

Corollary 3.3 *Take $1 < p_0 < p_1 < \infty$, and let φ be some admissible function. If T is a sublinear operator such that for every $p_0 < p \leq p_1$*

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu), \qquad C\varphi\left(\left[\frac{1}{p_0}-\frac{1}{p}\right]^{-1}\right),$$

and ν is a finite measure, then

$$T : L^{p_0,1}\varphi(\log L)(\mu) \to L^{p_0,\infty}(\nu), \qquad \frac{C}{p_0-1}.$$

Proof As a consequence of Theorem 3.1,

$$(Tf)_\nu^*(t) \lesssim \frac{1}{p_0-1}R_{p_0,p_1,\varphi}(f_\mu^*)(t), \qquad 0 < t < \nu(\mathcal{N}).$$

Further, by means of [8,Theorem 3.3],

$$\left\|R_{p_0,p_1,\varphi}\right\|_{L^{p_0,1}\varphi(\log L)(\mu)\to L^{p_0,\infty}(0,\nu(\mathcal{N}))} \lesssim \sup_{0<t<\nu(\mathcal{N})} t^{\frac{1}{p_0}}\left[\sup_{s>0}\frac{\int_0^s k(t,r)dr}{\int_0^s \varphi\left(1+\log^+\frac{1}{r}\right)\frac{dr}{r^{1-\frac{1}{p_0}}}}\right],$$

with $k(t,r)$ as in (3.6). Hence, if $0 < s \leq t$,

$$t^{\frac{1}{p_0}}\left[\frac{\int_0^s k(t,r)dr}{\int_0^s \varphi\left(1+\log^+\frac{1}{r}\right)\frac{dr}{r^{1-\frac{1}{p_0}}}}\right] = \frac{\int_0^s \varphi\left(1+\log\frac{t}{r}\right)\frac{dr}{r^{1-\frac{1}{p_0}}}}{\int_0^s \varphi\left(1+\log^+\frac{1}{r}\right)\frac{dr}{r^{1-\frac{1}{p_0}}}} \leq \max\left(1, \nu(\mathcal{N})^{\frac{1}{p_0}}\right),$$

while if $s > t$, we obtain

$$t^{\frac{1}{p_0}} \left[\frac{\int_0^s k(t,r)dr}{\int_0^s \varphi\left(1+\log^+ \frac{1}{r}\right) \frac{dr}{r^{1-\frac{1}{p_0}}}} \right] = \frac{t^{\frac{1}{p_0}} \left(\frac{1}{t^{\frac{1}{p_0}}} \int_0^t \varphi\left(1+\log \frac{t}{r}\right) \frac{dr}{r^{1-\frac{1}{p_0}}} + p_1 \left[\left(\frac{s}{t}\right)^{\frac{1}{p_1}} - 1 \right] \right)}{\int_0^s \varphi\left(1+\log^+ \frac{1}{r}\right) \frac{dr}{r^{1-\frac{1}{p_0}}}}$$
$$\leq \frac{t^{\frac{1}{p_0}} \left(C_\varphi + p_1 \left(\frac{s}{t}\right)^{\frac{1}{p_1}} \right)}{p_0 s^{\frac{1}{p_0}}} \leq \frac{C_\varphi + p_1}{p_0},$$

so that $\left\| R_{p_0,p_1,\varphi} \right\|_{L^{p_0,1}\varphi(\log L)(\mu) \to L^{p_0,\infty}((0,\nu(\mathcal{N})),\, dx)} < \infty.$ □

Remark 3.4 For $p_0 = 1$, we observe that following the lines of the sufficiency of the proof of Theorem 3.1, the only place where we could have problems is in (3.5), since this estimate blows up as p_0 approximates 1^+. Nevertheless, easy computations show that then, for every $t > 0$ and every μ-measurable function f,

$$(Tf)_\nu^*(t) \lesssim \frac{1}{t} \int_0^t \left(1 - \log \frac{r}{t}\right) \varphi\left(1 - \log \frac{r}{t}\right) f_\mu^*(r)\, dr + \frac{1}{t^{\frac{1}{p_1}}} \int_t^\infty f_\mu^*(r) \frac{dr}{r^{1-\frac{1}{p_1}}}.$$

However, when $\varphi(x) = x^\alpha$, $\alpha > 0$, it can be deduced that for an arbitrary measure μ and a finite measure ν,

$$T : L(\log L)^{\alpha+1}(\mu) \to L^{1,\infty}(\nu),$$

which, as we have seen on the introduction, is far from the best results known up to now (see, for instance, Theorem 1.3).

Open Question

Can we extend our result to the case $p_0 = 1$ in an optimal way?

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