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Dirac Operators with Delta-Interactions on Smooth Hypersurfaces in $\mathbb{R}^n$

Vladimir Rabinovich[1]

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Abstract

We consider the Dirac operators with singular potentials

$$D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma} = \mathfrak{D}_{\boldsymbol{A},\Phi,m} + \Gamma\delta_\Sigma \tag{1}$$

where

$$\mathfrak{D}_{\boldsymbol{A},\Phi,m} = \sum_{j=1}^{n}\alpha_j\left(-i\partial_{x_j} + A_j\right) + \alpha_{n+1}m + \Phi I_N \tag{2}$$

is a Dirac operator on $\mathbb{R}^n$ with variable magnetic and electrostatic potentials $A = (A_1, \dots, A_n) \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$ and $\Phi \in L^\infty(\mathbb{R}^n)$, and the variable mass $m \in L^\infty(\mathbb{R}^n)$. In formula (2) α_j are the $N \times N$ Dirac matrices, that is $\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_N$, I_N is the unit $N \times N$ matrix, $N = 2^{[(n+1)/2]}$. In formula (1) $\Gamma\delta_\Sigma$ is a singular delta-type potential supported by a C^2-hypersurface $\Sigma \subset \mathbb{R}^n$ which is the common boundary of the open sets $\Omega_\pm$. Let $H^1(\Omega^\pm, \mathbb{C}^N)$ be the Sobolev spaces of N-dimensional vector-valued distributions $\boldsymbol{u}$ on $\Omega^\pm$, and

$$H^1(\mathbb{R}^n\diagdown\Sigma, \mathbb{C}^N) = H^1(\Omega_+, \mathbb{C}^N) \oplus H^1(\Omega_-, \mathbb{C}^N).$$

We associate with the formal Dirac operator $D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}$ an unbounded in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ defined by the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi,m}$ with domain $dom\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} \subset H^1(\mathbb{R}^n\diagdown\Sigma, \mathbb{C}^N)$ defined by an interaction conditions. The main

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Vladimir Rabinovich
vladimir.rabinovich@gmail.com

[1] Instituto Politécnico Nacional, ESIME Zacatenco, Mexico City, Mexico

aims of the paper are the study of self-adjointmess of the operators $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ for uniformly regular C^2-hypersurfaces $\Sigma \subset \mathbb{R}^n$ and the essential spectra of $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ for closed C^2-hypersurfaces $\Sigma \subset \mathbb{R}^n$.

Keywords Dirac operators · Singular potentials · Delta-interactions · Self-adjointness · Essential spectrum

Mathematics Subject Classification 35J10 · 47A10 · 47A53 · 81Q10

1 Introduction

This paper is devoted to the study of n-dimensional Dirac operators ($n \geq 2$) with singular potentials of δ-type supported on both bounded and unbounded hypersurfaces Σ in $\mathbb{R}^n$. Such Dirac operators arise as approximations of the Hamiltonians of interactions of relativistic quantum particles with potentials localized in thin tubular neighborhoods of Σ (see, for instance, [14, 26, 30]). In physical formulations, such problems describe the transitions of relativistic particles with $spin(1/2)$ through the obstacles created by the potentials supported on the mentioned regions in $\mathbb{R}^3$. Moreover, these problems are associated with the MIT BAG models of the particle confinement in domains of $\mathbb{R}^3$ (see [6, 9, 13, 15, 16, 21, 22]).

The formal Dirac operators with singular potentials are realized as unbounded operators in Hilbert spaces with domains described by interaction conditions on the sets carrying the singular potentials. In the last time appeared many papers devoted to their spectral properties for the dimensions $n = 2, 3$, see for instance, [4, 7, 8, 10–12, 14, 20, 29–31, 36, 37]. We also note that the paper [28] establishes a connection between the Dirac operators on bounded domains in $\mathbb{R}^n$ and their complements to the Dirac operators on their boundaries.

We consider the formal Dirac operators on $\mathbb{R}^n$ with singular potentials

$$D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma} = \mathfrak{D}_{\boldsymbol{A},\Phi,m} + \Gamma\delta_\Sigma \tag{3}$$

where

$$\begin{aligned}\mathfrak{D}_{\boldsymbol{A},\Phi,m} &= \boldsymbol{\alpha}\cdot(\boldsymbol{D} + \boldsymbol{A}) + m\alpha_{n+1} + \Phi I_N \\ &= \sum_{j=1}^{n}\alpha_j(D_j + A_j) + m\alpha_{n+1} + \Phi I_N,\ D_j = -i\partial_{x_j},\ j = 1,\dots,n\end{aligned} \tag{4}$$

is the Dirac operator on $\mathbb{R}^n$ with regular magnetic potentials $\boldsymbol{A} = (A_1,\dots,A_n)$, electrostatic potentials Φ, and the "variable mass" $m = m(x) = m_0 + \mathfrak{L}(x)$ where $m_0 \in \mathbb{R}$ is the mass of the particle, $\mathfrak{L}$ is the scalar Lorentz potential. In formula (4) α_j, $j = 1,\dots,n+1$ are the $N \times N$ Dirac matrices, that is the Hermitian matrices satisfying the relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_N, \tag{5}$$

I_N is the $N \times N$ unit matrix, $N = N(n) = 2^{[(n+1)/2]}$, $\Gamma\delta_\Sigma$ is the singular potential, $\Gamma = \left(\Gamma_{i,j}\right)_{i,j=1}^N$ is the $N \times N$ strength matrix and δ_Σ is the delta-function with the support on C^2-hypersurface $\Sigma \subset \mathbb{R}^n$ being the common boundary of open sets $\Omega_\pm \subset \mathbb{R}^n$.

We assume that $A_j, \Phi, m \in L^\infty(\mathbb{R}^n)$, and elements $\Gamma_{i,j}$ of Γ belong to the space $C_b^1(\Sigma)$ of bounded on Σ functions with their first derivatives, Σ is a C^2-uniformly regular hypersurface (see, Definition 3 of this paper, and the papers [3, 19]). It should be noted that the class of C^2-uniformly regular hypersurfaces contains all closed C^2-hypersurfaces and a wide set of unbounded C^2-hypersurfaces with cylindrical, conical, and oscillating exits to infinity.

Let $H^1(\Omega_\pm, \mathbb{C}^N)$ be the Sobolev spaces of distributions on $\Omega_\pm$ with values in $\mathbb{C}^N$ and $H^1(\mathbb{R}^n\diagdown\Sigma, \mathbb{C}^N) = H^1(\Omega_+, \mathbb{C}^N) \oplus H^1(\Omega_-, \mathbb{C}^N)$. We associate with the formal Dirac operator $D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}$ the unbounded in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ defined by the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi,m}$ with domain

$$\begin{aligned} &dom\, \mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} \\ &\quad = \left\{\boldsymbol{u} \in H^1(\mathbb{R}^n\diagdown\Sigma, \mathbb{C}^N) : \mathfrak{B}_\Sigma \boldsymbol{u}(s) = a_+(s)\gamma_\Sigma^+ \boldsymbol{u}(s) + a_-(s)\gamma_\Sigma^- \boldsymbol{u}(s) = 0, s \in \Sigma\right\} \end{aligned} \tag{6}$$

where $\gamma_\Sigma^\pm : H^1(\Omega_\pm, \mathbb{C}^N) \to H^{1/2}(\Sigma, \mathbb{C}^N)$ are the trace operators, and

$$a_\pm = \frac{1}{2}\Gamma \mp i\boldsymbol{\alpha} \cdot \boldsymbol{\nu} \;\text{ on } \Sigma, \tag{7}$$

$\boldsymbol{\alpha} \cdot \boldsymbol{\nu} = \sum_{j=1}^n \alpha_j \nu_j$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ is the field of unit normal vectors to Σ directed into Ω_-.

We also associate with the formal Dirac operator $D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}$ the bounded operator of the interaction (transmission) problem

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} u = \begin{cases} \mathfrak{D}_{A,\Phi,m}\boldsymbol{u} \text{ on } \mathbb{R}^n\diagdown\Sigma \\ \mathfrak{B}_\Sigma \boldsymbol{u} \text{ on } \Sigma \end{cases}$$

acting from $H^1(\mathbb{R}^n\diagdown\Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^n)$.

We study the self-adjointness in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ of unbounded operators $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$. Our approach is based on the study of the parameter-dependent operators

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu) = \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} - i\mu I_N, \mu \in \mathbb{R}.$$

We introduce the local Lopatinsky–Shapiro condition for $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ at the point $x \in \Sigma$ as follows

$$\begin{aligned} &\det(\boldsymbol{\alpha} \cdot \xi_x + \frac{\Gamma(x)}{2} - i\mu I_N) \neq 0, \\ &\quad \text{for every } (\xi_x.\mu) \in \mathbb{T}_x^*(\Sigma) \times \mathbb{R} : |\xi_x|^2 + \mu^2 = 1, \end{aligned} \tag{8}$$

where $\alpha \cdot \xi_x = \sum_{j=1}^{n} \alpha_j \xi_x^j$ and $\mathbb{T}_x^*(\Sigma)$ is the cotangent space to the hypersurface Σ at the point x.

If the matrix $\Gamma(x)$ is Hermitian condition (8) holds if and only if the parameter-independent Lopatinsky–Shapiro condition

$$\det(\boldsymbol{\alpha} \cdot \xi_x + \frac{\Gamma(x)}{2}) \neq 0 \text{ for every } \xi_x \in \mathbb{T}_x^*(\Sigma) : |\xi_x| = 1 \tag{9}$$

holds.

Remark 1 Another forms of the Lopatinsky–Shapiro conditions for the dimensions $n = 2, 3$ were obtained in [36, 37]. However, the Lopatinsky–Shapiro conditions (8),(9) serve for every $n \geq 2$ and easily checked in important examples of singular potentials.

The following results are obtained in the paper.

- The operator

  $$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu) : H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$$

  is invertible for a large enough $|\mu|$ if $A_j, \Phi, m \in L^\infty(\mathbb{R}^n)$, $\Gamma_{i,j} \in C_b^1(\Sigma)$, Σ is a uniformly regular C^2-hypersurface and the local parameter-dependent Lopatinsky–Shapiro condition (8) is satisfied uniformly on Σ.
- Applying this result we obtained that the operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ is self-adjoint in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ if A_j, Φ, m are the real-valued functions, $\Gamma(x)$ is the Hermitian matrix for every $x \in \Sigma$, and the Lopatinsky–Shapiro condition (9) is satisfied uniformly on Σ.
- As an example, we consider the Dirac operator on $\mathbb{R}^n$ with singular potentials describing by the electrostatic and scalar Lorentz δ-shell interaction on the uniformly regular C^2-hypersurfaces $\Sigma \subset \mathbb{R}^n$ with the strength matrices

  $$\Gamma(s) = \eta(s) I_N + \tau(s)\alpha_{n+1}, s \in \Sigma$$

  where $\eta, \tau \in C_b^1(\Sigma)$ are real-valued functions. We proved that the condition

  $$\inf_{s \in \Sigma} \left| \eta^2(s) - \tau^2(s) - 4 \right| > 0 \tag{10}$$

  yields the uniform Lopatinsky–Shapiro condition (9) which insures the self-adjointness of $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ in $L^2(\mathbb{R}^n, \mathbb{C}^N), n \geq 2$ under above given conditions for $\boldsymbol{A}, \Phi, m, \Sigma$.
- It should be noted that the condition (10) of self-adjointnes was obtained earlier for $n = 3$ in the paper [11], see also the paper [10] for the electrostatic shell interactions in $\mathbb{R}^3$. The generalization of condition (10) for the electrostatic, scalar Lorentz, and magnetic δ-shell interaction was obtained in [12] for the dimension 2.

- We also study the essential spectrum of the operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ for closed C^2-hypersurfaces Σ under conditions: $A_j, \Phi, m \in C_b^1(\mathbb{R}^n)$ ($C_b^1(\mathbb{R}^n)$ is the space of differentiable functions on $\mathbb{R}^n$ bounded with their first partial derivatives), $\Gamma_{i,j} \in C^1(\Sigma)$, and local Lopatinsky–Shapiro condition (9) is satisfied at every point $x \in \Sigma$. In this case the essential spectrum of $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ depends on the behavior of the potentials $\mathbf{A}$, Φ, m at infinity, and is given by the formula

$$sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} = \bigcup_h sp\, \mathfrak{D}^h \tag{11}$$

where $\mathfrak{D}^h = \mathfrak{D}_{\mathbf{A}^h,\Phi^h,m^h}$ are the so-called limit operators for $\mathfrak{D}_{\mathbf{A},\Phi,m}$ which are the Dirac operators with the potentials $\mathbf{A}^h$, Φ^h, m^h defined by the sequences $\mathbb{Z}^n \ni h_k \to \infty$ as follows

$$\begin{aligned} \mathbf{A}^h(x) &= \lim_{k\to\infty} \mathbf{A}^h(x+h_k),\ \Phi^h(x\ ) = \lim_{k\to\infty} \Phi^h(x+h_k), \\ m^{\mathbf{h}}(x) &= \lim_{k\to\infty} m^h(x+h_k). \end{aligned} \tag{12}$$

The limits in (12) are understood in the sense of the uniform convergence on compact sets in $\mathbb{R}^n$ (see, for instance, [32, 33]).
- Let $A_j, \Phi, m \in C_b^1(\mathbb{R}^n)$ be the slowly oscillating at infinity functions, that is their partial derivatives tend to zero at infinity. Then $\mathfrak{D}^h = \mathfrak{D}_{\mathbf{A}^h,\Phi^h,m^h}$ are the Dirac operators on $\mathbb{R}^n$ with constant coefficients. Assuming that A_j, Φ, m are real-valued functions and applying formula (11) we obtained the explicit description of the essential spectrum of the operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$

$$\begin{aligned} &sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} \\ &= \left(-\infty, \limsup_{x\to\infty}(\Phi(x) - |m(x)|)\right] \bigcup \left[\liminf_{x\to\infty}(\Phi(x) + |m(x)|), +\infty\right). \end{aligned} \tag{13}$$

Remark 2 The approaches described above to the investigation of self-adjointness and essential spectra of Schrödinger operators on $\mathbb{R}^n$ and Dirac operators on $\mathbb{R}^2$, $\mathbb{R}^3$ with singular potentials were previously used in the papers [34–37].

2 Notations and Auxiliary Material

2.1 Notations

- If $\mathbf{X}, \mathbf{Y}$ are Banach spaces then we denote by $\mathcal{B}(\mathbf{X}, \mathbf{Y})$ the space of bounded linear operators acting from $\mathbf{X}$ into $\mathbf{Y}$ with the uniform operator topology, and by $\mathcal{K}(\mathbf{X}, \mathbf{Y})$ the subspace of $\mathcal{B}(\mathbf{X}, \mathbf{Y})$ of all compact operators. In the case $\mathbf{X} = \mathbf{Y}$ we write shortly $\mathcal{B}(\mathbf{X})$ and $\mathcal{K}(\mathbf{X})$.
- An operator $A \in \mathcal{B}(\mathbf{X}, \mathbf{Y})$ is called a Fredholm operator if $kerA$, and $cokerA = \mathbf{Y}/\operatorname{Im} A$ are finite dimensional spaces. Let $\mathcal{A}$ be a closed unbounded

operator in a Hilbert space $\mathcal{H}$ with a dense in $\mathcal{H}$ domain $dom\mathcal{A}$. Then $\mathcal{A}$ is called a Fredholm operator if $ker\mathcal{A} = \{u \in dom\mathcal{A} : \mathcal{A}u = 0\}$ and $cokerA = \mathcal{H}/\operatorname{Im}\mathcal{A}$ where $\operatorname{Im}\mathcal{A}=\{\boldsymbol{w} \in \mathcal{H} : \boldsymbol{w} = \mathcal{A}u, u \in \mathcal{D}_{\mathcal{A}}\}$ are the finite-dimensional spaces. Note that $\mathcal{A}$ is a Fredholm operator as the unbounded operator in $\mathcal{H}$ if and only if $\mathcal{A} : dom\mathcal{A} \to \mathcal{H}$ is a Fredholm operator as the bounded operator where $dom\mathcal{A}$ is equipped by the graph norm

$$\|u\|_{dom\mathcal{A}} = \left(\|u\|^2_{\mathcal{H}} + \|\mathcal{A}u\|^2_{\mathcal{H}}\right)^{1/2}, u \in dom\mathcal{A}$$

(see for instance [1]).

- The essential spectrum $sp_{ess}\mathcal{A}$ of an unbounded operator $\mathcal{A}$ is a set of $\lambda \in \mathbb{C}$ such that $\mathcal{A} - \lambda I$ is not the Fredholm operator as the unbounded operator, and the discrete spectrum $sp_{dis}\mathcal{A}$ of $\mathcal{A}$ is a set of isolated eigenvalues of finite multiplicity. It is well known that if $\mathcal{A}$ is a self-adjoint operator then $sp_{dis}\mathcal{A}=sp\mathcal{A}\diagdown sp_{ess}\mathcal{A}$.
- We denote by $L^2(\mathbb{R}^n, \mathbb{C}^N)$ the Hilbert space of N-dimensional vector-valued functions $\boldsymbol{u}(x) = (u^1(x), \dots, u^N(x)), x \in \mathbb{R}^n$ with the scalar product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = \int_{\mathbb{R}^n} \boldsymbol{u}(x) \cdot \boldsymbol{v}(x)dx,$$

and by $L^2(\measuredangle, \mathbb{C}^N)$ the Hilbert space with the scalar product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{L^2(\Sigma,\mathbb{C}^N)} = \int_{\Sigma} \boldsymbol{u}(s) \cdot \boldsymbol{v}(s)ds$$

where ds is the Lebesgue measure on Σ, and $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{j=1}^n u_j \bar{v}_j$.

- We denote by $H^s(\mathbb{R}^n, \mathbb{C}^N)$ the Sobolev space of vector-valued distributions $\boldsymbol{u} \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^N)$ such that

$$\|\boldsymbol{u}\|_{H^s(\mathbb{R}^n,\mathbb{C}^N)} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s \left\|\hat{\boldsymbol{u}}(\xi)\right\|^2_{\mathbb{C}^N} d\xi\right)^{1/2} < \infty, s \in \mathbb{R}$$

where $\hat{\boldsymbol{u}}$ is the Fourier transform of $\boldsymbol{u}$. If Ω is a domain in $\mathbb{R}^n$ then $H^s(\Omega, \mathbb{C}^N)$ is the space of restrictions of $\boldsymbol{u} \in H^s(\mathbb{R}^n, \mathbb{C}^N)$ on Ω with the norm

$$\|\boldsymbol{u}\|_{H^s(\Omega,\mathbb{C}^N)} = \inf_{l\boldsymbol{u}\in H^s(\mathbb{R}^n,\mathbb{C}^N)} \|l\boldsymbol{u}\|_{H^s(\mathbb{R}^n,\mathbb{C}^N)}$$

where $l\boldsymbol{u}$ is an extension of $\boldsymbol{u}$ on $\mathbb{R}^n$. If Σ is an enough smooth hypersurface in $\mathbb{R}^n$ we denote by $H^{s-1/2}(\Sigma, \mathbb{C}^N)$ the space of restrictions on Σ the distributions in $H^s(\mathbb{R}^n, \mathbb{C}^N), s > 1/2$.

- We denote by $C_b(\mathbb{R}^n)$ the class of bounded continuous functions on $\mathbb{R}^n$, $C_b^m(\mathbb{R}^n)$ the class of functions a on $\mathbb{R}^n$ such that $\partial^\alpha a \in C_b(\mathbb{R}^n)$ for all multi-indeces $\alpha : |\alpha| \leq m$. We denote by $C_b^1(\Sigma)$ the class of differentiable on Σ functions that are bounded with their first derivatives.

Definition 3 Let a C^2-hypersurface $\Sigma \subset \mathbb{R}^n, n \geq 2$ be the common boundary of the domains $\Omega_\pm$. We say that Σ is *uniformly regular* (see for instance [3, 19]) if : (i) there exists $r > 0$ such that for every point $x_0 \in \Sigma$ there exists a ball $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ and the diffeomorphism $\varphi_{x_0} : B_r(x_0) \to B_1(0)$ such that

$$\varphi_{x_0}(B_r(x_0) \cap \Omega_\pm) = B_1(0) \cap \mathbb{R}^n_\pm, \mathbb{R}^n_\pm = \left\{ y = (y', y_n) \in \mathbb{R}^{n-1}_{y'} \times \mathbb{R}_{y_n} : y_n \gtrless 0 \right\},$$
$$\varphi_{x_0}(B_r(x_0) \cap \Sigma) = B_1(0) \cap \mathbb{R}^{n-1}_{y'};$$

(ii) let $\varphi^i_{x_0}, \psi^i_{x_0}, i = 1, \ldots, n$ be the coordinate functions of the mappings $\varphi_{x_0}, \varphi^{-1}_{x_0}$. Then

$$\sup_{x_0 \in \Sigma} \sup_{|\alpha| \leq 2, x \in B_r(x_0)} \left| \partial^\alpha \varphi^i_{x_0}(x) \right| < \infty, i = 1, \ldots, n;$$
$$\sup_{x_0 \in \Sigma} \sup_{|\alpha| \leq 2, x \in B_1(0)} \left| \partial^\alpha \psi^i_{x_0}(x) \right| < \infty, i = 1, \ldots, n.$$

Note that each closed C^2-hypersurfaces are uniformly regular.

2.2 Free Dirac Operator

Let

$$\mathfrak{D}_{\boldsymbol{\alpha},m} = \boldsymbol{\alpha} \cdot \boldsymbol{D}_x + \alpha_{n+1} m = \sum_{j=1}^{n} \alpha_j D_{x_j} + \alpha_{n+1} m, \, D_{x_j} = -i \partial_{x_j} \tag{14}$$

be the free n-dimensional Dirac operator ($n \geq 2$) where α_j, $j = 1, \ldots, n+1$ are the Dirac matrices, that is α_j are Hermitian $N \times N$ matrices satisfying the relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_N; \; j, k = 1, \ldots, n+1, \tag{15}$$

I_N is the unit $N \times N$ matrix, $N = 2^{[(n+1)/2]}, m \in \mathbb{C}$. (see for instance [17, 24]). Note that the Dirac matrices can be obtained by the induction starting from the Pauli matrices in the dimension 2 (see [24], Appendix).

Property (15) implies that

$$\mathfrak{D}^2_{\boldsymbol{\alpha},m} = (-\Delta_n + m^2) I_{\mathbb{N}} \tag{16}$$

where Δ_n is the n-dimensional Laplacian. Moreover, if $m \in \mathbb{R}$ the operator $\mathfrak{D}_{\boldsymbol{\alpha},m}$ with domain $H^1(\mathbb{R}^n.\mathbb{C}^N)$ is self-adjoint in $L^2(\mathbb{R}^n.\mathbb{C}^N)$ and

$$sp\mathfrak{D}_{\boldsymbol{\alpha},m} = (-\infty, -|m|] \cup [|m|, +\infty)$$

Proposition 4 (see, [23], p. 150) *Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a linear mapping given by the orthogonal matrix $T=(T_{kl})_{k,l=1}^n$. Then the change of variables: $y=Tx$ transforms the Dirac operator $\mathfrak{D}_{\alpha,m}$ into the Dirac operator*

$$\mathfrak{D}_{\tilde{\boldsymbol{\alpha}},m}=\tilde{\boldsymbol{\alpha}}\cdot\boldsymbol{D}_y+\alpha_{n+1}m$$

with the Dirac matrices $\tilde{\boldsymbol{\alpha}}=(\tilde{\alpha}_1,\ldots,\tilde{\alpha}_n)$ defined as

$$\tilde{\alpha}_k=\sum_{k=1}^{n}T_{jk}\alpha_j,\ k=1,\ldots,n.$$

3 Dirac Operators with Singular Potentials as Unbounded Operators

Let

$$D_{\boldsymbol{A},\Phi,m,\Gamma\delta}\boldsymbol{u}(x)=\left(\mathfrak{D}_{\boldsymbol{A},\Phi,m}+\Gamma\delta_\Sigma\right)\boldsymbol{u}(x),\ x\in\mathbb{R}^n$$

be the formal Dirac operator defined by formulas (3),(4). We assume that Σ is the C^2-hypersurface in $\mathbb{R}^n$, $A_j,\Phi,m\in L^\infty(\mathbb{R}^n)$, $\Gamma=\left(\Gamma_{i,j}\right)_{i,j=1}^N$, $\Gamma_{i,j}\in C_b^1(\Sigma)$.

We define the product $\Gamma\delta_\Sigma\boldsymbol{u}$ where $\boldsymbol{u}\in H^1(\mathbb{R}^n\diagdown\Sigma,\mathbb{C}^N)$ as a distribution in $\mathcal{D}'(\mathbb{R}^n,\mathbb{C}^N)=\mathcal{D}'(\mathbb{R}^n)\otimes\mathbb{C}^N$ acting on the test functions $\boldsymbol{\varphi}\in C_0^\infty(\mathbb{R}^n,\mathbb{C}^N)$ as

$$(\Gamma\delta_\Sigma\boldsymbol{u})(\boldsymbol{\varphi})=\frac{1}{2}\int_\Sigma\Gamma(s)\left(\gamma_\Sigma^+\boldsymbol{u}(s)+\gamma_\Sigma^-\boldsymbol{u}(s)\right)\cdot\boldsymbol{\varphi}(s)ds. \tag{17}$$

Integrating by parts and taking into account (17) we obtain that

$$\begin{aligned}\left\langle D_{\boldsymbol{A},\Phi,m,\Gamma\delta}\boldsymbol{u},\boldsymbol{\varphi}\right\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}&=\int_{\Omega_+\cup\Omega_-}\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u}(x)\cdot\boldsymbol{\varphi}(x)dx\\&-\int_\Sigma i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}(s)(\gamma_\Sigma^+\boldsymbol{u}(s)-\gamma_\Sigma^-\boldsymbol{u}(s))\cdot\boldsymbol{\varphi}(s))ds\\&+\frac{1}{2}\int_\Sigma\left(\Gamma(s)\left(\gamma_\Sigma^+\boldsymbol{u}(s)+\gamma_\Sigma^-\boldsymbol{u}(s)\right)\cdot\boldsymbol{\varphi}(s)\right)ds,\\&\boldsymbol{\varphi}\in C_0^\infty(\mathbb{R}^n,\mathbb{C}^N)\end{aligned} \tag{18}$$

where $\gamma_\Sigma^\pm:H^1(\Omega_\pm,\mathbb{C}^N)\to H^{1/2}(\Omega_\pm,\mathbb{C}^N)$ are the trace operators, $\boldsymbol{\nu}(s)=(\nu_1(s),\ldots,\nu_n(s))$ is the field of unit normal vectors directed to Ω_-. Formula (18) yields that in the distribution sense

$$D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}\boldsymbol{u}=\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u}\left[-i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\left(\gamma_\Sigma^+\boldsymbol{u}-\gamma_\Sigma^-\boldsymbol{u}\right)+\frac{1}{2}\Gamma\left(\gamma_\Sigma^+\boldsymbol{u}+\gamma_\Sigma^-\boldsymbol{u}\right)\right]\delta_\Sigma, \tag{19}$$

where $\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u}$ is the regular distribution given by the function $\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u}\in L^2(\mathbb{R}^n,\mathbb{C}^N)$. Formula (19) yields that $\mathfrak{D}_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}\boldsymbol{u}\in L^2(\mathbb{R}^n,\mathbb{C}^N)$ if and only if

$$- i\boldsymbol{\alpha} \cdot \boldsymbol{\nu} \left(\gamma_\Sigma^+ \boldsymbol{u} - \gamma_\Sigma^- \boldsymbol{u}\right) + \frac{1}{2}\Gamma \left(\gamma_\Sigma^+ \boldsymbol{u} + \gamma_\Sigma^- \boldsymbol{u}\right) = 0 \text{ on } \Sigma. \tag{20}$$

Condition (20) can be written of the form

$$\mathfrak{B}_\Sigma \boldsymbol{u} = a_+ \gamma_\Sigma^+ \boldsymbol{u} + a_- \gamma_\Sigma^- \boldsymbol{u} = \boldsymbol{0} \text{ on } \Sigma \tag{21}$$

where $a_\pm$ are $N \times N$ matrices:

$$a_\pm = \frac{1}{2}\Gamma \mp i\boldsymbol{\alpha} \cdot \boldsymbol{\nu} \quad \text{on } \Sigma. \tag{22}$$

We associate with the formal Dirac operator $D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}$ the unbounded in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ defined by the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi,m}$ with the domain

$$\begin{aligned} dom\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} &= H^1_{\mathfrak{B}_\Sigma}(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N) \\ &= \left\{u \in H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N) : \mathfrak{B}_\Sigma \boldsymbol{u} = \boldsymbol{0} \text{ on } \Sigma\right\}, \end{aligned} \tag{23}$$

and the bounded operator of interaction (transmission) problem

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} \boldsymbol{u} = \begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{u} \text{ on } \mathbb{R}^n \diagdown \Sigma, \\ \mathfrak{B}_\Sigma \boldsymbol{u} = a_+ \gamma_\Sigma^+ \boldsymbol{u} + a_- \gamma_\Sigma^- \boldsymbol{u} \text{ on } \Sigma \end{cases} \tag{24}$$

acting from $H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$.

4 Parameter-Dependent Interaction Problems

We consider the parameter-dependent operator

$$\begin{aligned} \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)\boldsymbol{u} &= \left(\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} - i\mu I_N\right)\boldsymbol{u} \\ &= \begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi,m}(i\mu)\boldsymbol{u} = (\mathfrak{D}_{\boldsymbol{A},\Phi,m} - i\mu I_N)\boldsymbol{u} \text{ on } \mathbb{R}^n \diagdown \Sigma, \\ \mathfrak{B}_\Sigma \boldsymbol{u} = a_+ \gamma_\Sigma^+ \boldsymbol{u} + a_- \gamma_\Sigma^- \boldsymbol{u} \text{ on } \Sigma \end{cases}, \mu \in \mathbb{R} \end{aligned} \tag{25}$$

acting from $H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$.

We assume as above that A_j, $j = 1, \ldots, n$, $\Phi, m \in L^\infty(\mathbb{R}^n)$, $\Gamma_{ij} \in C_b^1(\Sigma)$, $i, j = 1, \ldots, N$, where $\Sigma \subset \mathbb{R}^n$ is a uniformly regular C^2-hypersurface being the common boundary of domains $\Omega_\pm$. We consider the invertibility of the operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ for large values of $|\mu|$. We follows the seminal paper [2] where the parameter-dependent boundary value problems for bounded domains in $\mathbb{R}^n$ have been considered (see also [1]). It is convenient to consider the operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ as acting from the spaces $\boldsymbol{X}_{|\mu|} = H^1_{|\mu|}(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ into the space $\boldsymbol{Y}_{|\mu|} = L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}_{|\mu|}(\Sigma, \mathbb{C}^N)$ depending on the parameter $|\mu|$, where the parameter-dependent

Sobolev spaces $H^1_{|\mu|}(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$, $H^{1/2}_{|\mu|}(\Sigma, \mathbb{C}^N)$ are provided by the norms induced by

$$\|\boldsymbol{u}\|_{H^s_{|\mu|}(\mathbb{R}^n,\mathbb{C}^N)} = \left(\int_{\mathbb{R}^n} (1+\mu^2+|\xi|^2)^s \left\|\hat{\boldsymbol{u}}(\xi)\right\|^2_{\mathbb{C}^N} d\xi\right)^{1/2}.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, and $\boldsymbol{u} = (\boldsymbol{f}, \boldsymbol{\psi}) \in \boldsymbol{Y}_{|\mu|}$.Then we set $\varphi(\boldsymbol{f}, \boldsymbol{\psi}) = (\boldsymbol{f}, \boldsymbol{\psi})\varphi = (\varphi \boldsymbol{f}, \varphi \boldsymbol{\psi})$.

Proposition 5 (Local principle) *Assume that there exist numbers: $\mu_0 > 0$, $r > 0$, $M > 0$ such that for every $x \in \mathbb{R}^n$, $\mu \in \mathbb{R} : |\mu| \geq \mu_0$, and every ball $B_r(x)$, $x \in \mathbb{R}^n$ there exist operators $L_x(\mu), R_x(\mu) \in \mathcal{B}(\boldsymbol{Y}_{|\mu|}, \boldsymbol{X}_{|\mu|})$ such that for every $\varphi_x \in C_0^\infty(B_r(x))$*

$$L_x(\mu)\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)\varphi_x I = \varphi_x I,\ \varphi_x \mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu) R_x(\mu) = \varphi_x I, \tag{26}$$

and

$$\sup_{x\in\mathbb{R}^n,|\mu|\geq\mu_0} \|L_x(\mu)\|_{\mathcal{B}(\boldsymbol{Y}_{|\mu|},\boldsymbol{X}_{|\mu|})} \leq M,\quad \sup_{x\in\mathbb{R}^n,|\mu|\geq\mu_0} \|R_x(\mu)\|_{\mathcal{B}(\boldsymbol{Y}_{|\mu|},\boldsymbol{X}_{|\mu|})} \leq M. \tag{27}$$

Then there exists $\mu_1 \geq \mu_0$ such that the operator $\mathbb{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu) : \boldsymbol{X}_{|\mu|} \to \boldsymbol{Y}_{|\mu|}$ is invertible for every $\mu : |\mu| \geq \mu_1$.

Proof It follows from conditions of Proposition 5 that there exist $r > 0$, $\mu_0 > 0$, $M > 0$, and a countable subsystem $\left\{B_r(x_j)\right\}_{j\in\mathbb{N}}$ of the finite multiplicity $d \in \mathbb{N}$ of the system $\{B_r(x)\}_{x\in\mathbb{R}^n}$ such that $\mathbb{R}^n = \bigcup_{j\in\mathbb{N}} B_r(x_j)$. We introduce a partition of unity

$$\sum_{j\in\mathbb{N}} \theta_j(x) = 1,\ x \in \mathbb{R}^n \tag{28}$$

subordinated to the system $\left\{B_r(x_j)\right\}_{j\in\mathbb{N}}$ with $\theta_j \in C_0^\infty(B_r(x_j))$, $0 \leq \theta_j(x) \leq 1$, such that the sum $\sum_{j\in\mathbb{N}} \theta_j(x)$ contains for every $x \in \mathbb{R}^n$ not more than d nonzero terms. Let $\varphi_j \in C_0^\infty(B_r(x_j))$, $0 \leq \varphi_j(x) \leq 1$, and $\theta_j\varphi_j = \theta_j$. We set

$$L(\mu)\boldsymbol{f} = \sum_{j\in\mathbb{N}} \theta_j L_{x_j}(\mu)\varphi_j \boldsymbol{f},\ R(\mu)\boldsymbol{f} = \sum_{j\in\mathbb{N}} \varphi_j R_{x_j}(\mu)\theta_j \boldsymbol{f},\ \boldsymbol{f} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N). \tag{29}$$

Taking into account that the coverage $\left\{B_r(x_j)\right\}_{j\in\mathbb{N}}$ has the finite multiplicity d we obtain the estimates

$$\|L(\mu)\boldsymbol{f}\|_{\boldsymbol{X}_{|\mu|}} \leq Md\,\|\boldsymbol{f}\|_{\boldsymbol{Y}_{|\mu|}},\ \|R(\mu)\boldsymbol{f}\|_{\boldsymbol{X}_{|\mu|}} \leq Md\,\|\boldsymbol{f}\|_{\boldsymbol{Y}_{|\mu|}} \tag{30}$$

for every $\mu : |\mu| \geq \mu_0 > 0$, $\boldsymbol{f} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$ and the constants M, d independent of μ and $\boldsymbol{f}$. Estimates (30) yield that the operators $L(\mu)$, $R(\mu)$ can be extended to bounded operators from $\boldsymbol{Y}_\mu$ into $\boldsymbol{X}_\mu$ for every $\mu : |\mu| \geq \mu_0 > 0$. Let $\psi_j \in C_0^\infty(B_r(x_j))$, $0 \leq \psi_j(x) \leq 1$, $\varphi_j \in C_0^\infty(B_r(x_j))$, and $\varphi_j \psi_j = \varphi_j$. Then

$$\begin{aligned} L(\mu)\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu) &= \sum_{j\in\mathbb{N}} \theta_j L_{x_j}(\mu)\varphi_j \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)\psi_j \\ &= \sum_{j\in\mathbb{N}} \theta_j L_{x_j}(\mu)\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)\varphi_j + T_1(\mu) \end{aligned} \tag{31}$$

where

$$T_1(\mu) = \sum_{j\in\mathbb{N}} \theta_j L_{x_j}(\mu)\left[\varphi_j I, \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)\right]\psi_j I,$$

$$\left[\varphi_j, \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\right] = \varphi_j \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} - \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\varphi_j I.$$

Note that

$$\sum_{j\in\mathbb{N}} \theta_j L_{x_j}(\mu)\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\varphi_j I_{\boldsymbol{X}_{|\mu|}} = I_{\boldsymbol{X}_{|\mu|}}, \tag{32}$$

and

$$\begin{aligned} \|T_1(\mu)\|_{\mathcal{B}(\boldsymbol{X}_{|\mu|})} &\leq \sum_{j\in\mathbb{N}} \left\|\theta_j L_{x_j}(\mu)\left[\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}, \varphi_j\right]\psi_j I\right\|_{\mathcal{B}(\boldsymbol{X}_{|\mu|})} \\ &\leq d \sup_{j\in\mathbb{N}} \left\|L_{x_j}(\mu)\right\|_{\mathcal{B}(\boldsymbol{Y}_{|\mu|},\boldsymbol{X}_{|\mu|})} \left\|\left[\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}, \varphi_j I\right]\right\|_{\mathcal{B}(\boldsymbol{X}_{|\mu|},\boldsymbol{Y}_{|\mu|})}. \end{aligned} \tag{33}$$

Taking into account the inequality

$$\left\|\left[\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}, \varphi_j I\right]\right\|_{\mathcal{B}(\boldsymbol{X}_{|\mu|},\boldsymbol{Y}_{|\mu|})} \leq \frac{C}{|\mu|}, \, |\mu| > \mu_0 \tag{34}$$

with the constant $C > 0$ independent of j and μ we obtain that

$$\|T_1(\mu)\|_{\mathcal{B}(\boldsymbol{X}_{|\mu|})} \leq \frac{CdM}{|\mu|}, \, |\mu| \geq \mu_0. \tag{35}$$

Estimate (35) yields that the operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ has the left inverse operator

$$\mathbb{L}(\precnsim) = (I + T_1(\mu))^{-1} L(\mu)$$

for every $\mu \in \mathbb{R} : |\mu| \geq \mu_1 > \mu_0$ where μ_1 is large enough. In the same way we prove that there exists a right inverse operator $\mathbb{R}(\precnsim)$ of $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ for $|\mu| \geq \mu_1 > \mu_0$. □

Since the norms in $H^s_{|\mu|}(\mathbb{R}^n, \mathbb{C}^N)$ and $H^s(\mathbb{R}^n, \mathbb{C}^N)$ are equivalent, the operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ is invertible from $H^1(\mathbb{R}^n\diagdown\Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N)\oplus H^{1/2}(\Sigma, \mathbb{C}^N)$ for large enough $|\mu|$.

Proposition 5 reduces the invertibility of operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ to the local invertibility of $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ at every point $x\in\mathbb{R}^n$.

1^0. *Local invertibility at the points $x\in\mathbb{R}^n\diagdown\Sigma$.* For $x_0\in\mathbb{R}^n\diagdown\Sigma$ there exists a ball $B_r(x_0)$ such that

$$\begin{aligned}\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)\varphi I &= \mathfrak{D}_{\boldsymbol{A}\Phi,m}(i\mu)\varphi I\\ \varphi\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu) &= \varphi\mathfrak{D}_{\boldsymbol{A}\Phi,m}(i\mu)\end{aligned}$$

for every function $\varphi\in C_0^\infty(B_r(x))$. The main part of the parameter-dependent operator $\mathfrak{D}_{\boldsymbol{A},\boldsymbol{\Phi},\boldsymbol{m}}(i\mu)$ is $\mathfrak{D}^0_{\boldsymbol{\alpha}}(i\mu)=\boldsymbol{\alpha}\cdot\boldsymbol{D}-i\mu I_N$. Since $\mathfrak{D}^0_{\boldsymbol{\alpha}}(i\mu)\mathfrak{D}^0_{\boldsymbol{\alpha}}(-i\mu)=\left(-\Delta_n+\mu^2\right)I_N$ the operator $\mathfrak{D}^0_{\boldsymbol{\alpha}}(i\mu):H^1(\mathbb{R}^n, \mathbb{C}^N)\to L^2(\mathbb{R}^n, \mathbb{C}^n)$ is invertible for every real $\mu\neq 0$. Moreover,

$$\begin{aligned}&\left\|\mathfrak{D}_{\boldsymbol{A},\Phi,m}(i\mu)-\mathfrak{D}^0_{\boldsymbol{\alpha}}(i\mu)\right\|_{\mathcal{B}(H^1_\mu(\mathbb{R}^n,\mathbb{C}^N),L^2(\mathbb{R}^n,\mathbb{C}^N))}\\ &\quad\le\|\boldsymbol{\alpha}\cdot\boldsymbol{A}+\alpha_{n+1}m+\Phi I_N\|_{\mathcal{B}(H^1_{|\mu|}(\mathbb{R}^n,\mathbb{C}^N),L^2(\mathbb{R}^n,\mathbb{C}^N))}\le\frac{C}{1+|\mu|},\end{aligned}\tag{36}$$

with a constant $C>0$ independent of $\mu\in\mathbb{R}$. Estimate (36) implies that the operator $\mathfrak{D}_{\boldsymbol{A},\Phi,m}(i\mu)$ is invertible for $|\mu|$ large enough. Hence, for every $x\in\mathbb{R}^n\diagdown\Sigma$ there exist the locally inverses operators $L_x(\mu)$, $R_x(\mu)$ for $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ satisfying conditions of Proposition 5.

2^0. *Local invertibility at the points $x\in\Sigma$.* Passing to the local coordinates at the point $x_0\in\Sigma$ we obtain in the standard way (see [2]) that the operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ is locally invertible at the point $x_0\in\Sigma$ if the interaction operator for the half-spaces

$$\mathbb{D}^{x_0}_{\mathfrak{B}_{\mathbb{T}_{x_0}(\Sigma)}}(i\mu)u=\begin{cases}\mathfrak{D}(i\mu)\boldsymbol{u}=(\alpha\cdot D_x-i\mu I_N)\,\boldsymbol{u}\text{ on }\mathbb{R}^n\diagdown\mathbb{T}_{x_0}(\Sigma)\\ \mathfrak{B}_\Sigma(x_0)\boldsymbol{u}=a_+(x_0)\gamma^+_{\mathbb{T}_{x_0}(\Sigma)}\boldsymbol{u}+a_-(x^0)\gamma^-_{\mathbb{T}_{x_0}(\Sigma)}\boldsymbol{u}\text{ on }\mathbb{T}_{x_0}(\Sigma)\end{cases}$$
$$a_\pm(x_0)=\frac{1}{2}\Gamma(x_0)\mp i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}(x_0)$$

is invertible from $H^1(\mathbb{R}^n\diagdown\mathbb{T}_{x_0}(\Sigma), \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N)\oplus H^{1/2}(\mathbb{T}_{x_0}(\Sigma), \mathbb{C}^N)$ for every $\mu\in\mathbb{R}\diagdown 0$ where $\mathbb{T}_{x_0}(\Sigma)$ is the tangent space to Σ at the point x_0.

Let $\mathfrak{U}:\mathbb{R}^n\to\mathbb{R}^n$ be the orthogonal transformation such that $\mathfrak{U}(\mathbb{T}_{x_0}(\Sigma))=\mathbb{R}^{n-1}$ and $\mathfrak{U}(\boldsymbol{\nu}(x_0))=(0,\ldots,1)$. We set

$$\boldsymbol{w}(y)=\left(\mathfrak{U}^*\boldsymbol{u}\right)(y)=\boldsymbol{u}(\mathfrak{U}^{-1}y),\,\boldsymbol{u}(x)=\left(\mathfrak{U}_*\boldsymbol{w}\right)(x)=\boldsymbol{w}(\mathfrak{U}x).$$

After the linear change of the variables $y = \mathfrak{U}x$ to the operator $\mathbb{D}^{x_0}_{\mathfrak{B}_{\mathbb{T}_{x_0}(\Sigma)}}(i\mu)$ we obtain the operator

$$\begin{aligned}\mathbb{D}^{x_0}_{\mathfrak{B}_{\mathbb{R}^{n-1}}}(i\mu)\boldsymbol{w}(\boldsymbol{y}) &= \left(\mathfrak{U}^*\mathbb{D}^{x_0}_{\mathfrak{B}_{T_{x_0}(\Sigma)}}(i\mu)\mathfrak{U}_*\right)\boldsymbol{w}(y) \\ &= \begin{cases}\mathfrak{D}_{\tilde{\alpha},\mu}\boldsymbol{w}(y) = \left(\tilde{\boldsymbol{\alpha}}\cdot\boldsymbol{D}_y - i\mu I_N\right)\boldsymbol{w}(y) \text{ on } \mathbb{R}^n\diagdown\mathbb{R}^{n-1} \\ \mathfrak{B}_{\mathbb{R}^{n-1}}(x_0)\boldsymbol{w} = \tilde{a}_+(x_0)\gamma^+_{\mathbb{R}^{n-1}_{y'}}\boldsymbol{w}(y') + \tilde{a}_-(x^0)\gamma^-_{\mathbb{R}^{n-1}_{y'}}\boldsymbol{w}(y') \text{ on } \mathbb{R}^{n-1}\end{cases}, \\ \tilde{a}_\pm(x_0) &= \frac{\Gamma(x_0)}{2} \mp i\tilde{\alpha}_n\end{aligned} \tag{37}$$

for the half-spaces

$$\mathbb{R}^n_\pm = \left\{y = (y', y_n) \in \mathbb{R}^n : y' = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}, y_n \gtrless 0\right\}$$

where $\tilde{\alpha}_i = \sum_{j=1}^n \mathfrak{U}(\mathrm{j.i})\alpha_j$, $i = 1, \ldots, n$. Note that $\tilde{\boldsymbol{\alpha}} = \{\tilde{\alpha}_i\}_{i=1}^n$ is a system of Dirac matrices since $\mathfrak{U}$ is the orthogonal matrix (see Proposition 4).

3^0.We study the invertibility of the operator $\mathbb{D}^{x_0}_{\mathfrak{B}_{\mathbb{R}^{n-1}}}(i\mu)$ for $\mu \in \mathbb{R}\diagdown\{0\}$. The general solution of the equation

$$\mathfrak{D}_{\tilde{\alpha},\mu}\boldsymbol{w} = \boldsymbol{f} \in L^2(\mathbb{R}^n, \mathbb{C}^N)$$

in the space $H^1(\mathbb{R}^n\diagdown\mathbb{R}^{n-1}, \mathbb{C}^N) = H^1(\mathbb{R}^n_+, \mathbb{C}^N) \oplus H^1(\mathbb{R}^n_-, \mathbb{C}^N)$ is

$$\boldsymbol{w}(y) = \boldsymbol{w}_0(y) + \mathfrak{D}^{-1}_{\tilde{\alpha},\mu}\boldsymbol{f}(y) \tag{38}$$

(see for instance [18], p. 268) where

$$\boldsymbol{w}_0(y) = \mathfrak{D}^{-1}_{\tilde{\alpha},\mu}(\boldsymbol{\varphi}(y') \otimes \delta(y_n)), \boldsymbol{\varphi} \in H^{1/2}(\mathbb{R}^{n-1}, \mathbb{C}^N) \tag{39}$$

is the general solution of the equation $\mathfrak{D}_{\tilde{\alpha},\mu}\boldsymbol{w} = \boldsymbol{0}$ in $H^1(\mathbb{R}^n\diagdown\mathbb{R}^{n-1}, \mathbb{C}^N)$. Note that

$$\mathfrak{D}^{-1}_{\tilde{\alpha},\mu} = \mathfrak{D}_{\tilde{\alpha},-\mu}(-\Delta + \mu^2)^{-1}I_N \tag{40}$$

is the pseudodifferential operator with the matrix symbol

$$\widehat{\mathfrak{D}^{-1}_{\tilde{\alpha},\mu}}(\xi) = \frac{\tilde{\alpha}\cdot\xi + i\mu I_N}{|\xi|^2 + \mu^2}, \xi \in \mathbb{R}^n. \tag{41}$$

Let

$$k_{\tilde{\alpha},\mu}(y) = F^{-1}_{\xi\to y}\left(\frac{\tilde{\alpha}\cdot\xi + i\mu I_N}{|\xi|^2 + \mu^2}\right)$$

where $F^{-1}_{\xi\to y}$ is the inverse Fourier transform. Then

$$\mathfrak{D}^{-1}_{\tilde{\alpha},\mu}\boldsymbol{\psi}(y) = \int_{\mathbb{R}^n} k_{\tilde{\alpha},\mu}(y-z)\boldsymbol{\psi}(z)dz,\ y\in\mathbb{R}^n,\ \boldsymbol{\psi}\in C_0^\infty(\mathbb{R}^n,\mathbb{C}^N). \tag{42}$$

We introduce the potential operator $K_{\tilde{\alpha},\mu}$ as follows

$$\begin{aligned}\left(K_{\tilde{\alpha},\mu}\boldsymbol{\varphi}\right)(y',y_n) &= \mathfrak{D}^{-1}_{\tilde{\alpha},\mu}(\boldsymbol{\varphi}(y')\otimes\delta(y_n)) \\ &= \int_{\mathbb{R}^{n-1}} k_{\tilde{\alpha},\mu}(y'-z',y_n)\text{'}(z')dz',\ y'\in\mathbb{R}^{n-1},\ y_n\gtrless 0\end{aligned} \tag{43}$$

bounded from $H^{1/2}(\mathbb{R}^{n-1},\mathbb{C}^N)$ into $H^1(\mathbb{R}^n\diagdown\mathbb{R}^{n-1},\mathbb{C}^N)$ (see, for instance, [18], Chap.3). Note that $\boldsymbol{u}_0 = K_{\tilde{\alpha},\mu}\boldsymbol{\varphi}\in\ker\mathfrak{D}_{\tilde{\alpha},\mu}\subset H^1(\mathbb{R}^n\diagdown\mathbb{R}^{n-1},\mathbb{C}^N)$.

The Fourier transform of $k_{\tilde{\alpha},\mu}(y',y_n)$ with respect to $y'\in\mathbb{R}^{n-1}$ is

$$\hat{k}_{\tilde{\alpha},\mu}(\xi',y_n) = \frac{\mathbf{1}}{2\pi}\int_{\mathbb{R}}\widehat{\mathfrak{D}^{-1}_{\tilde{\alpha},\mu}}(\xi',\xi_n)e^{iy_n\xi_n}d\xi_n,\ y_n\gtrless 0 \tag{44}$$

where the integral in (44) exists for every $y_n\neq 0$. Applying the residua theorem we obtain that for every $y_n\neq 0$

$$\begin{aligned}\hat{k}_{\tilde{\alpha},\mu}(\xi',y_n) &= \frac{\mathbf{1}}{2\pi}\int_{\mathbb{R}} e^{iy_n\xi_n}\frac{\tilde{\alpha}'\cdot\xi'+i\mu I_N+\tilde{\alpha}_n\xi_n}{|\xi|^2+\mu^2}d\xi_n \\ &= \frac{e^{-\sqrt{|\xi'|^2+\mu^2}|y_n|}\left(\tilde{\boldsymbol{\alpha}}'\cdot\xi'+i\mu I_N+sign(y_n)i\tilde{\alpha}_n\sqrt{|\xi'|^2+\mu^2}\right)}{2\sqrt{|\xi'|^2+\mu^2}} \\ &= e^{-\sqrt{|\xi'|^2+\mu^2}|y_n|}\left(sign(y_n)\frac{i\tilde{\alpha}_n}{2}+\frac{\tilde{\alpha}'\cdot\xi'+i\mu I_N}{2\sqrt{|\xi'|^2+\mu^2}}\right),\ y_n\gtrless 0.\end{aligned} \tag{45}$$

Formula (45) yields that there exist limits

$$\begin{aligned}K^{\pm}_{\tilde{\alpha},\mu}\varphi(y') &= \lim_{y_n\to\pm 0}\left(K_{\tilde{\alpha},\mu}\varphi\right)(y',y_n) \\ &= \pm\frac{i\tilde{\alpha}_n}{2}\varphi(y')+\mathcal{K}_{\tilde{\alpha},\mu}\varphi(y'),\ \varphi\in C_0^\infty(\mathbb{R}^{n-1},\mathbb{C}^N),\end{aligned} \tag{46}$$

where

$$\mathcal{K}_{\tilde{\alpha},\mu}\varphi(y') = \int_{\mathbb{R}^n} k_{\tilde{\alpha},\mu}(y'-z',0)\varphi(z')dz'$$

is a pseudodifferential operator with the symbol

$$\hat{\mathcal{K}}_{\tilde{\alpha},\mu}(\xi') = \frac{\tilde{\alpha}' \cdot \xi' + i\mu I_N}{2\sqrt{|\xi'|^2 + \mu^2}}.$$

Hence $K^{\pm}_{\tilde{\alpha},\mu}$ are the pseudodifferential operators on $\mathbb{R}^{n-1}$ of zero order with the symbols

$$\hat{K}^{\pm}_{\tilde{\alpha},\mu}(\xi') = \pm\frac{i\tilde{\alpha}_n}{2} + \frac{\tilde{\alpha}' \cdot \xi' + i\mu I_N}{2\sqrt{|\xi'|^2 + \mu^2}} \tag{47}$$

and with the main symbols

$$\hat{K}^{\pm}_{\tilde{\alpha},0}(\xi') = \pm\frac{1}{2}i\tilde{\alpha}_n + \frac{\tilde{\alpha}' \cdot \xi'}{2\,|\xi'|}, \xi' \in \mathbb{R}^{n-1}. \tag{48}$$

Remark 6 For $\mu = 0$ the integral operators $K^{\pm}_{\tilde{\alpha},0}$ have to understand as a singular integral operators on $\mathbb{R}^{n-1}$.

Note that the operators: $K_{\tilde{\alpha},\mu} : H^{1/2}(\mathbb{R}^{n-1}, \mathbb{C}^N) \to H^1(\mathbb{R}^n \diagdown \mathbb{R}^{n-1}, \mathbb{C}^N)$, and $K^{\pm}_{\tilde{\alpha},\mu} : H^{1/2}(\mathbb{R}^{n-1}, \mathbb{C}^N) \to H^{1/2}(\mathbb{R}^{n-1}, \mathbb{C}^N)$ are bounded (see, for instance, [18], Chap.3).

Substituting $\boldsymbol{w}$ given by formula (38) into the interaction condition

$$\tilde{a}_+(x_0)\gamma^+_{\mathbb{R}^{n-1}_{y'}}\boldsymbol{w}(y') + \tilde{a}_-(x^0)\gamma^-_{\mathbb{R}^{n-1}_{y'}}\boldsymbol{w}(y') = \boldsymbol{\psi}(y'), y' \in \mathbb{R}^{n-1}$$

and applying formula (46) we obtain a pseudodifferential equation on $\mathbb{R}^{n-1}$ with respect to $\boldsymbol{\varphi} \in H^{1/2}(\mathbb{R}^{n-1}, \mathbb{C}^N)$

$$\begin{aligned} &\Xi_{\tilde{\alpha},\mu}(x_0, D_{x'})\boldsymbol{\varphi}(y') \\ &= \frac{i\tilde{\alpha}_n}{2}(\tilde{a}_+(x_0) - \tilde{a}_-(x_0))\boldsymbol{\varphi}(y') + (\tilde{a}_+(x_0) + \tilde{a}_-(x_0))\left(\mathcal{K}_{\tilde{\alpha},\mu}\boldsymbol{\varphi}\right)(y') \\ &= \boldsymbol{\psi}(y') - \mathfrak{B}_{\mathbb{R}^{n-1}}\mathfrak{D}^{-1}_{\tilde{\alpha},\mu}\boldsymbol{f}(y'), y' \in \mathbb{R}^{n-1}. \end{aligned} \tag{49}$$

Taking into account that

$$\tilde{a}_+(x_0) - \tilde{a}_-(x_0) = -2i\tilde{\alpha}_n \text{ and } \tilde{a}_+(x_0) + \tilde{a}_-(x_0) = \Gamma(x_0)$$

we obtain that $\Xi_{\tilde{\alpha},\mu}(x_0, D_{x'})$ is the pseudodifferential operator on $\mathbb{R}^{n-1}$ with the matrix symbol

$$\Xi_{\tilde{\alpha},\mu}(x_0, \xi') = I_N + \Gamma(x_0)\frac{\tilde{\alpha}' \cdot \xi' + i\mu I_N}{2\sqrt{|\xi'|^2 + \mu^2}}, \xi' \in \mathbb{R}^{n-1}. \tag{50}$$

The operator $\Xi_{\tilde{\alpha},\mu}(x_0, D_{x'})$ is invertible in $H^{1/2}(\mathbb{R}^{n-1}, \mathbb{C}^N)$ for every $\mu \in \mathbb{R}$ if and only if

$$\det \Xi_{\tilde{\alpha},\mu}(x_0, \xi') \neq 0 \text{ for every } (\xi', \mu) \in S^{n-1}. \tag{51}$$

Taking into account that

$$\left(\tilde{\alpha}' \cdot \xi' + i\mu I_N\right)\left(\tilde{\alpha}' \cdot \xi' - i\mu I_N\right) = \left|\xi'\right|^2 + \mu^2$$

we write condidtin (51) as follows:

$$\det \Lambda_{\tilde{\alpha},\mu}(x_0, \xi') \neq 0 \text{ for every } (\xi', \mu) \in S^{n-1}$$

where

$$\Lambda_{\tilde{\alpha},\mu}(x_0, \xi') = \tilde{\alpha}' \cdot \xi' + \frac{\Gamma(x_0)}{2} - i\mu I_N, (\xi', \mu) \in S^{n-1}.$$

Proposition 7 *The operator*

$$\mathbb{D}^{x_0}_{\mathfrak{B}_{\mathbb{R}^{n-1}}}(i\mu) : H^1(\mathbb{R}^n \diagdown \mathbb{R}^{n-1}, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\mathbb{R}^{n-1}, \mathbb{C}^N)$$

is invertible for every $\mathbb{R} \ni \mu \neq 0$ *if and only if*

$$\det \Lambda_{\tilde{\alpha},\mu}(x_0, \xi') \neq 0 \textit{ for every } (\xi', \mu) \in S^{n-1}. \tag{52}$$

The inverse operator $\left(\mathbb{D}^{x_0}_{\mathfrak{B}_{\mathbb{R}^{n-1}}}(i\mu)\right)^{-1}$ *is*

$$\left(\mathbb{D}^{x_0}_{\mathfrak{B}_{\mathbb{R}^{n-1}}}(i\mu)\right)^{-1}(\boldsymbol{f}, \boldsymbol{\psi}) = \mathfrak{D}^{-1}_{\tilde{\alpha},\mu}\boldsymbol{f} + K_{\tilde{\alpha},\mu}\left(\boldsymbol{\psi} - \mathfrak{B}_{\mathbb{R}^{n-1}}\mathfrak{D}^{-1}_{\tilde{\alpha},\mu}\boldsymbol{f}\right). \tag{53}$$

Moreover,

$$\|u\|_{H^1_{|\mu|}(\mathbb{R}^n \diagdown \mathbb{R}^{n-1}, \mathbb{C}^N)} \leq C(\|\boldsymbol{f}\|_{L^2(\mathbb{R}^n, \mathbb{C}^N)} + \|\boldsymbol{\psi}\|_{H^{1/2}_{|\mu|}(\mathbb{R}^{n-1}, \mathbb{C}^N)}) \tag{54}$$

with a constant $C > 0$ *independent of* μ.

Returning to the variables $x = \mathfrak{U}^{-1}y$ we obtain the following result.

Corollary 8 *For every fix* $x \in \Sigma$ *the operator*

$$\mathbb{D}^{x}_{\mathfrak{B}_{T_x(\Sigma)}}(i\mu) : H^1(\mathbb{R}^n \diagdown \mathbb{T}_x(\Sigma), \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\mathbb{T}_x(\Sigma), \mathbb{C}^N)$$

is invertible for every $\mu \in \mathbb{R} \diagdown 0$ *if and only if*

$$\det \Lambda_{\alpha,\mu}(x, \xi_x) \neq 0 \textit{ for every } (\xi_x, \mu) \in \mathbb{T}^*_x(\Sigma) \times \mathbb{R} : |\xi_x|^2 + \mu^2 = 1 \tag{55}$$

where $\mathbb{T}_x^*(\Sigma)$ *is the cotangent space to* Σ *at the point* $x \in \Sigma$, *and*

$$\Lambda_{\alpha,\mu}(x,\xi_x) = \boldsymbol{\alpha}\cdot\xi_x + \frac{\Gamma(x)}{2} - i\mu I_N.$$

Hence, if condition (55) holds the operator $\mathbb{D}_{\boldsymbol{A},\boldsymbol{\Phi},\boldsymbol{m},\mathfrak{B}_\Sigma}(i\mu)$ *is locally invertible at the point* $x \in \Sigma$.

Condition (55) is called the **local parameter-dependent Lopatinsky–Shapiro condition,** and the condition

$$\inf_{x\in\Sigma,(\xi_x,\mu)\in T_x(\Sigma)\times\mathbb{R}:|\xi_x|^2+\mu^2=1}\left|\det\Lambda_{\alpha,\mu}(x,\xi_x)\right| > 0 \tag{56}$$

is called the **uniform parameter-dependent Lopatinsky–Shapiro condition.**

Therefore, Proposition 5 yields the following result.

Theorem 9 *Let* $A_j, j = 1,\ldots,n, \Phi, m \in L^\infty(\mathbb{R}^n), \Gamma_{ij} \in C_b^1(\Sigma), i,j = 1,\ldots,N, \Sigma$ *be the uniformly regular* C^2*-hypersurface, and the uniform parameter-dependent Lopatinsky–Shapiro condition (56) holds. Then there exists* $\mu_1 > 0$ *such that the operator*

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu): H^1(\mathbb{R}^n\diagdown\Sigma,\mathbb{C}^N) \to L^2(\mathbb{R}^n,\mathbb{C}^N)\oplus H^{1/2}(\Sigma,\mathbb{C}^N)$$

is invertible for all $\mu \in \mathbb{R} : |\mu| > \mu_1$. *If the conditions of theorem are satisfied, then the a priory estimate*

$$\|\boldsymbol{u}\|_{H^1(\mathbb{R}^n\diagdown\Sigma,\mathbb{C}^N)} \le C\left(\left\|\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u}\right\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \|\mathfrak{B}\mathbf{u}\|_{H^{1/2}(\Sigma,\mathbb{C}^N)} + \|\boldsymbol{u}\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\right) \tag{57}$$

holds for every function $\boldsymbol{u} \in H^1(\mathbb{R}^n\diagdown\Sigma,\mathbb{C}^N)$ *with a constant* $C > 0$ *independent of* $\boldsymbol{u}$.

Remark 10 If the matrix $\Gamma(x)$ is Hermitian then condition (55) holds if $\mu \in \mathbb{R}\diagdown 0$. Hence for the Hermitian matrix $\Gamma(x)$ condition (55) is equivalent to the condition

$$\det\left(\boldsymbol{\alpha}\cdot\xi_x + \frac{\Gamma(x)}{2}\right) \neq 0 \text{ for each } \xi_x \in \mathbb{T}_x^*(\Sigma) : |\xi_x| = 1. \tag{58}$$

Condition (58) is the **local Lopatinsky–Shapiro condition** for **the parameter-independent interaction operator** $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$.

5 Self-adjointness of Unbounded Operators $\mathcal{D}_{\boldsymbol{A},\boldsymbol{\Phi},\boldsymbol{m},\mathfrak{B}_\Sigma}$

Now we consider the self-adjointness of the unbounded operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ associated with the formal Dirac operators (3),(4) which defined by the Dirac operator

$$\mathfrak{D}_{\boldsymbol{A},\Phi,m} = \boldsymbol{\alpha}\cdot(\boldsymbol{D}_x + \boldsymbol{A}) + \alpha_{n+1}m + \Phi I_N$$

with the domain

$$
\begin{aligned}
&dom\left(\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\right) = H^1_{\mathfrak{B}_\Sigma}(\mathbb{R}^n\setminus\Sigma,\mathbb{C}^N)\\
&= \left\{\boldsymbol{u}\in H^1(\mathbb{R}^n\setminus\Sigma,\mathbb{C}^N) : \mathfrak{B}_\Sigma\boldsymbol{u} = a_+\gamma_\Sigma^+\boldsymbol{u} + a_-\gamma_\Sigma^-\boldsymbol{u} = \boldsymbol{0} \text{ on } \Sigma\right\}
\end{aligned}
$$

where $a_\pm = \frac{1}{2}\Gamma \mp i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}$.

Theorem 11 *Let*

(a) $A_j, j = 1,\ldots,n, \Phi, m \in L^\infty(\mathbb{R}^n)$, $\Gamma_{ij}\in C_b^1(\Sigma), i,j = 1,\ldots,N$, Σ *be the uniformly regular* C^2*-hypersurface;*

(b) The vector potential $\boldsymbol{A}$, *scalar potentials* Φ, *and variable mass* m *be real-valued, and* $\Gamma = \left(\Gamma_{ij}\right)_{i,j=1}^N$ *be an Hermitian matrix.*

(c) The uniform Lopatinsky–Shapiro condition

$$
\inf_{x\in\Sigma,\xi_x\in T_x^*(\Sigma):|\xi_x|=1}\left|\det\left(\boldsymbol{\alpha}\cdot\xi_x + \frac{\Gamma(x)}{2}\right)\right| > 0 \tag{59}
$$

is satisfied.

Then the operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ *is self-adjoint in* $L^2(\mathbb{R}^n,\mathbb{C}^N)$.

Proof At first, we prove that the operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ is symmetric. Indeed, let $\boldsymbol{u},\boldsymbol{v}\in dom\left(\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\right) = H^1_{\mathfrak{B}_\Sigma}(\mathbb{R}^n\setminus\Sigma,\mathbb{C}^N)$. Then, integrating by parts we obtain

$$
\begin{aligned}
&\left\langle\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u},\boldsymbol{v}\right\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} - \left\langle\boldsymbol{u},\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{v}\right\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\\
&= \left\langle(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\gamma_\Sigma^+\boldsymbol{u},\gamma_\Sigma^+\boldsymbol{v}\right\rangle_{L^2(\Sigma,\mathbb{C}^N)} - \left\langle(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\gamma_\Sigma^-\boldsymbol{u},\gamma_\Sigma^-\boldsymbol{v}\right\rangle_{L^2(\Sigma,\mathbb{C}^N)}\\
&= \frac{1}{2}\left\langle -i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\left(\gamma_\Sigma^+\boldsymbol{u}-\gamma_\Sigma^-\boldsymbol{u}\right),\gamma_\Sigma^+\boldsymbol{v}+\gamma_\Sigma^-\boldsymbol{v}\right\rangle_{L^2(\Sigma,\mathbb{C}^N)}\\
&\quad - \frac{1}{2}\left\langle\gamma_\Sigma^+\boldsymbol{u}+\gamma_\Sigma^-\boldsymbol{u},-i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\left(\gamma_\Sigma^+\boldsymbol{v}-\gamma_\Sigma^-\boldsymbol{v}\right)\right\rangle_{L^2(\Sigma,\mathbb{C}^N)}.
\end{aligned}
$$

Taking into account the equality

$$
-i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\left(\gamma_\Sigma^+\boldsymbol{u}-\gamma_\Sigma^-\boldsymbol{u}\right) + \frac{1}{2}\Gamma\left(\gamma_\Sigma^+\boldsymbol{u}+\gamma_\Sigma^-\boldsymbol{u}\right) = 0 \text{ on } \Sigma
$$

we obtain that

$$
\begin{aligned}
&\left\langle\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u},\boldsymbol{v}\right\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} - \left\langle\boldsymbol{u},\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{v}\right\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\\
&= -\frac{1}{4}\left\langle\Gamma\left(\gamma_\Sigma^+\boldsymbol{u}+\gamma_\Sigma^-\boldsymbol{u}\right),\gamma_\Sigma^+\boldsymbol{v}-\gamma_\Sigma^-\boldsymbol{v}\right\rangle_{L^2(\Sigma,\mathbb{C}^N)}\\
&\quad + \frac{1}{4}\left\langle\gamma_\Sigma^+\boldsymbol{u}+\gamma_\Sigma^-\boldsymbol{u},\Gamma(\gamma_\Sigma^+\boldsymbol{v}-\gamma_\Sigma^-\boldsymbol{v}\right\rangle_{L^2(\Sigma,\mathbb{C}^N)}.
\end{aligned} \tag{60}
$$

Since the matrix Γ is Hermitian for every $x \in \Sigma$ the right side part in (60) is 0. Hence,

$$\left\langle \mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{u}, \boldsymbol{v} \right\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = \left\langle \boldsymbol{u}, \mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{v} \right\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}$$

for every $\boldsymbol{u}, \boldsymbol{v} \in H^1_{\mathfrak{B}_\Sigma}(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$.

The uniform Lopatinsky–Shapiro condition (59) yields the a priori estimate (57). For $\boldsymbol{v} \in dom\left(\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\right)$ this estimate accepts the form

$$\|\boldsymbol{v}\|_{H^1(\mathbb{R}^n \diagdown \Sigma,\mathbb{C}^N)} \leq C\left(\left\|\mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{v}\right\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \|\boldsymbol{v}\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\right). \tag{61}$$

The estimate (61) yields the closedness of the operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$. Indeed, let $dom\left(\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\right) \ni \boldsymbol{u}_j \to \boldsymbol{u}$, and $\mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{u}_j \to \boldsymbol{f}$ in $L^2(\mathbb{R}^n,\mathbb{C}^N)$. Then (61) yields that $\boldsymbol{u} \in H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ and

$$\left\|\boldsymbol{u}_j - \boldsymbol{u}\right\|_{H^1(\mathbb{R}^n \diagdown \Sigma,\mathbb{C}^N)} \to 0. \tag{62}$$

The estimate

$$\left\|\mathfrak{B}_\Sigma \boldsymbol{u}_j\right\|_{H^{1/2}(\Sigma,\mathbb{C}^N)} \leq C\left\|\boldsymbol{u}_j\right\|_{H^1(\mathbb{R}^n \diagdown \Sigma,\mathbb{C}^N)} \tag{63}$$

and (62) yields that $\mathfrak{B}_\Sigma \boldsymbol{u} = \boldsymbol{0}$. Hence $\boldsymbol{u} \in dom\left(\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\right)$. Moreover,

$$\begin{aligned}
&\left\|\mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{u} - \boldsymbol{f}\right\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \\
&\quad \leq \left\|\mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{u} - \mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{u}_j\right\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \left\|\mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{u}_j - \boldsymbol{f}\right\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \\
&\quad \leq C\left\|\boldsymbol{u} - \boldsymbol{u}_j\right\|_{H^1(\mathbb{R}^n \diagdown \Sigma,\mathbb{C}^N)} + \left\|\mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{u}_j - \boldsymbol{f}\right\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \to 0.
\end{aligned}$$

Hence, $\mathfrak{D}_{\boldsymbol{A},\Phi,m} \boldsymbol{u} = \boldsymbol{f}$ and the operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ is closed.

Theorem 9 yields that the operator

$$\mathbb{D}^0_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)\, \boldsymbol{u} = \begin{cases} \left(\mathfrak{D}_{\boldsymbol{A},\Phi,m} - i\mu I_N\right) \boldsymbol{u} \text{ on } \mathbb{R}^n \diagdown \Sigma \\ \mathfrak{B}_\Sigma \boldsymbol{u} = \boldsymbol{0} \text{ on } \Sigma \end{cases}$$

is invertible from $H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N)$ for $|\mu|$ large enough. Therefore, $Range\left(\mathbb{D}^0_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)\right) = L^2(\mathbb{R}^n, \mathbb{C}^N)$ for $|\mu|$ large enough. Since

$$Range\left(\mathbb{D}^0_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)\right) = Range(\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} - i\mu I)$$

the deficiency indices of $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ equal zero. Hence (see for instance [5], page 100) the operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ is self-adjoint. □

6 Electrostatic and Lorentz Scalar δ-Shell Interactions in $\mathbb{R}^n$

Let $\Gamma = \eta I_N + \tau\alpha_{n+1}$, where $\eta, \tau \in C_b^1(\Sigma)$ be real-valued functions. The formal Dirac operator $D_{\boldsymbol{A},\Phi,m,\circ_\Sigma}$ is the Hamiltonian of relativistic particles in the field of the regular potentials $\boldsymbol{A},\Phi$, m and electrostatic and scalar Lorentz δ-shell potentials with supports on the uniformly regular C^2-hypersurface Σ.

Since

$$\left(\boldsymbol{\alpha}\cdot\xi_x + \frac{\eta I_N + \tau\alpha_{n+1}}{2}\right)\left(\boldsymbol{\alpha}\cdot\xi_x + \frac{-\eta I_N + \tau\alpha_{n+1}}{2}\right) = \left(|\xi_x|^2 - \frac{\eta^2-\tau^2}{4}\right) I_N,$$

the condition

$$\inf_{s\in\Sigma}\left|\eta^2(s) - \tau^2(s) - 4\right| > 0 \tag{64}$$

ensures the uniform Lopatinsky–Shapiro conditions

$$\inf_{x\in\Sigma}\ \inf_{\xi_x\in\mathbb{T}_x(\Sigma):|\xi_x|=1}\left|\det\left(\boldsymbol{\alpha}\cdot\xi_x + \frac{\eta(x)I_N + \tau(x)\alpha_{n+1}}{2}\right)\right| > 0. \tag{65}$$

By Theorem 11 the operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ is self-adjoint if the potentials $\boldsymbol{A}$, Φ, m are real-valued and condition (64) holds.

7 Splitting of Interaction Conditions

We consider the interaction problem

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\boldsymbol{u} = \begin{cases}\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u} = \boldsymbol{f}, \text{ on } \mathbb{R}^n\diagdown\Sigma,\\ \mathfrak{B}_\Sigma\boldsymbol{u} = a_+\bigcirc_\Sigma^+\boldsymbol{u} + a_-\bigcirc_\Sigma^-\boldsymbol{u} = \boldsymbol{\varphi} \text{ on } \Sigma\end{cases} \tag{66}$$

where $a_\pm = \frac{1}{2}\Gamma \mp i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}$. The interaction condition $\mathfrak{B}_\Sigma\boldsymbol{u} = \boldsymbol{\varphi}$ can be written as

$$\mathcal{P}^+\bigcirc_\Sigma^+\boldsymbol{u} + \mathcal{P}^-\bigcirc_\Sigma^-\boldsymbol{u} = i\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\boldsymbol{\varphi} \text{ on } \Sigma \tag{67}$$

where

$$\mathcal{P}^\pm = \frac{1}{2}(I_N \pm M),\ M = \frac{i}{2}\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\Gamma \text{ on } \Sigma. \tag{68}$$

Let

$$M^2 = I_N. \tag{69}$$

Then

$$\left(\mathcal{P}^{\pm}\right)^2 = \frac{1}{4}(I_N \pm M)^2 = \frac{1}{4}\left(I_N \pm 2M + M^2\right) = \mathcal{P}^{\pm},$$
$$\mathcal{P}^{+}\mathcal{P}^{-} = \frac{1}{4}(I_N - M^2) = 0.$$

Hence the operators $\mathcal{P}^{\pm}$ under condition (69) are the orthogonal projectors in $\mathbb{C}^N$ and interaction condition (66) splits into two independent boundary conditions

$$\mathcal{P}^{\pm}\gamma^{\pm}\boldsymbol{u} = \mathcal{P}^{\pm}i\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\boldsymbol{\varphi} \text{ on } \Sigma.$$

Hence, in this case the interaction problem (66) splits into two boundary problems for the Dirac operator

$$\mathbb{D}_{\boldsymbol{a},\Phi,m,\mathcal{P}^{\pm}}\boldsymbol{u}_{\pm} = \begin{cases} \mathfrak{D}_{\boldsymbol{a},\Phi,m}\boldsymbol{u}_{\pm} = \boldsymbol{f}_{\pm} \text{ on } \Omega_{\pm}, \\ \mathcal{P}^{\pm}\gamma_{\Sigma}^{\pm}\boldsymbol{u}_{\pm} = \mathcal{P}^{\pm}i\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\boldsymbol{\varphi} \text{ on } \Sigma \end{cases}. \tag{70}$$

Example 12 Let $\Gamma = \eta I_N + \tau\alpha_{n+1}$, $\eta, \tau \in \mathbb{R}$. Then

$$M = \frac{i}{2}\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\Gamma = \frac{i}{2}\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,(\eta I_N + \tau\alpha_{n+1}),$$

and

$$\begin{aligned} M^2 &= -\frac{1}{4}\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,(\eta I_N + \tau\alpha_{n+1})\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,(\eta I_N + \tau\alpha_{n+1}) \\ &= -\frac{1}{4}(\eta I_N - \tau\alpha_{n+1})\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})^2\,(\eta I_N + \tau\alpha_{n+1}) = -\frac{1}{4}(\eta^2 - \tau^2)I_N. \end{aligned} \tag{71}$$

Hence $M^2 = I_N$ if $\eta^2 - \tau^2 = -4$. Under this condition the interaction problem (66) splits into the orthogonal sum of the boundary problems

$$D^{\pm}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_{\Sigma}}\boldsymbol{u} = \begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u} \text{ on } \Omega_{\pm}, \\ \mathcal{P}^{\pm}\gamma_{\Sigma}^{\pm}\boldsymbol{u} = \mathcal{P}^{\pm}i\,(\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\,\boldsymbol{\varphi} \text{ on } \Sigma \end{cases}. \tag{72}$$

If $\eta = 0$ and $\tau^2 = 4$ the boundary value problems (72) for $n = 3$, $N = 4$ are called the *MIT Bag problems* which describes the confinement of the quarks in domains bounded by the hypersurfaces Σ (see for instance [15, 16, 21, 22]).

8 Essential Spectrum of Interaction Operators on Closed Hypersurfaces

In this chapter we consider the Fredholm property of the interaction operators

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\boldsymbol{u} = \begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u} \text{ on } \mathbb{R}^n\diagdown\Sigma, \\ \mathfrak{B}_\Sigma\boldsymbol{u} = a_+\gamma_\Sigma^+\boldsymbol{u} + a_-\gamma_\Sigma^-\boldsymbol{u} \text{ on } \Sigma \end{cases} \tag{73}$$

acting from $H^1(\mathbb{R}^n\diagdown\Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N)\oplus H^{1/2}(\Sigma, \mathbb{C}^N)$ and the essential spectrum of the unbounded operators $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$.

Our approach is based on the local principle of elliptic theory (see for instance [1, 25]) and the limit operators method (see [32, 33]. In this regard, we need additional smoothness conditions on the potentials. We assume that:

(a) Σ is a closed C^2-hypersurface;
(b) The potentials $\boldsymbol{A} = (A_1, \dots, A_n)$, Φ, and m are such that $A_j, \Phi, m \in C_b^1(\mathbb{R}^n)$, and $\Gamma = \left(\Gamma_{k,l}\right)_{k,l=1}^N$ is such that $\Gamma_{k,l} \in C_b^1(\Sigma)$;
(c) The Lopatinsky–Shapiro condition

$$\det\left(\boldsymbol{\alpha}\cdot\xi_x + \frac{\Gamma(x)}{2}\right) \neq 0 \text{ for each } \xi_x \in \mathbb{T}_x^*(\Sigma) : |\xi_x| = 1 \tag{74}$$

is satisfied at every point $x \in \Sigma$.

First we consider the Fredholm property of the operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$. For this aim we need some notations and definitions.

Let $\psi \in C_0^\infty(B_1(0))$, and $\psi(x) = 1$ for $x \in B_{1/2}(0)$, $0 \le \psi(x) \le 1$, $\chi(x) = 1 - \psi(x)$, $\psi_R(x) = \psi(x/R)$, $\chi_R(x) = \chi(x/R)$, $R > 0$.

Definition 13 Let $\boldsymbol{X} = H^1(\mathbb{R}^n\diagup\Sigma, \mathbb{C}^N)$, $\boldsymbol{Y} = L^2(\mathbb{R}^n, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$. (i) We say that the operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} : \boldsymbol{X} \to \boldsymbol{Y}$ is locally *Fredholm* at the point $x_0 \in \mathbb{R}^n$ if there exist a ball $B_\varepsilon(x_0)$, $\varepsilon > 0$ and operators $\mathcal{L}_{x_0}, \mathcal{R}_{x_0} \in \mathcal{B}(\boldsymbol{Y}, \boldsymbol{X})$ such that for every function $\varphi \in C_0^\infty(B_\varepsilon(x_0))$

$$\mathcal{L}_{x_0}\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\varphi I_{\boldsymbol{X}} = \varphi I_{\boldsymbol{X}} + K'_{x_0},$$
$$\varphi\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\mathcal{R}_{x_0} = \varphi I_{\boldsymbol{Y}} + K''_{x_0},$$

where $K'_{x_0} \in \mathcal{K}(\mathbf{X})$, $K''_{x_0} \in \mathcal{K}(\boldsymbol{Y})$; (ii) We say that the operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} : \boldsymbol{X} \to \boldsymbol{Y}$ is locally invertible at infinity if there exists $R > 0$ and operators $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}(\boldsymbol{Y}, \boldsymbol{X})$ such that

$$\mathcal{L}_R\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\chi_R I_{\boldsymbol{X}} = \chi_R I_{\boldsymbol{X}}, \tag{75}$$
$$\chi_R\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\mathcal{R}_R = \chi_R I_{\boldsymbol{Y}}.$$

The next statements follow from the standard elliptic theory [1] , [25].

Proposition 14 (Local Principle) *The operator* $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ *is a Fredholm operator if and only if* $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ *is a locally Fredholm operator at every point* $x \in \mathbb{R}^n$ *and* $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ *is locally invertible at infinity.*

Since the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi,m}$ is elliptic, and we assume that the Lopatinsky–Shapiro condition (74) is satisfied at each point $x \in \Sigma$, $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ is a locally

Fredholm operator at every point $x \in \mathbb{R}^n$. Hence, the operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ is a Fredholm operator if and only if $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ is locally invertible at infinity. For $R > 0$ large enough the operators $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}\chi_R I$, $\chi_R\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ coincide with the operators $\mathfrak{D}_{\boldsymbol{A},\Phi,m}\chi_R I.\chi_R\mathfrak{D}_{\boldsymbol{A},\Phi,m}$, respectively. Hence $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ is locally invertible at infinity if and only if the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi,m}$ is locally invertible at infinity.

Let the function $a \in C_b^1(\mathbb{R}^n)$ and $\mathbb{Z}^n \ni h_k \to \infty$. We consider the sequence $\{a(\cdot + h_k)\}_{k\in\mathbb{N}}$. Applying the Arzelá-Ascoli Theorem one can find a subsequence $\left\{a(x + h_{k_l})\right\}_{l\in\mathbb{N}}$ converging to a limit function $a^h(x) \in C_b(\mathbb{R}^n)$ uniformly on every compact set $K \subset \mathbb{R}^n$.

Thus every sequence $\mathbb{Z}^n \ni h_k \to \infty$ has a subsequence h_{k_l} such that there are limits

$$\boldsymbol{A}(x + h_{k_l}) \to \boldsymbol{A}^h(x), \Phi(x + h_{k_l}) \to \Phi^h(x), m(x + h_{k_l}) \to m^h(x)$$

in the sense of the uniformly convergence on the compact sets in $\mathbb{R}^n$.

We say that the Dirac operator $\mathfrak{D}^h_{\boldsymbol{A},\Phi,m} = \mathfrak{D}_{\boldsymbol{A}^h,\Phi^h,m^h}$ is the limit operator of $\mathfrak{D}_{\boldsymbol{A},\Phi,m}$ defined by the sequence $\mathbb{Z}^n \ni h_{m_k} \to \infty$.

Proposition 15 (see [32, 33]) *The operator* $\mathfrak{D}_{\boldsymbol{A},\Phi,m} : H^1(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n; \mathbb{C}^N)$ *is locally invertible at infinity if and only if all limit operators* $\mathfrak{D}^h_{\boldsymbol{A},\Phi,m} : H^1(\mathbb{R}^n; \mathbb{C}^N) \to L^2(\mathbb{R}^n; \mathbb{C}^N)$ *are invertible.*

This Proposition yields the following result.

Theorem 16 *Let conditions* (a), (b), (c) *be satisfied. Then* $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} : H^1(\mathbb{R}^n; \mathbb{C}^N) \to L^2(\mathbb{R}^n; \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$ *is a Fredholm operator if and only if all limit operators* $\mathfrak{D}^h_{\boldsymbol{A},\Phi,m} : H^1(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ *are invertible.*

Let $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ be unbounded operator associated with the interaction operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} : H^1(\mathbb{R}^n\diagdown\Sigma, \mathbb{C}^N) \to L^2(\mathbb{R}^n\diagdown\Sigma, \mathbb{C}^N) \oplus H^{1/2}(\Sigma, \mathbb{C}^N)$. Then Theorem 16 implies the following corollary.

Corollary 17 *Let conditions* (a), (b), (c) *be satisfied. Then the unbounded operator* $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ *is closed, and*

$$sp_{ess}\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} = \bigcup_h sp\mathfrak{D}^h_{\boldsymbol{A},\Phi,m} \tag{76}$$

where the union is taken with respect to all sequences $h = (h_m)$ *defining the limit operators* $\mathfrak{D}_{\boldsymbol{A}^h,\Phi^h,m^h}$.

Proof Since the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi,m}$ is uniformly elliptic on $\mathbb{R}^n$, and the Lopatinsky–Shapiro condition (74) is satisfied at every point $x \in \Sigma$, then in the spirite of the proof of Theorem 9 we obtain the a priori estimate for every vector-function $\boldsymbol{u} \in H^1(\mathbb{R}^n\diagdown\Sigma, \mathbb{C}^N)$

$$\begin{aligned}&\|\boldsymbol{u}\|_{H^1(\mathbb{R}^n\diagdown\Sigma,\mathbb{C}^N)}\\ &\quad\le C\left(\left\|\mathfrak{D}_{\boldsymbol{A},\Phi,m}\boldsymbol{u}\right\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \|\mathfrak{B}_\Sigma u\|_{H^{1/2}(\Sigma,\mathbb{C}^N)} + \|\boldsymbol{u}\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\right)\end{aligned} \tag{77}$$

with a constant $C > 0$ independent of $\boldsymbol{u}$ (see for instance [1]). This estimate implies that the operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is closed. Formula (76) follows from Theorem 16. □

Remark 18 There is another approach to studying the essential spectrum of self-adjoint operators $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ based on their resolvent property:

$$\left(\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma} - \lambda I\right)^{-1} - \left(\mathfrak{D}_{A,\Phi,m} - \lambda I\right)^{-1} = Q(\lambda), \lambda \in \mathbb{C}\diagdown\mathbb{R}, \tag{78}$$

where $Q(\lambda)$ is a compact in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ operator (see for instance [11]). The equality (78) yields that

$$sp_{ess}\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma} = sp_{ess}\mathfrak{D}_{A,\Phi,m}$$

(see for instance [38], Theorem XIII.14).

Definition 19 We say that a function $a \in C_b^1(\mathbb{R}^n)$ is slowly oscillating at infinity and belongs to the class $SO^1(\mathbb{R}^n)$ if

$$\lim_{x\to\infty} \partial_{x_j} a(x) = 0, j = 1, \ldots, n. \tag{79}$$

Proposition 20 ([32], Chap. 2) *If $a \in SO^1(\mathbb{R}^n)$ and the sequence $\mathbb{R}^n \ni h_k \to \infty$ is such that*

$$\lim_{k\to\infty} a(x + h_k) = a^h(x), x \in \mathbb{R}^n$$

in the sense of uniformly convergence on compact sets, then the limit function a^h is a constant.

Assume that A_j, Φ, and m belong to $SO^1(\mathbb{R}^n)$.Then the limit operators $\mathfrak{D}^h_{\mathbf{A},\boldsymbol{\Phi},\boldsymbol{m}}$ are of the form

$$\mathfrak{D}^h_{\mathbf{A},\boldsymbol{\Phi},\boldsymbol{m}} = \boldsymbol{\alpha}\cdot(\boldsymbol{D} + \boldsymbol{A}^h) + m^h\alpha_{n+1} + \Phi^h I_N \tag{80}$$

where $\boldsymbol{A}^h \in \mathbb{C}^n$, $m^h \in \mathbb{C}$, $\Phi^h \in \mathbb{C}$. Because $\boldsymbol{A}^h \in \mathbb{C}^n$ the operator $\mathfrak{D}^h_{\mathbf{A},\boldsymbol{\Phi},m}$ is unitary equivalent to the operator

$$\mathfrak{D}^h_{\Phi,m} = \boldsymbol{\alpha} \cdot \boldsymbol{D} + m^h\alpha_{n+1} + \Phi^h I_N.$$

Hence

$$sp\mathfrak{D}^h_{\mathbf{A},\Phi,m} = sp\mathfrak{D}^h_{\mathbf{0},\Phi,m} = \left\{\lambda \in \mathbb{C} : \lambda = \Phi^h \pm \sqrt{|\xi|^2 + \left(m^h\right)^2}, \xi \in \mathbb{R}^n\right\} \tag{81}$$

where the branch of the root $\sqrt{|\xi|^2 + z^2}$ is chosen such that $\sqrt{|\xi|^2 + z^2} \geq 0$ for $z \in \mathbb{R}$.

If the potentials Φ,and m are real-valued functions, then

$$sp\mathfrak{D}^h_{\mathbf{A},\Phi,m} = sp\mathfrak{D}^h_{\mathbf{0},\Phi,m} = \left(-\infty, \Phi^h - \left|m^h\right|\right] \bigcup \left[\Phi^h + \left|m^h\right|, +\infty\right). \tag{82}$$

Theorem 21 *Let A_j, $\Phi, m \in SO^1(\mathbb{R}^n)$ be real-valued functions. Then*

$$sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,m} = \left(-\infty, \limsup_{x\to\infty}(\Phi(x) - |m(x)|)\right] \bigcup \left[\liminf_{x\to\infty}(\Phi(x) + |m(x)|), +\infty\right). \tag{83}$$

Proof Formula (83) follows from formulas (76) and (82). □

Remark 22 Let conditions of Theorem 21 hold, and the mass of the particle be a constant $m \in \mathbb{R}$. Then the operator $\mathcal{D}_{\mathbf{A},\mathbf{\Phi},m,\mathcal{B}_\Sigma}$ can have the discrete spectrum if and only if

$$\limsup_{x\to\infty} \Phi(x) - \liminf_{x\to\infty} \Phi(x) < 2|m|.$$

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