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# $L^p$ Boundedness of Carleson & Hilbert Transforms Along Plane Curves with Certain Curvature Constraints

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## Abstract

This paper shows that for $p \in (1, \infty)$, a measurable function $u : \mathbb{R} \to \mathbb{R}$ and a generalized plane curve $\gamma$ with certain curvature constraints, not only the Carleson transform

$$\mathcal{C}_{u,\gamma} f(x) := \text{p.v.} \int_{-\infty}^{\infty} e^{iu(x)\gamma(t)} f(x-t) \frac{\mathrm{d}t}{t} \quad \forall \ x \in \mathbb{R}$$

is bounded on $L^p(\mathbb{R})$, but also the Hilbert transform

$$H_{u,\gamma} f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u(x_1)\gamma(t)) \frac{\mathrm{d}t}{t} \quad \forall \ (x_1, x_2) \in \mathbb{R}^2$$

is bounded on $L^p(\mathbb{R}^2)$, and especially $L^2(\mathbb{R})$-boundedness of $\mathcal{C}_{u,\gamma}$ induces indeed $L^2(\mathbb{R}^2)$-boundedness of $H_{u,\gamma}$.

**Keywords** Carleson transform · Hilbert transform · Shifted maximal operator · Plane curve

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# 1 Introduction

## 1.1 Principal Result and Remark

Let $u : \mathbb{R} \to \mathbb{R}$ be a measurable function and $\gamma$ be a generalized plane curve, the *Carleson transform* $\mathcal{C}_{u,\gamma}$ along the general curve $\gamma$ is defined by setting, for any function $f$ in the Schwartz class $\mathcal{S}(\mathbb{R})$,

$$\mathcal{C}_{u,\gamma} f(x) := \text{p.v.} \int_{-\infty}^{\infty} e^{iu(x)\gamma(t)} f(x-t) \frac{\mathrm{d}t}{t} \ \ \forall\ x \in \mathbb{R}. \tag{1.1}$$

Here and hereafter, p.v. $\int$ denotes the principal-value integral. The *Hilbert transform* $H_{u,\gamma}$ along the variable plane curve $u(x_1)\gamma$ is defined by setting, for any $f \in \mathcal{S}(\mathbb{R}^2)$ - the Schwarz class on $\mathbb{R}^2$,

$$H_{u,\gamma} f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u(x_1)\gamma(t)) \frac{\mathrm{d}t}{t} \ \ \forall\ (x_1, x_2) \in \mathbb{R}^2. \tag{1.2}$$

Below we establish $L^{1<p<\infty}$-boundedness of (1.1) and (1.2) for some generalized curves.

**Theorem 1.1** *Let $u : \mathbb{R} \to \mathbb{R}$ be a measurable function, $\gamma \in C^3(\mathbb{R})$ be either odd or even with $\gamma(0) = \gamma'(0) = 0$, and be convex on $(0, \infty)$ with the four curvature-oriented properties that:*

*(i) $\frac{\gamma'(2t)}{\gamma'(t)}$ is decreasing and bounded by a constant $C_1$ from above on $(0, \infty)$;*
*(ii) $\exists$ a positive constant $C_2$ such that $\frac{t\gamma''(t)}{\gamma'(t)} \le C_2$ on $(0, \infty)$;*
*(iii) $\exists$ a positive constant $C_3$ such that $|(\frac{\gamma''}{\gamma'})'(t)| \ge \frac{C_3}{t^2}$ on $(0, \infty)$;*
*(iv) $\frac{\gamma'''(t)}{\gamma''(t)}$ is strictly monotonic or equals to a constant on $(0, \infty)$.*

*Then, given $p \in (1, \infty)$ there exists a positive constant $C$ independent of $u$ such that*

$$\begin{cases} \|\mathcal{C}_{u,\gamma} f\|_{L^p(\mathbb{R})} \le C\|f\|_{L^p(\mathbb{R})} \ \ \forall\ f \in L^p(\mathbb{R}); \\ \|H_{u,\gamma} f\|_{L^p(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)} \ \ \forall\ f \in L^p(\mathbb{R}^2). \end{cases}$$

***Remark 1.2*** Here, it is worth saying more words on the conditions on $\gamma$ whose curvature is determined by

$$\kappa(t) := \frac{\gamma''(t)}{\left(1 + \left(\gamma'(t)\right)^2\right)^{\frac{3}{2}}} \ \ \forall\ t \in (0, \infty).$$

▷ From

$$\begin{cases} \gamma \in C^3(\mathbb{R}); \\ \gamma(0) = \gamma'(0) = 0; \\ \gamma \text{ being convex on } (0, \infty), \end{cases}$$

it follows that

$$\min \left\{ \gamma(t), \gamma'(t), \gamma''(t) \right\} \geq 0 \ \ \forall \ \ t \in (0, \infty)$$

and $\gamma'$ is increasing. We also know that

$$\gamma'(t) \geq \gamma'(1) \ \ \forall \ \ t \in [1, \infty),$$

which further implies

$$\lim_{t \to \infty} \gamma(t) = \infty.$$

Because $\gamma'$ is increasing on $(0, \infty)$ and $\gamma(0) = 0$, it is easy to check

$$1 \leq \frac{t\gamma'(t)}{\gamma(t)} \ \ \forall \ \ t \in (0, \infty).$$

On the other hand, since $\gamma(0) = 0$, by Cauchy's mean value theorem, for $t \in (0, \infty)$ there exists $\xi_t \in (0, t)$ such that

$$\frac{t\gamma'(t)}{\gamma(t)} = \frac{t\gamma'(t) - 0\gamma'(0)}{\gamma(t) - \gamma(0)} = \frac{\gamma'(\xi_t) + \xi_t \gamma''(\xi_t)}{\gamma'(\xi_t)}.$$

Thus, by Theorem 1.1(ii),

$$\exists \ C_4 := C_2 + 1 \text{ such that } 1 \leq \frac{t\gamma'(t)}{\gamma(t)} \leq C_4 \ \ \forall \ \ t \in (0, \infty).$$

▷ Since $\gamma'$ is increasing on $(0, \infty)$, Theorem 1.1(i) gives always

$$1 \leq \frac{\gamma'(2t)}{\gamma'(t)} \leq C_1 \ \ \ \forall \ \ \ t \in (0, \infty).$$

▷ The following are some curves satisfying all the conditions of Theorem 1.1. Here, we write only the part for any $t \in [0, \infty)$. For any $t \in (-\infty, 0]$, the curve is given by its even or odd property - e.g. -

(i) for any $t \in [0, \infty)$, $\gamma_1(t) := t^\alpha$ under $\alpha \in (1, \infty)$;
(ii) for any $t \in [0, \infty)$, $\gamma_2(t) := t^2 \log(1 + t)$;
(iii) for any $t \in [0, \infty)$, $\gamma_3(t) := \int_0^t \tau^\alpha \log(1 + \tau) \, d\tau$ under $\alpha \in (1, \infty)$.

### 1.2 Some Historical Notes

From now on, the assumption $p \in (1, \infty)$ will be made.

**Note 1.3** In [12, Theorem 1.2], Guo–Hickman–Lie–Roos obtained $L^p(\mathbb{R}^2)$-boundedness of $H_{u,\gamma}$ with the curve in Remark 1.2(i), but with $1 \neq \alpha \in (0, \infty)$. Thus, as a special case, Theorem 1.1 covers [12, Theorem 1.2] whenever $\alpha \in (1, \infty)$. The work [12] explains much more about the proof ideas, but we will still make several contributions to the argument.

▷ For a homogeneous curve, it is easy to see

$$\gamma(ab) = \gamma(a)\gamma(b) \ \forall \ a, b \in (0, \infty).$$

Since we seek $L^p(\mathbb{R}^2)$-boundedness of $H_{u,\gamma}$ with a bound independent of $u$, it is natural to absorb $u(x)$ by $\gamma$ for any fixed $x$, which can be easily obtained with $\gamma(t) := t^\alpha$ due to

$$|u(x)|\gamma(t) = \gamma(|u(x)|^{\frac{1}{\alpha}} t).$$

Furthermore, it is convenient to write

$$1 = \sum_{l \in \mathbb{Z}} \psi_l(|u(x)|^{\frac{1}{\alpha}} t)$$

for the homogeneous curve $t^\alpha$, which plays an important role in achieving [12, Theorem 1.2], where $\psi$ is a standard bump function supported on

$$\left\{t \in \mathbb{R} : \frac{1}{2} \leq |t| \leq 2\right\}.$$

Of course, this property cannot hold for a general curve $\gamma$. Therefore, we have to split our operator by a standard partition of unity; i.e.,

$$1 = \sum_{l \in \mathbb{Z}} \psi_l(t).$$

▷ Our demonstration originates from the classic $L^p(\mathbb{R}^2)$-theory for the Hilbert transform along a curve; it is usually assumed that $\gamma$ is convex on $(0, \infty)$. For the low-frequency part, we need to assume that $\frac{\gamma(t)}{t}$ is increasing on $(0, \infty)$, which leads to the case in which $\gamma(t) := t^\alpha$, where $\alpha \in (0, 1)$, which cannot be covered in this paper. For a further decomposition, motivated by [13], we introduce the map $n : \mathbb{R} \to \mathbb{Z}$ such that

$$\frac{1}{\gamma(2^{n(x)+1})} \leq |u(x)| \leq \frac{1}{\gamma(2^{n(x)})} \ \forall \ x \in \mathbb{R}.$$

One difficulty appears in $L^p(\mathbb{R})$-boundedness of $\mathcal{C}_{u,\gamma}$ and $L^2(\mathbb{R}^2)$-boundedness of $H_{u,\gamma}$. It is crucial to establish a decay estimate of an oscillatory integral as that in Proposition 2.4. If we have a homogeneous curve as in Remark 1.2(i), it is easy to calculate the derivatives of the phase functions, and the decay estimation will be easier to obtain. However, for a general curve $\gamma$, we need a more complicated analysis, and the assumptions (i), (ii), (iii) and (iv) of Theorem 1.1 for the curve appear naturally during the estimation.

Another difficulty appearing in $L^p(\mathbb{R}^2)$-boundedness of $H_{u,\gamma}$ for $p \in (1,2) \cup (2,\infty)$ is as follows. By the Littlewood-Paley theory and the commutative property

$$H_{u,\gamma} P_l = P_l H_{u,\gamma} \quad \forall \ l \in \mathbb{Z},$$

we need to establish a refined estimate for $H_{u,\gamma,k+n_l(x_1)} P_l$ by the shifted maximal operator. Here, $P_l$ denotes the Littlewood-Paley decomposition operator according to the second variable and $l \in \mathbb{Z}$. Guo–Hickman–Lie–Roos, who in [12] considered the homogeneous curve as in Remark 1.2(i), did not need $n_l(x_1)$, where the map $n_l : \mathbb{R} \to \mathbb{Z}$ for $l \in \mathbb{Z}$ is defined by

$$\frac{1}{\gamma(2^{n_l(x_1)+1})} \le 2^l |u(x_1)| \le \frac{1}{\gamma(2^{n_l(x_1)})} \quad \forall \ x_1 \in \mathbb{R}.$$

This new observation allows us to obtain the refined estimate for $H_{u,\gamma,k+n_l(x_1)} P_l$ with a great effort to control the dyadic pieces by the shifted maximal operator.

**Note 1.4** If $u : \mathbb{R} \to \mathbb{R}$ is a real number $\lambda$, then the operator in (1.2) is equivalent to the following *directional Hilbert transform* $H_{\lambda,\gamma}$ along a general curve $\gamma$ defined for a fixed direction $(1, \lambda)$ by

$$H_{\lambda,\gamma} f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \lambda\gamma(t)) \frac{\mathrm{d}t}{t} \quad \forall (x_1, x_2) \in \mathbb{R}^2,$$

whose $L^p(\mathbb{R}^2)$-boundedness can obviously be obtained by the *Hilbert transform* $H_\gamma$ along a general curve $\gamma$:

$$H_\gamma f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - \gamma(t)) \frac{\mathrm{d}t}{t} \quad \forall (x_1, x_2) \in \mathbb{R}^2.$$

This operator is of independent interest, and actually one of our major motivations. There are many works on this problem; see, for example, [4,5,8,18,24,25]. On the other hand, it is not hard to find

$$\sup_{\lambda \in \mathbb{R}} \left\| H_{\lambda,\gamma} f \right\|_{L^p(\mathbb{R}^2)} \le C \|f\|_{L^p(\mathbb{R}^2)}.$$

However, $L^p(\mathbb{R}^2)$-boundedness of the corresponding maximal operator

$$\sup_{\lambda \in \mathbb{R}} |H_{\lambda,\gamma} f(x_1, x_2)|$$

cannot be obtained readily. In fact, by linearization, this uniform estimate is equivalent to $L^p(\mathbb{R}^2)$-estimate for

$$H_{U,\gamma} f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - U(x_1, x_2)\gamma(t)) \,\frac{\mathrm{d}t}{t} \quad \forall\, (x_1, x_2) \in \mathbb{R}^2,$$

where the bound must be independent of the measurable function $U$. However, it is well known that $H_{U,\gamma}$ might not be bounded on any $L^p(\mathbb{R}^2)$ if we merely assume that $U$ is a measurable function (cf. [12]). Therefore, we cannot assume

$$\left\| \sup_{\lambda\in\mathbb{R}} |H_{\lambda,\gamma} f| \right\|_{L^p(\mathbb{R}^2)} \le C \, \|f\|_{L^p(\mathbb{R}^2)} .$$

Instead, Theorem 1.1 shows

$$\left\| \sup_{\lambda\in\mathbb{R}} \left\| H_{\lambda,\gamma} f(\cdot_1, \cdot_2) \right\|_{L^p(\mathbb{R}^1_{x_2})} \right\|_{L^p(\mathbb{R}^1_{x_1})} \le C \, \|f\|_{L^p(\mathbb{R}^2)} ,$$

which inserts the supremum between the two $L^p$-norms on the left-hand side of the equation. Here and hereafter, let $\cdot_1$ and $\cdot_2$ denote the first variable $x_1$ and the second variable $x_2$, respectively. As Stein-Wainger mentioned in [22], the curvature of the considered curve plays a crucial role in this problem, and the four conditions (i), (ii), (iii) and (iv) of Theorem 1.1 are used to describe the curvature $\kappa$ of the considered curve $\gamma$.

**Note 1.5** Bateman in [1] proved that if

$$\gamma(t) := t \ \ \forall \ \ t \in \mathbb{R},$$

then $H_{u,\gamma} P_k$ is bounded on $L^p(\mathbb{R}^2)$ uniformly for $k \in \mathbb{Z}$, where $P_k$ denotes the Littlewood-Paley projection operator in the second variable. Later, Bateman-Thiele in [2] proved $L^p(\mathbb{R}^2)$-boundedness of $H_{u,\gamma}$ for all $p \in (\frac{3}{2}, \infty)$. Moreover, let $\gamma$ be $|t|^\alpha$ or $\text{sgn}(t)|t|^\alpha$ for any $t \in \mathbb{R}$, where $1 \neq \alpha \in (0, \infty)$ ; in [12] Guo–Hickman–Lie–Roos obtained $L^p(\mathbb{R}^2)$-boundedness of $H_{u,\gamma}$. Furthermore, Carbery-Wainger-Wright in [6] obtained $L^p(\mathbb{R}^2)$-boundedness of $H_{u,\gamma}$, but with the restriction that $u(x_1) := x_1$ for any $x_1 \in \mathbb{R}$, where $\gamma \in C^3(\mathbb{R})$ is either an odd or even convex curve on $(0, \infty)$ satisfying $\gamma(0) = \gamma'(0) = 0$ and the quantity $\frac{t\gamma''(t)}{\gamma'(t)}$ is decreasing and bounded below on $(0, \infty)$. Under the same conditions, Bennett in [3] obtained $L^2(\mathbb{R}^2)$-boundedness of

$$H_{P,\gamma} f(x_1, x_2) := \text{p.v.} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - P(x_1)\gamma(t)) \,\frac{\mathrm{d}t}{t} \quad \forall\, (x_1, x_2) \in \mathbb{R}^2, \qquad (1.3)$$

for any general polynomial $P$. More recently, Chen-Zhu in [9] obtained $L^2(\mathbb{R}^2)$-boundedness of $H_{P,\gamma}$ in (1.3) by defining the curvature condition as

$$\Big(\frac{\gamma''}{\gamma'}\Big)'(t) \le -\frac{\lambda_1}{t^2} \text{ for any } t \in [0,\infty) \text{ and some positive constant } \lambda_1.$$

In [17], Li-Yu also obtained $L^2(\mathbb{R}^2)$-boundedness of $H_{P,\gamma}$ in (1.3) if the curvature condition for $\gamma \in C^2(\mathbb{R})$ is replaced with that:

(i) $\frac{\gamma''(t)}{\gamma'(t)}$ is decreasing on $(0,\infty)$;
(ii) $\exists$ a positive constant $\lambda_2$ such that $\frac{t\gamma''(t)}{\gamma'(t)} \ge \lambda_2\ \forall\, t \in (0,\infty)$;
(iii) $\gamma''(t)$ is monotonic on $(0,\infty)$.

**Note 1.6** Interestingly, all of these results for $H_{P,\gamma}$ are based on the iteration of the degree of polynomial $P$ and hence cannot extend to a general measurable function $u$. Accordingly, Theorem 1.1 is the first result for $H_{u,\gamma}$ with the generalized plane curve $\gamma$. Even more interestingly, $L^2(\mathbb{R})$-boundedness of (1.1) appears in the study of $L^2(\mathbb{R}^2)$-boundedness of (1.2). Indeed, from [19] it follows that

$$\|H_{u,\gamma}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \le \sup_{\lambda\in\mathbb{R}} \|S_\lambda\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})},$$

where

$$S_\lambda f(x) := \text{p.v.} \int_{-\infty}^{\infty} e^{-i\lambda u(x)\gamma(t)} f(x-t)\,\frac{\mathrm{d}t}{t} \quad \forall\, x \in \mathbb{R}.$$

Since $L^2(\mathbb{R}^2)$-boundedness of $H_{u,\gamma}$ will not depend on $u$, we need to establish only $L^2(\mathbb{R})$-estimate for

$$\mathcal{C}_{u,\gamma} f(x) = \text{p.v.} \int_{-\infty}^{\infty} e^{iu(x)\gamma(t)} f(x-t)\,\frac{\mathrm{d}t}{t} \quad \forall\, x \in \mathbb{R},$$

with a bound independent of $u$. This operator $\mathcal{C}_{u,\gamma}$ itself is also interesting. The original *Carleson transform* $\mathcal{C}$ is defined by setting

$$\mathcal{C} f(x) := \sup_{N\in\mathbb{R}} \left| \text{p.v.} \int_{-\infty}^{\infty} e^{iNt} f(x-t)\,\frac{\mathrm{d}t}{t} \right| \quad \forall\ (f,x) \in \mathcal{S}(\mathbb{R}) \times \mathbb{R}.$$

By linearization,

$$\|\mathcal{C} f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})} \Longleftrightarrow \|\mathcal{C}_u f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})},$$

where $u:\ \mathbb{R} \to \mathbb{R}$ is a measurable function and

$$\mathcal{C}_u f(x) := \text{p.v.} \int_{-\infty}^{\infty} e^{iu(x)t} f(x-t)\,\frac{\mathrm{d}t}{t} \quad \forall\, x \in \mathbb{R},$$

and the constant in the last inequality is independent of $u$. In [7], Carleson obtained $L^2(\mathbb{R})$-boundedness of $\mathcal{C}$, which plays an important role in obtaining almost everywhere convergence of a Fourier series of $L^2(\mathbb{R})$-functions and also confirmed the famous Luzin's conjecture. Hunt later obtained $L^p(\mathbb{R})$-boundedness in [14]. For further results about $\mathcal{C}$, we refer the reader to [10,15,16,20]. Stein-Wainger in [23] considered the *Carleson transform* $\mathcal{C}_{u,d}$ along a homogeneous curve $t^d$ with integer $d > 1$, namely,

$$\mathcal{C}_{u,d} f(x) := \text{p.v.} \int_{-\infty}^{\infty} e^{iu(x)t^d} f(x-t)\, \frac{\mathrm{d}t}{t} \quad \forall\ (f, x) \in \mathcal{S}(\mathbb{R}) \times \mathbb{R},$$

and showed that $L^p(\mathbb{R})$-boundedness is independent of $u$. Guo in [11] extended $\mathcal{C}_{u,d}$ further to a homogeneous curve $|t|^{\varepsilon_1}$ or $\text{sgn}(t)|t|^{\varepsilon_2}$, where $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$, $\varepsilon_1 \neq 1$ and $\varepsilon_2 \neq 0$. Thus, it is natural to consider $\mathcal{C}_{u,\gamma}$ along a more general curve as presented in Theorem 1.1.

### 1.3 Organization and Notation

The rest of this paper is organized as follows. Section 2.1 is used to collect three lemmas for (1.1). In Sect. 2.2, we prove Theorem 1.1 for Carleson transform. Section 3.1 is devoted to obtaining the single annulus $L^p(\mathbb{R}^2)$-estimate for (1.2). In Sect. 3.2 we verify Theorem 1.1 for Hilbert transform.

Throughout this paper, we use $C$ to denote a *positive constant* that is independent of the main parameters involved but whose value may vary from line to line. The *positive constants with subscripts*, such as $C_1$ and $C_2$, are the same in different occurrences. For two real functions $f$ and $g$, we use $f \lesssim g$ or $g \gtrsim f$ to denote $f \leq Cg$ and, if $f \lesssim g \lesssim f$, we then write $f \approx g$.

## 2 Verification of Theorem 1.1 for Carleson Transform

### 2.1 Three Lemmas

Before providing the proof of Theorem 1.1(i), we state three lemmas. Van der Corput's lemma is a useful tool to bound an oscillatory integral, but for the case of $k = 1$, a simple lower bound on $|\phi'|$ is not sufficient. We need to add a condition that $\phi'$ is monotonic such that

$$\int_a^b \left| \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{1}{\phi'(t)} \right) \right| \mathrm{d}t$$

is dominated by a constant. Lemma 2.1 is a slight variant of Van der Corput's lemma that replaces the additional condition with the condition that $\phi''$ is bounded from above. Lemma 2.2 is used to obtain an interesting fact: for the phase function $\phi$ of the considered oscillatory integral, we must have

$$\text{either } |\phi'| \gtrsim 1 \ \textbf{ or } \ |\phi''| \gtrsim 1.$$

However, it is not sufficient to complete our estimate even if we obtained the surprising lower bound on $|\phi'|$ or $|\phi''|$, since we can take an infinite number of intervals such that the lower bound is established on each of these intervals. Lemma 2.3 is used to ensure that such a case does not occur.

**Lemma 2.1** *Suppose that $\phi$ is real-valued and smooth on $(a, b)$ with two positive constants $\sigma_1$ & $\sigma_2$ obeying*

$$|\phi'(x)| \geq \sigma_1 \ \ \& \ \ |\phi''(x)| \leq \sigma_2 \ \ \forall \ \ x \in (a, b).$$

*Then*

$$\left| \int_a^b e^{i\phi(t)} \, dt \right| \leq \frac{2}{\sigma_1} + (b-a)\frac{\sigma_2}{\sigma_1^2}.$$

***Proof*** From the proof of van der Corput's lemma, see, for example, ( [21], P. 332, Proposition 2), which bounds the integral $\int_a^b e^{i\phi(t)} \, \mathrm{d}t$ by

$$\left| \frac{e^{i\phi(b)}}{i\phi'(b)} - \frac{e^{i\phi(a)}}{i\phi'(a)} \right| + \int_a^b \left| \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\phi'(t)}\right) \right| \mathrm{d}t \lesssim \frac{2}{\sigma_1} + \int_a^b \left| \frac{\phi''(t)}{\phi'(t)^2} \right| \mathrm{d}t \lesssim \frac{2}{\sigma_1} + (b-a)\frac{\sigma_2}{\sigma_1^2}.$$

It is easy to deduce the desired conclusion of Lemma 2.1. □

**Lemma 2.2** [13, Lemma 4.5] *Let $A$ be an invertible $n \times n$ matrix and $x \in \mathbb{R}^n$. Then,*

$$|Ax| \geq \frac{|detA||x|}{\|A\|^{n-1}},$$

*where $\|A\|$ denotes the matrix norm* $\sup_{|x|=1} |Ax|$.

**Lemma 2.3** *Let $\gamma$ be the same as in Theorem 1.1. For any $a, b, c, d \in \mathbb{R}$ and $d > 0$, there are at most a finite number of intervals such that*

$$|a\gamma'(t) - b\gamma'(t-c)| > d \tag{2.1}$$

*holds on each of these intervals, where $t \in \mathbb{R}$ and the number of intervals is independent of $a$, $b$, $c$, and $d$.*

***Proof*** Since (2.1) is equivalent to

$$a\gamma'(t) - b\gamma'(t-c) - d > 0$$

or

$$a\gamma'(t) - b\gamma'(t-c) + d < 0.$$

Notice that $\gamma \in C^3(\mathbb{R})$, it is enough to show that

$$a\gamma''(t) - b\gamma''(t-c) = 0 \tag{2.2}$$

has a finite number of solutions including there is no solution, or there are at most a finite number of intervals such that (2.2) is established on each of these intervals, or both, where the number is independent of $a$, $b$, and $c$. There are some cases:

If $b = 0$ and $a = 0$, then (2.1) does not exist; in other words, (2.1) has no solution.

If $b = 0$ and $a \neq 0$, and since $\gamma$ is either odd or even and $\gamma'$ is increasing on $(0, \infty)$, then Lemma 2.3 is easily obtained.

If $b \neq 0$, $c = 0$ and $a = b$, then (2.1) does not exist.

If $b \neq 0$, $c = 0$ and $a \neq b$, then (2.1) is equivalent to $|(a-b)\gamma'(t)| > d$; as stated above, it is easy to see that Lemma 2.3 is established.

If $b \neq 0$ and $c \neq 0$, then Theorem 1.1(iv) gives

$$\gamma''(t) \neq 0 \ \ \forall \ \ t \in (0, \infty).$$

Notice that $\gamma$ is either odd or even. So

$$\gamma''(t) \neq 0 \ \ \forall \ \ t \in (-\infty, 0) \cup (0, \infty).$$

It is easy to see that we should only consider $t \neq 0$ and $t \neq c$ for (2.2). Then, (2.2) is equivalent to

$$\frac{a}{b} = \frac{\gamma''(t-c)}{\gamma''(t)} \ \ \forall \ \ t \in \mathbb{R} \setminus \{0, c\}. \tag{2.3}$$

Let

$$F_c(t) := \frac{\gamma''(t-c)}{\gamma''(t)} \ \ \forall \ \ t \in \mathbb{R} \setminus \{0, c\}.$$

We see that if $t \in \mathbb{R} \setminus \{0, c\}$ then

$$F_c'(t) = \frac{\gamma'''(t-c)\gamma''(t) - \gamma''(t-c)\gamma'''(t)}{(\gamma''(t))^2} = \frac{\gamma''(t-c)\left[\frac{\gamma'''(t-c)}{\gamma''(t-c)} - \frac{\gamma'''(t)}{\gamma''(t)}\right]}{\gamma''(t)}. \tag{2.4}$$

From Theorem 1.1(iv) it follows that $\frac{\gamma'''(t)}{\gamma''(t)}$ is strictly monotonic or equal to a constant on $(0, \infty)$. Since $\gamma$ is either odd or even, the equation

$$\frac{\gamma'''(t-c)}{\gamma''(t-c)} = \frac{\gamma'''(t)}{\gamma''(t)} \quad \forall \ t \in \mathbb{R} \setminus \{0, c\}, \tag{2.5}$$

has a finite number of solutions including the situation that there is no solution, or $\exists$ at most a finite number of intervals such that (2.5) is established on each of these intervals, or both, where the number is independent of $c$. Therefore, $F_c'(t)$ in (2.4) has the same character as (2.5), and (2.3) also has the same character as (2.5). This completes the proof of Lemma 2.3. □

### 2.2 $L^p(\mathbb{R})$-Estimate for $\mathcal{C}_{u,\gamma}$

We now prove Theorem 1.1 for $\mathcal{C}_{u,\gamma}$. The main strategy of our proof is to decompose our operator into a low-frequency part and a high-frequency part. We want to bound the low frequency part by some classical operators, such as the Hardy-Littlewood maximal operator and the maximal truncated Hilbert transform. For the high-frequency part, which is further divided into a series of operators $\{S_k\}_{k=0}^{\infty}$, we want to obtain a decay estimate for each of $S_k$. The main tools are the $TT^*$-argument, the stationary phase method, and the lemmas introduced in §2.1.

***Proof of Theorem 1.1*** for $\mathcal{C}_{u,\gamma}$ Suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth function supported on

$$\left\{ t \in \mathbb{R} : \frac{1}{2} \le |t| \le 2 \right\}$$

and obeys

$$\begin{cases} 0 \le \psi(t) \le 1 \ \ \forall \ t \in \mathbb{R}; \\ \Sigma_{l \in \mathbb{Z}} \psi_l(t) = 1 \ \ \forall \ t \in \mathbb{R} \setminus \{0\}; \\ \psi_l(t) := \psi(2^{-l} t) \ \ \forall \ t \in \mathbb{R}. \end{cases}$$

From Remark 1.2, we have that $\gamma$ is increasing on $(0, \infty)$, and

$$\lim_{t \to \infty} \gamma(t) = \infty.$$

We can define $n : \mathbb{R} \to \mathbb{Z}$ such that

$$\frac{1}{\gamma(2^{n(x)+1})} \le |u(x)| \le \frac{1}{\gamma(2^{n(x)})} \quad \forall \ x \in \mathbb{R}. \tag{2.6}$$

For any $x \in \mathbb{R}$, let

$$\mathcal{C}_{u,\gamma,k} f(x) := \int_{-\infty}^{\infty} e^{iu(x)\gamma(t)} f(x-t) \psi_k(t) \, \frac{\mathrm{d}t}{t}$$

and decompose

$$\mathcal{C}_{u,\gamma} f(x) = \sum_{k \le n(x)-1} \mathcal{C}_{u,\gamma,k} f(x) + \sum_{k \ge n(x)} \mathcal{C}_{u,\gamma,k} f(x) =: \mathcal{C}^{(1)}_{u,\gamma} f(x) + \mathcal{C}^{(2)}_{u,\gamma} f(x).$$

▷ For the low-frequency part $\mathcal{C}^{(1)}_{u,\gamma} f(x)$ set

$$\sum_{k \le n(x)-1} \psi_k(t) =: \phi(t).$$

Then

$$\begin{aligned}\mathcal{C}^{(1)}_{u,\gamma} f(x) &= \text{p.v.} \int_{|t| \le 2^{n(x)}} \left[e^{iu(x)\gamma(t)} - 1\right] f(x-t)\phi(t) \frac{\mathrm{d}t}{t} \\ &\quad + \text{p.v.} \int_{|t| \le 2^{n(x)}} f(x-t)\phi(t) \frac{\mathrm{d}t}{t} \\ &=: T_1 f(x) + T_2 f(x).\end{aligned}$$

For $T_1 f$, since $\gamma'$ is increasing on $(0, \infty)$ and $\gamma(0) = 0$, we have that $\frac{\gamma(t)}{t}$ is increasing on $(0, \infty)$. These properties, combined with the fact that $\gamma$ is either odd or even and (2.6) is true, further implies that

$$\begin{aligned}T_1 f(x) &\le \int_{|t| \le 2^{n(x)}} |f(x-t)|\, |u(x)|\, \frac{\gamma(2^{n(x)})}{2^{n(x)}} \phi(t)\, \mathrm{d}t \\ &\le \frac{1}{2^{n(x)}} \int_{|t| \le 2^{n(x)}} |f(x-t)|\, \mathrm{d}t \\ &\lesssim Mf(x). \end{aligned} \tag{2.7}$$

Here and hereafter, $M$ denotes the *Hardy-Littlewood maximal operator* defined by setting

$$Mf(x) := \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x-t)|\, \mathrm{d}t \quad \forall\, x \in \mathbb{R}.$$

For $T_2 f$, we have

$$\begin{aligned}|T_2 f(x)| &= \left| \int_{|t| \le 2^{n(x)}} f(x-t) \frac{\phi(t)-1}{t} \mathrm{d}t + \text{p.v.} \int_{|t| \le 2^{n(x)}} f(x-t) \frac{\mathrm{d}t}{t} \right| \\ &\le \int_{2^{n(x)-1} \le |t| \le 2^{n(x)}} |f(x-t)| \left| \frac{\phi(t)-1}{t} \right| \mathrm{d}t + \mathcal{H}^* f(x) \\ &\le \frac{1}{2^{n(x)-1}} \int_{|t| \le 2^{n(x)}} |f(x-t)|\, \mathrm{d}t + \mathcal{H}^* f(x) \\ &\lesssim Mf(x) + \mathcal{H}^* f(x), \end{aligned} \tag{2.8}$$

where $\mathcal{H}^*$ is the *maximal truncated Hilbert transform*, which is defined by setting

$$\mathcal{H}^* f(x) := \sup_{\varepsilon, R>0} \left| \int_{\varepsilon<|t|<R} f(x-t) \frac{\mathrm{d}t}{t} \right| \quad \forall x \in \mathbb{R}.$$

Therefore, from (2.7) and (2.8) it follows that

$$\mathcal{C}^{(1)}_{u,\gamma} f(x) \lesssim Mf(x) + \mathcal{H}^* f(x).$$

It is well known that both $M$ and $\mathcal{H}^*$ are bounded on $L^p(\mathbb{R})$; therefore, we conclude

$$\|\mathcal{C}^{(1)}_{u,\gamma} f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

▷ For the high-frequency part $\mathcal{C}^{(2)}_{u,\gamma} f(x)$, we can then write

$$\mathcal{C}^{(2)}_{u,\gamma} f(x) = \sum_{k\geq 0} \int_{-\infty}^{\infty} e^{iu(x)\gamma(t)} f(x-t) \psi_{k+n(x)}(t) \frac{\mathrm{d}t}{t} =: \sum_{k\geq 0} S_k f(x).$$

For any given $k \geq 0$ we estimate

$$\begin{aligned} |S_k f(x)| &\leq \int_{2^{k+n(x)-1}\leq |t| \leq 2^{k+n(x)+1}} |f(x-t)| \frac{\left|\psi_{k+n(x)}(t)\right|}{|t|} \,\mathrm{d}t \\ &\leq \frac{1}{2^{k+n(x)-1}} \int_{|t|\leq 2^{k+n(x)+1}} |f(x-t)| \,\mathrm{d}t \\ &\lesssim Mf(x). \end{aligned}$$

From this and the well-known $L^p(\mathbb{R})$-boundedness of $M$ it follows that

$$\|S_k f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}, \tag{2.9}$$

and the bound depends only on $p$. To summarize all $k \geq 0$, we need a decay estimate for $\|S_k f\|_{L^p(\mathbb{R})}$. For this aim, we claim:

$$\exists \text{ a constant } \omega_0 > 0 \text{ such that } \|S_k f\|_{L^2(\mathbb{R})} \lesssim 2^{-\omega_0 k} \|f\|_{L^2(\mathbb{R})} \ \ \forall\ k \geq 0. \tag{2.10}$$

Then, by interpolating between (2.9) and (2.10), we obtain a positive constant $\omega_p$ such that

$$\|S_k f\|_{L^p(\mathbb{R})} \lesssim 2^{-\omega_p k} \|f\|_{L^p(\mathbb{R})},$$

which allows us to summarize all $k \geq 0$ and to obtain

$$\|\mathcal{C}^{(2)}_{u,\gamma} f\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

Therefore, it remains to verify (2.10). We use a $TT^*$-argument - the Stein-Wainger's approach in [23]. The dual operator of $S_k$ is given by

$$S_k{}^*g(y) = \text{p.v.} \int_{-\infty}^{\infty} e^{-iu(z)\gamma(z-y)} \psi(2^{-n(z)-k}(z-y)) \frac{g(z)}{z-y} \, \mathrm{d}z \quad \forall\, y \in \mathbb{R}.$$

Thus

$$S_k S_k{}^* f(x) = \text{p.v.} \int_{-\infty}^{\infty} \left( \text{p.v.} \int_{-\infty}^{\infty} e^{-iu(z)\gamma(z-x+t)} \frac{\psi(2^{-n(z)-k}(z-x+t))}{z-x+t} e^{iu(x)\gamma(t)} \frac{\psi(2^{-n(x)-k}t)}{t} \, \mathrm{d}t \right) f(z) \, \mathrm{d}z. \tag{2.11}$$

In the following calculation, without loss of generality we may assume that $2^{n(x)} \leq 2^{n(z)}$. Let $\xi := x - z$. Then, the kernel of $S_k S_k{}^*$ can be written as

$$\text{p.v.} \int_{-\infty}^{\infty} e^{-iu(z)\gamma(-\xi+t)} \frac{\psi(2^{-n(z)-k}(-\xi+t))}{-\xi+t} e^{iu(x)\gamma(t)} \frac{\psi(2^{-n(x)-k}t)}{t} \, \mathrm{d}t.$$

Upon replacing $2^{-n(x)-k}t$ with $t$, the last quantity is equivalent to

$$\text{p.v.} \int_{-\infty}^{\infty} e^{-iu(z)\gamma(-\xi+2^{n(x)+k}t)} \frac{\psi(-\xi 2^{-n(z)-k} + \frac{2^{n(x)}}{2^{n(z)}}t)}{-\xi + 2^{n(x)+k}t} e^{iu(x)\gamma(2^{n(x)+k}t)} \frac{\psi(t)}{t} \, \mathrm{d}t.$$

Now, we further set

$$0 < h := \frac{2^{n(x)}}{2^{n(z)}} \leq 1 \;\;\&\;\; s := \frac{\xi}{2^{n(z)+k}}.$$

Then, the kernel becomes

$$\frac{1}{2^{n(z)+k}} \text{p.v.} \int_{-\infty}^{\infty} e^{iu(x)\gamma(2^{n(x)+k}t) - iu(z)\gamma\left(2^{n(z)+k}[ht-s]\right)} \frac{\psi\,(ht-s)}{ht-s} \frac{\psi(t)}{t} \, \mathrm{d}t.$$

To evaluate the above integral, we use an estimate from the forthcoming Proposition 2.4. In fact, because $\frac{x-z}{2^{n(z)+k}} = s$, by (2.12) of Proposition 2.4, we therefore have

$$\begin{aligned} |S_k S_k{}^* f(x)| &= \left| \int_{-\infty}^{\infty} \frac{1}{2^{n(z)+k}} \text{p.v.} \int_{-\infty}^{\infty} e^{iu(x)\gamma(2^{n(x)+k}t) - iu(z)\gamma\left(2^{n(z)+k}[ht-s]\right)} \right. \\ &\qquad \left. \frac{\psi\,(ht-s)}{ht-s} \frac{\psi(t)}{t} \, \mathrm{d}t f(z) \, \mathrm{d}z \right| \\ &\lesssim \int_{-\infty}^{\infty} \frac{1}{2^{n(z)+k}} \left\{ \chi_{[-2^{-kr_1}, 2^{-kr_1}]}(s) + 2^{-kr_2} \chi_{[-4,4]}(s) \right\} |f(z)| \, \mathrm{d}z \end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{2^{-kr_1}}{2^{n(z)+k}2^{-kr_1}} \int_{\frac{|x-z|}{2^{n(z)+k}} \le 2^{-kr_1}} |f(z)|\,\mathrm{d}z + \frac{2^{-kr_2}}{2^{n(z)+k}} \int_{\frac{|x-z|}{2^{n(z)+k}} \le 4} |f(z)|\,\mathrm{d}z \\
&\lesssim 2^{-kr_1} Mf(x) + 2^{-kr_2} Mf(x) \\
&\lesssim 2^{-kr_0} Mf(x),
\end{aligned}$$

where $\gamma_0 := \min\{r_1, r_2\}$. Since $M$ is bounded on $L^2(\mathbb{R})$, we obtain the desired estimate

$$\|S_k\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \|S_k S_k{}^*\|^{\frac{1}{2}}_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \lesssim 2^{-\frac{r_0}{2}k},$$

thereby completing the proof of Theorem 1.1 for $\mathcal{C}_{u,\gamma}$.

□

**Proposition 2.4** *There exist positive constants $r_1$ and $r_2$ such that*

$$\begin{aligned}
&\left| \text{p.v.} \int_{-\infty}^{\infty} e^{iu(x)\gamma(2^{n(x)+k}t) - iu(z)\gamma\left(2^{n(z)+k}[ht-s]\right)} \frac{\psi\,(ht-s)}{ht-s} \frac{\psi(t)}{t}\,dt \right| \\
&\quad \le C\left(\chi_{[-2^{-kr_1}, 2^{-kr_1}]}(s) + 2^{-kr_2}\chi_{[-4,4]}(s)\right) \;\; \forall\, k \in \mathbb{N} \;\&\; (x, z, s) \in \mathbb{R}^3, \qquad (2.12)
\end{aligned}$$

*where $C$ is a positive constant independent of $k$, $x$, $z$, $s$, and $u$.*

***Proof*** Since $\psi : \mathbb{R} \to \mathbb{R}$ is smooth and supported on

$$\left\{ t \in \mathbb{R} : \frac{1}{2} \le |t| \le 2 \right\},$$

it follows from $0 < h \le 1$ that

$$\begin{cases} |t| \le 2; \\ |ht - s| \le 2; \\ |s| \le 4. \end{cases}$$

Let

$$Q(t) := u(x)\gamma(2^{n(x)+k}t) - u(z)\gamma(2^{n(z)+k}[ht-s]) \quad \forall\, t \in \mathbb{R}. \qquad (2.13)$$

It is clear that

$$\begin{cases} Q'(t) = u(x)2^{n(x)+k}\gamma'(2^{n(x)+k}t) - u(z)2^{n(z)+k}\gamma'(2^{n(z)+k}[ht-s])h \quad \forall\, t \in \mathbb{R}; \\ Q''(t) = u(x)2^{2(n(x)+k)}\gamma''(2^{n(x)+k}t) - u(z)2^{2(n(z)+k)}\gamma''(2^{n(z)+k}[ht-s])h^2 \quad \forall\, t \in \mathbb{R}. \end{cases}$$

To use Lemmas 2.1-2.2-2.3, we need some estimates for $Q'$ and $Q''$. For this purpose, we consider two cases. We want to remind the reader that the constants $C_1$ through $C_4$ are the same constants as in Theorem 1.1 and Remark 1.2.

**Case A**: $0 < h \le \frac{1}{4C_1^3 C_4}$.

Since $\frac{\gamma'(2t)}{\gamma'(t)}$ is decreasing on $(0, \infty)$, it follows that

$$\frac{\gamma'(2^k t)}{\gamma'(t)} = \frac{\gamma'(2^k t)}{\gamma'(2^{k-1} t)} \frac{\gamma'(2^{k-1} t)}{\gamma'(2^{k-2} t)} \cdots \frac{\gamma'(2t)}{\gamma'(t)}$$

is decreasing on $(0, \infty)$ for any $k \in \mathbb{N}$. By Remark 1.2, we know

$$1 \le \frac{t\gamma'(t)}{\gamma(t)} \le C_4 \ \ \forall \ \ t \in (0, \infty).$$

Noting that $\gamma$ is either odd or even, $\gamma'$ is increasing on $(0, \infty)$, (2.6) and

$$\begin{cases} |t| \le 2; \\ |ht - s| \le 2; \\ \frac{\gamma'(2t)}{\gamma'(t)} \le C_1; \end{cases} \qquad \forall \ \ t \in (0, \infty),$$

we obtain

$$\begin{aligned} |Q'(t)| &\ge \left|u(x) 2^{n(x)+k} \gamma'(2^{n(x)+k} t)\right| - \left|u(z) 2^{n(z)+k} \gamma'(2^{n(z)+k}[ht - s])\right| h \\ &\ge \left|\frac{1}{\gamma(2^{n(x)+1})} 2^{n(x)+k} \gamma'\left(2^{n(x)+k} \frac{1}{2}\right)\right| - \left|\frac{1}{\gamma(2^{n(z)})} 2^{n(z)+k} \gamma'(2^{n(z)+k} 2)\right| h \\ &= \left|\frac{2^{n(x)+1}\gamma'(2^{n(x)+1})}{\gamma(2^{n(x)+1})} \frac{2^{n(x)+k}}{2^{n(x)+1}} \frac{\gamma'(2^{n(x)+k}\frac{1}{2})}{\gamma'(2^{n(x)+k})} \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})} \frac{\gamma'(2^{n(x)})}{\gamma'(2^{n(x)+1})}\right| \\ &\quad - \left|\frac{2^{n(z)}\gamma'(2^{n(z)})}{\gamma(2^{n(z)})} \frac{2^{n(z)+k}}{2^{n(z)}} \frac{\gamma'(2^{n(z)+k}2)}{\gamma'(2^{n(x)+k})} \frac{\gamma'(2^{n(z)+k})}{\gamma'(2^{n(z)})}\right| h \\ &\ge \left(\frac{1}{2C_1^2}\right) 2^k \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})} - \left(C_1 C_4 2^k\right) \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})} h \\ &\ge \left(\frac{1}{4C_1^2}\right) 2^k \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})}. \end{aligned} \tag{2.14}$$

Using (2.14) and

$$\frac{t\gamma''(t)}{\gamma'(t)} \le C_2 \ \ \forall \ \ t \in (0, \infty) \ \ \& \ \ h \le 1,$$

we find

$$|Q''(t)| \le \left|u(x) 2^{2(n(x)+k)} \gamma''(2^{n(x)+k} t)\right| + \left|u(z) 2^{2(n(z)+k)} \gamma''(2^{n(z)+k}[ht - s]) h^2\right|$$

$$
\begin{aligned}
&= \left| u(x) 2^{2(n(x)+k)} \frac{(2^{n(x)+k}t)\gamma''(2^{n(x)+k}t)}{\gamma'(2^{n(x)+k}t)} \frac{\gamma'(2^{n(x)+k}t)}{(2^{n(x)+k}t)} \right| \\
&\quad + \left| u(z) 2^{2(n(z)+k)} \frac{(2^{n(z)+k}[ht-s])\gamma''(2^{n(z)+k}[ht-s])}{\gamma'(2^{n(z)+k}[ht-s])} \frac{\gamma'(2^{n(z)+k}[ht-s])}{(2^{n(z)+k}[ht-s])} h^2 \right| \\
&\le 2C_2 \left| u(x) 2^{(n(x)+k)} \gamma'(2^{n(x)+k}2) \right| + 2C_2 \left| u(z) 2^{(n(z)+k)} \gamma'(2^{n(z)+k}2) \right| \\
&\le 2C_1C_2 \left| \frac{1}{\gamma(2^{n(x)})} 2^{(n(x)+k)} \gamma'(2^{n(x)+k}) \right| + 2C_1C_2 \left| \frac{1}{\gamma(2^{n(z)})} 2^{(n(z)+k)} \gamma'(2^{n(z)+k}) \right| \\
&= 2C_1C_2 \left| \frac{2^{n(x)}\gamma'(2^{n(x)})}{\gamma(2^{n(x)})} \frac{2^{(n(x)+k)}}{2^{n(x)}} \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})} \right| \\
&\quad + 2C_1C_2 \left| \frac{2^{n(z)}\gamma'(2^{n(z)})}{\gamma(2^{n(z)})} \frac{2^{(n(z)+k)}}{2^{n(z)}} \frac{\gamma'(2^{n(z)+k})}{\gamma'(2^{n(z)})} \right| \\
&\le 2C_1C_2C_4 2^k \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})} + 2C_1C_2C_4 2^k \frac{\gamma'(2^{n(z)+k})}{\gamma'(2^{n(z)})} \\
&\le 4C_1C_2C_4 2^k \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})}.
\end{aligned} \tag{2.15}
$$

Combining (2.14) and (2.15), and using Lemma 2.1 and [21, p. 334, Corollary] and the fact that $\gamma'$ is increasing on $(0,\infty)$, we get

$$
\begin{aligned}
&\left| \text{p.v.} \int_{-\infty}^{\infty} e^{iu(x)\gamma(2^{n(x)+k}t) - iu(z)\gamma\left(2^{n(z)+k}[ht-s]\right)} \frac{\psi(ht-s)}{ht-s} \frac{\psi(t)}{t}\, \mathrm{d}t \right| \\
&\lesssim \frac{1}{\left(\frac{1}{4C_1^2}\right) 2^k \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})}} + \frac{4C_1C_2C_4 2^k \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})}}{\left[\left(\frac{1}{4C_1^2}\right) 2^k \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})}\right]^2} \\
&\lesssim \frac{1}{2^k}.
\end{aligned} \tag{2.16}
$$

Thus, in this case, (2.12) holds with $r_2 = 1$ and arbitrary positive constant $r_1$.

**Case B**: $\frac{1}{4C_1^3C_4} < h \le 1$.

If $|s| \le 2^{-\frac{k}{8}}$, and since $\psi : \mathbb{R} \to \mathbb{R}$ is supported on

$$
\left\{ t \in \mathbb{R} : \frac{1}{2} \le |t| \le 2 \right\},
$$

it follows that the integral in (2.12) is bounded by $C$. Thus, in this case, (2.12) holds with $r_1 = \frac{1}{8}$ and arbitrary positive constant $r_2$. In the remainder, we only consider the case $|s| \ge 2^{-\frac{k}{8}}$. We write

$$
\begin{pmatrix} Q'(t) \\ Q''(t) \end{pmatrix} = M_{t,s}\Upsilon, \tag{2.17}
$$

where $M_{t,s}$ is a $2 \times 2$ matrix and $\Upsilon$ is the vector:

$$\begin{cases} M_{t,s} := \begin{pmatrix} 1 & h \\ \frac{2^{n(x)+k}\gamma''(2^{n(x)+k}t)}{\gamma'(2^{n(x)+k}t)} & \frac{2^{n(z)+k}\gamma''(2^{n(z)+k}[ht-s])}{\gamma'(2^{n(z)+k}[ht-s])}h^2 \end{pmatrix}; \\ \Upsilon := \begin{pmatrix} u(x)2^{n(x)+k}\gamma'(2^{n(x)+k}t) \\ -u(z)2^{n(z)+k}\gamma'(2^{n(z)+k}[ht-s]) \end{pmatrix}. \end{cases} \tag{2.18}$$

We may compute immediately as in (2.14) that

$$|\Upsilon| \geq \left| u(x)2^{n(x)+k}\gamma'(2^{n(x)+k}t) \right| \geq \frac{1}{2C_1^2} 2^k \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})}. \tag{2.19}$$

Moreover, let

$$\begin{cases} a_0 := \frac{2^{n(x)+k}t\gamma''(2^{n(x)+k}t)}{\gamma'(2^{n(x)+k}t)}; \\ b_0 := \frac{2^{n(z)+k}(ht-s)\gamma''(2^{n(z)+k}[ht-s])}{\gamma'(2^{n(z)+k}[ht-s])}. \end{cases}$$

We can rewrite

$$M_{t,s} = \begin{pmatrix} 1 & h \\ a_0 \frac{1}{t} & b_0 \frac{h^2}{ht-s} \end{pmatrix}.$$

From

$$\frac{t\gamma''(t)}{\gamma'(t)} \leq C_2 \ \ \forall \ \ t \in (0, \infty),$$

it follows that

$$|a_0| \leq C_2 \ \ \& \ \ |b_0| \leq C_2,$$

and

$$\|M_{t,s}\| = \sup_{|x|=1} |M_{t,s}x| \lesssim 1. \tag{2.20}$$

From (2.18) and Theorem 1.1(iv), together with

$$|s| \leq 4 \ \ \& \ \ h = \frac{2^{n(x)}}{2^{n(z)}}$$

and the generalized mean value theorem, we have a positive constant $\theta \in [0, 1]$ such that

$$\begin{aligned} |\det M_{t,s}| =& h2^{n(x)+k} \left| \frac{\gamma''(2^{n(x)+k}t - 2^{n(z)+k}s)}{\gamma'(2^{n(x)+k}t - 2^{n(z)+k}s)} - \frac{\gamma''(2^{n(x)+k}t)}{\gamma'(2^{n(x)+k}t)} \right| \\ =& h2^{n(x)+k} \left| \left(\frac{\gamma''}{\gamma'}\right)' \left(2^{n(x)+k}t - 2^{n(z)+k}s\theta\right) 2^{n(z)+k}s \right| \\ \geq& C_3 h 2^{n(x)+k} \left[2^{n(x)+k}t - 2^{n(z)+k}s\theta\right]^{-2} \left|2^{n(x)+k}s\right| \\ =& \frac{C_3 h|s|}{\left(t - \frac{1}{h}s\theta\right)^2} \\ \gtrsim& 2^{-\frac{k}{8}}. \end{aligned} \tag{2.21}$$

Combining (2.19), (2.20), (2.21), and Lemma 2.2 with $n = 2$, we therefore have

$$M_{t,s}\Upsilon \geq |\det M_{t,s}| \|M_{t,s}\|^{-1} |\Upsilon| \gtrsim 2^{\frac{7k}{8}} \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})},$$

and consequently,

$$\sqrt{[Q'(t)]^2 + [Q''(t)]^2} \gtrsim 2^{\frac{7k}{8}} \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})}.$$

By the pigeonhole principle, there are two cases.

▷ If

$$|Q'(t)| \gtrsim 2^{\frac{7k}{8}} \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})},$$

then $h = \frac{2^{n(x)}}{2^{n(z)}}$ follows by Lemma 2.3. Let

$$\begin{cases} a := u(x)2^{n(x)+k}; \\ b := u(z)2^{n(z)+k}h; \\ c := 2^{n(z)+k}s; \\ d := 2^{\frac{7k}{8}} \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})}; \\ t := 2^{n(x)+k}t. \end{cases}$$

We see that this case only happens in at most a finite number of intervals, and the number of these intervals is independent of $x, z, s, k$ and $u$. Using (2.15), from Lemma 2.1, [21, p. 334, Corollary], and (2.16), we find that the integral in (2.12) on this portion is established with $r_2 = \frac{3}{4}$ and an arbitrary positive constant $r_1$.

▷ If

$$|Q''(t)| \gtrsim 2^{\frac{7k}{8}} \frac{\gamma'(2^{n(x)+k})}{\gamma'(2^{n(x)})},$$

then the argument for the first case shows that this second case also only happens in at most a finite number of intervals. By van der Corput's lemma, similarly to (2.16), we conclude that the integral in (2.12) on this portion is established with $r_2 = \frac{7}{16}$ and an arbitrary positive constant $r_1$.

Altogether, we now show that the integral in (2.12) is established with $r_2 = \frac{7}{16}$ and an arbitrary positive constant $r_1$, whence completing the proof of Proposition 2.4. □

## 3 Verification of Theorem 1.1 for Hilbert Transform

### 3.1 Annulus $L^p(\mathbb{R}^2)$-estimate for $H_{u,\gamma}$

Recall that $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth function supported on

$$\left\{t \in \mathbb{R} : \frac{1}{2} \le |t| \le 2\right\}$$

and enjoys

$$\begin{cases} 0 \le \psi(t) \le 1 \ \ \forall \ t \in \mathbb{R}; \\ \Sigma_{l\in\mathbb{Z}}\psi_l(t) = 1 \ \ \forall \ t \in \mathbb{R} \setminus \{0\}; \\ \psi_l(t) := \psi(2^{-l}t) \ \ \forall \ t \in \mathbb{R}. \end{cases}$$

For any $l \in \mathbb{Z}$, let $P_l$ denote the Littlewood-Paley projection in the second variable corresponding to $\psi_l$, namely,

$$P_l f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1, x_2 - z)\check{\psi}_l(z)\, dz \ \ \forall \ (x_1, x_2) \in \mathbb{R}^2.$$

**Theorem 3.1** *Let $u$ and $\gamma$ be the same as in Theorem 1.1. Then*

$$\|H_{u,\gamma} P_l f\|_{L^p(\mathbb{R}^2)} \le C\|P_l f\|_{L^p(\mathbb{R}^2)}$$

*holds uniformly in $l \in \mathbb{Z}$, and the bound $C$ is a positive constant independent of $u$.*

***Proof*** By the anisotropic scaling:

$$x_1 \to x_1 \ \ \& \ \ x_2 \to 2^{-l}x_2,$$

we consider only the case $l = 0$. Set

$$H_{u,\gamma,k} P_0 f(x_1, x_2) := \int_{-\infty}^{\infty} P_0 f(x_1 - t, x_2 - u(x_1)\gamma(t))\psi_k(t) \frac{\mathrm{d}t}{t}.$$

Let $n : \mathbb{R} \to \mathbb{Z}$ be such that

$$\frac{1}{\gamma(2^{n(x_1)+1})} \leq |u(x_1)| \leq \frac{1}{\gamma(2^{n(x_1)})} \ \ \forall \ x_1 \in \mathbb{R}. \tag{3.1}$$

We make the following decomposition:

$$\begin{aligned} H_{u,\gamma} P_0 f(x_1, x_2) &= \sum_{k \leq n(x_1)-1} H_{u,\gamma,k} P_0 f(x_1, x_2) + \sum_{k \geq n(x_1)} H_{u,\gamma,k} P_0 f(x_1, x_2) \\ &=: H_{u,\gamma}^{(1)} P_0 f(x_1, x_2) + H_{u,\gamma}^{(2)} P_0 f(x_1, x_2). \end{aligned} \tag{3.2}$$

For $H_{u,\gamma}^{(1)} P_0 f$, let $\rho$ be a non-negative smooth function with

$$\begin{cases} \mathrm{supp}\rho \subseteq \{\xi \in \mathbb{R} : \ \frac{1}{4} \leq |\xi| \leq 4\}; \\ \rho = 1 \ \text{ on } \ \{\xi \in \mathbb{R} : \ \frac{1}{2} \leq |\xi| \leq 2\}, \end{cases}$$

and

$$\mathbb{P}_0 f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1, x_2 - s)\check{\rho}(s) \, \mathrm{d}s.$$

By a Fourier transform, it is easy to check

$$\mathbb{P}_0 P_0 f = P_0 f. \tag{3.3}$$

- $\triangleright$ We first consider $H_{u,\gamma}^{(1)} \mathbb{P}_0 f$. If

  $$\sum_{k \leq n(x_1)-1} \psi_k(t) =: \phi(t),$$

  then

  $$H_{u,\gamma}^{(1)} \mathbb{P}_0 f(x_1, x_2) = \text{p.v.} \int_{|t| \leq 2^{n(x_1)}} \mathbb{P}_0 f(x_1 - t, x_2 - u(x_1)\gamma(t))\phi(t) \frac{\mathrm{d}t}{t}.$$

  Let us consider an approximate operator

  $$\tilde{H} \mathbb{P}_0 f(x_1, x_2) := \text{p.v.} \int_{|t| \leq 2^{n(x_1)}} \mathbb{P}_0 f(x_1 - t, x_2)\phi(t) \frac{\mathrm{d}t}{t}.$$

As shown in (2.8), we have

$$\tilde{H}\mathbb{P}_0 f(x_1, x_2) \lesssim M_1 \mathbb{P}_0 f(x_1, x_2) + \tilde{\mathcal{H}}_1^* \mathbb{P}_0 f(x_1, x_2). \tag{3.4}$$

Here and hereafter, $\tilde{\mathcal{H}}_1^*$ denotes the maximal truncated Hilbert transform applied in the first variable, $M_1$ and $M_2$ denote the Hardy-Littlewood maximal operators applied in the first variable and the second variable, respectively. Since both $M_1$ and $\tilde{\mathcal{H}}_1^*$ are bounded on $L^p(\mathbb{R}^2)$, from (3.4) we may conclude

$$\|\tilde{H}\mathbb{P}_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|\mathbb{P}_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}. \tag{3.5}$$

- $\triangleright$ Now we turn to the difference between $H_{u,\gamma}^{(1)}\mathbb{P}_0 f$ and $\tilde{H}\mathbb{P}_0 f$, which can be written as

$$\text{p.v.} \int_{|t|\le 2^{n(x_1)}} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - z) \left[\check{\rho}(z - u(x_1)\gamma(t)) - \check{\rho}(z)\right] \mathrm{d}z \phi(t) \frac{\mathrm{d}t}{t}.$$

Since $\gamma$ is increasing on $(0, \infty)$ and $|t| \le 2^{n(x_1)}$, we have

$$|u(x_1)\gamma(t)| \le |u(x_1)|\gamma(2^{n(x_1)}) \le 1.$$

Then, apply the mean value theorem to obtain

$$|\check{\rho}(z - u(x_1)\gamma(t)) - \check{\rho}(z)| \lesssim \sum_{m\in\mathbb{Z}} \frac{1}{(|m-1|+1)^2} \chi_{[m,m+1]}(z)|u(x_1)\gamma(t)|.$$

Because

$$\sum_{m\in\mathbb{Z}} \frac{1}{(|m-1|+1)^2} \lesssim 1,$$

it suffices to dominate the operator defined by setting for any fixed $m \in \mathbb{Z}$,

$$K_m f(x_1, x_2) := \int_m^{m+1} \int_{|t|\le 2^{n(x_1)}} |f(x_1 - t, x_2 - z)| \frac{|u(x_1)\gamma(t)|}{|t|} \phi(t)\, \mathrm{d}t\, \mathrm{d}z$$

with a bound independent of $m$ and $u$. By Minkowski's inequality, (3.1) and noticing that $\frac{\gamma(t)}{t}$ is increasing on $(0, \infty)$, we have that if $1 < p < \infty$ then

$$\begin{aligned}
&\|K_m f(\cdot_1, \cdot_2)\|_{L^p(\mathbb{R}^2)}^p \\
&\le \int_{-\infty}^{\infty} \left( \int_m^{m+1} \int_{|t|\le 2^{n(x_1)}} \|f(x_1 - t, \cdot_2)\|_{L^p(\mathbb{R}^1_{x_2})} \frac{|u(x_1)\gamma(t)|}{|t|} \phi(t)\, \mathrm{d}t\, \mathrm{d}z \right)^p \mathrm{d}x_1 \\
&\le \int_{-\infty}^{\infty} \left( \int_{|t|\le 2^{n(x_1)}} \|f(x_1 - t, \cdot_2)\|_{L^p(\mathbb{R}^1_{x_2})} \frac{|u(x_1)\gamma(2^{n(x_1)})|}{|2^{n(x_1)}|} \phi(t)\, \mathrm{d}t \right)^p \mathrm{d}x_1
\end{aligned}$$

$$\le \int_{-\infty}^{\infty} \left( \frac{1}{2^{n(x_1)}} \int_{|t|\le 2^{n(x_1)}} \|f(x_1-t,\cdot_2)\|_{L^p(\mathbb{R}^1_{x_2})}\, \mathrm{d}t \right)^p \mathrm{d}x_1$$
$$\lesssim \int_{-\infty}^{\infty} \left( M(\|f(\cdot,\cdot_2)\|_{L^p(\mathbb{R}^1_{x_2})})(x_1) \right)^p \mathrm{d}x_1$$
$$\lesssim \|f\|^p_{L^p(\mathbb{R}^2)} \tag{3.6}$$

and hence

$$\|H^{(1)}_{u,\gamma}\mathbb{P}_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

follows from (3.5) and (3.6). Accordingly, (3.3) implies

$$\|H^{(1)}_{u,\gamma} P_0 f\|_{L^p(\mathbb{R}^2)} \lesssim \|P_0 f\|_{L^p(\mathbb{R}^2)}.$$

- ▷ For $H^{(2)}_{u,\gamma} P_0 f$ let $f := P_0 f$. Then we can write

$$H^{(2)}_{u,\gamma} f(x_1,x_2) = \sum_{k\ge 0} \int_{-\infty}^{\infty} f(x_1-t, x_2-u(x_1)\gamma(t))\psi_{n(x_1)+k}(t)\,\frac{\mathrm{d}t}{t}.$$

By Minkowski's inequality and (3.6), we have

$$\left\| \int_{-\infty}^{\infty} f(\cdot_1-t,\cdot_2-u(\cdot_1)\gamma(t))\psi_{n(\cdot_1)+k}(t)\,\frac{\mathrm{d}t}{t} \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)} \tag{3.7}$$

and then use (2.10) to get

$$\left\| \int_{-\infty}^{\infty} e^{iu(\cdot)\gamma(t)} f(\cdot - t)\psi_{k+n(\cdot)}(t)\,\frac{\mathrm{d}t}{t} \right\|_{L^2(\mathbb{R})} \lesssim 2^{-\omega_0 k}\|f\|_{L^2(\mathbb{R})},$$

which ensures

$$\left\| \int_{-\infty}^{\infty} f(\cdot_1-t,\cdot_2-u(\cdot_1)\gamma(t))\psi_{n(\cdot_1)+k}(t)\,\frac{\mathrm{d}t}{t} \right\|_{L^2(\mathbb{R}^2)} \lesssim 2^{-\omega_0 k}\|f\|_{L^2(\mathbb{R}^2)}. \tag{3.8}$$

By interpolating between (3.7) and (3.8) and making a sum over $k \ge 0$, we obtain

$$\|H^{(2)}_{u,\gamma} f\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)} \ \text{ under } \ p \in (1,\infty),$$

thereby completing the proof of Theorem 3.1.

□

### 3.2 $L^p(\mathbb{R}^2)$-estimate for $H_{u,\gamma}$

As explained in Note (1.6), the case $p = 2$ can be obtained by the $L^2(\mathbb{R})$-boundedness of (1.1). So, it remains to handle the case $p \in (1, 2) \cup (2, \infty)$. Our argument (actually for any case $p \in (1, \infty)$) crucially relies on the commutative property between $H_{u,\gamma}$ and $P_l$. Accordingly, we can turn our attention to a square function. As before, we also decompose our operator into a low-frequency part and a high-frequency part. The low-frequency part is controlled by the Hardy-Littlewood maximal operator and the maximal truncated Hilbert transform. The high-frequency part is also represented by a series of operators. Building on the already proved $L^2(\mathbb{R}^2)$-estimate with bound $2^{-\omega_0 k}$ and the interpolation strategy, it suffices to obtain an $L^p(\mathbb{R}^2)$-estimate with bound $k^2$. This unusual $L^p(\mathbb{R}^2)$-boundedness can be achieved by the shifted maximal operator, which forms a pointwise estimate for taking the average along the variable plane curve $u(x_1)\gamma$.

***Proof of Theorem 1.1*** for $H_{u,\gamma}$ We note that the commutative property

$$H_{u,\gamma} P_l = P_l H_{u,\gamma}$$

holds for any $l \in \mathbb{Z}$. By the Littlewood-Paley theory, it is enough to show

$$\left\| \left[ \sum_{l\in\mathbb{Z}} \left| H_{u,\gamma} P_l f \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}. \tag{3.9}$$

Regarding (3.1), for any $l \in \mathbb{Z}$, let $n_l : \mathbb{R} \to \mathbb{Z}$ be such that

$$\frac{1}{\gamma(2^{n_l(x_1)+1})} \le 2^l |u(x_1)| \le \frac{1}{\gamma(2^{n_l(x_1)})} \quad \forall\ x_1 \in \mathbb{R}. \tag{3.10}$$

In a similar way to handle (3.2), we decompose $H_{u,\gamma} P_l$ as

$$\begin{aligned} H_{u,\gamma} P_l f(x_1, x_2) &= \sum_{k\le n_l(x_1)-1} \int_{-\infty}^{\infty} P_l f(x_1 - t, x_2 - u(x_1)\gamma(t))\psi_k(t) \frac{\mathrm{d}t}{t} \\ &\quad + \sum_{k\ge 0} \int_{-\infty}^{\infty} P_l f(x_1 - t, x_2 - u(x_1)\gamma(t))\psi_{k+n_l(x_1)}(t) \frac{\mathrm{d}t}{t} \\ &=: H_{u,\gamma}^{(I)} P_l f(x_1, x_2) + \sum_{k\ge 0} H_{u,\gamma,k+n_l(x_1)} P_l f(x_1, x_2). \end{aligned} \tag{3.11}$$

Using the triangle inequality, the left term of (3.9) can be controlled by

$$\left\| \left[ \sum_{l\in\mathbb{Z}} \left| H_{u,\gamma}^{(I)} P_l f \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} + \sum_{k\ge 0} \left\| \left[ \sum_{l\in\mathbb{Z}} \left| H_{u,\gamma,k+n_l(\cdot_1)} P_l f \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} . \tag{3.12}$$

▷ For the low-frequency part in (3.12), let

$$\begin{cases} \sum_{k\le n_l(x_1)-1} \psi_k(t) =: \phi(t); \\ \tilde{H} f(x_1, x_2) := \text{p.v.} \int_{|t|\le 2^{n_l(x_1)}} f(x_1 - t, x_2)\phi(t)\, \frac{dt}{t}. \end{cases}$$

As done in (2.8), we may obtain

$$\tilde{H} P_l f(x_1, x_2) \lesssim M_1 P_l f(x_1, x_2) + \tilde{\mathcal{H}}_1^* P_l f(x_1, x_2), \tag{3.13}$$

The vector-valued estimate for $M_1$ follows from the corresponding estimate for the one-dimensional Hardy-Littlewood maximal function. Similarly, the vector-valued estimate for $\tilde{\mathcal{H}}_1^*$ follows from Cotlar's inequality and the vector-valued estimate for the Hilbert transform and the maximal function. Then, from (3.13) and the Littlewood-Paley theory it follows that

$$\left\| \left[ \sum_{l\in\mathbb{Z}} \left| \tilde{H} P_l f \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \lesssim \left\| \left[ \sum_{l\in\mathbb{Z}} |P_l f|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}. \tag{3.14}$$

▷ Concerning the difference between $H_{u,\gamma}^{(I)} P_l f$ and $\tilde{H} P_l f$, we recall that $\rho$ is a non-negative smooth function obeying

$$\begin{cases} \text{supp}\rho \subseteq \{s \in \mathbb{R} : \frac{1}{4} \le |\xi| \le 4\}; \\ \rho(t) = 1 \ \ \forall \ \ t \in \{s \in \mathbb{R} : \frac{1}{2} \le |s| \le 2\}. \end{cases}$$

Let

$$\begin{cases} \rho_l(s) := \rho(2^{-l}s) \quad \forall \quad l \in \mathbb{Z}; \\ \mathbb{P}_l f(x_1, x_2) := \int_{-\infty}^{\infty} f(x_1, x_2 - s)\check{\rho}_l(s)\, ds. \end{cases}$$

Then, taking Fourier transform gives

$$\mathbb{P}_l P_l f = P_l f. \tag{3.15}$$

The difference between $H_{u,\gamma}^{(I)} \mathbb{P}_l f$ and $\tilde{H} \mathbb{P}_l f$ can be written as

$$\text{p.v.} \int_{|t|\le 2^{n_l(x_1)}} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - s) \left[\check{\rho}_l(s - u(x_1)\gamma(t)) - \check{\rho}_l(s)\right] ds\phi(t)\, \frac{dt}{t}. \tag{3.16}$$

By the mean value theorem, we have

$$|\check{\rho}_l(s - w) - \check{\rho}_l(s)| \lesssim |w| 2^{2l} 2^{-2j} \ \ \forall \ \ |w| \le 2^{-l}$$

if $s$ is in the annulus

$$2^{-l+j-1} \le |s| \le 2^{-l+j} \ \forall \ j \in \mathbb{N}.$$

Meanwhile, for $j = 0$, the estimate holds for all $|s| \le 2^{-l}$. Because $\gamma$ is increasing on $(0, \infty)$ and $\gamma$ is either odd or even, from (3.10) it follows that

$$2^l |u(x_1)\gamma(t)| \le 2^l |u(x_1)|\gamma(2^{n_l(x_1)}) \le 1 \ \forall \ |t| \le 2^{n_l(x_1)}.$$

Thus, the absolute value of (3.16) can be estimated by a positive constant multiplied by

$$\sum_{j\in\mathbb{N}} \int_{|t|\le 2^{n_l(x_1)}} \int_{|s|\le 2^{-l+j}} |f(x_1 - t, x_2 - s)| 2^{2l} 2^{-2j} |u(x_1)| \left| \frac{\gamma(t)}{t} \right| \mathrm{d}s\, \mathrm{d}t. \quad (3.17)$$

Notice that $\frac{\gamma(t)}{t}$ is increasing on $(0, \infty)$, and $\gamma$ is either odd or even. So we can use (3.10) to control (3.17) via

$$\begin{aligned} &\sum_{j\in\mathbb{N}} \int_{|t|\le 2^{n_l(x_1)}} \int_{|s|\le 2^{-l+j}} |f(x_1 - t, x_2 - s)| 2^{2l} 2^{-2j} |u(x_1)| \left| \frac{\gamma(2^{n_l(x_1)})}{2^{n_l(x_1)}} \right| \mathrm{d}s\, \mathrm{d}t \\ &\lesssim \sum_{j\in\mathbb{N}} \frac{2^{-j}}{2^{n_l(x_1)}} \int_{|t|\le 2^{n_l(x_1)}} \frac{1}{2^{-l+j}} \int_{|s|\le 2^{-l+j}} |f(x_1 - t, x_2 - s)|\, \mathrm{d}s\, \mathrm{d}t \\ &\lesssim M_1 M_2 f(x_1, x_2). \end{aligned} \quad (3.18)$$

Therefore, the vector-valued estimates for $M_1$ and $M_2$, the Littlewood-Paley theory, (3.15), the triangle inequality and (3.18) yield

$$\begin{aligned} \left\| \left[ \sum_{l\in\mathbb{Z}} \left| H_{u,\gamma}^{(I)} P_l f - \tilde{H} P_l f \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} &= \left\| \left[ \sum_{l\in\mathbb{Z}} \left| H_{u,\gamma}^{(I)} \mathbb{P}_l P_l f - \tilde{H} \mathbb{P}_l P_l f \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \\ &\lesssim \left\| \left[ \sum_{l\in\mathbb{Z}} |M_1 M_2 P_l f|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \\ &\lesssim \|f\|_{L^p(\mathbb{R}^2)}. \end{aligned} \quad (3.19)$$

From (3.14) and (3.19) it follows that

$$\left\| \left[ \sum_{l\in\mathbb{Z}} \left| H_{u,\gamma}^{(I)} P_l f \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}.$$

▷ For the high-frequency part in (3.12), it is enough to show that there exists a convergent series $\{C_k\}_{k=0}^{\infty}$ such that for any $k \geq 0$,

$$\left\| \left[ \sum_{l\in\mathbb{Z}} \left| H_{u,\gamma,k+n_l(\cdot_1)} P_l f \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \lesssim C_k \|f\|_{L^p(\mathbb{R}^2)} . \tag{3.20}$$

If $p = 2$, then noting that the bound in (2.10) is independent of $u$, we can replace $u$ with $2^l u$ in (2.10). By the Littlewood-Paley theory we have

$$\left\| \left[ \sum_{l\in\mathbb{Z}} \left| H_{u,\gamma,k+n_l(\cdot_1)} P_l f \right|^2 \right]^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^2)} \lesssim 2^{-\omega_0 k} \|f\|_{L^2(\mathbb{R}^2)} \tag{3.21}$$

for some positive constant $\omega_0$. So, it remains to verify that

$$\left\| \left[ \sum_{l\in\mathbb{Z}} \left| H_{u,\gamma,k+n_l(\cdot_1)} P_l f \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \lesssim k^2 \|f\|_{L^p(\mathbb{R}^2)} \tag{3.22}$$

holds for all $2 \neq p \in (1, \infty)$, since (3.20) follows from the interpolation between (3.21) and (3.22). Notice that

$$\begin{aligned} & H_{u,\gamma,k+n_l(x_1)} \mathbb{P}_l f(x_1, x_2) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1 - t, x_2 - u(x_1)\gamma(t) - s) \frac{\psi_{k+n_l(x_1)}(t)}{t} \check{\rho}_l(s) \, dt \, ds \\ &\leq \int_{\frac{1}{2}2^{k+n_l(x_1)} \leq |t| \leq 2\cdot 2^{k+n_l(x_1)}} \int_{-\infty}^{\infty} |f(x_1 - t, x_2 \\ &- u(x_1)\gamma(t) - s)| \left| \frac{\psi_{k+n_l(x_1)}(t)}{t} \right| |\check{\rho}_l(s)| \, ds \, dt \\ &\lesssim \frac{1}{2^{k+n_l(x_1)}} \int_{\frac{1}{2}2^{k+n_l(x_1)} \leq |t| \leq 2\cdot 2^{k+n_l(x_1)}} \int_{-\infty}^{\infty} |f(x_1 - t, x_2 \\ &- u(x_1)\gamma(t) - 2^{-l}s)||\check{\rho}(s)| \, ds \, dt \\ &\lesssim \sum_{\tau\in\mathbb{Z}} \frac{(1+|\tau|)^{-4}}{2^{k+n_l(x_1)}} \int_{\frac{1}{2}2^{k+n_l(x_1)} \leq |t| \leq 2\cdot 2^{k+n_l(x_1)}} \int_{\tau}^{\tau+1} |f(x_1 - t, x_2 \\ &- u(x_1)\gamma(t) - 2^{-l}s)| \, ds \, dt \\ &\lesssim \sum_{\tau\in\mathbb{Z}} \frac{(1+|\tau|)^{-4}}{2^{k+n_l(x_1)}} \int_{\frac{1}{2}2^{k+n_l(x_1)} \leq |t| \leq 2\cdot 2^{k+n_l(x_1)}} \int_{0}^{1} |f(x_1 - t, x_2 \\ &- u(x_1)\gamma(t) - 2^{-l}(s+\tau))| \, ds \, dt. \end{aligned} \tag{3.23}$$

So, we are led to control the last term in (3.23) by

$$\sum_{\tau\in\mathbb{Z}} \frac{1}{(1+|\tau|)^4 N_k} \sum_{m=0}^{N_k-1} \frac{1}{|I_m|} \int_{I_m} M_2^{(\sigma_m^{(2)})} f(x_1-t, x_2)\,\mathrm{d}t,$$

where $\{I_m\}_{m=0}^{N_k-1}$ and the shifted maximal operator $M_2^{(\sigma_m^{(2)})}$ will be given below. By a scaling argument, it suffices to prove

$$\begin{aligned}&\sum_{\tau\in\mathbb{Z}} \frac{(1+|\tau|)^{-4}}{2^{k+n_l(x_1)}} \int_{\frac{1}{2}2^{k+n_l(x_1)}\le|t|\le 2\cdot 2^{k+n_l(x_1)}} \int_0^1 |f(x_1-t, x_2 \\ &\quad - 2^l u(x_1)\gamma(t) - s - \tau)|\,\mathrm{d}s\,\mathrm{d}t \\ &\lesssim \sum_{\tau\in\mathbb{Z}} \frac{1}{(1+|\tau|)^4 N_k} \sum_{m=0}^{N_k-1} \frac{1}{|I_m|} \int_{I_m} M_2^{(\sigma_m^{(2)})} f(x_1-t, x_2)\,\mathrm{d}t. \end{aligned} \tag{3.24}$$

We cover the region

$$\left\{t\in\mathbb{R} : \frac{1}{2}2^{k+n_l(x_1)} \le |t| \le 2\cdot 2^{k+n_l(x_1)}\right\}$$

by intervals $\{I_m\}_{m=0}^{N_k-1}$, where

$$\begin{aligned} I_m := \Bigg\{t\in\mathbb{R} : \ & \frac{1}{2}2^{k+n_l(x_1)} + \frac{m}{2^l|u(x_1)|\gamma'(2^{k+n_l(x_1)})} \le |t| \\ & \le \frac{1}{2}2^{k+n_l(x_1)} + \frac{m+1}{2^l|u(x_1)|\gamma'(2^{k+n_l(x_1)})}\Bigg\} \end{aligned}$$

and $N_k\in\mathbb{N}$ enjoys

$$3\cdot 2^{k+n_l(x_1)-1} \le \frac{N_k}{2^l|u(x_1)|\gamma'(2^{k+n_l(x_1)})} \le 2\cdot 2^{k+n_l(x_1)}. \tag{3.25}$$

Therefore,

$$|I_m| = \frac{1}{2^l|u(x_1)|\gamma'(2^{k+n_l(x_1)})},$$

which implies

$$\frac{1}{2^{1+k+n_l(x_1)}} \le \frac{1}{N_k\cdot|I_m|} \le \frac{1}{3\cdot 2^{k+n_l(x_1)-1}}. \tag{3.26}$$

Thus, the first term in (3.24) can be controlled by

$$\sum_{\tau\in\mathbb{Z}} \frac{1}{(1+|\tau|)^4 N_k} \sum_{m=0}^{N_k-1} \frac{1}{|I_m|} \int_{I_m} \int_0^1 |f(x_1-t, x_2-2^l u(x_1)\gamma(t)-s-\tau)|\, \mathrm{d}s\, \mathrm{d}t. \tag{3.27}$$

Without loss of generality, we may denote

$$\Re_m := \left\{(t, 2^l u(x_1)\gamma(t)+s+\tau) \in \mathbb{R}^2 : t \in I_m, s \in (0,1)\right\} \subseteq I_m \times J_m,$$

where

$$\begin{cases} J_m := [Ja, Jb]; \\ Ja := 2^l |u(x_1)|\gamma\left(\frac{1}{2}2^{k+n_l(x_1)} + \frac{m}{2^l|u(x_1)|\gamma'(2^{k+n_l(x_1)})}\right) + \tau; \\ Jb := 2^l |u(x_1)|\gamma\left(\frac{1}{2}2^{k+n_l(x_1)} + \frac{m+1}{2^l|u(x_1)|\gamma'(2^{k+n_l(x_1)})}\right) + 1 + \tau. \end{cases}$$

We can show

$$|J_m| \approx 1. \tag{3.28}$$

In fact, the mean value theorem implies

$$|J_m| = 1 + \frac{1}{\gamma'(2^{k+n_l(x_1)})}\gamma'\left(\frac{1}{2}2^{k+n_l(x_1)} + \frac{m+\theta}{2^l|u(x_1)|\gamma'(2^{k+n_l(x_1)})}\right)$$
$$\text{for some } \theta \in [0,1].$$

It is easy to see

$$|J_m| \geq 1. \tag{3.29}$$

Also, from $\theta \in [0,1]$ and (3.25) it follows that

$$m+\theta \leq N_k - 1 + \theta \leq N_k \leq 2 \cdot 2^{k+n_l(x_1)} 2^l |u(x_1)|\gamma'(2^{k+n_l(x_1)}).$$

Since $\gamma'$ is increasing on $(0,\infty)$ and

$$\frac{\gamma'(2t)}{\gamma'(t)} \leq C_1 \;\; \forall \;\; t \in (0,\infty),$$

we obtain

$$|J_m| \leq 1 + \frac{\gamma'(4 \cdot 2^{k+n_l(x_1)})}{\gamma'(2^{k+n_l(x_1)})} = 1 + \frac{\gamma'(4 \cdot 2^{k+n_l(x_1)})}{\gamma'(2 \cdot 2^{k+n_l(x_1)})} \frac{\gamma'(2 \cdot 2^{k+n_l(x_1)})}{\gamma'(2^{k+n_l(x_1)})} \leq 1 + C_1^2. \tag{3.30}$$

Now, both (3.29) and (3.30) yield (3.28). Furthermore, (3.27) is bounded by

$$\sum_{\tau \in \mathbb{Z}} \frac{1}{(1+|\tau|)^4 N_k} \sum_{m=0}^{N_k-1} \frac{1}{|I_m|} \int_{I_m} \frac{1}{|J_m|} \int_{J_m} |f(x_1 - t, x_2 - s)| \,\mathrm{d}s \,\mathrm{d}t. \tag{3.31}$$

Given a non-negative parameter $\sigma$, the *shifted maximal operator* is defined by

$$M^{(\sigma)} f(z) := \sup_{z \in I \subset \mathbb{R}} \frac{1}{|I|} \int_{I^{(\sigma)}} |f(\zeta)| \,\mathrm{d}\zeta,$$

where $I^{(\sigma)}$ denotes a shift of the interval $I := [a, b]$ given by

$$I^{(\sigma)} := [a - \sigma \cdot |I|, b - \sigma \cdot |I|] \cup [a + \sigma \cdot |I|, b + \sigma \cdot |I|].$$

Upon observing

$$\frac{1}{|J_m|} \int_{J_m} |f(x_1 - t, x_2 - s)| \mathrm{d}s \leq M_2^{(\sigma_m^{(2)})} f(x_1 - t, x_2), \tag{3.32}$$

where $M_2^{(\sigma_m^{(2)})}$ is a shifted maximal operator applied to the second variable and

$$\sigma_m^{(2)} := \frac{2^l |u(x_1)|}{|J_m|} \gamma \left( \frac{1}{2} 2^{k+n_l(x_1)} + \frac{m}{2^l |u(x_1)| \gamma'(2^{k+n_l(x_1)})} \right) + \frac{\tau}{|J_m|},$$

and a combination of (3.31) and (3.32) derives (3.24). Altogether, we obtain

$$|H_{u,\gamma,k+n_l(x_1)} \mathbb{P}_l f(x_1, x_2)| \lesssim \sum_{\tau \in \mathbb{Z}} \frac{1}{(1+|\tau|)^4 N_k} \sum_{m=0}^{N_k-1} \frac{1}{|I_m|} \int_{I_m} M_2^{(\sigma_m^{(2)})} f(x_1 - t, x_2) \,\mathrm{d}t \tag{3.33}$$

for any $l \in \mathbb{Z}$, thereby using (3.15) to reach

$$|H_{u,\gamma,k+n_l(x_1)} P_l f(x_1, x_2)| \lesssim \sum_{\tau \in \mathbb{Z}} \frac{1}{(1+|\tau|)^4 N_k} \sum_{m=0}^{N_k-1} \frac{1}{|I_m|} \int_{I_m} M_2^{(\sigma_m^{(2)})} P_l f(x_1 - t, x_2) \,\mathrm{d}t \tag{3.34}$$

for any $l \in \mathbb{Z}$.

Since $\gamma(0) = 0$, Remark 1.2 and Cauchy's mean value theorem imply

$$\frac{\gamma(2t)}{\gamma(t)} \leq 2C_1 \ \ \forall \ \ t \in (0, \infty).$$

Notice that $\gamma$ is increasing on $(0, \infty)$. So combining

$$m \leq N_k - 1 \leq N_k$$

and (3.10), (3.25), (3.28), we obtain

$$\begin{aligned}
\sigma_m^{(2)} &\leq \left(\frac{1}{|J_m|\gamma(2^{n_l(x_1)})}\right)\gamma\left(\frac{1}{2}2^{k+n_l(x_1)} + \frac{m}{2^l|u(x_1)|\gamma'(2^{k+n_l(x_1)})}\right) + \frac{\tau}{|J_m|} \\
&\lesssim \left(\frac{1}{\gamma(2^{n_l(x_1)})}\right)\gamma\left(\frac{1}{2}2^{k+n_l(x_1)} + \frac{2\cdot 2^{k+n_l(x_1)}2^l|u(x_1)|\gamma'(2^{k+n_l(x_1)})}{2^l|u(x_1)|\gamma'(2^{k+n_l(x_1)})}\right) + \tau \\
&\lesssim \frac{\gamma(2^{n_l(x_1)+k+2})}{\gamma(2^{n_l(x_1)})} + \tau \\
&\lesssim (2C_1)^{k+2} + \tau. 
\end{aligned} \tag{3.35}$$

From [12, Theorem 3.1], (3.35) and the Littlewood-Paley theory, we obtain the following vector-valued estimate for the one-dimensional shifted maximal operator:

$$\begin{aligned}
&\left\| \left[\sum_{l\in\mathbb{Z}} \left| M_2^{(\sigma_m^{(2)})} P_l f(\cdot_1 - t, \cdot_2)\right|^2\right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^1_{x_2})} \\
&\quad\lesssim \left[\log(2 + |\sigma_m^{(2)}|)\right]^2 \left\| \left[\sum_{l\in\mathbb{Z}} |P_l f(\cdot_1 - t, \cdot_2)|^2\right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^1_{x_2})} \\
&\quad\lesssim \left[\log(2 + (2C_1)^{k+2} + |\tau|)\right]^2 \|f(\cdot_1 - t, \cdot_2)\|_{L^p(\mathbb{R}^1_{x_2})} \\
&\quad\lesssim k^2(1 + |\tau|)^2 \|f(\cdot_1 - t, \cdot_2)\|_{L^p(\mathbb{R}^1_{x_2})}.
\end{aligned}$$

Combining (3.34), the triangle inequality and Minkowski's inequality yields that the left-hand side of (3.22) is controlled by

$$\sum_{\tau\in\mathbb{Z}} \frac{1}{(1 + |\tau|)^4} \left\| \frac{1}{N_k} \sum_{m=0}^{N_k-1} \frac{1}{|I_m|} \int_{I_m} \left\| \left[\sum_{l\in\mathbb{Z}} \left| M_2^{(\sigma_m^{(2)})} P_l f(\cdot_1 - t, \cdot_2)\right|^2\right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^1_{x_2})} \mathrm{d}t \right\|_{L^p(\mathbb{R}^1_{x_1})}$$

Consequently, the above expression is bounded by

$$k^2 \sum_{\tau \in \mathbb{Z}} \frac{1}{(1+|\tau|)^2} \left\| \frac{1}{N_k} \sum_{m=0}^{N_k-1} \frac{1}{|I_m|} \int_{I_m} \|f(\cdot_1 - t, \cdot_2)\|_{L^p(\mathbb{R}^1_{x_2})} \, \mathrm{d}t \right\|_{L^p(\mathbb{R}^1_{x_1})}$$

With the help of (3.25) and (3.26), we can control the above term by

$$\begin{aligned} & k^2 \sum_{\tau \in \mathbb{Z}} \frac{1}{(1+|\tau|)^2} \left\| \frac{1}{2^{k+n_l(\cdot_1)}} \int_{2^{k+n_l(\cdot_1)-1} \le |t| \le 5 \cdot 2^{k+n_l(\cdot_1)-1}} \|f(\cdot_1 - t, \cdot_2)\|_{L^p(\mathbb{R}^1_{x_2})} \, \mathrm{d}t \right\|_{L^p(\mathbb{R}^1_{x_1})} \\ & \lesssim k^2 \sum_{\tau \in \mathbb{Z}} \frac{1}{(1+|\tau|)^2} \left\| M_1 \left( \|f(\cdot, \cdot_2)\|_{L^p(\mathbb{R}^1_{x_2})} \right)(\cdot_1) \right\|_{L^p(\mathbb{R}^1_{x_1})} \\ & \lesssim k^2 \sum_{\tau \in \mathbb{Z}} \frac{1}{(1+|\tau|)^2} \left\| \|f(\cdot_1, \cdot_2)\|_{L^p(\mathbb{R}^1_{x_2})} \right\|_{L^p(\mathbb{R}^1_{x_1})} \\ & \lesssim k^2 \|f\|_{L^p(\mathbb{R}^2)} . \end{aligned}$$

Accordingly, we obtain (3.22), thereby completing the proof of Theorem 1.1 for $H_{u,\gamma}$. □

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