Characterization of Smooth Symbol Classes by Gabor Matrix Decay

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Abstract

For $m \in \mathbb{R}$ we consider the symbol classes S^m, $m \in \mathbb{R}$, consisting of smooth functions σ on $\mathbb{R}^{2d}$ such that $|\partial^\alpha \sigma(z)| \leq C_\alpha (1 + |z|^2)^{m/2}$, $z \in \mathbb{R}^{2d}$, and we show that can be characterized by an intersection of different types of modulation spaces. In the case $m = 0$ we recapture the Hörmander class $S^0_{0,0}$ that can be obtained by intersection of suitable Besov spaces as well. Such spaces contain the Shubin classes Γ^m_ρ, $0 < \rho \leq 1$, and can be viewed as their limit case $\rho = 0$. We exhibit almost diagonalization properties for the Gabor matrix of τ-pseudodifferential operators with symbols in such classes, extending the characterization proved by Gröchenig and Rzeszotnik (Ann Inst Fourier 58(7):2279–2314, 2008). Finally, we compute the Gabor matrix of a Born–Jordan operator, which allows to prove new boundedness results for such operators.

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1 Introduction and Results

Modulation spaces were originally introduced by Feichtinger [17] in 1983 and have revealed to be very useful in many different frameworks, which include harmonic analysis, quantum mechanics, pseudodifferential and Fourier integral operators, partial differential equations (we refer the reader to Sect. 2 for their definitions and main properties).

Several authors have studied inclusion relations of such spaces with other classical function spaces such as Besov, Triebel–Lizorkin Gelfand–Shilov spaces [25,35,39,44]. In particular, when they are considered as symbol classes for pseudodifferential or Fourier integral operators, their relationship with classical symbol spaces such as the Hörmander classes or the Shubin–Sobolev spaces has been investigated in many contributions (see e.g., [4,11,29,40] and the references therein).

In 1994 Sjöstrand [33] introduced the first symbol class via time-frequency concentration on the phase-space, the Sjöstrand class, which later revealed to be a type of modulation space. This rough symbol class have been inspired many works on pseudodifferential operators with symbols in modulation spaces (see, e.g., [2,3,11,22,23,30,36–39] and the book [11]). The contributions are so many that it is not possible to cite them all.

In [34] Sjöstrand continued his study on pseudodifferential operators with rough symbols and he also considered the symbol class object of our study. Namely, for $m \in \mathbb{R}$, let us define

$$S^m(\mathbb{R}^{2d}) = \left\{ \sigma \in \mathcal{C}^{\infty}(\mathbb{R}^{2d}) : |\partial^{\alpha}\sigma(z)| \leq C_{\alpha}\langle z \rangle^m, \quad \alpha \in \mathbb{N}^{2d}, \ z \in \mathbb{R}^{2d} \right\}, \quad (1)$$

for the definition of $\langle z \rangle^m$ see (12). Notice that this is a special instance of the class $S(w)$ introduced in [34, Formula (3.2)].

There were several papers/books in the seventies and eighties where this symbol class were considered. For example, the whole theory of the Weyl calculus, e.g. in [6] can be applied on this class.

Another work on pseudodifferential operators with symbols of the type above is due to Rochberg and Tachizawa [31]. Later, these classes were considered as spaces for symbols of Fourier integral operators [12, Remark 3.2].

For $m = 0$ we recapture the standard Hörmander class $S_{0,0}^0(\mathbb{R}^{2d})$: pseudodifferential operators with these symbols are an algebra which is closed under inversion. This claim was originally proved by Beals [1] and later recaptured by Gröchenig and Rzeszotnik [24], using time-frequency analysis; key tool was the almost diagonalization property of the related Gabor matrix.

We continue this spirit of investigation and present a characterization of pseudodifferential operators with symbols in $S^m(\mathbb{R}^{2d})$ in terms of the decay properties of the related Gabor matrix. Let us introduce the main features of this work.

For $\tau \in [0, 1]$, the (cross-)τ-Wigner distribution is the time-frequency representation defined by

$$W_\tau(f, g)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i y \omega} f(x + \tau y)\overline{g(x - (1 - \tau)y)}\, dy, \quad f, g \in \mathcal{S}(\mathbb{R}^d),$$

$$(2)$$

cf. [27]. Given any tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, the τ-pseudodifferential operator $\mathrm{Op}_\tau(\sigma)$ can be introduced weakly as

$$\langle \mathrm{Op}_\tau(\sigma)f, g \rangle = \langle \sigma, W_\tau(g, f) \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

$$(3)$$

The Weyl form $\mathrm{Op}_W(\sigma)$ of a pseudodifferential operator can be recaptured when $\tau = 1/2$, the Kohn–Nirenberg case $\mathrm{Op}_{KN}(\sigma)$ corresponds to $\tau = 0$.

Given $z = (x, \omega) \in \mathbb{R}^{2d}$, we define the related time-frequency shift acting on a function or distribution f on $\mathbb{R}^d$ as

$$\pi(z)f(t) = e^{2\pi i \omega t} f(t - x), \quad t \in \mathbb{R}^d.$$

$$(4)$$

Let us recall the definition of a Gabor frame. Given a lattice $\Lambda = A\mathbb{Z}^{2d}$, with $A \in GL(2d, \mathbb{R})$, and a non-zero window function $g \in L^2(\mathbb{R}^d)$, we define the *Gabor system*:

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}.$$

The Gabor system $\mathcal{G}(g, \Lambda)$ is called a Gabor frame, if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \le B\|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d).$$

$$(5)$$

Fix $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. The *Gabor matrix* of a linear continuous operator T from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ is defined to be

$$\langle T\pi(z)g, \pi(u)g \rangle, \quad z, u \in \mathbb{R}^{2d}.$$

$$(6)$$

This Gabor matrix can be viewed as the kernel of an integral operator, cf. Sect. 2 for details.

For $\tau \in [0, 1]$, define the change of variables

$$\mathcal{T}_\tau(z, u) = ((1 - \tau)z_1 + \tau u_1, \tau z_2 + (1 - \tau)u_2), \quad z = (z_1, z_2), u = (u_1, u_2) \in \mathbb{R}^{2d}.$$

$$(7)$$

We possess all the instruments for the characterization of $S^m(\mathbb{R}^{2d})$:

Theorem 1.1 *Consider $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and a lattice Λ such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Fix $m \in \mathbb{R}$. For any $\tau \in [0, 1]$, the following properties are equivalent:*

(i) $\sigma \in S^m\left(\mathbb{R}^{2d}\right)$.

(ii) $\sigma \in \mathcal{S}'\left(\mathbb{R}^{2d}\right)$ and for every $s \geq 0$, $0 < q \leq \infty$, there exists a function $H_\tau \in L^q_{\langle \cdot \rangle^s}(\mathbb{R}^{2d})$, with

$$\|H_\tau\|_{L^q_{\langle \cdot \rangle^s}} \leq C, \quad \forall \tau \in [0,1], \tag{8}$$

such that

$$\left|\langle \mathrm{Op}_\tau\left(\sigma\right)\pi\left(z\right)g, \pi\left(u\right)g\rangle\right| \leq H_\tau(u-z)\langle \mathcal{T}_\tau(z,u)\rangle^m, \quad \forall u, z \in \mathbb{R}^{2d}. \tag{9}$$

(iii) $\sigma \in \mathcal{S}'\left(\mathbb{R}^{2d}\right)$ and for every $s \geq 0$ there exists a sequence $h_\tau \in \ell^q_{\langle \cdot \rangle^s}(\Lambda)$ with $\|h_\tau\|_{\ell^q_{\langle \cdot \rangle^s}} \leq C$ for every $\tau \in [0,1]$, such that

$$\left|\langle \mathrm{Op}_\tau\left(\sigma\right)\pi\left(\mu\right)g, \pi\left(\lambda\right)g\rangle\right| \leq h_\tau(\lambda-\mu)\langle \mathcal{T}_\tau(\mu,\lambda)\rangle^m, \quad \forall \lambda, \mu \in \Lambda. \tag{10}$$

For the Hörmander class $S^0(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d})$, the Gabor matrix characterization for Weyl operators was shown by Gröchenig and Rzeszotnik in [24, Theorem 6.2] (see also [31]) in the case $q = \infty$. So this result can be viewed as an extension to any $0 < q \leq \infty$ and $\tau \in [0,1]$.

The central role in the proof of the result above is the characterization of the class $S^m(\mathbb{R}^{2d})$ by an intersection of weighted modulation spaces (in particular, weighted Sjöstrand classes): for $0 < q \leq \infty$,

$$S^m(\mathbb{R}^{2d}) = \bigcap_{s \geq 0} M^{\infty,q}_{\langle \cdot \rangle^{-m} \otimes \langle \cdot \rangle^s}(\mathbb{R}^{2d}),$$

cf. Lemma 2.2.

For the special case $m = 0$, the Hörmander class $S^0(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d})$ can also be represented as the intersection of Besov spaces and Hölder–Zygmund classes:

$$S^0_{0,0}(\mathbb{R}^{2d}) = \bigcap_{s \geq 0} \mathcal{C}^s(\mathbb{R}^{2d}) = \bigcap_{s \geq 0} B^{\infty,q}_s(\mathbb{R}^{2d}) = \bigcap_{s \geq 0} M^{\infty,q}_{1 \otimes \langle \cdot \rangle^s}(\mathbb{R}^{2d}),$$

cf. Lemma 2.3, which extends the characterization in [24].

Observe that S^m contains the Shubin classes Γ^m_ρ, $0 < \rho \leq 1$, defined as [32]

$$\Gamma^m_\rho(\mathbb{R}^{2d}) = \{\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2d}) : |\partial^\alpha \sigma(z)| \leq C_\alpha \langle z \rangle^{m-\rho|\alpha|}, \quad \alpha \in \mathbb{N}^{2d}, \; z \in \mathbb{R}^{2d}\},$$

and can be viewed as their limit case $\rho = 0$. The Shubin classes enjoy a symbolic calculus very useful when dealing with the corresponding pseudodifferential operators. This is not the case of $S^m(\mathbb{R}^{2d})$. Hence, the characterization in Theorem 1.1 might be an instrument to infer boundedness, composition, inversion properties of the corresponding operators in suitable function spaces, such as the modulation ones.

As a byproduct, Theorem 1.1 allows to compute the Gabor matrix decay of a Born–Jordan operator. We present some continuity properties of the latter on weighted modulation spaces, extending the work [8].

This study paves the way to other possible investigations. For instance, when the symbol σ on $\mathbb{R}^{2d}$ satisfies a Geverey-type regularity of order $s > 0$:

$$|\partial^\alpha \sigma(z)| \lesssim M(z) C^{|\alpha|} (\alpha!)^s, \quad \alpha \in \mathbb{N}^{2d}, \ z \in \mathbb{R}^{2d}, \tag{11}$$

with M any possible v-moderate weight (see Sect. 2 for its definition). These symbols were applied in [13] to investigate the sparsity of the Gabor-matrix representation of Fourier integral operators. In this case we conjecture that the right modulation spaces to be considered are of the type $M^{\infty,q}_{M \otimes e^{-\epsilon|\cdot|^{1/s}}}(\mathbb{R}^{2d})$.

Eventually, one might extend the characterization exhibited in Theorem 1.1 to Fourier integral operators of Schrödinger-type with symbols in S^m and suitable phases as in [10]. This will be the object of a further work.

The paper is organized as follows. In Sect. 2 we present the function spaces object of our study. In particular, we focus on modulation spaces and present the properties needed for our results. We then prove the characterization of the classes $S^m(\mathbb{R}^{2d})$ and in particular of the Hörmander classes $S^0_{0,0}(\mathbb{R}^{2d})$. Section 2.1 is devoted to the study of the Gabor matrix for τ-operators and Born–Jordan operators. As an application, boundedness results on modulation spaces are exhibited.

2 Function Spaces and Preliminaries

In this manuscript $\hookrightarrow$ denotes the continuous embeddings of function spaces. Recall that the conjugate exponent p' of $p \in [1, \infty]$ is defined by $1/p + 1/p' = 1$.

The notation $y\omega$ means the inner product $y \cdot \omega$, $|x|$ stands for the Euclidean norm of x and x^2 means $|x|^2$.

We denote by v a continuous, positive, submultiplicative weight function on $\mathbb{R}^d$, i.e., $v(z_1 + z_2) \leq v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^d$. We say that $w \in \mathcal{M}_v(\mathbb{R}^d)$ if w is a positive, continuous weight function on $\mathbb{R}^d$ v-moderate: $w(z_1 + z_2) \leq Cv(z_1)w(z_2)$ for all $z_1, z_2 \in \mathbb{R}^d$ (or for all $z_1, z_2 \in \mathbb{Z}^d$). We will mainly work with polynomial weights of the type

$$v_s(z) = \langle z \rangle^s = (1 + |z|^2)^{s/2}, \quad s \in \mathbb{R}, \quad z \in \mathbb{R}^d \ (\text{or } \mathbb{Z}^d). \tag{12}$$

Moreover, we limit to weights w with at most polynomial growth, that is there exist $C > 0$ and $s > 0$ such that

$$w(z) \leq C\langle z \rangle^s, \quad z \in \mathbb{R}^d. \tag{13}$$

We shall work mostly with weights on $\mathbb{R}^{2d}$ or $\mathbb{Z}^{2d}$; we define $(w_1 \otimes w_2)(x, \omega) := w_1(x)w_2(\omega)$, for w_1, w_2 weights on $\mathbb{R}^d$.

Spaces of Sequences For $0 < p \leq \infty$, $w \in \mathcal{M}_v(\mathbb{Z}^d)$, the space $\ell_w^p(\mathbb{Z}^d)$ consists of all sequences $a = (a_k)_{k \in \mathbb{Z}^d}$ for which the (quasi-)norm

$$\|a\|_{\ell_w^p} = \left(\sum_{k \in \mathbb{Z}^d} |a_k|^p w(k)^p \right)^{\frac{1}{p}}$$

(with obvious modification for $p = \infty$) is finite.

We are going to use the following inclusion relations for $w(k) = \langle k \rangle^s$, $s \geq 0$: If $0 < p_1, p_2 \leq \infty$, with

$$s_2 \leq s_1, \quad \frac{1}{p_2} + \frac{s_2}{d} < \frac{1}{p_1} + \frac{s_1}{d},$$

then

$$\ell_{\langle k \rangle^{s_2}}^{p_2}(\mathbb{Z}^d) \hookrightarrow \ell_{\langle k \rangle^{s_1}}^{p_1}(\mathbb{Z}^d). \tag{14}$$

The so-called translation and modulation operators are defined by $T_x g(y) = g(y - x)$ and $M_\omega g(y) = e^{2\pi i \omega y} g(y)$, respectively. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz class and consider $g \in \mathcal{S}(\mathbb{R}^d)$ a non-zero window function. The the short-time Fourier transform (STFT) $V_g f$ of a function/tempered distribution f in $\mathcal{S}'(\mathbb{R}^d)$ with respect to the the window g is defined by

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int e^{-2\pi i \omega y} f(y) \overline{g(y - x)} \, dy,$$

(i.e., the Fourier transform $\mathcal{F}$ applied to $f \overline{T_x g}$).

Modulation Spaces For $1 \leq p, q \leq \infty$ such spaces were introduced by H. Feichtinger in [17], then extended to $0 < p, q \leq \infty$ by Y.V. Galperin and S. Samarah in [20]. Their main properties and applications are now available in several textbooks, see for instance [11].

Definition 2.1 Fix a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a weight $w \in \mathcal{M}_v(\mathbb{R}^{2d})$ and $0 < p, q \leq \infty$. The modulation space $M_w^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the (quasi-)norm

$$\|f\|_{M_w^{p,q}} = \|V_g f\|_{L_w^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p w(x, \omega)^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}} \tag{15}$$

(obvious changes with $p = \infty$ or $q = \infty$) is finite.

They are quasi-Banach spaces (Banach spaces whenever $1 \leq p, q \leq \infty$), whose (quasi-)norm does not depend on the window g, in the sense that different non-zero window functions in $\mathcal{S}(\mathbb{R}^d)$ yield equivalent (quasi-)norms. Moreover, if

$1 \leq p, q \leq \infty$, the window class $\mathcal{S}(\mathbb{R}^d)$ can be extended to the modulation space $M_v^{1,1}(\mathbb{R}^d)$ (so-called Feichtinger algebra).

To be short, we write $M_w^p(\mathbb{R}^d)$ in place of $M_w^{p,p}(\mathbb{R}^d)$ and $M^{p,q}(\mathbb{R}^d)$ if $w \equiv 1$.

We recall the inversion formula for the STFT: assume $g \in M_v^1(\mathbb{R}^d) \setminus \{0\}$, $f \in M_w^{p,q}(\mathbb{R}^d)$, with $w \in \mathcal{M}_v(\mathbb{R}^{2d})$, then

$$f = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}^{2d}} V_g f(z) \pi(z) g \, dz , \qquad (16)$$

and the equality holds in $M_w^{p,q}(\mathbb{R}^d)$. The adjoint operator of V_g, defined by

$$V_g^* F(t) = \int_{\mathbb{R}^{2d}} F(z) \pi(z) g \, dz ,$$

maps the mixed-norm space $L_w^{p,q}(\mathbb{R}^{2d})$ into $M_w^{p,q}(\mathbb{R}^d)$. In particular, if $F = V_g f$ the inversion formula (16) can be rephrased as

$$\mathrm{Id}_{M_w^{p,q}} = \frac{1}{\|g\|_2^2} V_g^* V_g. \qquad (17)$$

We need to introduce an alternative definition of modulation spaces we shall use in the sequel. For $k \in \mathbb{Z}^d$, we denote by $\mathcal{Q}_k$ the unit closed cube centred at k. The family $\{\mathcal{Q}_k\}_{k \in \mathbb{Z}^d}$ is a covering of $\mathbb{R}^d$. We define $|\omega|_\infty := \max_{i=1,\dots,d} |\omega_i|$, for $\omega \in \mathbb{R}^d$. Consider now a smooth function $\rho : \mathbb{R}^d \to [0,1]$ satisfying $\rho(\omega) = 1$ for $|\omega|_\infty \leq 1/2$ and $\rho(\omega) = 0$ for $|\omega|_\infty \geq 3/4$. Define

$$\rho_k(\omega) = T_k \rho(\omega) = \rho(\omega - k), \quad k \in \mathbb{Z}^d, \qquad (18)$$

that is, ρ_k is the translation of ρ at k. By the assumption on ρ, we infer that $\rho_k(\omega) = 1$ for $\omega \in \mathcal{Q}_k$ and

$$\sum_{k \in \mathbb{Z}^d} \rho_k(\omega) \geq 1, \quad \forall \omega \in \mathbb{R}^d.$$

Denote by

$$\sigma_k(\omega) = \frac{\rho_k(\omega)}{\sum_{l \in \mathbb{Z}^d} \rho_l(\omega)}, \quad \omega \in \mathbb{R}^d, \, k \in \mathbb{Z}^d. \qquad (19)$$

Observe that $\sigma_k(\omega) = \sigma_0(\omega - k) \in \mathcal{D}(\mathbb{R}^d)$ and the sequence $\{\sigma_k\}_{k \in \mathbb{Z}^d}$ is a smooth partition of unity

$$\sum_{k \in \mathbb{Z}^d} \sigma_k(\omega) = 1, \quad \forall \omega \in \mathbb{R}^d.$$

For $k \in \mathbb{Z}^d$, we define the frequency-uniform decomposition operator by

$$\Box_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}. \tag{20}$$

The previous operators allow to introduce an alternative (quasi-)norm on the weighted modulation spaces $M^{p,q}_{h \otimes w}(\mathbb{R}^d)$ inspired by [43] as follows.

Proposition 2.2 *For $0 < p, q \le \infty$, $h, w \in \mathcal{M}_v(\mathbb{R}^d)$ have*

$$\|f\|_{M^{p,q}_{h \otimes w}(\mathbb{R}^d)} \asymp \left(\sum_{k \in \mathbb{Z}^d} \|\Box_k f\|^q_{L^p_h} w(k)^q \right)^{\frac{1}{q}}, \quad f \in \mathcal{S}'(\mathbb{R}^d), \tag{21}$$

with obvious modification for $q = \infty$.

Proof The case $p, q \ge 1$ is well known, see for example [11, Proposition 2.3.25]. The cases $0 < p < 1$ or $0 < q < 1$ are an easy modification of that proof. Namely, let us point out the main changes. If $0 < p \le 1$, we consider

$$\Box_k f = \mathcal{F}^{-1} \sigma_k \mathcal{F} f = \mathcal{F}^{-1} \sigma_k T_\omega \bar{\hat{\phi}} \mathcal{F} f, \quad \text{for} \quad \omega \in \mathcal{Q}_k,$$

since $T_\omega \bar{\hat{\phi}} = 1$ in supp σ_k for $\omega \in \mathcal{Q}_k$. Using Young's inequality for distributions compactly supported in the frequencies (see [28, Lemma 2.6], which holds also for $L^p_h, 0 < p \le 1$, with h being v-moderate), for $\omega \in \mathcal{Q}_k$, we obtain

$$\|\Box_k f\|_{L^p_h} \lesssim \|\mathcal{F}^{-1} \sigma_k\|_{L^p_v} \|\mathcal{F}^{-1} T_\omega \bar{\hat{\phi}} \mathcal{F} f\|_{L^p_h} \lesssim \|\mathcal{F}^{-1} T_\omega \bar{\hat{\phi}} \mathcal{F} f\|_{L^p_h}.$$

The rest of the proof is analogous to the Banach case and we leave the details to the interested reader. $\qquad\square$

An useful embedding is contained in what follows.

Proposition 2.3 *Given $0 < p_1, p_2, q_1, q_2 \le \infty$, with m, s_1, s_2 in $\mathbb{R}$, one has*

$$M^{p_1, q_1}_{\langle \cdot \rangle^m \otimes \langle \cdot \rangle^{s_1}}(\mathbb{R}^d) \hookrightarrow M^{p_2, q_2}_{\langle \cdot \rangle^m \otimes \langle \cdot \rangle^{s_2}}(\mathbb{R}^d) \tag{22}$$

if and only if

$$p_1 \le p_2 \tag{23}$$

and

$$q_1 \le q_2, \ s_1 \ge s_2 \quad or \quad q_1 > q_2, \ \frac{s_1}{d} + \frac{1}{q_1} > \frac{s_2}{d} + \frac{1}{q_2}. \tag{24}$$

Proof The Banach case when $m = 0$ was originally shown by H. Feichtinger in [17]. We use similar arguments as in that proof. The discrete modulation norm defined in (21) is given by

$$\|f\|_{M^{p,q}_{\langle\cdot\rangle^m\otimes\langle\cdot\rangle^s}} \asymp \left(\sum_{k\in\mathbb{Z}^d} \|\Box_k f\|^q_{L^p_{\langle\cdot\rangle^m}} \langle k\rangle^{sq}\right)^{\frac{1}{q}}.$$

The necessity of (23) follows from the fact that $\mathcal{F}L^{p_1}$ is locally contained in $\mathcal{F}L^{p_2}$ if and only if $p_1 \leq p_2$ (with strict inclusion if $p_1 < p_2$), cf. [5,18,28,42]. The set of conditions in (24) in turn describes the inclusions between weighted ℓ^q spaces: $\ell^{q_1}_{\langle\cdot\rangle^{s_1}} \subset \ell^{q_2}_{\langle\cdot\rangle^{s_2}}$ if and only if the indices' relations in (24) are satisfied, cf. for instance [25, Lemma 2.10]. This concludes the proof. □

We also recall the following inclusion relations, see e.g. [11, Theorem 2.4.17] or [20, Theorem 3.4]: If $p_1 \leq p_2$, $q_1 \leq q_2$ and $w_2 \lesssim w_1$, then

$$M^{p_1,q_1}_{w_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}_{w_2}(\mathbb{R}^d). \tag{25}$$

Corollary 2.4 *For $0 < q_1 \leq q_2 \leq \infty$, $d \in \mathbb{N}_+$, $m, s, r \in \mathbb{R}$, $r > s + d(1/q_1 - 1/q_2)$, we have the following continuous embeddings:*

$$M^{\infty,q_1}_{\langle\cdot\rangle^m\otimes\langle\cdot\rangle^r}(\mathbb{R}^d) \hookrightarrow M^{\infty,q_2}_{\langle\cdot\rangle^m\otimes\langle\cdot\rangle^r}(\mathbb{R}^d) \hookrightarrow M^{\infty,q_1}_{\langle\cdot\rangle^m\otimes\langle\cdot\rangle^s}(\mathbb{R}^d). \tag{26}$$

Proof The first embedding is a straightforward application of the inclusion relations in (25). The second one follows by the embedding in Proposition 2.3. □

Besov Spaces The Besov spaces are denoted by $B^{p,q}_s(\mathbb{R}^d)$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$, and defined as follows. Suppose that $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^d)$ satisfy supp $\psi_0 \subset \{\omega \in \mathbb{R}^d : |\omega| \leq 2\}$, supp $\psi \subset \{\omega \in \mathbb{R}^d : 1/2 \leq |\omega| \leq 2\}$ and $\psi_0(\omega) + \sum_{j=1}^{\infty} \psi(2^{-j}\omega) = 1$ for every $\omega \in \mathbb{R}^d$. Set $\psi_j(\omega) := \psi(2^{-j}\omega)$, $\omega \in \mathbb{R}^d$. Then the Besov space $B^{p,q}_s(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the (quasi-)norm

$$\|f\|_{B^{p,q}_s} = \left(\sum_{j=0}^{\infty} 2^{jsq}\|\mathcal{F}^{-1}(\psi_j\mathcal{F}f)\|^q_p\right)^{1/q} < \infty \tag{27}$$

(with usual modifications when $q = \infty$). Besov spaces are generalizations of both Hölder–Zygmund and Sobolev spaces, see e.g. [42]. Precisely, we recapture the Sobolev spaces when $p = q = 2$, $s \in \mathbb{R}$: $B^{2,2}_s(\mathbb{R}^d) = H^s(\mathbb{R}^d)$. For $s > 0$, $B^{\infty,\infty}_s(\mathbb{R}^d) = \mathcal{C}^s(\mathbb{R}^d)$, the Hölder–Zygmund classes, whose definition is as follows. For $s > 0$, we can write $s = n + \epsilon$, with $n \in \mathbb{N}$ and $\epsilon < 1$. Then $\mathcal{C}^s(\mathbb{R}^d)$ is the space of functions $f \in \mathcal{C}^n(\mathbb{R}^d)$ such that for each multi-index $\alpha \in \mathbb{N}^d$, with $|\alpha| = n$, the derivative $\partial^\alpha f$ satisfies the Hölder condition $|\partial^\alpha f(x) - \partial^\alpha f(y)| \leq K|x - y|^\epsilon$, for a suitable $K > 0$.

Inclusion relations between modulation and Besov spaces $B_s^{\infty,q}$ were first obtained for $1 \le q \le \infty$ (the Banach setting) in [39, Theorem 2.10] and then for $0 < q \le \infty$ in [43]: for $0 < q \le \infty$, set $\theta(q) = \min\{0, 1/q - 1\}$, then

$$B_{s+d/q}^{\infty,q}(\mathbb{R}^d) \hookrightarrow M_{1\otimes\langle\cdot\rangle^s}^{\infty,q}(\mathbb{R}^d) \hookrightarrow B_{s+d\theta(q)}^{\infty,q}(\mathbb{R}^d), \quad s \in \mathbb{R}. \tag{28}$$

2.1 Gabor Analysis of τ-Pseudodifferential Operators

For any fixed $m \in \mathbb{R}$, the class $S^m(\mathbb{R}^{2d})$ in (1) is a Fréchet space when endowed with the sequence of norms $\{|\cdot|_{N,m}\}_{N\in\mathbb{N}}$,

$$|\sigma|_{N,m} := \sup_{|\alpha|\le N} \sup_{z\in\mathbb{R}^{2d}} |\partial^\alpha \sigma(z)| \langle z \rangle^{-m}, \quad N \in \mathbb{N}. \tag{29}$$

For $n \in \mathbb{N}$, $m \in \mathbb{R} \setminus \{0\}$, we define by $\mathcal{C}_m^n(\mathbb{R}^{2d})$ the space of functions having n derivatives and satisfying (29) for $N = n$, whereas $\mathcal{C}^n(\mathbb{R}^{2d})$ is the space of functions with n bounded derivatives. Clearly we have the equalities

$$S^m(\mathbb{R}^{2d}) = \bigcap_{n\ge 0} \mathcal{C}_m^n(\mathbb{R}^{2d}), \, m \in \mathbb{R} \setminus \{0\}, \quad S^0(\mathbb{R}^{2d}) = \bigcap_{n\ge 0} \mathcal{C}^n(\mathbb{R}^{2d}).$$

A characterization of the class $S^0(\mathbb{R}^{2d}) = S_{0,0}^0(\mathbb{R}^{2d})$ with modulation spaces was announced by Toft in [41, Remark 3.1] and proved in [24, Lemma 6.1].

Lemma 2.1 *We have the equalities*

$$\bigcap_{n\ge 0} \mathcal{C}^n(\mathbb{R}^d) = \bigcap_{s\ge 0} M_{1\otimes\langle\cdot\rangle^s}^{\infty}(\mathbb{R}^d) = \bigcap_{s\ge 0} M_{1\otimes\langle\cdot\rangle^s}^{\infty,1}(\mathbb{R}^d). \tag{30}$$

Hence $S^0(\mathbb{R}^{2d}) = \bigcap_{s\ge 0} M_{1\otimes\langle\cdot\rangle^s}^{\infty}(\mathbb{R}^{2d}) = \bigcap_{s\ge 0} M_{1\otimes\langle\cdot\rangle^s}^{\infty,1}(\mathbb{R}^{2d}).$

In what follows we extend the previous outcome to all the classes $S^m(\mathbb{R}^{2d})$, $m \in \mathbb{R}$.

Lemma 2.2 *For $m \in \mathbb{R}$, $0 < q \le \infty$, $n \in \mathbb{N}$, $s \in (0, +\infty)$, we have the equalities of Fréchet spaces*

$$S^m(\mathbb{R}^{2d}) = \bigcap_{n\ge 0} \mathcal{C}_m^n(\mathbb{R}^{2d}) = \bigcap_{n\ge 0} M_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}^{\infty,q}(\mathbb{R}^{2d}) = \bigcap_{s\ge 0} M_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}^{\infty,q}(\mathbb{R}^d) \tag{31}$$

with equivalent families of (quasi-)norms

$$\{|\cdot|_{n,m}\}_{n\in\mathbb{N}}, \quad \left\{\|\cdot\|_{M_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}^{\infty,q}}\right\}_{n\in\mathbb{N}}, \quad \left\{\|\cdot\|_{M_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}^{\infty,q}}\right\}_{s\ge 0}. \tag{32}$$

In particular, for every $n \in \mathbb{N}$,

$$\|f\|_{M_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}^{\infty}} \le C(n,m) |f|_{n,m}. \tag{33}$$

Proof The equality $S^m(\mathbb{R}^{2d}) = \bigcap_{n\geq 0} M^{\infty,1}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}(\mathbb{R}^{2d})$ was proved in [26, Remark 2.18]. The embeddings in (26) then give the equalities in (31) with the equivalent families of (quasi-)norms in (32). Let us show the estimate (33). For $f \in \mathcal{C}^n_m(\mathbb{R}^d)$ ($\mathcal{C}^n(\mathbb{R}^d)$ if $m = 0$) and any multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq n$, we consider the function $\partial^\alpha(f T_x \bar{g})$. Taking its Fourier transform we get

$$\mathcal{F}(\partial^\alpha(f T_x \bar{g}))(\omega) = (2\pi i\omega)^\alpha \mathcal{F}(f T_x \bar{g})(\omega) = (2\pi i\omega)^\alpha V_g f(x,\omega). \qquad (34)$$

In what follows we use the boundedness of $\mathcal{F}: L^1(\mathbb{R}^d) \to \mathcal{C}_0(\mathbb{R}^d)$, Peetre's inequality $\langle x\rangle^{-m} \leq 2^{-m}\langle x-t\rangle^{|m|}\langle t\rangle^{-m}$, and Leibniz' formula:

$$
\begin{aligned}
\langle x\rangle^{-m}\|\mathcal{F}(\partial^\alpha(f T_x\bar{g}))\|_\infty &\leq \langle x\rangle^{-m}\|\partial^\alpha(f T_x\bar{g})\|_1 \\
&= \left\|\langle x\rangle^{-m}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial^\beta f\, T_x\partial^{\alpha-\beta}\bar{g}\right\|_1 \\
&\leq 2^{-m}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\|(\partial^\beta f)\langle\cdot\rangle^{-m}\|_\infty\|(\partial^{\alpha-\beta}\bar{g})\langle\cdot\rangle^{|m|}\|_1 \\
&\leq 2^{-m}\sup_{|\beta|\leq n}\|(\partial^\beta f)\langle\cdot\rangle^{-m}\|_\infty M_\alpha\max_{\beta\leq\alpha}\binom{\alpha}{\beta}\|(\partial^{\alpha-\beta}\bar{g})\langle\cdot\rangle^{|m|}\|_1 \\
&= C_{\alpha,g,m}|f|_{n,m},
\end{aligned}
$$

where $C_{g,\alpha,m} = 2^{-m}M_\alpha\max_{\beta\leq\alpha}\binom{\alpha}{\beta}\|(\partial^{\alpha-\beta}\bar{g})\langle\cdot\rangle^{|m|}\|_1$ with $M_\alpha = \#\{\beta \in \mathbb{N}^d, \beta \leq \alpha\}$. The estimate above and formula (34) yield

$$\sup_{x\in\mathbb{R}^d} |V_g f(x,\omega)|\langle x\rangle^{-m} \leq C_{g,\alpha,m}|f|_{n,m}|\omega^\alpha|^{-1}, \quad |\omega| \neq 0, \quad \forall|\alpha| \leq n. \qquad (35)$$

Now if $f \in \bigcap_{n\geq 0}\mathcal{C}^n_m(\mathbb{R}^d)$ then for every $\alpha \in \mathbb{N}^d$ there exists $C = C_\alpha > 0$ such that the estimate in (35) holds true. Since $\langle\omega\rangle^n \leq \sum_{|\alpha|\leq n}c_\alpha|\omega^\alpha|$ for suitable $c_\alpha \geq 0$, we obtain

$$\sup_{x,\omega\in\mathbb{R}^d} |V_g f(x,\omega)|\langle x\rangle^{-m}\langle\omega\rangle^n \leq C|f|_{n,m}, \quad \forall n \geq 0$$

for a suitable $C = C(n,m) > 0$ that is (33). $\qquad\square$

     In particular, for $m = 0$ we recapture the outcome of Lemma 2.1.

     For the case $m = 0$ we can characterize the Hörmander class $S^0(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d})$ by Hölder–Zygmund classes $\mathcal{C}^s(\mathbb{R}^{2d}) = B^{\infty,\infty}_s(\mathbb{R}^{2d})$ and by Besov spaces.

Lemma 2.3 *For $0 < q \leq \infty$, we have the equalities*

$$S^0_{0,0}(\mathbb{R}^{2d}) = \bigcap_{s\geq 0}\mathcal{C}^s(\mathbb{R}^{2d}) = \bigcap_{s\geq 0}B^{\infty,q}_s(\mathbb{R}^{2d}) = \bigcap_{s\geq 0}M^{\infty,q}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d}), \qquad (36)$$

with equivalent families of (quasi-)norms

$$\{\|\cdot\|_{B_s^{\infty,\infty}}\}_{s\geq 0}, \quad \{\|\cdot\|_{B_s^{\infty,q}}\}_{s\geq 0}, \quad \{\|\cdot\|_{M_{1\otimes\langle\cdot\rangle^s}^{\infty,q}}\}_{s\geq 0}. \tag{37}$$

Proof It is a straightforward consequence of Lemma 2.2 and the inclusion relations in (28). □

3 Gabor Matrix Decay

Let us first represent the Gabor matrix as a kernel of an integral operator. Consider a linear and bounded operator T from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$. The inversion formula (17) for $g \in M_v^1(\mathbb{R}^d)$, $\|g\|_2 = 1$ is simply $V_g^* V_g = \mathrm{Id}$. The operator T can be written as

$$T = V_g^* V_g T V_g^* V_g. \tag{38}$$

The linear transformation $V_g T V_g^*$ is an integral operator with kernel K_T given by the Gabor matrix of T:

$$K_T(u, z) = \langle T\pi(z)g, \pi(u)g\rangle, \quad u, z \in \mathbb{R}^{2d}.$$

By definition and the inversion formula, V_g is bounded from $M_w^{p,q}(\mathbb{R}^d)$ to $L_w^{p,q}(\mathbb{R}^{2d})$ and V_g^* from $L_w^{p,q}(\mathbb{R}^{2d})$ to $M_w^{p,q}(\mathbb{R}^d)$. Hence the continuity properties of T on modulation spaces can be obtained by the corresponding ones of the operator $V_g T V_g^*$ on mixed-norm $L_w^{p,q}$ spaces. These issues will be studied in Proposition 3.5 and Corollary 3.10 and can be achieved by studying the Gabor matrix decay of T.

First, we focus on the characterization of the Gabor matrix of $\mathrm{Op}_\tau(\sigma)$.

Proposition 3.1 *Consider* $0 < p, q \leq \infty$, $\tau \in [0, 1]$, $w \in \mathcal{M}_v(\mathbb{R}^{4d})$ *satisfying* (13), $G \in \mathcal{S}(\mathbb{R}^{2d}) \setminus \{0\}$, $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ *and define* $\Phi_\tau := W_\tau(g, g)$. *Then there exist* $A = A(v, g, G) > 0$, $B = B(v, g, G) > 0$ *such that*

$$A\|V_G\sigma\|_{L_w^{p,q}} \leq \|V_{\Phi_\tau}\sigma\|_{L_w^{p,q}} \leq B\|V_G\sigma\|_{L_w^{p,q}}, \tag{39}$$

for every $\tau \in [0, 1]$ *and* $\sigma \in M_w^{p,q}(\mathbb{R}^{2d})$.

Proof By Proposition 2.2 and Remark 2.3 in [16] the mapping

$$(\tau, f, g) \mapsto W_\tau(f, g)$$

is continuous from $\mathbb{R} \times \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^{2d})$ and locally uniformly bounded. Since Φ_τ for $\tau \in [0, 1]$ belongs to a bounded set in $\mathcal{S}(\mathbb{R}^{2d})$, the result follows immediately from [21, Theorem 11.3.7] for $p, q \geq 1$ and [20, Theorem 3.1] for $0 < p, q \leq \infty$. □

Finally, we need the following result for τ-pseudodifferential operators [14, Lemma 4.1].

Lemma 3.1 *Fix a window $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and define $\Phi_\tau = W_\tau(g, g)$ for $\tau \in [0, 1]$. Then, for $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$,*

$$\left| \langle \mathrm{Op}_\tau(\sigma) \pi(z) g, \pi(u) g \rangle \right| = \left| V_{\Phi_\tau} \sigma(\mathcal{T}_\tau(z, u), J(u - z)) \right|. \tag{40}$$

where $z = (z_1, z_2)$, $u = (u_1, u_2)$, the operator $\mathcal{T}_\tau$ is defined in (7) and J is given by

$$J(z) = (z_2, -z_1).$$

We are ready to state the characterization of τ-operators with symbols in $M^{\infty, q}_{\langle \cdot \rangle^{-m} \otimes \langle \cdot \rangle^s}(\mathbb{R}^{2d})$.

Theorem 3.2 *Consider $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and a lattice $\Lambda \subset \mathbb{R}^{2d}$ such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. For $\tau \in [0, 1]$, let $\mathcal{T}_\tau$ be the linear transformation defined in (7). For any $s, m \in \mathbb{R}$, $0 < q \leq \infty$, the following properties are equivalent:*

(i) $\sigma \in M^{\infty, q}_{\langle \cdot \rangle^{-m} \otimes \langle \cdot \rangle^s}(\mathbb{R}^{2d})$.
(ii) $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a function $H_\tau \in L^q_{\langle \cdot \rangle^s}(\mathbb{R}^{2d})$ satisfying (8) such that*

$$\left| \langle \mathrm{Op}_\tau(\sigma) \pi(z) g, \pi(u) g \rangle \right| \leq H_\tau(u - z) \langle \mathcal{T}_\tau(z, u) \rangle^m, \qquad \forall u, z \in \mathbb{R}^{2d}. \tag{41}$$

(iii) $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a sequence $h_\tau \in \ell^q_{\langle \cdot \rangle^s}(\Lambda)$ with $\|h_\tau\|_{\ell^q_{\langle \cdot \rangle^s}} \leq C$, for every $\tau \in [0, 1]$ such that*

$$\left| \langle \mathrm{Op}_\tau(\sigma) \pi(\mu) g, \pi(\lambda) g \rangle \right| \leq C h_\tau(\lambda - \mu) \langle \mathcal{T}_\tau(\mu, \lambda) \rangle^m, \qquad \forall \lambda, \mu \in \Lambda. \tag{42}$$

Proof The proof follows the pattern of the corresponding one for Weyl operators with symbols in weighted Sjöstrand's classes [22, Theorem 3.2].

$(i) \Rightarrow (ii)$ This implication comes easily from the characterization (40). In details, observing that $\langle Ju \rangle = \langle u \rangle$,

$$
\begin{aligned}
\left| \langle \mathrm{Op}_\tau(\sigma) \pi(z) g, \pi(u) g \rangle \right| &= \left| V_{\Phi_\tau} \sigma(\mathcal{T}_\tau(z, u), J(u - z)) \right| \\
&\leq \sup_{w \in \mathbb{R}^{2d}} \left(|V_{\Phi_\tau} \sigma|(w, J(u - z)) |\langle w \rangle^{-m}) \langle \mathcal{T}_\tau(z, u) \rangle^m \right. \\
&= H_\tau(u - z) \langle \mathcal{T}_\tau(z, u) \rangle^m,
\end{aligned}
$$

where

$$H_\tau(u) := \sup_{w \in \mathbb{R}^{2d}} \left(|V_{\Phi_\tau} \sigma|(w, Ju) |\langle w \rangle^{-m} \right).$$

For $0 < q < \infty$,

$$\|H_\tau\|_{L^q_{\langle \cdot \rangle^s}} = \left(\int_{\mathbb{R}^{2d}} \left[\sup_{w \in \mathbb{R}^{2d}} \left(|V_{\Phi_\tau} \sigma|(w, Ju) |\langle w \rangle^{-m} \right) \right]^q \langle u \rangle^{qs} du \right)^{\frac{1}{q}} \asymp \|\sigma\|_{M^{\infty, q}_{\langle \cdot \rangle^{-m} \otimes \langle \cdot \rangle^s}},$$

Hence by Proposition 3.1 we obtain the estimate (8). The case $q = \infty$ is analogous. $(ii) \Rightarrow (i)$ Consider the change of variables $y = \mathcal{T}_\tau(z, u)$ and $t = J(u - z)$, so that

$$\begin{cases} z(y, t) = y - U_\tau J^{-1} t \\ u(y, t) = y + (I_{2d} - U_\tau) J^{-1} t \end{cases}, \qquad U_\tau z := \begin{bmatrix} \tau I_d & 0 \\ 0 & (1 - \tau) I_d \end{bmatrix} z = \mathcal{T}_\tau(0, z) \tag{43}$$

and $u(y, t) - z(y, t) = J^{-1} t$. For $0 < q < \infty$, using (40) and (41),

$$\|\sigma\|_{M^{\infty,q}_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^s}} \asymp \left(\int_{\mathbb{R}^{2d}} \left(\sup_{y \in \mathbb{R}^{2d}} \left| V_{\Phi_\tau} \sigma(y, t) \right| \langle y \rangle^{-m} \right)^q \langle t \rangle^{qs} dt \right)^{\frac{1}{q}}$$

$$= \left(\int_{\mathbb{R}^{2d}} \left(\sup_{y \in \mathbb{R}^{2d}} \left| \langle \mathrm{Op}_\tau(\sigma) \pi(z(y, t)) g, \pi(u(y, t)) g \rangle \right| \langle \mathcal{T}_\tau(z, u) \rangle^{-m} \right)^q \right.$$

$$\left. \times \langle t \rangle^{qs} dt \right)^{\frac{1}{q}}$$

$$\leq \left(\int_{\mathbb{R}^{2d}} \left| H_\tau(J^{-1} t) \right|^q \langle t \rangle^{qs} dt \right)^{\frac{1}{q}}$$

$$\leq C,$$

where we used (8). The case $q = \infty$ is analogous.
$(ii) \Leftrightarrow (iii)$ The argument requires that $\mathcal{G}(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Then the equivalence can be proved similarly to [10, Theorem 3.1] and [22, Theorem 3.2]. $\qquad \square$

The proof of the characterization of the symbol classes $S^m(\mathbb{R}^{2d})$ claimed in Theorem 1.1, can be inferred easily from the result above.

Proof of Theorem 1.1 The proof is a direct application of the characterization of the classes $S^m(\mathbb{R}^{2d})$ presented in (31) and Theorem 3.2. $\qquad \square$

The following issue is an improvement of [7, Theorem 2.4] and relies on the new characterization of $S^m(\mathbb{R}^{2d})$ proved in Lemma 2.2.

Proposition 3.3 *Consider* $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, $m \in \mathbb{R}$ *and* $\sigma \in S^m(\mathbb{R}^{2d})$. *For any* $n \in \mathbb{N}$ *there exists* $C = C(n) > 0$, *which does not depend on* σ *or* τ, *such that*

$$\left| \langle \mathrm{Op}_\tau(\sigma) \pi(z) g, \pi(u) g \rangle \right| \leq C |\sigma|_{n,m} \frac{\langle \mathcal{T}_\tau(z, u) \rangle^m}{\langle u - z \rangle^n}, \qquad \forall \tau \in [0, 1], \ \forall u, z \in \mathbb{R}^{2d}. \tag{44}$$

Proof Using the characterization of the Hörmander classes $S^m(\mathbb{R}^{2d})$ in (31) we infer that $\sigma \in M^\infty_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}(\mathbb{R}^{2d})$ and, for any $n \in \mathbb{N}$, the norm estimate in (33) says that there exists $C = C(n, m)$ such that

$$\|\sigma\|_{M^\infty_{\langle\cdot\rangle^{-m}\otimes\langle\cdot\rangle^n}} \leq C(n, m) |\sigma|_{n,m}, \tag{45}$$

where $C(n, m) > 0$ is independent of σ. For $z, w \in \mathbb{R}^{2d}$ we use Lemma 3.1 and the norm estimate in (45) which yield

$$
\left| \langle \mathrm{Op}_\tau (\sigma) \pi(z) g, \pi(u) g \rangle \right| = \left| V_{\Phi_\tau} \sigma \left(\mathcal{T}_\tau (z, u), J(u - z) \right) \right|
$$
$$
\leq C |\sigma|_{n,m} \frac{\langle \mathcal{T}_\tau(z, u) \rangle^m}{\langle u - z \rangle^n},
$$

that is the desired result. $\qquad\square$

For $s \in [0, +\infty) \setminus \mathbb{N}$, the estimate reads as follows.

Proposition 3.4 *Consider* $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, $\tau \in [0, 1]$, $m \in \mathbb{R}$ *and* $\sigma \in S^m (\mathbb{R}^{2d})$. *For any* $s \in [0, +\infty) \setminus \mathbb{N}$ *there exists* $C = C(s, m) > 0$, *which does not depend on* σ *or* τ, *such that*

$$
\left| \langle \mathrm{Op}_\tau (\sigma) \pi(z) g, \pi(u) g \rangle \right| \leq C |\sigma|_{n+1,m} \frac{\langle \mathcal{T}_\tau(z, u) \rangle^m}{\langle u - z \rangle^s}, \qquad \forall u, z \in \mathbb{R}^{2d}, \quad (46)
$$

where $n = [s]$ *is the integer part of* s.

Proof The result is attained by the the same argument as Proposition 3.3 and the inclusion relations between modulation spaces in (25). $\qquad\square$

3.1 Boundedness Results

The characterization of the class S^m in Lemma 2.2 and Theorem 3.1 are the key tool for boundedness properties of τ-operators on weighted modulation spaces.

Proposition 3.5 *Consider* $\tau \in [0, 1]$, $m \in \mathbb{R}$, $\sigma \in S^m(\mathbb{R}^{2d})$, $0 < p, q \leq \infty$. *Then* $\mathrm{Op}_\tau(\sigma)$, *from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{S}'(\mathbb{R}^d)$, *extends uniquely to a bounded operator*

$$
\mathrm{Op}_\tau(\sigma) \colon M^{p,q}_{\langle \cdot \rangle^{r+m}}(\mathbb{R}^d) \to M^{p,q}_{\langle \cdot \rangle^r}(\mathbb{R}^d),
$$

for every $r \in \mathbb{R}$.

Proof Choose $g \in \mathcal{S}(\mathbb{R}^d)$ and a lattice Λ such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Define $t := \min\{1, p, q\}$ and choose $s > (2d + |r|)/t$. Using the equivalent discrete (quasi-)norm for the modulation space, see e.g. [40, Proposition 1.5], the estimate in (42) and Young's convolution inequality in [19, Theorem 3.1], we obtain the result. Namely,

$$
\| \mathrm{Op}_\tau(\sigma) f \|_{M^{p,q}_{\langle \cdot \rangle^r}} \asymp \| V_g(\mathrm{Op}_\tau(\sigma) f) \|_{\ell^{p,q}_{\langle \cdot \rangle^r}(\Lambda)} \leq \left\| h_\tau * |V_g f| \langle \cdot \rangle^{|m|} \right\|_{\ell^{p,q}_{\langle \cdot \rangle^r}(\Lambda)}
$$
$$
\leq \| h_\tau \|_{\ell^t_{\langle \cdot \rangle^s}(\Lambda)} \left\| V_g f \langle \cdot \rangle^m \right\|_{\ell^{p,q}_{\langle \cdot \rangle^r}(\Lambda)} \leq C \| f \|_{M^{p,q}_{\langle \cdot \rangle^{r+m}}}.
$$

Alternatively, since $\sigma \in S^m = \bigcap_{s \geq 0} M^{\infty,q}_{\langle \cdot \rangle^{-m} \otimes \langle \cdot \rangle^s}(\mathbb{R}^{2d})$ by Lemma 2.2, one can use [40, Theorem 3.1] with $p = \infty$ and $q \leq 1$ small enough to yield the claim. $\qquad\square$

Remark 3.6 (i) For $\sigma \in S^0(\mathbb{R}^{2d}) = S_{0,0}^0(\mathbb{R}^{2d})$ and we recapture the continuity of

$$\mathrm{Op}_\tau(\sigma) \colon M_{\langle \cdot \rangle^r}^{p,q}(\mathbb{R}^d) \to M_{\langle \cdot \rangle^r}^{p,q}(\mathbb{R}^d).$$

This was already shown in [39] for $p, q \geq 1$, for the quasi-Banach cases see [40]. (ii) For $p = q = 2$ we have the continuity between the Shubin–Sobolev spaces $Q_{r+m}(\mathbb{R}^d)$ and $Q_r(\mathbb{R}^d)$.

Corollary 3.7 *Consider* $\tau \in [0, 1]$, $m, r \in \mathbb{R}$, $\sigma \in S^m(\mathbb{R}^{2d})$, $0 < p, q \leq \infty$. *Let* $\|\mathrm{Op}_\tau(\sigma)\|$ *denote the norm of* $\mathrm{Op}_\tau(\sigma)$ *in* $B(M_{\langle \cdot \rangle^{r+m}}^{p,q}(\mathbb{R}^d), M_{\langle \cdot \rangle^r}^{p,q}(\mathbb{R}^d))$. *Then there exists a constant* $C > 0$ *such that*

$$\|\mathrm{Op}_\tau(\sigma)\| \leq C, \qquad \forall \tau \in [0, 1]. \tag{47}$$

Proof The claim is evident from proof of Proposition 3.5. $\qquad\qquad\square$

3.2 Born–Jordan Operators

The Born–Jordan operator with symbol $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ can be defined as

$$\langle \mathrm{Op}_{BJ}(\sigma) f, g \rangle = \langle \sigma, W_{BJ}(g, f) \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d),$$

where the Born–Jordan distribution $W_{BJ}(g, f)$ is

$$W_{BJ}(g, f) = \int_0^1 W_\tau(g, f) \, d\tau,$$

see, e.g., the textbook [15]. In what follows we study the Gabor matrix decay for Born–Jordan operators.

Theorem 3.8 *Consider* $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. *For* $m \in \mathbb{R}$ *consider* $\sigma \in S^m(\mathbb{R}^{2d})$. *Then for every* $s \geq 0$, $0 < q \leq \infty$, $\tau \in [0, 1]$ *there exists a function* $H_\tau \in L_{\langle \cdot \rangle^s}^q(\mathbb{R}^{2d})$ *which satisfies* (8) *and such that*

$$\left| \langle \mathrm{Op}_{BJ}(\sigma) \pi(z) g, \pi(u) g \rangle \right| \leq \langle z \rangle^m \int_0^1 H_\tau(u - z) \, d\tau, \qquad \forall u, z \in \mathbb{R}^{2d}. \tag{48}$$

Proof For $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, $\mathrm{Op}_{BJ}(\sigma)$ is linear and continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$, see [16]. For $z, u \in \mathbb{R}^{2d}$, $\sigma \in S^m(\mathbb{R}^{2d})$ and $g \in \mathcal{S}(\mathbb{R}^d)$ we compute

$$\langle \mathrm{Op}_{BJ}(\sigma) \pi(z) g, \pi(u) g \rangle = \langle \sigma, W_{BJ}(\pi(u)g, \pi(z)g) \rangle$$
$$= \int_{\mathbb{R}^{2d}} \sigma(y) \int_0^1 \overline{W_\tau(\pi(u)g, \pi(z)g)(y)} \, d\tau dy =: I.$$

From [16, Proposition 2.2, Remark 2.3] we have that the mapping

$$\mathbb{R} \times \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^{2d}), \quad (t, \varphi, \psi) \mapsto W_t(\varphi, \psi)$$

is continuous and locally uniformly bounded. Thus $W_{BJ}(\varphi, \psi) \in \mathcal{S}(\mathbb{R}^{2d})$ and the integral I is absolutely convergent, so that

$$I = \int_0^1 \int_{\mathbb{R}^{2d}} \sigma(y) \overline{W_\tau(\pi(u)g, \pi(z)g)(y)} \, dy d\tau = \int_0^1 \langle \mathrm{Op}_\tau(\sigma) \pi(z) g, \pi(u) g \rangle \, d\tau.$$

By Peetre's inequality:

$$\langle \mathcal{T}_\tau(z, u) \rangle^m = \langle z_1 + \tau(u_1 - z_1), z_2 + (1 - \tau)(u_2 - z_2) \rangle^m$$
$$\lesssim \langle z \rangle^m \langle u - z \rangle^{|m|},$$

for every $u = (u_1, u_2), z = (z_1, z_2) \in \mathbb{R}^{2d}$. Hence, using Theorem 1.1,

$$|I| \leq \int_0^1 \left| \langle \mathrm{Op}_\tau(\sigma) \pi(z) g, \pi(u) g \rangle \right| \, d\tau \lesssim \int_0^1 H_\tau(u - z) \langle u - z \rangle^{|m|} \, d\tau \langle z \rangle^m.$$

Then the function $H_\tau(z) \langle z \rangle^{|m|}$ satisfies condition (8). $\qquad\square$

Remark 3.9 (i) For $q \geq 1$, we can define $H(z) := \int_0^1 H_\tau(z) d\tau$. Using Minkowski's integral inequality we infer $H \in L^q_{(\cdot)^s}(\mathbb{R}^{2d})$ and the estimate (48) becomes

$$\left| \langle \mathrm{Op}_{BJ}(\sigma) \pi(z) g, \pi(u) g \rangle \right| \leq H(u - z) \langle z \rangle^m, \qquad \forall u, z \in \mathbb{R}^{2d}.$$

Notice that for $0 < q < 1$ Minkowski's integral inequality is not true in general. (ii) Arguing as in Theorem 3.8, we may discretize the Gabor matrix decay in (48) as follows: consider $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and a lattice Λ in $\mathbb{R}^{2d}$ such that $\mathcal{G}(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. If $\sigma \in S^m(\mathbb{R}^{2d})$ then for every $s \geq 0, 0 < q \leq \infty$, there exists a sequence $h_\tau \in \ell^q_{(\cdot)^s}(\Lambda)$ with $\|h_\tau\|_{\ell^q_{(\cdot)^s}} \leq C$ for every $\tau \in [0, 1]$ such that

$$\left| \langle \mathrm{Op}_{BJ}(\sigma) \pi(\mu) g, \pi(\lambda) g \rangle \right| \leq \langle \mu \rangle^m \int_0^1 h_\tau(\lambda - \mu) d\tau, \qquad \forall \lambda, \mu \in \Lambda.$$

Corollary 3.10 *Consider $m \in \mathbb{R}, \sigma \in S^m(\mathbb{R}^{2d}), 0 < p, q \leq \infty$. Then $\mathrm{Op}_{BJ}(\sigma)$, from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, extends uniquely to a bounded operator*

$$\mathrm{Op}_{BJ}(\sigma) \colon M^{p,q}_{(\cdot)^{r+m}}(\mathbb{R}^d) \to M^{p,q}_{(\cdot)^r}(\mathbb{R}^d),$$

for every $r \in \mathbb{R}$.

Proof The proof is similar to the one of Proposition 3.5, using the decay for Gabor matrix of $\mathrm{Op}_{BJ}(\sigma)$ found in Theorem 3.8, with h_τ replaced by $\int_0^1 h_\tau(\cdot)d\tau$. Then, for $t \geq 1$ we use Minkowski's inequality to write

$$\left\| \int_0^1 h_\tau(\cdot)d\tau \right\|_{\ell^t_{(\cdot)^s}} \leq \int_0^1 \|h_\tau\|_{\ell^t_{(\cdot)^s}} d\tau \leq C.$$

For $t < 1$ we use the inclusion relations (14) and majorize

$$\left\| \int_0^1 h_\tau(\cdot)d\tau \right\|_{\ell^t_{(\cdot)^s}} \lesssim \left\| \int_0^1 h_\tau(\cdot)d\tau \right\|_{\ell^1_{(\cdot)^{\tilde{s}}}},$$

with $\tilde{s} \geq 0$ such that $1/t + s/(2d) < 1 + \tilde{s}/(2d)$, that is

$$\tilde{s} > \frac{2d}{t}(1 - t),$$

and we proceed as above. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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