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Global Fourier Integral Operators in the Plane and the Square Function

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Abstract
We prove the local smoothing estimate for general Fourier integral operators with phase function of the form $\phi(x, t, \xi) = x \cdot \xi + t\, q(\xi)$, with $q \in C^\infty(\mathbb{R}^2 \setminus \{0\})$, homogeneous of degree one, and amplitude functions in the symbol class of order $m \leq 0$. The result is global in the space variable, and also improves our previous work in this direction (Manna et al (in: Georgiev et al., Advances in harmonic analysis and partial differential equations, Trends in Mathematics. Birkhäuser, Cham, pp. 1–35, 2020)). The approach involves a reduction to operators with amplitude function depending only on the covariable, and a new estimate for square function based on angular decomposition.

Keywords Fourier integral operator · Wave front set · Local smoothing · Square function

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1 Introduction

We consider Fourier integral operators of the form

$$\mathcal{F}f(x,t) = \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t\,q(\xi))}\, a(x,t,\xi)\, \hat{f}(\xi)\, d\xi,\ \ f \in \mathcal{S}(\mathbb{R}^2), \tag{1.1}$$

where $q \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ is non vanishing and homogeneous of degree 1, and the amplitude function $a \in S^m(\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2)$, $m \leq 0$, i.e., $a(x,t,\xi)$ is a smooth function satisfying the estimate $|\partial_{x,t}^{\beta}\partial_\xi^\alpha a(x,t,\xi)| \leq C_{\alpha,\beta}\,(1+|\xi|)^{m-|\alpha|}$ for all multi-indices α, β.

The general theory of Fourier integral operators was developed by Hörmander in [14] in 1971, soon after the work of Eskin [12] who studied such operators as degenerate elliptic pseudo differential operators. Hörmander established the local L^2 regularity estimate for operators as above with $a \in S^0$, under certain geometric conditions on the phase function. The local L^p regularity estimates has been proved by Seeger, Sogge and Stein [20] for Fourier integral operators of the form (1.1), but with more general phase functions, and amplitude function $a \in S^m$, $-\frac{n-1}{2} < m \leq 0$, for the range $m \leq -(n-1)|1/p - 1/2|$, $1 < p < \infty$. It is well known that the condition $a \in S^m$ with $-\frac{n-1}{2} < m \leq 0$, is necessary for the local L^p boundedness for $1 < p < \infty$, see [24].

The above range of p is also optimal in view of the result of Peral [18] and Miyachi [16] on wave equation, which corresponds to the phase function $\phi(x,t,\xi) = x\cdot\xi + t|\xi|$, $x, \xi \in \mathbb{R}^n$, $t > 0$ and $a \equiv 1$.

A global L^2 regularity estimate has been obtained by Asada and Fujiwara in 1978 [1], under certain assumptions on amplitude and phase functions. This result has been extended to a larger class of Fourier integral operators by Ruzhansky and Sugimoto in [19]. A global L^p estimate has been obtained by Coriasco and Ruzhansky, under additional assumptions, see [7, 8]. In particular they use a decay assumption on all derivatives of the amplitude function. See also [4, 10] for some recent development in this direction. In connection with the study of wave equation, Sogge observed certain gain in regularity for the associated Fourier integral operators in [21, 22], the so called local smoothing estimate.

The aim of this article is to establish a local smoothing estimate for operators of the form (1.1), global with respect to the space variable, and with very mild decay assumption on the amplitude function and a few of its derivatives with respect to the space time variables. In fact we only assume that $a(x,t,\xi) \in S^m$, $m \leq 0$ satisfies the estimate

$$\left|\partial_{x,t}^{\beta}\partial_\xi^\alpha a(x,t,\xi)\right| \leq C_{\alpha,\beta}\frac{(1+|\xi|)^{m-|\alpha|}}{1+|(x,t)|^4} \tag{1.2}$$

for some constant $C_{\alpha,\beta}$ for all multi indices α, β with $|\beta| \leq 4$. We also show that the above space time decay assumption can be completely dispensed with, for an interesting subclass of symbols $a \in S^m$.

In [15], we have extended the local smoothing estimate of Mockenhaupt et al. [17], to more general amplitude functions assuming a decay condition as in (1.2) involving derivatives with respect to the (x, t) variables upto order 8. The present work generalises the result obtained in [15] to more general phase functions of the form $\phi(x, t, \xi) = x \cdot \xi + t\, q(\xi)$ and also extend it to a much larger class of symbols.

Recall that the fractional Sobolev space L^p_α of order $\alpha > 0$ is defined by $L^p_\alpha := (-\Delta + I)^{-\frac{\alpha}{2}} L^p(\mathbb{R}^n)$, which is the Sobolev space of L^p functions on $\mathbb{R}^n$ with α derivatives in L^p, see [23]. L^p_α is a Banach space with norm $\|f\|_{L^p_\alpha} := \|(-\Delta + I)^{\alpha/2} f\|_{L^p}$. Note that L^p_α is also defined for complex α, and are spaces of tempered distributions when $\mathrm{Re}(\alpha) < 0$. Our main result is the following.

Theorem 1.1 *Let $\mathcal{F}$ be the Fourier integral operator given by* (1.1) *with amplitude function $a \in S^m$, $m \leq 0$ satisfying* (1.2). *Then for any compact t-interval I, the following estimate holds true*

$$\|\mathcal{F} f\|_{L^p(\mathbb{R}^2 \times I)} \leq C_\sigma \|f\|_{L^p_{-\sigma+m}(\mathbb{R}^2)},$$

for all $f \in L^p(\mathbb{R}^2)$, with a constant C_σ depending on the length of I, where

$$\begin{cases} Re(\sigma) < \frac{1}{2}\left(\frac{1}{p} - \frac{1}{2}\right), & \text{for } 2 < p \leq 4, \\ Re(\sigma) < \frac{3}{2p} - \frac{1}{2}, & \text{for } 4 \leq p < \infty. \end{cases}$$

As an interesting byproduct of our approach, we also obtain the local smoothing estimate associated with a class of symbols in S^m, without any decay assumption in space time variables, see Theorem 6.3.

Theorem 1.1 gives the local smoothing of order up to $\epsilon(p) = \frac{1}{2p}$ for $4 \leq p < \infty$. Since the above estimate for $\mathcal{F} f$ is local in the t variable, it is enough to work with Fourier integral operators of the form

$$\mathcal{F} f(x, t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x \cdot \xi + t q(\xi))}\, a(x, t, \xi)\, \hat{f}(\xi)\, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^2), \tag{1.3}$$

where $\rho_1 \in C_c^\infty(\mathbb{R})$.

A crucial step in our approach is the use of a duality argument, which requires the introduction of a three dimensional square function, based on angular decomposition in the plane, as in [15]. Namely,

$$S(g)(x, t) = \left(\sum_{\nu=0}^{N-1} \left| T^\delta_{\nu, j} g \right|^2 \right)^{\frac{1}{2}}, \tag{1.4}$$

where $T^\delta_{\nu,j}$, $j \in \mathbb{N}$, $\delta > 0$ and $\nu = 0, 1, \ldots, N-1$ are Fourier multiplier operators on $\mathbb{R}^3$ given by

$$\widehat{T^\delta_{\nu,j} g}\,(\xi, \tau) = \rho(2^{-j}|\xi|)\, \tilde{\chi}_\nu(\xi)\, \psi\left(\frac{q(\xi) - \tau}{\delta}\right) \hat{g}(\xi, \tau), \;\; g \in \mathcal{S}(\mathbb{R}^3), \tag{1.5}$$

with $\rho \in C_c^\infty[1/2, 2]$, as in (3.1) and $\tilde{\chi}_\nu$ is a homogeneous function (smooth and compactly supported as a function on $\mathbb{S}^1$).

Notice that the employment of refined decompositions, after a first dyadic one, to obtain estimates for Fourier integral operators, dates back to the celebrated paper [20]. Compared to the techniques adopted there, here we follow a different approach (see [15] and Sects. 2 and 3 below for the details). In particular, our duality argument requires the boundedness of $S(g)$ on $L^{4/3}(\mathbb{R}^3)$. In fact, we prove the following

Theorem 1.2 *Let Sg be defined by* (1.4)*. Then, there exists constants C and b such that the inequality*

$$\|Sg\|_{L^p(\mathbb{R}^3)} \le C\, 2^{j/8}\, j^b\, \delta^{1/4}\, \|g\|_{L^p(\mathbb{R}^3)}$$

holds true for all $g \in \mathscr{S}(\mathbb{R}^3)$, for $p \in [4/3, 4]$.

The above square function estimate is new for the range $4/3 \le p \le 2$ and extends the one obtained in [15] to a larger class of $T^\delta_{\nu,j}$, corresponding to general homogeneous function q.

Remark 1.3 As in the case of wave equation, the Fourier intergral operators with phase function $\phi(x, t, \xi) = x \cdot \xi + tq(\xi)$ arises in the solution of initial value problem for strictly hyperbolic partial differential equations. The result of Beals in [2] (Theorem 5.4) gives the fixed time estimates, and Theorem 1.1 of the present article gives the local smoothing for the solutions of such Cauchy problems.

2 Decomposition of the Fourier Integral Operator

The proof of Theorem 1.1 involves several decompositions of the operator $\mathcal{F}$, which we discuss in detail in this section. The first decomposition is to express $\mathcal{F}$ as a sum of operators with symbols independent of the space time variables. In fact, we reduce the analysis to a family of Fourier integral operators $\{\mathcal{F}^{n,k}\}_{n,k\in\mathbb{Z}^3}$ of the form

$$\mathcal{F}^{n,k} f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + tq(\xi))}\, a_n^k(\xi)\, \hat{f}(\xi)\, d\xi,\ \ f \in \mathcal{S}(\mathbb{R}^2), \tag{2.1}$$

with amplitude function $a_n^k \in S^0(\mathbb{R}^2)$ independent of (x, t). In [15], we have employed Hermite expansion to get the estimate, that is global in x variable. Here we show that we can actually use Fourier series expansion in (x, t) variables and improve that result. In fact, if $(x, t) \to a(\cdot, \cdot, \xi)$ is supported in a cube Q_k of side length 2 and centered at the integer lattice point $k \in \mathbb{Z}^2 \times \mathbb{Z}$, then clearly we can expand the amplitude function $a(x, t, \xi)$ as a Fourier series in (x, t) variables, to write the Fourier integral operator (1.3) as an infinite sum of Fourier integral operators of the form (2.1). Interestingly, the general case can also be reduced to this case by a partition of unity argument, leading to a family of operators $\mathcal{F}^{n,k}$ with amplitude functions a_n^k, the Fourier coefficients of $a(\cdot, \cdot, \xi)$. For notational simplicity, we will obtain basic L^p estimates working with operator of the form (2.1) with amplitude function a independent of (x, t), suppressing the indices n and k. The Fourier series approach enables us to allow more general amplitude functions, requiring less decay conditions thereby improving our previous work [15] based on Hermite expansion.

We perform a further dyadic decomposition in the ξ variable, to reduce the analysis to the simpler case of operators with compactly supported kernels, as follows. Choose $\rho_0 \in C_c^\infty([\frac{1}{2}, 2])$ such that $1 = \sum_{j\in\mathbb{Z}} \rho_0(2^{-j}|\xi|)$, (see [11], page 162 for the construction of such a $\rho_0 \geq 0$). For technical reasons, we take ρ_0 to be of the form $\rho_0 = \rho^2$ with $\rho \in C_c^\infty([\frac{1}{2}, 2])$. Setting $\varphi_0 = \sum_{j\leq 0} \rho_0(2^{-j}|\xi|)$, we can write $1 = \varphi_0 + \sum_{j\in\mathbb{N}} \rho_0(2^{-j}|\xi|)$, where φ_0 is a smooth function supported in the ball $|\xi| \leq 2$. Then for each $j \in \mathbb{N}$, we define the operators $\mathcal{F}_j$, such that

$$\mathcal{F}_j f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + tq(\xi))}\, \rho_0(2^{-j}|\xi|)\, a(\xi)\, \hat{f}(\xi)\, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^2), \tag{2.2}$$

so that

$$\mathcal{F}f(x,t) = \mathcal{F}_0 f(x,t) + \sum_{j\in\mathbb{N}} \mathcal{F}_j f(x,t) \tag{2.3}$$

as a tempered distribution. Note that $\mathcal{F}_0$ is a Fourier integral operator with amplitude function $b(\xi) := a(\xi)\varphi_0(\xi)$ supported in $|\xi| \leq 2$. It turns out that $f \to \mathcal{F}_0 f(\cdot, t)$ is an infinitely smoothing operator, see Proposition 5.7.

We use the wave front set analysis as in [17] to isolate the region where the Fourier transform of $\mathcal{F}_j f$ has rapid decay. In fact, by Proposition 2.5.7, in [14], p. 123, the wave front set of each of the distributions $\mathcal{F}_j f,\ j \in \mathbb{N}$, given by (2.2) is actually contained in the conic set

$$C = \{(x,t,\xi,\tau) : \tau = q(\xi), \quad x + t\nabla q(\xi) = 0, \quad \xi \in \mathbb{R}^2 \setminus 0\}.$$

Choose an even function $\psi \in C_c^\infty(-2, 2)$ such that $0 \leq \psi \leq 1$, $\psi = 1$ on $[-1, 1]$. For $\delta > 0$ this gives a cut off function ψ^δ supported near the cone $\tau = q(\xi)$ in $\mathbb{R}^3$ defined by

$$\psi^\delta(\xi,\tau) = \psi\left(\frac{q(\xi) - \tau}{\delta}\right), \quad (\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R}. \tag{2.4}$$

This leads to two Fourier multiplier operators Q_δ and R_δ on $\mathbb{R}^3$:

$$\begin{aligned} \widehat{Q_\delta(\mathcal{F}_j f)}(\xi,\tau) &= \psi^\delta(\xi,\tau)\, \widehat{\mathcal{F}_j f}(\xi,\tau), \\ \widehat{\mathcal{R}_\delta(\mathcal{F}_j f)}(\xi,\tau) &= [1 - \psi^\delta(\xi,\tau)]\, \widehat{\mathcal{F}_j f}(\xi,\tau). \end{aligned} \tag{2.5}$$

Since $\mathcal{F}_j f = Q_\delta(\mathcal{F}_j f) + \mathcal{R}_\delta(\mathcal{F}_j f)$, the L^p estimate for $\mathcal{F}_j f$ follows from the corresponding estimates for $Q_\delta(\mathcal{F}_j f)$ and $\mathcal{R}_\delta(\mathcal{F}_j f)$.

$\mathcal{R}_\delta \mathcal{F}_j$ turns out to be a smoothing operator, and the estimate follows by standard kernel estimates, see Proposition 5.3. The operator $Q^\delta \mathcal{F}_j$ is more delicate. To deal with

$Q^\delta(\mathcal{F}_j f)$ we do a further decomposition of the operators $\mathcal{F}_j$ in terms of the angular variable, as in [17, 20]. For fixed $j \geq 1$ let $N = N(j)$ be the largest integer less than or equal to $2^{j/2}$, so that $2^{j/2} - 1 < N \leq 2^{j/2}$. We now choose N equally spaced points $\xi_0, \xi_1, \ldots, \xi_{N-1}$ on the unit circle $\mathbb{S}^1 = \{\xi \in \mathbb{R}^2 : |\xi| = 1\}$ with $\xi_0 = e_1$. In fact, we take $\xi_\nu = O\xi_0$ for $1 \leq \nu \leq N-1$, where O is the counterclockwise rotation by an angle $2\pi\nu/N$.

With $N = N(j)$ as above, let $\{\chi_\nu\}_{\nu=0}^{N-1}$ be a partition of unity on $\mathbb{R}^2 \setminus \{0\}$ with respect to the angular variable, see [15, 24]. Note that the functions χ_ν are homogeneous functions of degree zero on $\mathbb{R}^2 \setminus \{0\}$ with the following properties:

$$\chi_\nu(\xi) = \chi_0(O^{-1}\xi), \; 1 \leq \nu \leq N-1 \tag{2.6}$$

where $\xi/|\xi| = (\cos\theta, \sin\theta)$ and O is the counterclockwise rotation by an angle $2\pi\nu/N$, and

$$|\partial_{\xi_1}^k \chi_0(\xi)| \leq C_k, \quad |\partial_{\xi_2}^k \chi_0(\xi)| \leq C_k N^k \approx C_k\, 2^{\frac{jk}{2}} \text{ for } |\xi| = 1, \tag{2.7}$$

for all $k \in \mathbb{N}$, with a constant C_k independent of ν (hence independent of j).

Using the homogeneous partitions of unity $\{\chi_\nu\}_\nu$, we define the operators $\mathcal{F}_{j,\nu}$ by

$$\mathcal{F}_{j,\nu} f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + tq(\xi))}\, \rho_0(2^{-j}|\xi|)\, a(\xi)\, \chi_\nu(\xi)\, \hat{f}(\xi)\, d\xi \tag{2.8}$$

for $j \geq 1,\; 0 \leq \nu \leq N-1$, so that $Q_\delta \mathcal{F}_j f = \sum_{\nu=0}^{N-1} Q_\delta \mathcal{F}_{j,\nu} f$.

Remark 2.1 Note that in [17] the local smoothing estimates for operators of the form (2.1) has been established for $q(\xi) = |\xi|$. Unfortunately, we cannot appeal to the estimate in [17] even in this case, as we need more refined estimate with precise dependence of a on the constants involved. Our basic decompositions are similar, but we do a slightly different approach to estimate $Q_\delta \mathcal{F}_j f$ using duality as in [15] and an estimate for the square function associated with q, based on angular decomposition, proved in Theorem 1.2.

3 Auxiliary Estimates

For $0 \leq \rho_1, \rho \in \mathbb{C}_c^\infty(\mathbb{R}), \rho \geq 0$, and symbol $a \in S^0(\mathbb{R}^2)$, consider the Fourier integral operator $\tilde{\mathcal{F}}_{j,\nu}$ given by

$$\tilde{\mathcal{F}}_{j,\nu} f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + tq(\xi))}\, \rho(2^{-j}|\xi|)\, a(\xi)\, \chi_\nu(\xi)\, \hat{f}(\xi)\, d\xi, \tag{3.1}$$

with χ_ν as in (2.6). Then we have

$$\tilde{\mathcal{F}}_{j,\nu} f(x,t) = \int_{y\in\mathbb{R}^2} \tilde{K}_{j,\nu}(x-y,t)\, f(y)\, dy,$$

where

$$\tilde{K}_{j,\nu}(x,t) = \tilde{K}^{a,q}_{j,\nu}(x,t) = \rho_1(t) \int_\xi e^{i(x\cdot\xi + tq(\xi))} \rho(2^{-j}|\xi|)\, a(\xi)\, \chi_\nu(\xi)\, d\xi. \quad (3.2)$$

The following kernel estimate is a refinement of the one obtained in [20], and is crucial in our argument for dealing with general amplitude functions depending on (x, t).

Proposition 3.1 *Let $\tilde{K}_{j,\nu}$ be as in (3.2), with $\rho_1 \in C_c^\infty(\mathbb{R})$, $0 \le \rho \in \mathbb{C}_c^\infty([\frac{1}{2}, 2])$ and $a \in S^0(\mathbb{R}^2)$, $j \in \mathbb{N}, 0 \le \nu \le N \approx 2^{j/2}$. Then, $\|\tilde{K}_{j,\nu}\|_{L^1(\mathbb{R}^2\times\mathbb{R})}$ is uniformly bounded in j. More precisely, there exists a constant $C = C_{\rho_1}$ such that*

$$\|\tilde{K}_{j,\nu}\|_{L^1(\mathbb{R}^2\times\mathbb{R})} \le C \sup_{|\alpha|\le l} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}.$$

The proof relies on appropriate pointwise estimate for the kernel, which requires the following technical result.

Lemma 3.2 *Let $q \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ be homogeneous of degree 1. Then the function $h(\xi) = q(\xi) - \xi \cdot \nabla q(e_1)$ satisfies*

$$|\partial^k_{\xi_1} h(\xi)| \le A_k\, 2^{-kj},\ \ |\partial^k_{\xi_2} h(\xi)| \le B_k\, 2^{-\frac{kj}{2}},\ \ \textit{for } k \ge 1,$$

on the set $E = \{\xi \in \mathbb{R}^2 : 2^{j-1} \le |\xi| \le 2^{j+1}, 0 \le \arg(\xi) < \frac{2\pi}{N}\}$ with $N = [2^{j/2}]$.

Remark 3.3 Note that the above estimates have been obtained in [24] for more general h depending also on x. In our special case, we give a proof using a geometric argument as in [15], where we considered the case $q(\xi) = |\xi|$.

Proof of Lemma 3.2 We first consider the case $k = 1$. Since q is homogeneous of degree 1, writing $\xi = r(\cos\theta, \sin\theta)$, we see that

$$q(\xi) = rq(\cos\theta, \sin\theta) := r\tilde{q}(\theta). \quad (3.3)$$

Differentiating (3.3) with respect to r and θ, we get

$$\begin{aligned}\tilde{q}(\theta) &= \cos\theta\ \partial_1 q(\xi) + \sin\theta\ \partial_2 q(\xi),\\ \partial_\theta\tilde{q}(\theta) &= -\sin\theta\ \partial_1 q(\xi) + \cos\theta\ \partial_2 q(\xi)\end{aligned} \quad (3.4)$$

where $\partial_i = \partial_{\xi_i}, i = 1, 2$. From this we see that

$$\partial_1 q(\cos\theta, \sin\theta) = \cos\theta\ \tilde{q}(\theta) - \sin\theta\ \partial_\theta\tilde{q}(\theta), \quad (3.5)$$

$$\partial_2 q(\cos\theta, \sin\theta) = \sin\theta\ \tilde{q}(\theta) + \cos\theta\ \partial_\theta\tilde{q}(\theta). \quad (3.6)$$

For future reference, we also note that the same argument leads to the identity

$$\partial_1 g = -\sin\theta\ \ \partial_\theta\tilde{g} \quad (3.7)$$

on $\|\xi\| = 1$, when $g(\xi)$ is a homogeneous function of degree zero.

Writing

$$h(\xi) = q(\xi) - \alpha_1\xi_1 - \alpha_2\xi_2,\ \xi = (\xi_1, \xi_2) \tag{3.8}$$

with $\alpha_1 = \partial_1 q(e_1),\ \alpha_2 = \partial_2 q(e_1)$ and using (3.5), we see that

$$\begin{aligned}\partial_1 h(\xi) &= \partial_1 q(\xi) - \alpha_1 = \cos\theta\ \tilde{q}(\theta) - \sin\theta\ \partial_\theta\tilde{q}(\theta) - \partial_1 q(e_1)\\ &= \cos\theta\ \tilde{q}(\theta) - \sin\theta\ \partial_\theta\tilde{q}(\theta) - \tilde{q}(0),\end{aligned}$$

as $\partial_1 q(e_1) = q(e_1) = \tilde{q}(0)$ in view of the Euler identity $\xi\cdot\nabla q(\xi) = q(\xi)$ for homogeneous function of degree 1, choosing $\xi = e_1$. By the mean value theorem applied to $Q_1(\theta) = \cos\theta\ \tilde{q}(\theta) - \sin\theta\ \partial_\theta\tilde{q}(\theta)$, we get

$$\partial_1 h(\xi) = \theta\cdot Q_1'(\theta_1) = \theta\cdot[-\sin\theta_1\ \tilde{q}(\theta_1) - \sin\theta_1\ \partial_\theta^2\tilde{q}(\theta_1)],$$

for some $\theta_1 \in (0,\theta)$. Thus,

$$|\partial_{\xi_1} h(\xi)| \le |\theta||\sin\theta_1||\tilde{q}(\theta_1) + \partial_\theta^2\tilde{q}(\theta_1)| \le |\theta|^2\cdot M, \tag{3.9}$$

since $|\sin\theta_1| \le |\theta_1| \le |\theta|$, where $M = \sup\limits_{\theta_1\in[0,2\pi]} |\tilde{q}(\theta_1) + \partial_\theta^2\tilde{q}(\theta_1)| < \infty$ as $q \in C^\infty(\mathbb{S}^1)$.

A similar mean value theorem argument using the identity (3.6) yields

$$\begin{aligned}|\partial_{\xi_2} h(\xi)| &\le |\partial_{\xi_2} q(\xi) - \alpha_2| = |\sin\theta\ \tilde{q}(\theta) + \cos\theta\ \partial_\theta\tilde{q}(\theta) - \partial_\theta\tilde{q}(0)|\\ &\le |\theta|\cdot M_1,\end{aligned} \tag{3.10}$$

where $M_1 = \sup\limits_{\theta_1\in[0,2\pi]} |\cos\theta_1\ \tilde{q}(\theta_1) + \cos\theta_1\ \partial_\theta^2\tilde{q}(\theta_1)| < \infty$.

From (3.9) and (3.10), the result follows for the case $k = 1$ as $|\theta| \le \frac{2\pi}{N} \le 8\pi 2^{-j/2}$ since $N = [2^{j/2}] \ge 2^{j/2} - 1,\ j \ge 1$.

To deal with the case $k > 1$, we write $\partial_{\xi_1}^k h(\xi) = \partial_{\xi_1}^{k-1} g(\xi)$, where $g = \partial_{\xi_1} h$, which is a function homogeneous of degree zero on $\mathbb{R}^2$, hence $\partial_{\xi_1}^{k-1} g$ is homogeneous of degree $1-k$. It follows that

$$\partial_{\xi_1}^k h(\xi) = |\xi|^{1-k}(\partial_{\xi_1}^{k-1} g)(\xi/|\xi|). \tag{3.11}$$

Recall that $g(\xi) = \cos\theta\ \tilde{q}(\theta) - \sin\theta\ \partial_\theta\tilde{q}(\theta) - \tilde{q}(0) := \tilde{g}(\theta)$ as computed above and also $\partial_{\xi_1} = -\sin\theta\,\partial_\theta$ on homogeneous functions, on $|\xi| = 1$. An easy induction argument shows that

$$(-\sin\theta\ \partial_\theta)^{k-1}\tilde{g}(\theta) = F_k(\cos\theta, \tilde{q}(\theta), \ldots, \partial_\theta^k\tilde{q}(\theta))\ \sin^2\theta,$$

where F_k is a smooth function. Now for $\xi = (r\cos\theta, r\sin\theta) \in E$, we have $|\theta| \leq 2\pi/N$, and hence $|F_k(\cos\theta, \tilde{q}(\theta), \ldots, \partial^k\tilde{q}(\theta))\,\sin^2\theta| \leq c_k|\sin^2\theta| \leq c_k 4\pi^2 N^{-2} \approx C_k 2^{-j}$ for some constant C_k independent of j. It follows from (3.11) that, for $k > 1$

$$\left|\partial^k_{\xi_1} h(\xi)\right| \leq C_k 2^{-kj}$$

as $|\xi| \approx 2^j$ on E.

For $k \geq 2$, note that $\partial^k_{\xi_2} h(\xi) = \partial^k_{\xi_2}(q(\xi) - \alpha_2\xi_2)$. Since the function $g_1(\xi) = q(\xi) - \alpha_2\xi_2$ is homogeneous of degree 1, these derivatives are homogeneous functions of degree $1-k$. It follows that $|\partial^k_{\xi_2} h(\xi)| \leq C_k|\xi|^{1-k} \leq C_k|\xi|^{-k/2}$ on E, for $k \geq 2$ and hence the required inequality holds true on E. □

Lemma 3.4 *Let $\tilde{K}_{j,\nu}$ be as in (3.2) with $a \in S^m(\mathbb{R}^2)$, $m \leq 0$. Then for each $l \in \mathbb{N}$, the kernel $\tilde{K}_{j,\nu}$ satisfies the estimates*

$$|\tilde{K}_{j,\nu}(x,t)| \leq C_l 2^{3j/2}|\rho_1(t)| \sup_{|\alpha|\leq l} \|\partial^\alpha a\|_\infty \,\Psi_j(Tx + t\nabla q_\nu(e_1)), \tag{3.12}$$

with constants C_l independent of j and ν, and

$$\Psi_j(x) = \Psi_{j,l}(x) = \left[1 + 2^{2j}|x_1|^2\right]^{-l}\left[1 + 2^j|x_2|^2\right]^{-l}, l \in \mathbb{N}$$

$T \in SO(2)$ is such that $T\xi_\nu = e_1$, $0 \leq \nu \leq N-1$ and $q_\nu = q \circ T^{-1}$.

Proof The proof follows by arguments similar to the ones in [15]. We first consider the case $\xi_\nu = \xi_0 = e_1$ and estimate $\tilde{K}_{j,0}(x,t)$ by oscillatory integral techniques as in [15, 20]. From (3.2) we have

$$\tilde{K}_{j,0}(x,t) = \rho_1(t) \int_\xi e^{i(x\cdot\xi + tq(\xi))}\,\rho(2^{-j}|\xi|)\,a(\xi)\,\chi_0(\xi)\,d\xi. \tag{3.13}$$

Let $L_j = \left(I - 2^{2j}\partial^2_{\xi_1}\right)\left(I - 2^j\partial^2_{\xi_2}\right)$, so that for each $l \in \mathbb{N}$

$$\begin{aligned} &L^l_j\, e^{i(x+t\nabla q(e_1))\cdot\xi} \\ &\quad= \left[1 + 2^{2j}|x_1 + t(\partial_{\xi_1}q)(e_1)|^2\right]^l \left[1 + 2^j|x_2 + t(\partial_{\xi_2}q)(e_1)|^2\right]^l e^{i(x+t\nabla q(e_1))\cdot\xi}. \end{aligned}$$

Re writing $e^{i(x\cdot\xi+tq(\xi))}$ as $e^{it(q(\xi)-\nabla q(e_1)\cdot\xi)}\,e^{i(x+t\nabla q(e_1))\cdot\xi}$ and using the above formula, we get

$$\begin{aligned} e^{i(x\cdot\xi+tq(\xi))} &= \left[1 + 2^{2j}|x_1 + t(\partial_{\xi_1}q)(e_1)|^2\right]^{-l}\left[1 + 2^j|x_2 + t(\partial_{\xi_2}q)(e_1)|^2\right]^{-l} \\ &\quad\times e^{it(q(\xi)-\nabla q(e_1)\cdot\xi)}\,L^l_j e^{i(x+t\nabla q(e_1))\cdot\xi}. \end{aligned}$$

Using this formula in (3.13), an integration by parts argument shows that

$$\tilde{K}_{j,0}(x,t) = A^a_{j,0}(x,t)\,\rho_1(t)\,\Psi_j(x + t\nabla q(e_1)), \tag{3.14}$$

where $\Psi_j(x) = \left[1 + 2^{2j}|x_1|^2\right]^{-l}\left[1 + 2^j|x_2|^2\right]^{-l}$ and

$$A^a_{j,0}(x,t) = \int_\xi e^{i[x+t\nabla q(e_1)]\cdot\xi}\; L^l_j\left[e^{it(q(\xi)-\nabla q(e_1)\cdot\xi)}\,\rho(2^{-j}|\xi|)\,a(\xi)\,\chi_0(\xi)\right]d\xi. \tag{3.15}$$

Note that the integrand in (3.15) is supported in the set

$$E = \operatorname{supp}\chi_0 \cap \{\xi : 2^{j-1} \le |\xi| \le 2^{j+1}\}.$$

We need to show that $|A^a_{j,0}(x,t)| \le C_k \sup_{|\alpha|\le l}\|\partial^\alpha a\|_\infty\, 2^{3j/2}$, to complete the proof for $\nu = 0$. For this it is enough to verify the following;

- The measure of E is bounded by a constant times $2^{3j/2}$,
- $L^k_j\left[e^{it(q(\xi)-\nabla q(e_1)\cdot\xi)}\,\rho(2^{-j}|\xi|)\,a(\xi)\,\chi_0(\xi)\right] \le C_k \sup_{|\alpha|\le l}\|\partial^\alpha a\|_\infty$ for some constant C_k independent of j.

Since $|\xi_2| \le \xi_1\sin(2\pi/N) \lesssim 2^{j/2}$ and $2^{j-1} \le \xi_1 \le 2^{j+1}$ on E, the first statement is clear.

For the second, we observe that L^m_j is a linear combination of various derivatives $(2^{2j}\partial^2_{\xi_1})^{k_1}(2^j\partial^2_{\xi_2})^{k_2}$ with $k_1 + k_2 \le 2m$. In view of (2.7) and the fact that $\chi_0(\xi)$ is homogeneous of degree zero, the above derivatives of $[\chi_0(\xi)\;\rho(2^{-j}|\xi|)]$ are uniformly bounded in $j \in \mathbb{N}$. Also each of these $2m$ derivatives of a are bounded by $\sup_{|\alpha|\le m}\|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}$, which is independent of j. All the above derivatives applied to $e^{it(q(\xi)-\nabla q(e_1)\cdot\xi)}$ also give functions bounded uniformly in j on E, in view of Lemma 3.2.

To estimate $A^a_{j,\nu}$ for general ν, first note that $\chi_\nu(\xi) = \chi_0(O^{-1}\xi)$ by (2.6) where $O \in SO(2)$ is such that $\xi_\nu = Oe_1$. Thus, using the change of variable $\xi \to O\xi$ in (3.2), we see that

$$\tilde{K}_{j,\nu}(x,t) = \rho_1(t)\int_\xi e^{i(O^{-1}x\cdot\xi + t[q\circ O](\xi))}\,\rho(2^{-j}|\xi|)\,[a\circ O](\xi)\,\chi_0(\xi)\,d\xi.$$

It follows that, $\tilde{K}_{j,\nu}(x,t) := \tilde{K}^{a,q}_{j,\nu}(x,t) = K^{a^\nu,q^\nu}_{j,0}(O^{-1}x,t)$ where $a^\nu(\xi) = a(O\xi)$ and $q^\nu(\xi) = q(O\xi)$. Notice that the estimate for $|A^{a^\nu,q^\nu}_{j,0}(x,t)|$ depends on the derivatives of $a^\nu = a\circ O$ and $q^\nu = q\circ O$, which have the same bounds as a and q respectively. Hence, the proof follows with $T = O^{-1}$. □

Proof of Proposition 3.1 The proof is an immediate consequence of the pointwise estimate for the kernel $\tilde{K}_{j,\nu}$ given by Lemma 3.4. Since $\|\Psi_j\|_{L^1(\mathbb{R}^2)} = 2^{-3j/2}\|\Psi_1\|_{L^1(\mathbb{R}^2)}$ and $\Psi_1 = \Psi_{1,l} \in L^1(\mathbb{R}^2)$ for the choice of $l = 2$ in Lemma 3.4. □

4 Square Function Estimate

In this section we will prove Theorem 1.2. The proof essentially follows the same argument as in Theorem 5.1, in [15], with appropriate modification for general homogeneous function $q(\xi)$. We first establish the case $p = 4$, which will be used to establish the case $4/3 \le p < 4$. We start with the following auxiliary estimate.

Proposition 4.1 *Let $T^{\delta}_{\nu,j}$ be as in (1.5). Then the square function estimate*

$$\|Sg\|_{L^4(\mathbb{R}^3)} = \left\| \left(\sum_{\nu=0}^{N-1} |T^{\delta}_{\nu,j} g|^2 \right)^{\frac{1}{2}} \right\|_{L^4(\mathbb{R}^3)} \le C\, \delta^{1/4} j^b \, \|g\|_{L^4(\mathbb{R}^3)} \tag{4.1}$$

holds true for all $g \in \mathcal{S}(\mathbb{R}^3)$, with constants C and b independent of j.

Note that,

$$T^{\delta}_{\nu,j} g = \int_{\mathbb{R}^3} \tilde{k}^{\delta}_{j,\nu}(x-y, t-s)\, g(y,s)\, dy\, ds,$$

where

$$\begin{aligned} \tilde{k}^{\delta}_{j,\nu}(x,t) &= \int_{\mathbb{R}^3} e^{i(x\cdot\xi + t\tau)} \left[\tilde{\chi}_{\nu}(\xi)\, \rho(2^{-j}|\xi|)\, \psi\left(\frac{q(\xi)-\tau}{\delta} \right) \right] d\xi\, d\tau \\ &= \delta \psi^{\vee}(\delta t)\, K_{j,\nu}(x,t), \end{aligned}$$

with $K_{j,\nu}(x,t) = \int_{\mathbb{R}^2} e^{i(x\cdot\xi + tq(\xi))} \left[\tilde{\chi}_{\nu}(\xi)\, \rho(2^{-j}|\xi|) \right] d\xi$ and $\tilde{\chi}_{\nu}$ is a homogeneous function (smooth and compactly supported as a function on $\mathbb{S}^1$), such that $\tilde{\chi}_{\nu}\chi_{\nu} = \chi_{\nu}$. Note that $\rho_1(t) K_{j,\nu}$ is same as $\tilde{K}_{j,\nu}$ in (3.2) with $a \equiv 1$. Thus, by the argument as in Lemma 3.4, we see that $\tilde{k}^{\delta}_{j,\nu}(x,t)$ also satisfies the estimate (3.12), but with $\delta\psi^{\vee}(\delta t)$ instead of $\rho_1(t)$ and $a \equiv 1$, on the right hand side. Hence the proof of the above proposition follows from the same argument as in Proposition 5.1 in [15], where the special case $q(\xi) = |\xi|$ is considered.

Proof of Theorem 1.2 We use the Rademacher function argument as in Stein [23], p. 106, to reduce the proof of the square function estimate (4.1) to a multiplier problem. Recall that the Rademacher functions $\{r_k\}_{k\ge 0}$ are functions on $\mathbb{R}$ defined as follows. First, let r_0 be the periodic function on $\mathbb{R}$ with period 1 defined by

$$r_0(s) = \chi_{[0,1/2]}(s) - \chi_{(1/2,1)}(s), \text{ for } 0 \le s < 1.$$

Recall that here χ_A denotes the characteristic function of the set $A \subset [0,1]$. Then, for $k \in \mathbb{N}$, define $r_k(s) = r_0(2^k s),\ k \ge 1$.

The Rademacher functions have the following interesting property: if $F(s) = \sum_{\nu} a_{\nu} r_{\nu}(s) \in L^2([0,1])$, there exist positive constants c_1, c_2, depending only on

p (and not on the particular function F), such that

$$c_1\|F\|_{L^2([0,1])} \le \|F\|_{L^p([0,1])} \le c_2\|F\|_{L^2([0,1])}, \tag{4.2}$$

for all $p \in (1, \infty)$, see [23], p. 277.

For each $s \in [0, 1)$, setting $P(s,x,t) = \sum_{\nu=0}^{N-1} r_\nu(s)\, T^\delta_{\nu,j} g(x,t)$, where $T^\delta_{\nu,j}$ is as in (1.5), we see that $|Sg(x,t)| = \left(\int_0^1 |P(s,x,t)|^2 ds\right)^{1/2}$, by the orthonormality of the collection $\{r_\nu\}$. Thus, in view of (4.2), we see that

$$|Sg(x,t)| = \left(\int_0^1 |P(s,x,t)|^2 ds\right)^{1/2} \le C_p\, \|P(\cdot,x,t)\|_{L^p([0,1])},$$

for $1 < p < \infty$, for each $(x,t) \in \mathbb{R}^3$, with a constant C_p independent of (x,t). It follows that

$$\int_{\mathbb{R}^3} |Sg(x,t)|^p dx\, dt \le C_p^p \int_{\mathbb{R}^3}\int_0^1 |P(s,x,t)|^p ds\, dxdt. \tag{4.3}$$

Note that $P(s,x,t) = T_s g(x,t)$, where T_s is the multiplier operator on $\mathbb{R}^3$, defined by

$$\widehat{T_s g}(\xi,\tau) = \tilde{m}_j^{\delta,s}(\xi,\tau)\widehat{g}(\xi,\tau) \tag{4.4}$$

where $\tilde{m}_j^{\delta,s}(\xi,\tau) = \sum_{\nu=0}^{N-1} r_\nu(s)\, \tilde{\chi}_\nu(\xi)\, \rho(2^{-j}|\xi|)\, \psi\left(\frac{q(\xi)-\tau}{\delta}\right)$, for given j and δ.

It follows that (4.3) can be re written as

$$\int_{\mathbb{R}^3} |Sg(x,t)|^p dx\, dt \le c_2^p \int_0^1 \int_{\mathbb{R}^3} |T_s g(x,t)|^p dx\, dt\, ds. \tag{4.5}$$

Thus, the L^p-boundedness of S for $4/3 \le p \le 4$ follows once we prove the estimate

$$\|T_s g\|_{L^p(\mathbb{R}^3)} \le C\, 2^{j/8}\delta^{1/4} j^b \|g\|_{L^p(\mathbb{R}^3)},\ 4/3 \le p \le 4.$$

Recall that $T_s g = \sum_{\nu=0}^{N-1} r_\nu(s) T^\delta_{\nu,j} g$. By considering the operator $\tilde{T}_s : L^2(\mathbb{R}^3 : \mathbb{R}^N) \to L^2(\mathbb{R}^3)$ given by

$$\tilde{T}_s(h) = \sum_{\nu=0}^{N-1} r_\nu(s)\, T^\delta_{\nu,j}(h_\nu),\ h = (h_0, h_1, \ldots, h_{N-1}) \tag{4.6}$$

we can see, as in Proposition 5.2 of [15], that the following inequalities hold true.

$$\|\tilde{T}_s h\|_{L^2(\mathbb{R}^3)} \le \sqrt{5} \left\| \left(\sum_\nu |T^\delta_{\nu,j} h_\nu|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^3)}, \tag{4.7}$$

$$\|\tilde{T}_s h\|_{L^\infty(\mathbb{R}^3)} \leq C 2^{j/4} \left\| \left(\sum_\nu |T^\delta_{\nu,j} h_\nu|^2 \right)^{1/2} \right\|_{L^\infty(\mathbb{R}^3)}. \quad (4.8)$$

Relying on vector-valued interpolation (see [9], Theorem 1.19), (4.7) and (4.8) yield the estimate

$$\|\tilde{T}_s h\|_{L^p(\mathbb{R}^3)} \leq 5^{1/p}\, 2^{\frac{j}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \left\| \left(\sum_\nu |T^\delta_{\nu,j} h_\nu|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^3)}, \; 2 \leq p \leq \infty. \quad (4.9)$$

Note that for $g \in \mathcal{S}(\mathbb{R}^3)$, we have $T_s(g) = \tilde{T}_s(h)$ with $h = (g, g, \ldots, g)$, in view of (4.4) and (4.6). Hence the inequality (4.9) with $p = 4$ gives

$$\|T_s g\|_{L^4(\mathbb{R}^3)} \leq C 2^{j/8} \left\| \left(\sum_\nu |T^\delta_{\nu,j} g|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3)} \leq C 2^{j/8} \delta^{1/4} j^b \|g\|_{L^4(\mathbb{R}^3)}$$

by Proposition 4.1. Since T_s, given by (4.4) is a multiplier operator, the above estimate holds true for the dual index $p = 4/3$ as well. Hence the required estimate follows by Riesz-Thorin interpolation theorem between these two estimates. This completes the proof. □

5 L^p Estimates for $\mathcal{F}_j f$

In this section we prove the L^p regularity estimate for $\mathcal{F}_j f$, for $4 \leq p \leq \infty$. This follows by interpolation, once we prove the L^4 and L^∞ estimates. We start with the L^∞ estimate.

Proposition 5.1 *Let $\mathcal{F}_j$ be the operator given by (2.2) for $j \in \mathbb{N}$. Then $\mathcal{F}_j$ satisfies the inequality*

$$\|\mathcal{F}_j f\|_{L^\infty(\mathbb{R}^3)} \leq C\, 2^{j/2} \sup_{|\alpha|\leq 2} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} \|f\|_{L^\infty(\mathbb{R}^2)}$$

with a constant C independent of j.

Proof We have $\mathcal{F}_j = \sum_{\nu=0}^{N-1} \mathcal{F}_{j,\nu}$ where $\mathcal{F}_{j,\nu}$ is the operator given by (2.8), which is convolution in x-variable, with kernel

$$K_{j,\nu}(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + tq(\xi))} \rho_0(2^{-j}|\xi|)\, a(\xi)\, \chi_\nu(\xi)\, d\xi.$$

Since $\rho_0 = \rho^2$ by assumption, by Proposition 3.1 we have the uniform bound

$$\|K_{j,\nu}\|_{L^1(\mathbb{R}^2\times\mathbb{R})} \leq C \sup_{|\alpha|\leq 2} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}.$$

It follows that for each $0 \le \nu \le N-1$, the estimate

$$\|\mathcal{F}_{j,\nu} f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})} \le C \sup_{|\alpha|\le 2} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} \|f\|_{L^\infty(\mathbb{R}^2)}$$

holds true with a constant C independent of j. Summing over ν, this gives the required estimate as there are $N = N(j) \approx 2^{j/2}$ terms in the sum. □

We next prove the L^4 estimate. For this, we write $\mathcal{F}_j f = \mathcal{Q}_\delta(\mathcal{F}_j f) + \mathcal{R}_\delta(\mathcal{F}_j f)$ and estimate the norm of each of these terms separately. We start with $\mathcal{R}_\delta(\mathcal{F}_j f)$, which is easier and follows via standard kernel estimate, once we make the following observation:

Lemma 5.2 *Let q be as in (1.1). For $j \in \mathbb{N}$ and $0 < \delta < 2^j$, consider the set*

$$A_j^\delta = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : 2^{j-1} < |\xi| \le 2^{j+1}, |\tau - q(\xi)| > \delta\}.$$

Then, for each $0 < \epsilon \le \frac{1}{2}$, there exists a constant C_ϵ independent of j and δ such that the estimate

$$|\tau - q(\xi)| > C_{j,\epsilon,\delta}(|\tau| + |q(\xi)|)^\epsilon \tag{5.1}$$

holds true for all $(\xi, \tau) \in A_j^\delta$ with $C_{j,\epsilon,\delta} = C_\epsilon \frac{\delta}{2^{j\epsilon}}$.

Proof Since q is nonvanishing, we have either $q(\xi) > 0$ or $q(\xi) < 0$ for all ξ. We first prove the estimate for $q(\xi) > 0$. In this case, it clearly follows if $\tau \le 0$. In fact, $\frac{|\tau - q(\xi)|}{||\tau| + q(\xi)|^\epsilon} \ge (|\tau| + q(\xi))^{1-\epsilon} = (-\tau + q(\xi))^{1-\epsilon} > \delta^{1-\epsilon}$ and the claim is proved, since $\delta < 2^j$.

Now, assume $\tau > 0$ and $q(\xi) > 0$. We write $A_j^\delta = B_1 \cup B_2$ where

$$B_1 = \{(\xi, \tau) \in A_j^\delta : \tau > 2q(\xi)\}, \quad B_2 = \{(\xi, \tau) \in A_j^\delta : \tau \le 2q(\xi)\}$$

We show that $\inf_{(\xi,\tau)\in B_i} \frac{|\tau - q(\xi)|}{(\tau + |q(\xi)|)^\epsilon} \ge C_{j,\epsilon,\delta}$ for $i = 1, 2$. Since $\tau > 2q(\xi)$ on B_1, we have

$$\frac{|\tau - q(\xi)|}{(\tau + |q(\xi)|)^\epsilon} = \frac{\tau - q(\xi)}{(\tau + |q(\xi)|)^\epsilon} > \tau^{1-\epsilon} \frac{1-\theta}{(1+\theta)^\epsilon} > \left(\frac{2}{3}\right)^\epsilon \frac{\tau^{1-\epsilon}}{2}.$$

as $\theta = q(\xi/\tau) < 1/2$ on B_1. Writing $\tau = \tau - q(\xi) + q(\xi)$, we see that $|\tau|^{1-\epsilon} > \delta^{1-\epsilon}$ as $q(\xi) > 0$, hence the required estimate holds true on B_1, as $\delta < 2^j$.

On the other hand on B_2, we have

$$\frac{|\tau - q(\xi)|}{(\tau + |q(\xi)|)^\epsilon} > \frac{\delta}{[3q(\xi)]^\epsilon} \ge \frac{\delta}{[3|\xi| q(\xi/|\xi|)]^\epsilon}$$

as q is homogeneous of degree 1. Setting $C_2 = \sup_{|\xi|=1} |q(\xi)|$, we see that the last term above is bounded from below by $\frac{\delta}{(6C_2)^\epsilon 2^{j\epsilon}}$ as $|\xi| \le 2^{j+1}$ on B_2. This completes the proof in the case $q > 0$. For $q < 0$, one can work with $-q$ as in the previous case, since the right hand side of (5.1) is given in terms of $|q|$. Hence the proof. □

5.1 L^4 Estimates for $\mathcal{R}_\delta(\mathcal{F}_j f)$

The operator $\mathcal{R}_\delta$ was defined as a multiplier operator. Thus in view of (2.5) and (2.2), we have

$$\widehat{\mathcal{R}_\delta(\mathcal{F}_j f)}(\xi, \tau) = [1 - \psi^\delta(\xi, \tau)]\, \hat{f}(\xi)\, a(\xi)\, \rho_0(2^{-j}|\xi|)\, \hat{\rho}_1(\tau - q(\xi)). \tag{5.2}$$

Thus for $f \in \mathcal{S}(\mathbb{R}^2)$, by Fourier inversion formula

$$\mathcal{R}_\delta(\mathcal{F}_j f)(x, t) \\ = \int_{(\xi,\tau)\in\mathbb{R}^2\times\mathbb{R}} e^{i(x\cdot\xi+t\tau)}\, [1 - \psi^\delta(\xi, \tau)]\, \hat{f}(\xi)\, a(\xi)\, \rho_0(2^{-j}|\xi|)\, \hat{\rho}_1(\tau - q(\xi)) d\xi d\tau. \tag{5.3}$$

Proposition 5.3 *For $j \in \mathbb{N}$ and $0 < \delta < 2^{j/2}$, let $\mathcal{R}_\delta(\mathcal{F}_j f)$ be as in (5.3) with $a \in S^0$. Then for each $\epsilon > 0$ and $N \in \mathbb{N}$, there exist a constant $C_{\epsilon,N}$ independent of j, such that the inequality*

$$\|\mathcal{R}_\delta(\mathcal{F}_j f)\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \le C_{\epsilon,N} \sup_{|\alpha|\le 2N} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} \left(\frac{2^{\epsilon j}}{\delta}\right)^{3/\epsilon} \|f\|_{L^p} \tag{5.4}$$

holds true for all $f \in L^p(\mathbb{R}^2)$, $1 \le p < \infty$.

Proof Since $\mathcal{R}_\delta \mathcal{F}_j$ is a linear map, by density of $\mathcal{S}(\mathbb{R}^2)$ in $L^p(\mathbb{R}^2)$, $1 \le p < \infty$, it is enough to estimate (5.4) for $f \in \mathcal{S}(\mathbb{R}^2)$. Expanding $\hat{f}$ in (5.3), we see that

$$\mathcal{R}_\delta(\mathcal{F}_j f)(x, t) = \int_{\mathbb{R}^2} \mathcal{K}_j^\delta(x - y, t) f(y) dy$$

where, $\mathcal{K}_j^\delta(x, t) =$

$$\int_{\mathbb{R}^2\times\mathbb{R}} e^{i(x\cdot\xi+t\tau)}\, [1 - \psi^\delta(\xi, \tau)] a(\xi) \rho_0(2^{-j}|\xi|) \hat{\rho}_1(\tau - q(\xi))\, d\xi d\tau. \tag{5.5}$$

In view of Young's inequality [13], it is enough to prove the estimate

$$\|\mathcal{K}_j^\delta\|_{L^1(\mathbb{R}^2\times\mathbb{R})} \lesssim C_\epsilon \sup_{|\alpha|\le 2N} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} \left(\frac{2^{\epsilon j}}{\delta}\right)^{3/\epsilon} \tag{5.6}$$

for each $t \in \mathbb{R}$, with C_ϵ independent of j and t. The rest of the proof follows as in Lemma 3.4, observing that for any $N \in \mathbb{N}$,

$$(1+|x|^2)^N (1+|t|^2)^N e^{i(x\cdot\xi+t\tau)} = (I-\Delta_\xi)^N (I-\partial_\tau^2)^N e^{i(x\cdot\xi+t\cdot\tau)}.$$

An integration by parts in (5.5) shows that

$$\begin{aligned}&(1+|x|^2)^N (1+|t|^2)^N \mathcal{K}_j^\delta(x,t)\\ &= \int_{\xi,\tau} e^{i(x\cdot\xi+t\tau)} (I-\Delta_\xi)^N (I-\partial_\tau^2)^N b_j(\xi,\tau)\, d\xi d\tau, \qquad (5.7)\end{aligned}$$

where $b_j(\xi,\tau) = [1-\psi^\delta(\xi,\tau)]a(\xi)\rho_0(2^{-j}|\xi|)\hat{\rho}_1(\tau - q(\xi))$.

Note that $(I-\Delta_\xi)^N (I-\partial_\tau^2)^N b_j(\xi,\tau)$ is a sum of terms that involves various partial derivatives $\partial_\xi^\alpha \partial_\tau^\beta$, with $|\alpha| \le 2N$ and $|\beta| \le 2N$ acting on functions $\psi^\delta(\xi,\tau)$, $a(\xi)$, $\rho_0(2^{-j}|\xi|)$ and $\hat{\rho}_1(\tau - q(\xi))$. Each derivative on ψ^δ brings in a negative power of δ and since $\delta = 2^{\epsilon j}$, all these derivatives are uniformly bounded in ϵ and j. Same is the case with ρ_0. All partial derivatives of a upto order $2N$ are bounded by $\sup_{|\alpha|\le 2N} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}$. Since $\hat{\rho}_1$ is a Schwartz class function for each $M \in \mathbb{N}$, there is a constant $C_{N,M}$ such that the inequality $|\partial^\alpha \hat{\rho}_1(y)| \le C_{M,N}(1+|y|)^{-M}$ holds true for $|\alpha| \le N$, for all $y \in \mathbb{R}$. It follows that for each $N, M \in \mathbb{N}$, there is a constant $C_{M,N}$ independent of j such that

$$\begin{aligned}&|(I-\Delta_\xi)^N (I-\partial_\tau^2)^N b_j(\xi,\tau)|\\ &\le C_{M,N} \sup_{|\alpha|\le 2N} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} (1+|\tau - q(\xi)|)^{-M}\\ &\le C_{M,N} \sup_{|\alpha|\le 2N} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} (1+C_{j,\epsilon,\delta}(|\tau| + |q(\xi)|)^\epsilon)^{-M}\end{aligned}$$

for $|\tau - q(\xi)| > \delta$, by Lemma 5.2. Note that the integral in (5.5) and hence in (5.7) is actually over the set $|\tau - q(\xi)| > \delta$, as $\psi^\delta(\xi,\tau) = 1$ on $|\tau - q(\xi)| \le \delta$. Hence, Lemma 5.2 can be applied. Since q is homogeneous of degree 1, we have $|q(\xi)| \ge C_1|\xi|$ where $C_1 = \inf_{|\xi|=1} |q\left(\frac{\xi}{|\xi|}\right)| > 0$, as q is non vanishing. Thus the above inequality reads as

$$\begin{aligned}&|(I-\Delta_\xi)^N (I-\partial_\tau^2)^N b_j(\xi,\tau)|\\ &\le C_{M,N} \sup_{|\alpha|\le 2N} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} (1+C_{j,\epsilon,\delta}(|\tau| + C_1|\xi|)^\epsilon)^{-M}. \qquad (5.8)\end{aligned}$$

Using (5.8), the right hand side of (5.7) is bounded by $C_{M,N}$ times

$$\begin{aligned}&\sup_{|\alpha|\le 2N} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2\times\mathbb{R}} \left(1+C_{j,\epsilon,\delta}(|\tau| + C_1|\xi|)^\epsilon\right)^{-M} d\xi d\tau\\ &= \sup_{|\alpha|\le 2N} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} C_1^2 (C_{j,\epsilon,\delta})^{-3/\epsilon} \int_{\mathbb{R}^2\times\mathbb{R}} (1+(|\tau| + |\xi|)^\epsilon)^{-M} d\xi d\tau\end{aligned}$$

by a change of scale in the variables ξ and τ. Choosing $M > 3/\epsilon$, the last integral is finite and (5.7) translates to the inequality

$$|\mathcal{K}_j^\delta(x,t)| \leq C_{M(\epsilon),N} \sup_{|\alpha|\leq 2N} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} \frac{(C_{j,\epsilon,\delta})^{-3/\epsilon}}{(1+|x|^2)^N(1+|t|^2)^N}. \tag{5.9}$$

Choosing $N = 2$, we see that $\mathcal{K}_j(\cdot,\cdot) \in L^1(\mathbb{R}^2 \times \mathbb{R})$ and the estimate (5.6) holds true. The proof is complete. □

5.2 L^4 Estimates for $Q_\delta(\mathcal{F}_j f)$

We prove the L^4 estimate of $Q_\delta(\mathcal{F}_j f)$. The estimate we obtain here is a refinement of the one proved in [17], in terms of the precise dependence on a of the constants in the estimate, which is crucial in our argument. Using Theorem 1.2, we first obtain the following estimate.

Proposition 5.4 *Let $Q_\delta(\mathcal{F}_j f)$ and $\tilde{\mathcal{F}}_{j,\nu} f$ be as in (2.5) and (3.1), respectively. Then the inequality*

$$\|Q_\delta(\mathcal{F}_j f)\|_{L^4(\mathbb{R}^3)} \leq C\,\delta^{1/4}\, j^b\, 2^{j/8} \left\| \left(\sum_{\nu=0}^{N-1} |\tilde{\mathcal{F}}_{j,\nu} f|^2 \right)^{\frac{1}{2}} \right\|_{L^4(\mathbb{R}^3)},$$

holds true for all $f \in \mathcal{S}(\mathbb{R}^2)$, with constants C and b independent of j.

Proof We use duality to estimate the L^4 norm. For $H \in L^{4/3}(\mathbb{R}^3)$, writing $Q_\delta(\mathcal{F}_j f) = \sum_{\nu=0}^{N-1} Q_\delta(\mathcal{F}_{j,\nu} f)$, we have

$$\langle Q_\delta(\mathcal{F}_j f), H\rangle = \int_{\mathbb{R}^3} \sum_\nu Q_\delta(\mathcal{F}_{j,\nu} f)(x,t)\,\overline{H}(x,t)\,dxdt. \tag{5.10}$$

By Parseval's theorem for the Fourier transform, in view of (2.5) we see that

$$\begin{aligned} \int_{\mathbb{R}^3} Q_\delta(\mathcal{F}_{j,\nu} f)(x,t)\,\overline{H}(x,t)\,dxdt &= \int_{\mathbb{R}^3} \widehat{\mathcal{F}_{j,\nu} f}(\xi,\tau)\,\overline{\widehat{Q_\delta(H)}}(\xi,\tau)\,d\xi d\tau \\ &= \int_{\mathbb{R}^3} (\tilde{\mathcal{F}}_{j,\nu} f)(x,t)\,\overline{T^\delta_{\nu,j} H(x,t)}\,dxdt, \end{aligned} \tag{5.11}$$

where $T^\delta_{\nu,j}$ is the multiplier operator given by (1.5) with $\rho^2 = \rho_0$, and $\tilde{\mathcal{F}}_{j,\nu}$ is as in (3.1).

Now, summing over ν and using Cauchy-Schwarz inequality with respect to ν on the right-hand side of (5.11), followed by an application of Hölder's inequality, yields

$$|\langle Q_\delta(\mathcal{F}_j f), H\rangle| \leq \left\| \left(\sum_\nu |(\tilde{\mathcal{F}}_{j,\nu} f)|^2 \right)^{\frac{1}{2}} \right\|_4 \left\| \left(\sum_\nu \left| T^\delta_{\nu,j} H \right|^2 \right)^{\frac{1}{2}} \right\|_{4/3}. \tag{5.12}$$

By Theorem 1.2, the second term on the right hand side of (5.12) is bounded by $C\,\delta^{1/4}\,j^b\,2^{j/8}\,\|H\|_{4/3}$. Taking the supremum over $\|H\|_{4/3} \leq 1$ yields the required estimate. □

Next we estimate the L^4 norm of the square function in Proposition 5.4, using arguments very similar to those in [5].

Proposition 5.5 *Let $\tilde{\mathcal{F}}_{j,\nu} f$ be as in (3.1). Then, there exist constants b and C, independent of j, such that the following square function estimate holds true*

$$\left\| \left(\sum_{\nu=0}^{N-1} |\tilde{\mathcal{F}}_{j,\nu} f|^2 \right)^{\frac{1}{2}} \right\|_{L^4(\mathbb{R}^3)} \leq C j^{b+3/4} \sup_{|\alpha|\leq 2} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} \, \|f\|_{L^4(\mathbb{R}^2)}, \;\; f \in \mathcal{S}(\mathbb{R}^2).$$

Proof The proof follows as in [15]. For the sake of completeness, we sketch it here. Let Φ_ν denote the characteristic function of the support of χ_ν defined in (2.7) so that $\chi_\nu = \Phi_\nu \chi_\nu$. Setting $\widehat{f_\nu}(\xi) = \Phi_\nu(\xi)\,\hat{f}(\xi)$, we have $\tilde{\mathcal{F}}_{j,\nu} f = \tilde{\mathcal{F}}_{j,\nu} f_\nu$. Thus we see that

$$\tilde{\mathcal{F}}_{j,\nu} f = \int_{\mathbb{R}^2} \tilde{K}_{j,\nu}(x-y,t)\, f_\nu(y)\, dy, \tag{5.13}$$

with $\tilde{K}_{j,\nu}$ as in (3.2). In view of Lemma 3.4 with $l = 2$, using Cauchy-Schwarz inequality in (5.13) and summing over ν, we get

$$\begin{aligned} &\sum_{\nu=0}^{N-1} |\tilde{\mathcal{F}}_{j,\nu} f_\nu(x,t)|^2 \\ &\quad \leq C \sup_{|\alpha|\leq 2} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} \sum_{\nu} |f_\nu(y)|^2 |\tilde{K}_{j,\nu}(x-y,t)| dy \end{aligned} \tag{5.14}$$

for some constant C independent of t and j. Squaring, integrating and taking square root in (5.14) leads to the inequality

$$\begin{aligned} &\left\| \left(\sum_{\nu=0}^{N-1} |\tilde{\mathcal{F}}_{j,\nu} f_\nu|^2 \right)^{\frac{1}{2}} \right\|^2_{L^4(\mathbb{R}^3)} \leq C \sup_{|\alpha|\leq 2} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)} \\ &\times \sup_{\|g\|_{L^2}=1} \left| \int_{\mathbb{R}^2} \sum_{\nu} |f_\nu(y)|^2 \left[\int_{\mathbb{R}^3} |\tilde{K}_{j,\nu}(x-y,t)|\,|g(x,t)|\, dxdt \right] dy \right| \end{aligned} \tag{5.15}$$

where we used duality in the above inequality for the $L^2(\mathbb{R}^3)$ norm and Fubini's theorem. By Cauchy-Schwarz inequality in y variable, the term inside the modulus sign in the right-hand side is at most

$$\left[\int_{\mathbb{R}^2}\left(\sum_{\nu=0}^{N-1}|f_\nu(y)|^2\right)^2 dy\right]^{1/2}\left[\int_{\mathbb{R}^2}\sup_\nu\left|\int_{\mathbb{R}^3}|\tilde{K}_{j,\nu}(x-y,t)|\,|g(x,t)|\,dxdt\right|^2 dy\right]^{\frac{1}{2}}.$$

Note that the first factor satisfies

$$\left\|\left(\sum_{\nu=0}^{N-1}|f_\nu|^2\right)^{1/2}\right\|^2_{L^4(\mathbb{R}^2)} \leq C[\log N]^{2b}\,\|f\|^2_{L^4(\mathbb{R}^2)}$$

for constants $b > 0$ and C independent of N, as shown by A. Cordoba in [6]. Since $N \approx 2^{j/2}$, we see that the first term above is at most $Cj^{2b}\,\|f\|^2_{L^4}$, with C and b independent of j.

In view of the pointwise estimate for $\tilde{K}_{j,\nu}$ given by (3.12), we have the maximal inequality

$$\left[\int_{\mathbb{R}^2}\sup_\nu\left|\int_{\mathbb{R}^3}|\tilde{K}_{j,\nu}(x-y,t)g(x,t)|\,dxdt\right|^2 dy\right]^{\frac{1}{2}}$$
$$\leq Cj^{3/2}\sup_{|\alpha|\leq 2}\|\partial^\alpha a\|_\infty\|g\|_{L^2(\mathbb{R}^3)}$$

with C independent of j. This is a restatement of the maximal inequality (1.11) in [17], in view of Lemma 1.4 with $\delta = 2^{-j/2}$ in the quoted paper. Using these estimates in (5.15) and taking the square root, the claim follows. □

Now, we proceed to estimate the L^4 norm of $\mathcal{F}_j f$.

Proposition 5.6 *Let $\mathcal{F}_j f$ be as in (2.1), with amplitude function a depending only on ξ. Then, for each $0 < \epsilon \leq 1/2$, there exists a constant C_ϵ, independent of j, such that the estimate*

$$\|\mathcal{F}_j f\|_4 \leq C_\epsilon \sup_{|\alpha|\leq 4}\|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}\,2^{j(3\epsilon+1/8)}\,\|f\|_{L^4(\mathbb{R}^2)},$$

holds true for all $f \in L^4(\mathbb{R}^2)$.

Proof We have $\mathcal{F}_j f = Q_\delta(\mathcal{F}_j f) + \mathcal{R}_\delta(\mathcal{F}_j f)$ with $\delta = 2^{\epsilon j}$, where $Q_\delta\mathcal{F}_j f$ and $\mathcal{R}_\delta(\mathcal{F}_j f)$ are as in (2.5). In view of Propositions 5.4 and 5.5, the estimate

$$\|Q_\delta\mathcal{F}_j f\|_4 \leq C_\epsilon\,\delta^{1/4}\,2^{j/8}\,j^{2b+3/4}\sup_{|\alpha|\leq 2}\|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}\|f\|_{L^4(\mathbb{R}^2)}, \tag{5.16}$$

holds true for all $f \in L^4(\mathbb{R}^2)$, where $b > 0$ and C_ϵ is independent of j. Since $j^{2b+3/4} \leq C_{b,\epsilon}2^{11\epsilon j/4}$, the required estimate follows from (5.16) and Proposition 5.3 with $N = 2$. The proof is complete. □

Since the homogeneous function q has a singularity at the origin, our approach for estimating $\mathcal{F}_j f$ for $j \geq 1$ will not work for the case $j = 0$. However, $\mathcal{F}_0$ is a smoothing operator: For $\rho_1 \in C_c^\infty(I)$ and $a_0 \in C_c^\infty(\mathbb{R}^2)$ set

$$\mathcal{F}_0 f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + tq(\xi))} a_0(\xi)\, \hat{f}(\xi)\, d\xi, \;\; f \in \mathcal{S}(\mathbb{R}^2). \tag{5.17}$$

Proposition 5.7 *Let $\mathcal{F}_0 f$ be as in (5.17), with supp $a_0 \subset \{\xi \in \mathbb{R}^2 : |\xi| \leq 2\}$. Then the operator $f \to \mathcal{F}_0 f(\cdot, t)$ is a smoothing operator. In fact, for each $\sigma \in \mathbb{C}$, the estimate*

$$\|(I - \Delta_x)^\sigma \mathcal{F}_0 f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \lesssim C_{\sigma,n}\, \|f\|_{L^p(\mathbb{R}^2)}, \;\; 1 < p < \infty. \tag{5.18}$$

holds true for all $f \in L^p(\mathbb{R}^2)$ with $C_\sigma = \sup_{|\alpha|\leq 2} \|\partial^\alpha a_0(\xi)\|_\infty$.

Proof The proof follows as in [15], employing the Hörmander–Mihilin multiplier theorem. In fact, for each $t \in \mathbb{R}$ and $\mathrm{Re}(\sigma) < 0$, the operator $T_t^\sigma : f(x) \to (I - \Delta_x)^\sigma \mathcal{F}_0 f(\cdot, t)$ is a multiplier operator on $L^2(\mathbb{R}^2)$ with multiplier function

$$M_t^\sigma(\xi) = \rho_1(t)\, e^{itq(\xi)}\, (1 + |\xi|^2)^\sigma\, a_0(\xi) \;\in L^\infty(\mathbb{R}^2).$$

Since q is homogeneous of degree 1, we have that $|\xi|^{|\alpha|}|\partial^\alpha q(\xi)|$ is bounded for $|\xi| \leq 2$ for each α. Thus, since $a_0 \in C_c^\infty$, it follows that

$$|\xi|^{|\alpha|}|\partial_\xi^\alpha M_t^\sigma(\xi)| \leq C\, \rho_1(t) \sup_{|\alpha|\leq 2} \|\partial^\alpha a_0(\xi)\|_\infty, \;\; |\alpha| \leq 2,$$

with C independent of t. Hence, in view of the Hörmander–Mihilin multiplier theorem [3], followed by a $t-$integration yields the required estimate, for $1 < p < \infty$. □

Remark 5.8 Note that $\mathcal{F}_0$ commutes with $(I - \Delta_x)^\sigma$, since b is independent of x. Hence, the inequality (5.18) is equivalent to the Sobolev estimate

$$\|\mathcal{F}_0 f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \lesssim C_{\sigma,n}\, \|(I - \Delta_x)^{-\sigma} f\|_{L^p(\mathbb{R}^2)} := C_{\sigma,n}\, \|f\|_{L^p_\sigma(\mathbb{R}^2)},$$

for $1 < p < \infty$.

The regularity estimate for $\mathcal{F}_j f$ given by (2.1) also follows from the L^p estimates for $j \geq 1$, as the amplitude a is independent of (x, t) variables, using the following Lemma.

Lemma 5.9 *For $\sigma \in \mathbb{C}$ and ρ as in (3.1), define $f_{\sigma,j}$ by*

$$\widehat{f_{\sigma,j}}(\xi) = \hat{f}(\xi)\, \rho(2^{-j}|\xi|)\, (1 + |\xi|^2)^{\sigma/2}, \;\; f \in \mathcal{S}(\mathbb{R}^2).$$

Then the estimate

$$\|f_{\sigma,j}\|_{L^p(\mathbb{R}^2)} \leq C_\sigma\, 2^{jRe(\sigma)}\, \|f\|_{L^p(\mathbb{R}^2)}, \;\; 1 \leq p \leq \infty$$

holds true for all $f \in \mathcal{S}(\mathbb{R}^2)$ *with* $Re(\sigma) \leq 0$*, where* C_σ *is independent of* j*.*

The operator $f \to f_{\sigma,j}$ is a convolution operator, whose kernel is given by the inverse Fourier transform of the function $\rho(2^{-j}|\xi|)\,(1+|\xi|^2)^{\sigma/2}$. A simple integration by parts gives a favorable pointwise estimate for the kernel, which leads to the proof, see [15].

6 Local Smoothing Estimates

Now we proceed to prove the local smoothing estimate for the Fourier integral operators of the form (1.1) with general amplitude function $a(x,t,\xi)$ satisfying (1.2). We work with operators of the form (1.3) with $\rho_1 \in C_c^\infty(\mathbb{R})$, and complete the proof in three steps, discussed in the next three subsections:

6.1 Case of *a* Not Depending on (*x*, *t*)

For $0 < \rho \in C_c^\infty(\mathbb{R}_+)$, and $a \in S^m(\mathbb{R}^2)$, $m \leq 0$, consider the Fourier integral operator $\tilde{\mathcal{F}}_j$ defined as

$$\tilde{\mathcal{F}}_j f(x,t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x\cdot\xi + tq(\xi))}\, \rho(2^{-j}|\xi|)\, a(\xi)\, \hat{f}(\xi)\, d\xi, \;\; f \in \mathcal{S}(\mathbb{C}^n) \quad (6.1)$$

which differs from $\mathcal{F}_j$ given by (2.2) only in the power of ρ, namely $\rho^2 = \rho_0$. Note that, $\tilde{\mathcal{F}}_j f$ also satisfies the same norm estimates as in Propositions 5.6 and 5.1:

$$\|\tilde{\mathcal{F}}_j f\|_{L^4(\mathbb{R}^2\times\mathbb{R})} \leq C_\epsilon\, 2^{j(3\epsilon+1/8)} \sup_{|\alpha|\leq 4} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}\, \|f\|_{L^4(\mathbb{R}^2)}, \quad (6.2)$$

valid for $0 < \epsilon \leq 1/2$, and

$$\|\tilde{\mathcal{F}}_j f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})} \leq C\, 2^{j/2} \sup_{|\alpha|\leq 2} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}\, \|f\|_{L^\infty(\mathbb{R}^2)} \quad (6.3)$$

with constants C, C_ϵ independent of $j \in \mathbb{N}$.

In fact, the L^4 and L^∞ estimates for $\mathcal{F}_j^n f$ involve the bound for ρ_0 and its derivatives, which in turn depend only on the bound for ρ and its derivatives, since $\rho_0 = \rho^2$, as seen in the proofs of Lemma 3.4, and Propositions 5.3, 5.4 and 5.5.

Proposition 6.1 *Let* $\mathcal{F}_j$ *be the Fourier integral operator as in* (2.2) *with* $a \in S^m$, $m \leq 0$*, independent of* (x,t)*. Then for each* $\epsilon > 0$*, there exist constants* θ *and* $C_\epsilon > 0$ *independent of* $j \in \mathbb{N}$ *such that*

$$\|\mathcal{F}_j f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq C_\epsilon\, 2^{j\theta} \sup_{|\alpha|\leq 4} \|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}\, \|f\|_{L^p_{m-\sigma}(\mathbb{R}^2)},\; Re(\sigma) \leq 0,$$

for all $f \in L^p(\mathbb{R}^2)$, $4 \leq p \leq \infty$*, where* $\theta = 12\epsilon/p + (1/2 - 3/2p) + Re(\sigma)$*.*

Proof Since a is independent of (x,t), we have

$$(I-\Delta_x)^{(\sigma-m)/2}(\mathcal{F}_j f)(x,t)=\mathcal{F}_j[(I-\Delta_x)^{(\sigma-m)/2}f](x,t),$$

which can be seen by taking the Fourier transform of both sides with respect to x. Hence, it is enough to prove the inequality

$$\begin{aligned}&\|(I-\Delta_x)^{(\sigma-m)/2}(\mathcal{F}_j f)\|_{L^p(\mathbb{R}^2\times\mathbb{R})}\\ &\leq C_\epsilon 2^{j\theta}\sup_{|\alpha|\leq 4}\|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}\ \|f\|_{L^p(\mathbb{R}^2)}.\end{aligned} \tag{6.4}$$

To this end, we start with the case $m=0$. Setting $\mathcal{L}=(I-\Delta_x)^{1/2}$, we have

$$\mathcal{L}^\sigma(\mathcal{F}_j f)=\tilde{\mathcal{F}}_j(f_{\sigma,j}), \tag{6.5}$$

where $\tilde{\mathcal{F}}_j$ and $f_{\sigma,j}$ are as in (6.1) and Lemma 5.9 respectively. This follows by taking the Fourier transform in the x-variable and using the fact that $\rho_0=\rho^2$.

By Riesz–Thorin interpolation, (6.2) and (6.3) yields

$$\|\tilde{\mathcal{F}}_j f\|_{L^p(\mathbb{R}^2\times\mathbb{R})}\leq C_\epsilon\sup_{|\alpha|\leq 4}\|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}\ 2^{j(3\epsilon+1/8)(1-t)}\ 2^{tj/2}\ \|f\|_{L^p(\mathbb{R}^2)}, \tag{6.6}$$

for $4\leq p\leq\infty$, where $\frac{1}{p}=\frac{1-t}{4}$. This inequality with f replaced by $f_{\sigma,j}$, for $\mathrm{Re}(\sigma)\leq 0$ reads as

$$\|\mathcal{L}^\sigma(\mathcal{F}_j f)\|_{L^p(\mathbb{R}^2\times\mathbb{R})}\leq C_\epsilon\sup_{|\alpha|\leq 4}\|\partial^\alpha a\|_{L^\infty(\mathbb{R}^2)}\ 2^{j\theta}\ \|f\|_{L^p(\mathbb{R}^2)}, \tag{6.7}$$

in view of 6.5 and Lemma 5.9, where $\theta=12\epsilon/p+(1/2-3/2p)+\mathrm{Re}(\sigma)$. This completes the proof in the case $m=0$.

Now if $a\in S^m$, $m<0$, then $\mathcal{L}^{-m}\mathcal{F}_j$ is a Fourier integral operator with amplitude function $(1+|\xi|^2)^{-m/2}a(\xi)\in S^0$. Thus since $\mathcal{L}^{\sigma-m}\mathcal{F}_j=\mathcal{L}^\sigma(\mathcal{L}^{-m}\mathcal{F}_j)$, the proof follows from the case $m=0$. □

6.2 Case of $a(\cdot,\cdot,\xi)$ Compactly Supported in the Cube Q_k

If for each fixed ξ, $a(\cdot,\cdot,\xi)$, is compactly supported in the open cube $Q_k=(-1,1)^3+k$ centered at an integer lattice point $k\in\mathbb{R}^3$, then for each fixed ξ, we have the Fourier series expansion

$$a(x,t,\xi)=\sum_{n\in\mathbb{Z}^3}a_n^k(\xi)\,e^{i\pi\langle n,(x,t)\rangle}, \tag{6.8}$$

valid for $(x,t)\in Q_k$, with

$$a_n^k(\xi)=e^{-i\pi\langle n,k\rangle}\int_{Q_k}a\,(x,t,\xi)\,e^{-i\pi\langle n,(x,t)\rangle}\,dxdt. \tag{6.9}$$

Thus, the Fourier integral operator $\mathcal{F}$ in (1.3) becomes a sum of Fourier integral operators as in (2.1) with amplitude function $a_n = a_n^k$ independent of the (x, t) variables. Writing

$$e^{-i\pi\langle n,(x,t)\rangle} = (1+\pi^2|n|^2)^{-2}(1-\Delta_{x,t})^2 e^{-i\pi\langle n,(x,t)\rangle}$$

an integration by parts shows that $|a_n^k(\xi)| \lesssim \frac{\|(1-\Delta_{x,t})^2 a\|_{L^\infty(Q_k)}}{1+|n|^4}$. Moreover, using the above arguments on $\partial_\xi^\alpha a$ give the estimate

$$|\partial_\xi^\alpha a_n^k(\xi)| \le \frac{\|(1-\Delta_{x,t})^2\partial_\xi^\alpha a\|_{L^\infty(Q_k)}}{1+|n|^4} \le \frac{B_\alpha}{(1+|k|)^4}\frac{(1+|\xi|)^{m-|\alpha|}}{(1+|n|^4)} \tag{6.10}$$

for all multi indices $\alpha = (\alpha_1, \alpha_2)$ with a constant B_α, in view of (1.2), and the fact that $|(x, t)| \approx |k|$ on Q_k. Thus it follows that each $a_n^k \in S^m$. Also, from the decay estimate (6.10) with $\alpha = 0$, it follows that the series on the right hand side of (6.8) converges absolutely and uniformly in (x, t, ξ), as $a \in S^m$, $m \le 0$. We use these observations to prove Theorem 1.1.

Proposition 6.2 *Let $\mathcal{F}f$ be as in (1.1) with $a(\cdot,\cdot,\xi)$ supported in the cube Q_k centered at the integer lattice point $k \in \mathbb{R}^3$. Then there exists a constant C_σ independent of k such that the inequality*

$$\|\mathcal{F}f\|_p \le C_\sigma\,(1+|k|)^{-4}\,\|f\|_{L^p_{m-\sigma}}$$

holds true for $Re(\sigma) < \frac{3}{2p} - \frac{1}{2}$ if $4 \le p < \infty$, and for $Re(\sigma) < 1/2(1/p - 1/2)$ if $2 < p \le 4$, for any $f \in \mathcal{S}(\mathbb{R}^2)$.

Proof Since $a(\cdot,\cdot,\xi)$ is supported on the cube Q_k centred at k, in view of (6.8) and the decomposition (2.3) involving the dyadic one, we have for $f \in \mathscr{S}(\mathbb{R}^2)$,

$$\mathcal{F}f(x,t) = \sum_{j=0}^{\infty}\sum_{n\in\mathbb{Z}^3} e^{i\langle n,(x,t)\rangle}\,\mathcal{F}_j^n f(x,t), \tag{6.11}$$

where $\mathcal{F}_j^n f := \mathcal{F}_j^{n,k} f$ is as in (2.2), 2.3 for $j \in \mathbb{N}_0$, but with a replaced by a_n^k given by (6.9). The above step involves an interchange of integral and sum, which is justified by the dominated convergence theorem whenever $f \in \mathscr{S}(\mathbb{R}^2)$ and the fact that $\sum_{n,j}|a_n^k(\xi)|\,\rho(2^{-j}|\xi|)$ is bounded uniformly in ξ, which follows from (6.10) with $\alpha = 0$. Since $|e^{i\langle n,(x,t)\rangle}| = 1$, taking the L^p norm on both sides of (6.11) yields

$$\|\mathcal{F}f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \le \sum_{n\in\mathbb{Z}^3}\sum_{j=0}^{\infty}\|\mathcal{F}_j^n f\|_{L^p(\mathbb{R}^2\times\mathbb{R})}.$$

In view of Propositions 6.1, 5.7 and (6.10), there exist $C_\epsilon = C_{\epsilon(\sigma)}$ such that

$$\|\mathcal{F}_j^n f(x,t)\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \le \frac{C_\epsilon}{(1+|k|)^4}\frac{2^{j\theta}}{(1+|n|)^4}\|f\|_{L^p_{m-\sigma}(\mathbb{R}^2)}, \tag{6.12}$$

for $4 \le p \le \infty$, for $n \in \mathbb{Z}^3$ and $j \in \mathbb{N}_0$, where $\theta = 12\epsilon/p + (1/2 - 3/2p) + \mathrm{Re}(\sigma)$. Since $\sum_{n\in\mathbb{Z}^3} \frac{1}{(1+|n|^4)} < \infty$, and $\theta < 0$ whenever $\mathrm{Re}(\sigma) < \frac{3}{2p} - \frac{1}{2} - 12\epsilon/p$, it follows that $\sum_{n,j} \mathcal{F}_j^n f$ is absolutely summable in $L^p(\mathbb{R}^3)$ and

$$\|\mathcal{F}f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \le \sum_{n\in\mathbb{Z}^3}\sum_{j=0}^{\infty}\|\mathcal{F}_j^n f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \le \frac{C_\epsilon(\sigma)}{1+|k|^4}\|f\|_{L^p_{m-\sigma}(\mathbb{R}^2)}, \tag{6.13}$$

for $\mathrm{Re}(\sigma) < \sigma_\epsilon = \frac{3}{2p} - \frac{1}{2} - 12\epsilon/p$ with

$$C_\epsilon(\sigma) = C_\epsilon \sum_{n\in\mathbb{Z}^3}\frac{1}{(1+|n|)^4}\sum_{j=0}^{\infty} 2^{j\theta} < \infty.$$

Note that $\epsilon > 0$ is arbitrary, and $\sigma_\epsilon \to \frac{3}{2p} - \frac{1}{2}$ as $\epsilon \to 0$. Thus for any given σ with $\mathrm{Re}(\sigma) < \frac{3}{2p} - \frac{1}{2}$, we have $\mathrm{Re}(\sigma) < \sigma_\epsilon$ for some small $\epsilon > 0$. It follows that the estimate (6.13) holds true for $\mathrm{Re}(\sigma) < \frac{3}{2p} - \frac{1}{2}$, and $4 \le p < \infty$.

The case $2 < p \le 4$ follows as in [15]. Writing $\mathcal{F}f = \sum_{n\in\mathbb{Z}^3}\mathcal{F}^n f$ where

$$\mathcal{F}^n f(x,t) = \rho_1(t)\int_{\mathbb{R}^2} e^{i(x\cdot\xi + tq(\xi))}\, a_n^k(\xi)\,\hat{f}(\xi)\,d\xi, \tag{6.14}$$

with a_n^k as in (6.9). Using Plancheral theorem and (6.10) with $\alpha = 0$, we get for $\mathrm{Re}(\sigma) \le 0$

$$\|\mathcal{F}^n f(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le \frac{C}{(1+|n|)^4}\frac{\rho_1(t)}{(1+|k|)^4}\|f\|_{L^2_{m-\sigma}(\mathbb{R}^2)}$$

for $f \in \mathcal{S}(\mathbb{R}^2)$. A further t-integration gives

$$\|\mathcal{F}^n f\|_{L^2(\mathbb{R}^2\times\mathbb{R})} \le \frac{C}{(1+|n|)^4}\frac{C_1}{(1+|k|)^4}\|f\|_{L^2_{m-\sigma}(\mathbb{R}^2)},\ \mathrm{Re}(\sigma) \le 0.$$

This is equivalent to

$$\|(I-\Delta_x)^{(\sigma-m)/2}\mathcal{F}^n f\|_{L^2(\mathbb{R}^2\times\mathbb{R})} \lesssim \frac{C_1}{(1+|n|)^4}\frac{\|f\|_{L^2(\mathbb{R}^2)}}{(1+|k|)^4}, \tag{6.15}$$

valid for $\mathrm{Re}(\sigma) \le 0$.

Also for $p = 4$, (6.12) is equivalent to

$$\|(I - \Delta_x)^{(\sigma - m)/2} \mathcal{F}_j^n f\|_{L^4(\mathbb{R}^2 \times \mathbb{R})} \leq \frac{C_\epsilon}{(1 + |k|)^4} \frac{2^{j\theta}}{(1 + |n|)^4} \|f\|_{L^4(\mathbb{R}^2)},$$

for $n \in \mathbb{Z}^3$ and $j \in \mathbb{N}_0$, Thus writing $\mathcal{F}^n f = \sum_{j \in \mathbb{N}_0} \mathcal{F}_j^n f$, we see that

$$\|(I - \Delta_x)^{(\sigma - m)/2} \mathcal{F}^n f\|_{L^4(\mathbb{R}^2 \times \mathbb{R})} \lesssim \frac{C_{2,\epsilon}}{(1 + |k|)^4 (1 + |n|)^4} \|f\|_{L^4(\mathbb{R}^2)}, \tag{6.16}$$

for $n \in \mathbb{Z}^3$ and $j \in \mathbb{N}_0$, as $\sum_j 2^{j\theta} < \infty$ for $\mathrm{Re}(\sigma) < \frac{1}{8}$. Note that $C_{2,\epsilon} = C_\sigma$ as the choice of ϵ is determined by σ. Thus by analytic interpolation (see [25]), between (6.15) and (6.16) we get

$$\|(I - \Delta_x)^{(\sigma - m)/2} \mathcal{F}^n f\|_{L^p(\mathbb{R}^2 \times \mathbb{R})} \lesssim \frac{C_\sigma}{(1 + |k|)^4 (1 + |n|)^4} \|f\|_{L^p(\mathbb{R}^2)}, \tag{6.17}$$

for $\mathrm{Re}(\sigma) < \frac{1}{2}(\frac{1}{p} - \frac{1}{2}),\ 2 \leq p \leq 4$, which is same as

$$\|\mathcal{F}^n f\|_{L^p(\mathbb{R}^2 \times \mathbb{R})} \leq \frac{C_\sigma}{(1 + |k|)^4 (1 + |n|)^4} \|f\|_{L^p_{m-\sigma}(\mathbb{R}^2)}. \tag{6.18}$$

for each $n \in \mathbb{Z}^3$. From this we conclude:

$$\|\mathcal{F} f\|_{L^p(\mathbb{R}^2 \times \mathbb{R})} \leq \frac{C_\sigma}{(1 + |k|)^4} \|f\|_{L^p_{m-\sigma}(\mathbb{R}^2)}, \tag{6.19}$$

for $2 < p \leq 4$ and $\mathrm{Re}(\sigma) < 1/2(1/p - 1/2)$, when $a(\cdot, \cdot, \xi)$ is supported in the cube Q_k. This completes the proof. □

6.3 The General Case

The local smoothing estimates in the case of general amplitude function can be deduced from the above case, via a partition of unity argument. Let Ψ be a smooth function on $\mathbb{R}^3$ supported on the open cube $Q = (-1, 1)^3$ such that $\sum_{k \in \mathbb{Z}^3} \Psi^k = 1$, where $\Psi^k(y) = \Psi(y - k),\ y = (x, t) \in \mathbb{R}^3,\ k \in \mathbb{Z}^3$. Then $a^k(x, t, \xi) = a(x, t, \xi)\, \Psi^k(x, t)$ is compactly supported in Q_k in (x, t) variable, for each ξ. Then for each $k \in \mathbb{Z}^3$, we define the operator

$$\mathcal{F}^k f(x, t) = \rho_1(t) \int_{\mathbb{R}^2} e^{i(x \cdot \xi + t q(\xi))}\, a^k(x, t, \xi)\, \hat{f}(\xi)\, d\xi. \tag{6.20}$$

Note that $a^k \in S^m$ as $a \in S^m$.

Proof of Theorem 1.1 Using the partition of unity $\{\Psi^k\}_{k\in\mathbb{Z}^3}$ discussed above, we have

$$\mathcal{F} f(x,t) = \sum_{k\in\mathbb{Z}^3} \mathcal{F}^k f(x,t), \tag{6.21}$$

as a tempered distribution, where $\mathcal{F}^k$ is the Fourier integral operator defined in (6.20) with amplitude function $a^k(x,t,\xi) = a(x,t,\xi)\,\Psi^k(x,t) \in S^m$. In view of (6.10), the n-th Fourier coefficient of a^k satisfies the estimate

$$|\partial_\xi^\alpha a_n^k(\xi)| \leq \frac{C_\alpha}{(1+|k|)^4} \frac{(1+|\xi|)^{m-|\alpha|}}{(1+|n|)^4} \tag{6.22}$$

for all α, with a constant C_α independent of k. Thus Proposition 6.2 yields

$$\|\mathcal{F}^k f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq \frac{C_\sigma}{(1+|k|)^4} \|f\|_{L^p_{m-\sigma}(\mathbb{R}^2)}, \tag{6.23}$$

for each k. Since $\sum\limits_{k\in\mathbb{Z}^3} \frac{1}{(1+|k|)^4} < \infty$, we see that $\sum\limits_k \mathcal{F}^k f$ is absolutely summable in $L^p(\mathbb{R}^3)$ and

$$\|\mathcal{F} f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq \sum_{k\in\mathbb{Z}^3} \|\mathcal{F}^k f\|_{L^p(\mathbb{R}^2\times\mathbb{R})} \leq C_\sigma \|f\|_{L^p_{m-\sigma}(\mathbb{R}^2)}, \tag{6.24}$$

for the same range of σ as in Proposition 6.2. This completes the proof. □

Note that Theorem 1.1 assumes some decay assumptions on amplitude functions a and a few of its space-time derivatives. However, for local smoothing to hold, such a decay assumptions is not necessary, as is clear from the case of amplitude functions of the form $a(x,t,\xi) = a(\xi)$. However, this is a trivial example in the sense that all the space time derivatives for such a function are identically zero.

So it is natural to ask, if the local smoothing estimate holds for Fourier integral operators with symbols having no decay in (x,t) variables, on any of its derivatives? Interestingly, the answer to the above question is affirmative and is already contained in the proof of Theorem 1.1. Since this fact does not follow as a corollary of the above theorem, we state it as

Theorem 6.3 *Let $\mathcal{F}$ be a Fourier integral operator as in (1.1) with amplitude function $a \in S^m(\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}^2)$, $m \leq 0$, which is periodic in (x,t) variables. Then the local smoothing estimate*

$$\|\mathcal{F} f\|_p \leq C_\sigma \, \|f\|_{L^p_{m-\sigma}}$$

holds true with a constant C_σ, for $Re(\sigma) < \frac{3}{2p} - \frac{1}{2}$ if $4 \leq p < \infty$, and for $Re(\sigma) < 1/2(1/p - 1/2)$ if $2 < p \leq 4$.

Recall that in Sect. 6.2, we deal with Fourier integral operators with amplitude functions compactly supported in an open cube Q_k in (x, t) variables, centered at $k \in \mathbb{Z}^3$. In fact, there we were actually using the Fourier series of $a(x, t, \xi)$ for $(x, t) \in Q_k$. We can use the same idea for periodic functions. So we only sketch the main points of the proof here.

Proof By a scaling, we can assume that $a(x, t, \xi)$ has period 2 in each of the variables x_1, x_2 and t. Thus for each fixed ξ, we have the Fourier series expansion

$$a(x, t, \xi) = \sum_{n\in\mathbb{Z}^3} a_n(\xi)\, e^{i\pi\langle n,(x,t)\rangle}, \tag{6.25}$$

valid for $(x, t) \in [-1, 1]^3 = Q_0$. Hence we can write $\mathcal{F}f = \sum_{n\in\mathbb{Z}^3} \mathcal{F}_n f$ where $\mathcal{F}_n$ is the Fourier integral operator as in (1.1) with amplitude function

$$a_n(\xi) = \int_{[-1,1]^3} a\,(x, t, \xi)\, e^{-i\pi\langle n,(x,t)\rangle}\, dxdt. \tag{6.26}$$

This is same as the formula given by (6.9) for $k = 0$, as $[-1, 1]^3 = Q_0$, Hence we get by integration by parts (valid also in the periodic case)

$$|\partial_\xi^\alpha a_n(\xi)| \le B_\alpha \frac{(1 + |\xi|)^{m-|\alpha|}}{(1 + |n|^4)} \tag{6.27}$$

as a special case of (6.10) with $k = 0$, valid for all multi indices $\alpha = (\alpha_1, \alpha_2)$. In particular, this shows that $a_n \in S^m$, the same symbol class as a, for all $n \in \mathbb{Z}^3$. Hence the proof follows, using the special case $k = 0$ of Proposition 6.2, since $\sum_{n\in\mathbb{Z}^3} \frac{1}{(1+|n|^4)} < \infty$. □

Remark 6.4 Note that the estimate of Theorem 6.3 is also valid if we assume periodicity only in the space variable. In fact, for local smoothing estimate, we can always multiply the Fourier integral operator with $\rho_1 \in C_c^\infty(\mathbb{R})$, hence can assume by scaling, that the t support of a is contained in $(-1, 1)$, in which case we can periodize a in t-variable and appeal to Theorem 6.3.

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