



Localised Module Frames and Wannier Bases from Groupoid Morita Equivalences

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Received: 10 September 2020 / Revised: 5 May 2021 / Accepted: 24 June 2021 /
Published online: 4 August 2021

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Abstract

Following the operator algebraic approach to Gabor analysis, we construct frames of translates for the Hilbert space localisation of the Morita equivalence bimodule arising from a groupoid equivalence between Hausdorff groupoids, where one of the groupoids is étale and with a compact unit space. For finitely generated and projective submodules, we show these frames are orthonormal bases if and only if the module is free. We then apply this result to the study of localised Wannier bases of spectral subspaces of Schrödinger operators with atomic potentials supported on (aperiodic) Delone sets. The noncommutative Chern numbers provide a topological obstruction to fast-decaying Wannier bases and we show this result is stable under deformations of the underlying Delone set.

Keywords Operator algebras · Groupoid and Morita equivalence · Gabor analysis · Wannier basis

Mathematics Subject Classification 46L08 · 46L55 · 81R60

1 Introduction

A key question in time-frequency analysis and related fields is the reconstruction of elements in a Hilbert space \mathfrak{h} via a set of vectors $\{T_j w_1, \dots, T_j w_m\}_{j \in J}$ spanning \mathfrak{h} and where $\{T_j\}_{j \in J} \subset \mathcal{B}(\mathfrak{h})$ is some canonically defined set of operators. An important

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example are (multi-window) Gabor frames, where given a locally compact and abelian group G , a frame on $L^2(G)$ is constructed via translation and modulation operators from a closed subset $\Lambda \subset G \times \widehat{G}$ on a window function. Morita equivalence bimodules of C^* -algebras, also called imprimitivity bimodules, have been shown to be a useful tool in the construction of such Gabor frames [1,2,27]. One may also consider frames generated from translations by elements of a discrete group or étale groupoid, which we call *frames of translates*, cf. [14, Chap. 9] or [8].

Closely related to frames of translates and arising from physics are Wannier bases. Given a Schrödinger-type operator H with periodic potential acting on $L^2(\mathbb{R}^d, \mathbb{C}^n)$, a Wannier basis is an orthonormal basis of a spectral subspace of H constructed from a finite set of functions along with their translations in \mathbb{Z}^d . Because the operator H has a periodic potential, such bases exist by the Bloch–Floquet transform. The regularity of Wannier bases changes drastically depending on the topological properties of the spectral band of the Schrödinger operator, where a delocalised Wannier basis can be used as an indicator that the system has a non-trivial topological phase, see [15,16,24,29,32] for example. Wannier bases with exponential decay can be constructed for periodic and aperiodic Hamiltonians such that the compression of a position operator by the Fermi projection has uniform spectral gaps [40,41].

In the context of periodic systems with a space group translation symmetry $G \subset \mathbb{R}^d$, it was shown by Ludewig and Thiang that the existence of a fast-decaying Wannier basis is equivalent to whether a finitely generated and projective $C_r^*(G)$ -module is free or not [26].

The purpose of this paper is twofold. First, using a similar framework to [2], we study the relation between frames of translates and Morita equivalence bimodules arising from groupoid equivalences. We then use this relation to extend the results of Ludewig and Thiang [26] on fast-decaying Wannier bases to étale groupoids. Regularity of frames is examined using pre-Morita equivalence bimodules of algebras defined from derivations and differential seminorms.

For both of our aims, our guiding example is the étale groupoid \mathcal{G}_{Del} constructed from a Delone set $\Lambda \subset \mathbb{R}^d$ and which is equivalent to the crossed product groupoid of the translation action on the orbit space of Λ , $\Omega_\Lambda \rtimes \mathbb{R}^d$. In previous work [11,12], the index theory of \mathcal{G}_{Del} and its application to aperiodic topological phases was studied. In [23], Gabor frames of $L^2(\mathbb{R}^d)$ were constructed from Delone subsets of \mathbb{R}^{2d} with finite local complexity and the groupoid \mathcal{G}_{Del} using results from [19]. Gabor duality was shown for Gabor frames constructed from model sets in [31]. We do not consider the important question of Gabor duality in this work.

Outline and Main Results

Because our results bring together constructions from time-frequency analysis, groupoids, C^* -modules and Morita equivalence, we give a brief overview of these concepts in Sect. 2. In Sect. 2.3, using the framework of differential seminorms (cf. [10]), we construct pre-Morita equivalence bimodules for pre- C^* -subalgebras obtained from a finite family of commuting unbounded $*$ -derivations.

In Sect. 3 we consider a groupoid equivalence $\mathcal{G} \leftarrow Z \rightarrow \mathcal{H}$ of Hausdorff, second countable and locally compact groupoids \mathcal{G} and \mathcal{H} , where \mathcal{H} is étale with a compact unit space $\mathcal{H}^{(0)}$. By choosing the evaluation state for some $x \in \mathcal{H}^{(0)}$ we obtain a translation action of the fibre $r^{-1}(x)$ on the Hilbert space localisation \mathfrak{h}_x of the Morita equivalence bimodule between $C_r^*(\mathcal{G})$ and $C_r^*(\mathcal{H})$. This allows us to construct a normalised tight frame of translates for \mathfrak{h}_x from the C^* -module frame of the Morita equivalence bimodule. Restricting to finitely generated and projective $C_r^*(\mathcal{H})$ -submodules, we obtain a frame of translates with a finite generating set for a subspace of the Hilbert space localisation \mathfrak{h}_x . This frame is an orthonormal basis for all $x \in \mathcal{H}^{(0)}$ if and only if the finitely generated and projective module is free, and thus its class in the reduced K -theory of $C_r^*(\mathcal{H})$ is trivial. These results are extended in Sect. 4 to the case of abstract transversals with a normalised 2-cocycle twist.

We apply these results to the étale Delone groupoid in Sect. 5. We consider a magnetic Schrödinger operator with an atomic potential v arranged on a Delone set $\Lambda \subset \mathbb{R}^d$,

$$H_\Lambda = \sum_{j=1}^d \left(-i \frac{\partial}{\partial x_j} - A_j \right)^2 + \sum_{p \in \Lambda} v(\cdot - p),$$

where A is the magnetic potential. Results by Bellissard et al. show that for sufficiently regular v , H_Λ and its magnetic translates are affiliated to the crossed product C^* -algebra $C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)$ with θ a magnetic twist [5,6]. We show that for any Delone set \mathcal{L} in the transversal subset $\Omega_0 \subset \Omega_\Lambda$, a gapped spectral subspace of $H_{\mathcal{L}}$ has a normalised tight frame built from the magnetic translations $\{U_y\}_{y \in \mathcal{L}}$. This frame is an orthonormal basis if and only if the corresponding finitely generated and projective $C_r^*(\mathcal{G}_{\text{Del}}, \theta)$ -module is free. Using derivations of the groupoid algebras and differential seminorms, we show this normalised tight frame has faster than polynomial decay. We therefore see that topological properties of spectral subspaces of the Delone Schrödinger operator can be related to the regularity of (aperiodic) Wannier bases.

The regularity of such Wannier basis is closely related to the Localisation Dichotomy Conjecture for non-periodic insulators raised in [29, Section 5 (arXiv version)] and further studied in [25,30]. We prove a weaker version of this conjecture in Sect. 5.3 and show that the existence of Wannier bases with faster than polynomial decay is equivalent to the existence of Wannier bases such that $\sum_j (1+|x|^2)|w_j(x)|^2 \in L^1(\mathbb{R}^d)$. This in turn is equivalent to the spectral projection defining a freely generated $C_r^*(\mathcal{G}_{\text{Del}}, \theta)$ -module. Our results are weaker than those posed in [29] as we consider a family of Schrödinger operators and Hilbert spaces rather than a single Hamiltonian. Similarly we do not consider Wannier bases with exponential decay.

By relating the existence of a localised Wannier basis to a K -theoretic property, the noncommutative Chern numbers for $C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)$ and $C_r^*(\mathcal{G}_{\text{Del}}, \theta)$ studied in [11–13] give a topological obstruction to a Wannier basis with fast decay. We also show that these Chern number formulas are continuous under deformations of the magnetic field or Delone atomic potential provided the spectral gap stays open. This

implies that a non-localised Wannier basis is stable under deformations of the atomic potential (e.g. from a periodic lattice to an aperiodic point pattern).

2 Preliminaries

2.1 Hilbert Space Frames

Let us recall a few basic definitions from time-frequency analysis.

Definition 2.1 Let \mathfrak{h} be a Hilbert space and $\{g_j\}_{j \in J}$ is a collection of elements in \mathfrak{h} . We say that $\{g_j\}_{j \in J}$ form a Hilbert space frame if there are constants $C, D > 0$ such that

$$C \|\psi\|^2 \leq \sum_{j \in J} |\langle g_j, \psi \rangle|^2 \leq D \|\psi\|^2 \quad \text{for all } \psi \in \mathfrak{h}.$$

If $C = D$, then $\{g_j\}_{j \in J}$ is called a tight frame. If $C = D = 1$, then $\{g_j\}_{j \in J}$ is called a normalised tight frame or Parseval frame.

Orthonormal bases are obvious examples of normalised tight frames. The restriction of an orthonormal basis of a Hilbert space to a closed subspace yields a normalised tight frame for the subspace. Normalised tight frames always arise as a compression of an orthonormal basis.

Proposition 2.2 [20, Sect. 1] *Let $\{g_j\}_{j \in J}$ be a normalised tight frame of a Hilbert space \mathfrak{h}_1 . Then there is Hilbert space \mathfrak{h}_2 and an orthonormal basis $\{e_j\}_{j \in J}$ of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ such that $g_j = \text{pr}_1(e_j)$.*

Any Hilbert space frame yields an invertible frame operator

$$S : \mathfrak{h} \rightarrow \mathfrak{h}, \quad S(\psi) = \sum_{j \in J} g_j \langle g_j, \psi \rangle.$$

We note that, in contrast to the Gabor analysis literature, we work with inner-products that are linear on the right. This is to make our results more easily compatible with right C^* -modules and their Hilbert space localisations (Definition 2.11 below). Because S is invertible, one obtains a reconstruction formula for elements in \mathfrak{h} .

$$\psi = \sum_{j \in J} S^{-1} g_j \langle g_j, \psi \rangle, \quad \psi \in \mathfrak{h}.$$

The elements $\{S^{-1} g_j\}_{j \in J}$ are called the dual frame to $\{g_j\}_{j \in J}$.

2.2 Pre- C^* -Modules and Morita Equivalence

Following a similar framework to [2], we will use C^* -modules and Morita equivalence bimodules to study questions around frames of translates for their Hilbert space localisations. We now recall some basic definitions and establish notation. Further details can be found in [9,35].

We will also be interested in the case where the C^* -algebras A and B have dense $*$ -subalgebras \mathcal{A} and \mathcal{B} .

Definition 2.3 Let B be a C^* -algebra, $\mathcal{B} \subset B$ a dense $*$ -subalgebra and $\mathcal{E}_{\mathcal{B}}$ a vector space that is also a right \mathcal{B} -module. We say that $\mathcal{E}_{\mathcal{B}}$ is a right inner product \mathcal{B} -module if there is a sesquilinear pairing $(e_1, e_2) \mapsto (e_1 | e_2)_{\mathcal{B}} \in \mathcal{B}$ linear in the second variable such that for $e_1, e_2, e \in \mathcal{E}$

$$\begin{aligned} (e_1 | e_2)_{\mathcal{B}} &= (e_2 | e_1)_{\mathcal{B}}^*, & (e_1 | e_2 \cdot b)_{\mathcal{B}} &= (e_1 | e_2)_{\mathcal{B}} b, \\ (e | e)_{\mathcal{B}} &\geq 0 \in B, & (e | e)_{\mathcal{B}} = 0 &\Rightarrow e = 0. \end{aligned}$$

An inner product module $\mathcal{E}_{\mathcal{B}}$ is called full if $\text{span}\{(e_1 | e_2) : e_1, e_2 \in \mathcal{E}_{\mathcal{B}}\}$ is C^* -norm dense in B . If B is a C^* -algebra, a C^* -module is a right inner product B -module that is complete in the norm $\|e\|^2 := \|(e | e)_B\|_B$.

Given a C^* -algebra A and dense $*$ -subalgebra \mathcal{A} , a left inner product \mathcal{A} -module ${}_{\mathcal{A}}\mathcal{E}$ can be analogously defined, where the sesquilinear pairing $(e_1, e_2) \mapsto {}_{\mathcal{A}}(e_1 | e_2)$ is linear in the first variable and ${}_{\mathcal{A}}(a \cdot e_1 | e_2) = a {}_{\mathcal{A}}(e_1 | e_2)$. In case the algebras \mathcal{A} and \mathcal{B} and left/right inner product module structure are clear from context, we will suppress subscripts and write \mathcal{E} for ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}$.

On an inner product module $\mathcal{E}_{\mathcal{B}}$, the norm $\|e\|^2 := \|(e | e)_B\|_B$ is defined and the completion of $\mathcal{E}_{\mathcal{B}}$ in this norm is a right C^* -module E_B over B . For a right inner product \mathcal{B} -module, the $*$ -algebra of finite rank operators $\text{Fin}_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$ is defined to be the algebraic span of the finite-rank operators $\{\Theta_{e_1, e_2}^R\}_{e_1, e_2 \in \mathcal{E}}$, where

$$\Theta_{e_1, e_2}^R(e_3) = e_1 \cdot (e_2 | e_3)_{\mathcal{B}}, \quad \left(\Theta_{e_1, e_2}^R\right)^* := \Theta_{e_2, e_1}^R, \quad e_1, e_2, e \in \mathcal{E}.$$

When E_B is a C^* -module over a $*$ -algebra B , the compact endomorphisms $\mathbb{K}_B(E)$ are defined as the operator norm closure of $\text{Fin}_B(E)$. It is a closed two-sided ideal in the C^* -algebra $\text{End}_B^*(E)$ of adjointable operators on E_B .

Definition 2.4 Let $\mathcal{E}_{\mathcal{B}}$ be a right inner product \mathcal{B} -module. A set $\{e_1, \dots, e_n\}$ is called a finite module frame if

$$\text{Id}_{\mathcal{E}} = \sum_{k=1}^n \Theta_{e_k, e_k}^R.$$

If E_B is a right C^* -module, a countable subset $\{e_j\}_{j \in J} \subset E_B$ is a (right) C^* -module frame if $\sum_j \Theta_{e_j, e_j}^R$ converges strictly to Id_E .

We remark that any countably generated right C^* -module over a σ -unital algebra B admits a C^* -module frame [9,35]. If an inner product module \mathcal{E}_B admits a finite frame $\{v_j\}_{j=1}^n$, then there is a projection $p = p^* = p^2 \in M_n(B)$ and right module maps

$$S : \mathcal{E} \rightarrow \mathcal{B}^n, \quad R : \mathcal{B}^n \rightarrow \mathcal{E}, \tag{1}$$

$$S(e) = ((v_j | e)_B)_{j=1}^n, \quad R((b_j)_{j=1}^n) = \sum_{j=1}^n v_j \cdot b_j, \tag{2}$$

that restrict to isomorphisms $S : \mathcal{E} \rightarrow p\mathcal{B}^n$ and $R : p\mathcal{B}^n \rightarrow \mathcal{E}$. In particular

$$p = (v_i | v_j)_B \in M_n(B), \quad R \circ S = \text{Id}_{\mathcal{E}}, \quad e = R \circ S(e) = \sum_{j=1}^n v_j \cdot (v_j | e)_B = \sum_{j=1}^n \Theta_{v_j, v_j}^R(e).$$

Similar formulas hold for left modules with a finite frame.

Definition 2.5 Let A and B be C^* -algebras. An A - B Morita equivalence bimodule is a full right- B C^* -module and full left- A C^* -module ${}_A E_B$ such that

$$(a \cdot e_1 | e_2)_B = (e_1 | a^* \cdot e_2)_B, \quad A(e_1 | e_2 \cdot b) = A(e_1 \cdot b^* | e_2), \quad A(e_1 | e_2) \cdot e_3 = e_1 \cdot (e_2 | e_3)_B$$

for all $a \in A, b \in B$ and $e_1, e_2, e_3 \in E$. We say that A and B are Morita equivalent if there is a Morita equivalence bimodule ${}_A E_B$.

To distinguish left and right inner products, for $e_1, e_2 \in E$ we use the notation

$$\Theta_{e_1, e_2}^L(e_3) = A(e_3 | e_1) \cdot e_2, \quad \Theta_{e_1, e_2}^R(e_3) = e_1 \cdot (e_2 | e_3)_B.$$

A full right- B C^* -module is a Morita equivalence bimodule between $\mathbb{K}_B(E)$ and B with the $\mathbb{K}_B(E)$ -valued inner product $\mathbb{K}_B(E)(e_1 | e_2) = \Theta_{e_1, e_2}^R$. Hence A is Morita equivalent to B if and only if there is a full right- B C^* -module E_B and a $*$ -isomorphism $\phi : A \rightarrow \mathbb{K}_B(E)$.

Definition 2.6 Let \mathcal{A} and \mathcal{B} be dense $*$ -subalgebras of C^* -algebras A and B . A pre-Morita equivalence bimodule is an \mathcal{A} - \mathcal{B} bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}$, equipped with a full left- \mathcal{A} valued and a full right- \mathcal{B} valued inner product such that for any $a \in \mathcal{A}, b \in \mathcal{B}$ and $e, e_1, e_2, e_3 \in \mathcal{E}$ the compatibility conditions

$${}_{\mathcal{A}}(e \cdot b | e \cdot b) \leq \|b\|_{\mathcal{B}}^2 {}_{\mathcal{A}}(e | e), \quad (a \cdot e | a \cdot e)_{\mathcal{B}} \leq \|a\|_A^2 (e | e)_{\mathcal{B}}, \quad {}_{\mathcal{A}}(e_1 | e_2) \cdot e_3 = e_1 \cdot (e_2 | e_3)_{\mathcal{B}},$$

are satisfied. Here $\|\cdot\|$ denotes the C^* -norm on the algebras \mathcal{A} and \mathcal{B} , and the inequalities are in the C^* -algebras A and B .

As expected, a pre-Morita equivalence bimodule ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}$ can be completed to a Morita equivalence bimodule ${}_A E_B$, using either of the norms $\|e\|^2 := \|(e \mid e)_{\mathcal{B}}\|_B$ or $\|e\|^2 := \|{}_{\mathcal{A}}(e \mid e)\|_A$ see [35, Sect. 3.1]. We proceed with some definitions and results concerning frames in dense submodules of C^* -modules. For this we need our dense $*$ -subalgebras to be equipped with additional analytic structure.

We provide a definition of smooth subalgebra of a C^* -algebra (see for instance [18, Definitions 3.25 and 3.26]) that is flexible enough for our purposes.

Definition 2.7 We say that a $*$ -algebra \mathcal{A} is a pre- C^* -algebra if it is

- (i) Fréchet, i.e. complete and metrizable such that the multiplication is jointly continuous;
- (ii) Isomorphic to a proper dense $*$ -subalgebra $\iota(\mathcal{A})$ of a C^* -algebra A , where $\iota : \mathcal{A} \hookrightarrow A$ is the inclusion map, and $\iota(\mathcal{A})$ is stable under the holomorphic functional calculus. That is, if f is a holomorphic function on a neighbourhood of the spectrum of $a \in \iota(\mathcal{A})$, then $f(a) \in \iota(\mathcal{A})$.

Stability under the holomorphic functional calculus extends to nonunital algebras, since the spectrum of an element in a nonunital algebra is defined to be the spectrum of this element in the one-point unitization, though we must restrict to functions satisfying $f(0) = 0$. Similarly, the definition of a Fréchet algebra does not require a unit.

The K -theory groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ can be defined for a pre- C^* -algebra \mathcal{A} , see [18, Sect. 3.8] or [9] for example. A useful feature of pre- C^* -algebras is that they contain all the K -theoretic information of their C^* -completion.

Proposition 2.8 [37] *If \mathcal{A} is a pre- C^* -algebra with C^* -completion A , then the map induced by the inclusion $\iota_* : K_j(\mathcal{A}) \rightarrow K_j(A)$ is an isomorphism for $j = 0, 1$.*

Lemma 2.9 *Let \mathcal{A} and \mathcal{B} be pre- C^* -algebras with \mathcal{B} unital and ${}_{\mathcal{A}}\mathcal{E}_{\mathcal{B}}$ a pre-Morita equivalence bimodule.*

1. *There is a finite left module frame $\{g_1, \dots, g_n\} \subset \mathcal{E}$ and $1_{\mathcal{B}} = \sum_{k=1}^n (g_k \mid g_k)_{\mathcal{B}}$.*
2. *For any $p = p^* = p^2 \in M_n(\mathcal{A})$, $p\mathcal{E}^{\oplus n}$ is a finitely generated and projective \mathcal{B} -module and there exists a finite right module frame $\{v_1, \dots, v_m\} \subset p\mathcal{E}^{\oplus n}$.*

Proof We write A and B for the C^* -algebra closures of \mathcal{A} and \mathcal{B} and E for the C^* -module closure of \mathcal{E} , which is a Morita equivalence bimodule for the C^* -algebras A and B .

(1) The $*$ -algebra of finite rank operators $\text{span}\{\Theta_{e_1, e_2}^L\}_{e_1, e_2 \in \mathcal{E}}$ is C^* -norm dense in ${}_A\mathbb{K}(E)$. Thus for $\varepsilon < 1$ we can find $\{e_1, \dots, e_n\} \subset \mathcal{E}$ such that the operator

$$g := \sum_{k=1}^n \Theta_{e_k, e_k}^L = \sum_{k=1}^n (e_k \mid e_k)_{\mathcal{B}} \in \mathcal{B},$$

satisfies $\|1_B - g\|_B < \varepsilon$. Hence the positive element g is invertible in B . Because the spectrum of g is contained in the region of analyticity of $f(z) = z^{-1}$ and \mathcal{B} is stable under the holomorphic functional calculus, g^{-1} and $g^{-1/2} \in \mathcal{B}$. Define $g_k := e_k \cdot g^{-1/2} \in \mathcal{E}$, for $1, \dots, n$ so that

$$\begin{aligned} \sum_{k=1}^n (g_k | g_k)_{\mathcal{B}} &= \sum_{k=1}^n \Theta_{g_k, g_k}^L = \sum_{k=1}^n \Theta_{e_k g^{-\frac{1}{2}}, e_k g^{-\frac{1}{2}}}^L \\ &= g^{-\frac{1}{2}} \left(\sum_{k=1}^n \Theta_{e_k, e_k}^L \right) g^{-\frac{1}{2}} = g^{-\frac{1}{2}} g g^{-\frac{1}{2}} = 1_{\mathcal{B}}, \end{aligned}$$

which proves the claim.

(2) Since p is a compact operator on $E^{\oplus n}$, it is finite rank by [18, Corollary 3.10], so the module $W := pE^{\oplus n}$ is finitely generated and projective over B . Write $\mathcal{W} := p\mathcal{E}^{\oplus n}$ for the dense \mathcal{B} -submodule defined by $p \in M_n(\mathcal{A})$.

The $*$ -algebra of finite rank operators $\text{span}\{\Theta_{e_1, e_2}^R\}_{e_1, e_2 \in \mathcal{W}}$ is C^* -norm dense in $\mathbb{K}_{\mathcal{B}}(W)$. Hence for $\varepsilon < 1$ there exist $w_1, \dots, w_m \in \mathcal{W}$ such that $w := \sum \Theta_{w_k, w_k}^R$ satisfies $\|p - w\|_{\mathbb{K}_{\mathcal{B}}(W)} < \varepsilon$. It follows that w is invertible in the unital C^* -algebra $\mathbb{K}_{\mathcal{B}}(W) \cong pM_n(\mathcal{A})p$ and has spectrum contained in $B(1; \varepsilon)$. The spectrum of w in $M_n(\mathcal{A})$ is thus contained in the disconnected set $B(1; \varepsilon) \cup \{0\}$. By spectral invariance, the same holds for the spectrum of $w \in M_n(\mathcal{A})$. Thus there is a function f , holomorphic on a neighborhood of the spectrum of w such that $f(0) = 0$ and $f(z) = z^{-\frac{1}{2}}$ for $z \in B(1; \varepsilon)$. Hence $f(w) \in M_n(\mathcal{A}) \cap pM_n(\mathcal{A})p = pM_n(\mathcal{A})p$ and satisfies $f(w)wf(w) = p$. Now put $v_k := f(w)w_k$ so that, as above,

$$\sum_{k=1}^m \Theta_{v_k, v_k}^R = f(w) \left(\sum_{k=1}^m \Theta_{w_k, w_k}^R \right) f(w) = p,$$

which proves the claim. □

2.3 Derivations, Pre-Morita Equivalence Bimodules and Localisation

We now describe a general method to construct pre-Morita equivalence bimodules and pre- C^* -algebras from a family of densely defined derivations on a given C^* -algebra. For instance, the algebra $C^\infty(M)$, with M a compact manifold, can be constructed in this way.

As a technical tool we will use the notion of differential seminorms introduced in [10, Definition 3.1]. The space $\ell^1(\mathbb{N})$ is an algebra in the convolution product $f * g(n) := \sum_{k \leq n} f(k)g(n-k)$. The subspace $\ell^1_+(\mathbb{N}) := \ell^1(\mathbb{N}, \mathbb{R}_+) \subset \ell^1(\mathbb{N})$ inherits the coordinatewise ordering from \mathbb{R}_+ and satisfies $\ell^1_+(\mathbb{N})\ell^1_+(\mathbb{N}) \subset \ell^1_+(\mathbb{N})$. Following [10], a differential seminorm on a subalgebra $\mathfrak{A} \subset A$ is a map $T : \mathfrak{A} \rightarrow \ell^1_+(\mathbb{N})$ such that $T(a)(0) \leq C\|a\|$ and $T(\lambda a) = |\lambda|T(a)$ and $T(ab) \leq T(a) * T(b)$. The functional $f : \ell^1(\mathbb{N}) \rightarrow \mathbb{C}, f \mapsto \sum_{k \in \mathbb{N}} f(k)$, is positive and multiplicative, and if $T : \mathfrak{A} \rightarrow \ell^1_+(\mathbb{N})$ is a differential seminorm, then $f \circ T : \mathfrak{A} \rightarrow \mathbb{R}_+$ is a submultiplicative seminorm.

Proposition 2.10 (Cf. [10]) *Let A be a C^* -algebra with norm $\|\cdot\|$, $\mathfrak{A} \subset A$ a dense $*$ -subalgebra and $\{\partial_j : \mathfrak{A} \rightarrow \mathfrak{A}\}_{j=1}^d$ a finite family of commuting $*$ -derivations. Then*

for $\alpha \in \mathbb{N}^d$ a multi-index,

$$\|a\|_n = \sum_{|\alpha| \leq n} \frac{\|\partial^\alpha a\|}{\alpha!}, \quad \partial^\alpha = \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}, \quad |\alpha| := \sum_{k=1}^n \alpha_k, \quad \alpha! := \prod_{k=1}^d (\alpha_k!),$$

is a sequence of submultiplicative seminorms on \mathfrak{A} . Let \mathcal{A}_n denote the closure of \mathfrak{A} in the seminorms $\|\cdot\|_k, k \leq n$ and $\mathcal{A}_\infty := \varinjlim \mathcal{A}_n$ the Fréchet closure of \mathfrak{A} in all the seminorms $\|\cdot\|_n$. Then for $n = 0, \dots, \infty, \mathcal{A}_n$ is a pre- C^* -algebra.

Proof Defining $\binom{\alpha}{\beta} := \prod_{k=1}^d \binom{\alpha_k}{\beta_k}$, one proves that $\partial^\alpha(ab) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta(a) \partial^{\alpha-\beta}(b)$, by induction on α . Following [10], for $n \in \mathbb{N}$ we consider the maps

$$T_n : \mathfrak{A} \rightarrow \ell^1_+(\mathbb{N}), \quad T_n(a)(k) := \frac{\sum_{|\alpha|=k} \|\partial^\alpha a\|}{\alpha!} \text{ for } k \leq n, \quad T_n(a)(k) = 0 \text{ for } k > n.$$

Indeed the T_n satisfy $T_n(a)(0) = \|a\|$ and $T_n(\lambda a) = |\lambda|T_n(a)$ as well as

$$\begin{aligned} T_n(ab)(k) &= \sum_{|\alpha|=k} \frac{1}{\alpha!} \|\partial^\alpha(ab)\| \leq \sum_{|\alpha|=k} \sum_{\beta \leq \alpha} \frac{1}{\beta!} \|\partial^\beta(a)\| \frac{1}{(\alpha-\beta)!} \|\partial^{\alpha-\beta}(b)\| \\ &\leq \sum_{|\alpha|+|\beta|=k} \frac{1}{\beta!} \|\partial^\beta(a)\| \frac{1}{\alpha!} \|\partial^\alpha(b)\| \leq T_n(a) * T_n(b)(k), \end{aligned}$$

which shows that T_n is a differential algebra norm. Moreover for $k \geq 1$

$$\begin{aligned} T_n(ab)(k) &\leq \sum_{|\alpha|+|\beta|=k} \frac{1}{\beta!} \|\partial^\beta(a)\| \frac{1}{\alpha!} \|\partial^\alpha(b)\| \\ &= \|a\| \left(\sum_{|\beta|=k} \frac{1}{\beta!} \|\partial^\beta(b)\| \right) + \left(\sum_{|\alpha|=k} \frac{1}{\alpha!} \|\partial^\alpha(a)\| \right) \|b\| \\ &\quad + \sum_{|\alpha|, |\beta| \geq 1} \frac{1}{\beta!} \|\partial^\beta(a)\| \frac{1}{\alpha!} \|\partial^\alpha(b)\|, \end{aligned}$$

so that each T_n is of logarithmic order ≤ 1 (see [10, Definition 3.4]). We then have $\|a\|_n = \int T_n(a)$ and the result for \mathcal{A}_n follows from [10, Propositions 3.3 and 3.12]. The result for \mathcal{A}_∞ then follows since $\mathcal{A}_\infty = \bigcap_{n=0}^\infty \mathcal{A}_n$ and the result holds for each \mathcal{A}_n . □

Now let $\mathfrak{A} \subset A$ and $\mathfrak{B} \subset B$ be dense $*$ -subalgebras and \mathfrak{E} an \mathfrak{A} - \mathfrak{B} pre-Morita equivalence bimodule. Suppose we are given a families of commuting $*$ -derivations

$$\{\partial_j^L\}_{j=1}^d : \mathfrak{A} \rightarrow \mathfrak{A}, \quad \{\partial_j^R\}_{j=1}^d : \mathfrak{B} \rightarrow \mathfrak{B},$$

as well as a commuting family of operators $\nabla_j : \mathfrak{E} \rightarrow \mathfrak{E}$ such that for each j and all $a \in C_c(\mathcal{G}), b \in C_c(\mathcal{H}), \xi, \eta \in C_c(\mathcal{Z})$ and $b \in \mathfrak{B}$ we have

$$\nabla_j(a \cdot \xi \cdot b) = \partial_j^L(a) \cdot \xi \cdot b + a \cdot \nabla_j(\xi) \cdot b + a \cdot \xi \cdot \partial_j^R(b), \tag{3}$$

$$\partial_j^R(\xi \mid \eta)_{\mathfrak{B}} = (\xi \mid \nabla_j(\eta))_{\mathfrak{B}} - (\nabla_j(\xi) \mid \eta)_{\mathfrak{B}}. \tag{4}$$

It is worth noting that the identity

$$\partial_j^L \mathfrak{A}(\xi \mid \eta) = \mathfrak{A}(\xi \mid \nabla_j(\eta)) - \mathfrak{A}(\nabla_j(\xi) \mid \eta), \tag{5}$$

is satisfied as well. By using Eqs. (3), (4) and the compatibility of left and right inner products, we find for $\xi, \eta, e \in \mathcal{E}$:

$$\begin{aligned} \mathfrak{A}(\xi \mid \nabla_j(\eta)) \cdot e - \mathfrak{A}(\nabla_j(\xi) \mid \eta) \cdot e &= \xi \cdot (\nabla_j(\eta) \mid e)_{\mathfrak{B}} - \nabla_j(\xi) \cdot (\eta \mid e)_{\mathfrak{B}} \\ &= \xi \cdot (\eta \mid \nabla_j(e))_{\mathfrak{B}} - \xi \cdot \partial_j^R(\eta \mid e) \\ &\quad - \nabla_j(\xi) \cdot (\eta \mid e)_{\mathfrak{B}} \\ &= \xi \cdot (\eta \mid \nabla_j(e))_{\mathfrak{B}} - \nabla_j(\xi \cdot (\eta \mid e)_{\mathfrak{B}}) \\ &= \mathfrak{A}(\xi \mid \eta) \cdot \nabla_j(e) - \nabla_j(\mathfrak{A}(\xi \mid \eta) \cdot e) \\ &= \partial_j^L \mathfrak{A}(\xi \mid \eta) \cdot e, \end{aligned}$$

so that (5) follows.

We write $\mathcal{A}_n, \mathcal{B}_n$ for the pre- C^* -algebra completions obtained through Proposition 2.10. Similarly we write

$$\nabla^\alpha := \nabla_1^{\alpha_1} \cdots \nabla_d^{\alpha_d}, \quad \alpha \in \mathbb{N}^d, \quad \|e\|_n := \sum_{|\alpha| \leq n} \frac{\|\nabla^\alpha(e)\|}{\alpha!}$$

as well as \mathcal{E}_n for the completion of \mathfrak{E} in the seminorms up to degree n , and $\mathcal{E} = \mathcal{E}_\infty$ for the completion of \mathfrak{E} in all these seminorms.

Definition 2.11 Suppose that E_B is a right C^* -module with B a unital C^* -algebra and $\omega_B : B \rightarrow \mathbb{C}$ a state. The localisation \mathfrak{h}_ω is the Hilbert space that comes from the completion of E in the inner product

$$\langle e_1, e_2 \rangle_\omega = \omega_B((e_1 \mid e_2)_B).$$

Remark 2.12 If ${}_A E_B$ is a Morita equivalence bimodule and the state $\tau : B \rightarrow \mathbb{C}$ is a trace, then there is a canonical dual tracial weight Tr_τ on $\mathbb{K}_B(E) \cong A$ such that $\text{Tr}_\tau(\Theta_{e_1, e_2}^R) = \tau((e_2 \mid e_1)_B)$ for any $e_1, e_2 \in E$. We can localise ${}_A E_B$ with a left-linear inner-product from Tr_τ to obtain the dual localisation space \mathfrak{h}_ω^* . Hence in this special case, the localisation Hilbert space directly agrees with the Gabor analysis literature. See [2] for more details.

Given any state $\omega : B \rightarrow \mathbb{C}$, the seminorms $\|\cdot\|_n$ induce a family of seminorms $\|\cdot\|_{n,\omega}$ on the Hilbert space localisation \mathfrak{h}_ω ,

$$\|\xi\|_{n,\omega} := \sum_{|\alpha| \leq n} \frac{\|\nabla^\alpha(\xi)\|_{\mathfrak{h}_\omega}}{\alpha!}, \quad n \in \mathbb{N}.$$

Proposition 2.13 *For $n = 0, \dots, \infty$ the space \mathcal{E}_n is a \mathcal{A}_n - \mathcal{B}_n pre-Morita equivalence bimodule. Moreover, for any state $\omega : B \rightarrow \mathbb{C}$, the image of \mathcal{E}_n in \mathfrak{h}_ω is bounded in the seminorms $\|\cdot\|_{k,\omega}$ for $0 \leq k \leq n$.*

Proof The space

$$\mathcal{L}(\mathfrak{E}) := \left\{ \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} : a \in \mathfrak{A}, \quad b \in \mathfrak{B}, \quad \xi, \eta \in \mathfrak{E} \right\}$$

with multiplication and involution

$$\begin{aligned} \begin{pmatrix} a_1 & \xi_1 \\ \eta_1^* & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & \xi_2 \\ \eta_2^* & b_2 \end{pmatrix} &:= \begin{pmatrix} a_1 a_2 + \mathfrak{A}(\xi_1 | \eta_2) & a_1 \xi_2 + \xi_1 b_2 \\ (a_2^* \eta_1)^* + (\eta_2 b_2)^* & (\eta_1 | \xi_2) \mathfrak{B} + b_1 b_2 \end{pmatrix}, \quad \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix}^* \\ &:= \begin{pmatrix} a^* & \eta^* \\ \xi & b^* \end{pmatrix}, \end{aligned}$$

is an associative $*$ -algebra, the linking algebra of \mathfrak{E} . It is a dense $*$ -subalgebra of the linking C^* -algebra $\mathcal{L}(E)$ of the C^* -module closure E of \mathfrak{E} . Using the identities (3), (4) and (5), one shows that the maps ∇_j induce a family of commuting $*$ -derivations on the linking algebra via

$$\Delta_j \begin{pmatrix} a & \xi \\ \eta^* & b \end{pmatrix} := \begin{pmatrix} \partial_j^L a & \nabla_j(\xi) \\ -\nabla_j(\eta)^* & \partial_j^R b \end{pmatrix},$$

and the norms $\|e\|_n$ defined above are the restrictions of the norms obtained from the derivations Δ_j . This proves that \mathcal{E}_n is an \mathcal{A}_n - \mathcal{B}_n bimodule. Since \mathcal{E}_n is a subspace of E , the properties of Definition 2.6 will follow once we show that the left and right inner products on ${}_A E_B$ take values in \mathcal{A}_n and \mathcal{B}_n , respectively, when restricted to \mathcal{E}_n . This in turn follows since the inner products are realised as multiplications in the linking algebra. Lastly we have $\|e\|_{k,\omega} \leq \|e\|_k$ for any state ω , so the image of \mathcal{E}_n is bounded in each of the seminorms $\|e\|_{k,\omega}$ with $k \leq n$. □

Remark 2.14 The pre-Morita equivalence bimodule ${}_{\mathcal{A}_\infty} \mathcal{E}_\infty {}_{\mathcal{B}_\infty}$ is the ‘smoothest’ bimodule over pre- C^* -algebras that we can consider from the derivations $\{\partial_j^L\}_{j=1}^d$ and $\{\partial_j^R\}_{j=1}^d$. The lower-order pre-Morita equivalence bimodules ${}_{\mathcal{A}_n} \mathcal{E}_n {}_{\mathcal{B}_n}$ for $1 \leq n < \infty$ will allow us to consider C^* -module and Hilbert space frames of differing regularity. This will be of use to us when considering the localisation dichotomy of Wannier bases in Sect. 5.3.

2.4 Groupoid C^* -Algebras and Equivalence

Groupoids provide a useful generalisation of groups and group actions on spaces. A standard reference for groupoid C^* -algebras is [36].

Definition 2.15 A groupoid is a set \mathcal{G} with a subset $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$, a multiplication map $\mathcal{G}^{(2)} \rightarrow \mathcal{G}$, $(\gamma, \xi) \mapsto \gamma\xi$ and an inverse $\mathcal{G} \rightarrow \mathcal{G}$, $\gamma \mapsto \gamma^{-1}$ such that

- (i) $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in \mathcal{G}$,
- (ii) if $(\gamma, \xi), (\xi, \eta) \in \mathcal{G}^{(2)}$, then $(\gamma\xi, \eta), (\gamma, \xi\eta) \in \mathcal{G}^{(2)}$,
- (iii) $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$ for all $\gamma \in \mathcal{G}$,
- (iv) for all $(\gamma, \xi) \in \mathcal{G}^{(2)}$, $(\gamma\xi)\xi^{-1} = \gamma$ and $\gamma^{-1}(\gamma\xi) = \xi$.

Given a groupoid we denote by $\mathcal{G}^{(0)} = \{\gamma\gamma^{-1} : \gamma \in \mathcal{G}\}$ the space of units and define the source and range maps $r, s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ by the equations

$$r(\gamma) = \gamma\gamma^{-1}, \quad s(\gamma) = \gamma^{-1}\gamma$$

for all $\gamma \in \mathcal{G}$. The source and range maps allow us to characterise

$$\mathcal{G}^{(2)} = \{(\gamma, \xi) \in \mathcal{G} \times \mathcal{G} : s(\gamma) = r(\xi)\}.$$

Throughout this work, we will assume that \mathcal{G} is equipped with second countable, locally compact and Hausdorff topology such that the multiplication, inversion, source and range maps are all continuous. A groupoid \mathcal{G} is étale if the range map $r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ is a local homeomorphism. Étale groupoids have the useful property that for all $x \in \mathcal{G}^{(0)}$, the fibres $r^{-1}(x)$ and $s^{-1}(x)$ are discrete.

- Examples 2.16** (i) Let G be a group, then it is also a groupoid such that $G^{(0)} = \{e\}$ with multiplication and inverse given by the group operation. If G is discrete, then it is étale as a groupoid.
- (ii) Let X be a locally compact Hausdorff space, G a locally compact group and suppose there is a continuous left-action $G \rightarrow \text{Homeo}(X)$ so that $(g, x) \mapsto g \cdot x$ is jointly continuous. We can define the locally compact and Hausdorff transformation groupoid $X \rtimes G$ given by pairs $(x, g) \in X \times G$ such that $(X \rtimes G)^{(0)} = X$,

$$\begin{aligned} (x, g)^{-1} &= (g^{-1} \cdot x, g^{-1}), & (x, g)(g^{-1} \cdot x, h) &= (x, gh), \\ s(x, g) &= g^{-1} \cdot x, & r(x, g) &= x. \end{aligned}$$

Definition 2.17 Let \mathcal{G} be a locally compact and Hausdorff groupoid. A continuous map $\sigma : \mathcal{G}^{(2)} \rightarrow \mathbb{T}$ is a 2-cocycle if

$$\sigma(\gamma, \xi)\sigma(\gamma\xi, \eta) = \sigma(\gamma, \xi\eta)\sigma(\xi, \eta)$$

for any $(\gamma, \xi), (\xi, \eta) \in \mathcal{G}^{(2)}$, and

$$\sigma(\gamma, s(\gamma)) = 1 = \sigma(r(\gamma), \gamma)$$

for all $\gamma \in \mathcal{G}$. We will call a groupoid 2-cocycle normalised if $\sigma(\gamma, \gamma^{-1}) = 1$ for all $\gamma \in \mathcal{G}$.

The following result is well-known, though for completeness we provide a proof.

Lemma 2.18 *If a groupoid 2-cocycle $\sigma : \mathcal{G}^{(2)} \rightarrow \mathbb{T}$ is normalised, then $\sigma(\gamma, \xi) = \overline{\sigma(\gamma\xi, \xi^{-1})}$ and $\sigma(\gamma, \xi) = \overline{\sigma(\xi^{-1}, \gamma^{-1})}$ for all $(\gamma, \xi) \in \mathcal{G}^{(2)}$.*

Proof Using the 2-cocycle identity,

$$\sigma(\gamma, \xi)\sigma(\gamma\xi, \xi^{-1}) = \sigma(\gamma, s(\xi^{-1}))\sigma(\xi, \xi^{-1}) = 1$$

so $\sigma(\gamma, \xi) = \overline{\sigma(\gamma\xi, \xi^{-1})}$. Next we compute that

$$\sigma(\gamma\xi, \xi^{-1})\sigma(\gamma\xi\xi^{-1}, \gamma^{-1}) = \sigma(\gamma\xi, \xi^{-1}\gamma^{-1})\sigma(\xi^{-1}, \gamma^{-1}) = \sigma(\xi^{-1}, \gamma^{-1})$$

and using the first identity

$$\begin{aligned} \sigma(\gamma\xi, \xi^{-1})\sigma(\gamma\xi\xi^{-1}, \gamma^{-1}) \\ = \overline{\sigma(\gamma, \xi)\sigma(\gamma\xi\xi^{-1}\gamma^{-1}, \gamma)} = \overline{\sigma(\gamma, \xi)\sigma(r(\gamma\xi), \gamma)} = \overline{\sigma(\gamma, \xi)} \end{aligned}$$

which gives that $\overline{\sigma(\gamma, \xi)} = \sigma(\xi^{-1}, \gamma^{-1})$. □

We briefly review the construction of groupoid C^* -algebras.

Definition 2.19 A Haar system on a locally compact Hausdorff groupoid \mathcal{G} is a set of measures $\{\nu^x : x \in \mathcal{G}^{(0)}\}$ on \mathcal{G} such that $\text{supp}(\nu^x) = r^{-1}(x)$ and for all $f \in C_c(\mathcal{G})$,

$$\int_{\mathcal{G}} f(\xi) d\nu^{r(\eta)}(\xi) = \int_{\mathcal{G}} f(\eta\xi) d\nu^{s(\eta)}(\xi), \quad g(x) := \int_{\mathcal{G}} f(\xi) d\nu^x(\xi) \in C(\mathcal{G}^{(0)}).$$

Étale groupoids always have a Haar system given by the counting measure on the (discrete) fibres $r^{-1}(x)$. Given \mathcal{G} with a 2-cocycle σ and Haar system $\{\nu^x\}_{x \in \mathcal{G}^{(0)}}$, we can define a $*$ -algebra structure on $C_c(\mathcal{G}, \sigma)$,

$$(f_1 * f_2)(\eta) = \int_{\mathcal{G}} f(\xi)g(\xi^{-1}\eta) \sigma(\xi, \xi^{-1}\eta) d\nu^{r(\eta)}(\xi), \quad f^*(\xi) = \sigma(\xi, \xi^{-1}) \overline{f(\xi^{-1})}.$$

We will restrict ourselves to normalised cocycles, $\sigma(\xi, \xi^{-1}) = 1$ as it covers all examples of interest to us. See [36] for the general construction. The algebra $C_c(\mathcal{G}, \sigma)$ has a right $C_0(\mathcal{G}^{(0)})$ -module structure, where $(f \cdot g)(\xi) = f(\xi)g(s(\xi))$ for $f \in C_c(\mathcal{G}, \sigma)$ and $g \in C_0(\mathcal{G}^{(0)})$. We can define a $C_0(\mathcal{G}^{(0)})$ -valued inner-product

$$\begin{aligned} (f_1 | f_2)_{\mathcal{G}^{(0)}}(x) &= \int_{\mathcal{G}} \overline{f_1(\xi^{-1})} f_2(\xi^{-1}) \sigma(\xi, \xi^{-1}) d\nu^x(\xi) \\ &= \int_{\mathcal{G}} \overline{f_1(\xi^{-1})} f_2(\xi^{-1}) d\nu^x(\xi) \end{aligned}$$

as σ is normalised. Completing this space in the norm of $C_0(\mathcal{G}^{(0)})$ gives a right- $C_0(\mathcal{G}^{(0)})$ -module, which we denote by $L^2(\mathcal{G}, \nu)_{\mathcal{G}^{(0)}}$. There is a canonical left-action of $C_c(\mathcal{G}, \sigma)$ on $L^2(\mathcal{G}, \nu)_{\mathcal{G}^{(0)}}$ by the (twisted) convolution product.

Definition 2.20 (cf. [22]) The reduced groupoid C^* -algebra $C_r^*(\mathcal{G}, \sigma)$ is the completion of $C_c(\mathcal{G}, \sigma)$ in the norm inherited from the embedding $C_c(\mathcal{G}, \sigma) \hookrightarrow \text{End}_{C_0(\mathcal{G}^{(0)})}^*(L^2(\mathcal{G}, \nu))$.

If there is a topological space Z with (continuous) map $\rho : Z \rightarrow \mathcal{G}^{(0)}$, we denote by $\mathcal{G} \times_{\rho} Z$ and $Z \rtimes_{\rho} \mathcal{G}$ the pullback with respect to the source and range maps respectively.

Definition 2.21 (\mathcal{G} -space) A Hausdorff topological space Z is a left \mathcal{G} -space if there is a continuous map, called the anchor or moment map, $\rho : Z \rightarrow \mathcal{G}^{(0)}$ and a continuous map

$$\mathcal{G} \times_{\rho} Z \rightarrow Z, \quad (\gamma, z) \mapsto \gamma \cdot z \in Z$$

such that for $(\gamma, z) \in \mathcal{G} \times_{\rho} Z$ and $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$,

$$\rho(\gamma \cdot z) = r(\gamma), \quad \rho(z) \cdot z = z, \quad (\gamma_1 \gamma_2) \cdot z = \gamma_1 \cdot (\gamma_2 \cdot z)$$

Unless otherwise stated, we will always assume that the moment map $\rho : Z \rightarrow \mathcal{G}^{(0)}$ is open and surjective. One may also consider a right \mathcal{H} -space from $\phi : Z \rightarrow \mathcal{H}^{(0)}$, where the definition is analogous to the above but instead using a map $Z \rtimes_{\phi} \mathcal{H} \rightarrow Z$ such that $\rho(z \cdot \eta) = s(\eta)$. When the context is clear, we will write left/right-actions as γz or $z\eta$.

We say that Z is a proper \mathcal{G} -space (or that \mathcal{G} acts properly) if the map

$$\mathcal{G} \times_{\rho} Z \rightarrow Z \times Z, \quad (\gamma, z) \mapsto (\gamma \cdot z, z)$$

is proper. If the map $(\gamma, z) \mapsto (\gamma \cdot z, z)$ is injective, then we say that Z is a free \mathcal{G} -space or that \mathcal{G} acts freely.

Definition 2.22 Let \mathcal{G} and \mathcal{H} be (locally compact, Hausdorff) groupoids and assume that Z carries both a left- \mathcal{G} and right- \mathcal{H} action via moment maps $\rho : Z \rightarrow \mathcal{G}^{(0)}$ and $\phi : Z \rightarrow \mathcal{H}^{(0)}$. We say that Z is a \mathcal{G} - \mathcal{H} -bibundle if the actions commute, i.e.,

- (1) for all $(\gamma, z) \in \mathcal{G} \times_{\rho} Z$ and $(z, \eta) \in Z \rtimes_{\phi} \mathcal{H}$, $(\gamma \cdot z) \cdot \eta = \gamma \cdot (z \cdot \eta)$,
- (2) for all $(z, \eta) \in Z \rtimes_{\phi} \mathcal{H}$, $\rho(z \cdot \eta) = \rho(z)$,
- (3) for all $(\gamma, z) \in \mathcal{G} \times_{\rho} Z$, $\phi(\gamma \cdot z) = \phi(z)$.

A \mathcal{G} - \mathcal{H} bibundle is a groupoid equivalence if the maps

$$\begin{aligned} \mathcal{G} \times_{\rho} Z &\rightarrow Z *_{\mathcal{H}^{(0)}} Z, & (\gamma, z) &\mapsto (\gamma \cdot z, z), \\ Z \rtimes_{\phi} \mathcal{H} &\rightarrow Z *_{\mathcal{G}^{(0)}} Z, & (z, \eta) &\mapsto (z, z \cdot \eta) \end{aligned}$$

are homeomorphisms.

At the level of operator algebras, groupoid equivalence implies Morita equivalence of the (full or reduced) groupoid C^* -algebras [34,38]. Let $\mathcal{G} \xleftarrow{\rho} Z \xrightarrow{\phi} \mathcal{H}$ be a groupoid equivalence such that \mathcal{G} and \mathcal{H} have Haar systems $\{\nu^x\}_{x \in \mathcal{G}^{(0)}}$ and $\{\lambda^y\}_{y \in \mathcal{H}^{(0)}}$ respectively. Then there is a left (resp. right) action of $C_c(\mathcal{G})$ (resp. $C_c(\mathcal{H})$) on $C_c(Z)$ given by

$$\begin{aligned} (f \cdot \xi)(z) &= \int_{\mathcal{G}} f(\gamma)\xi(\gamma^{-1}z) \, d\nu^{\rho(z)}(\gamma), \quad \xi \in C_c(Z), \quad f \in C_c(\mathcal{G}), \\ (\xi \cdot g)(z) &= \int_{\mathcal{H}} \xi(z\eta)g(\eta^{-1}) \, d\lambda^{\phi(z)}(\eta), \quad \xi \in C_c(Z), \quad g \in C_c(\mathcal{H}). \end{aligned} \tag{6}$$

There is also the $C_c(\mathcal{H})$ -valued inner product

$$(\xi_1 \mid \xi_2)_{\mathcal{H}}(\eta) = \int_{\mathcal{G}} \overline{\xi_1(\gamma^{-1}z)}\xi_2(\gamma^{-1}z\eta) \, d\nu^{\rho(z)}(\gamma) \tag{7}$$

where $z \in Z$ is chosen such that $\phi(z) = r(\eta)$.

Proposition 2.23 [38, Theorem 4.1] *Let Z be a \mathcal{G} - \mathcal{H} groupoid equivalence. Then $C_c(Z)$ is a pre-Morita equivalence bimodule for $C_c(\mathcal{G})$ and $C_c(\mathcal{H})$. Consequently, $C_r^*(\mathcal{G})$ and $C_r^*(\mathcal{H})$ are Morita equivalent.*

We denote by ${}_G L^2(Z)_{\mathcal{H}}$ the Morita equivalence bimodule that links $C_r^*(\mathcal{G})$ and $C_r^*(\mathcal{H})$. Because we work with reduced groupoid C^* -algebras, the Morita equivalence bimodule completion of $C_c(Z)$ is constructed from the linking groupoid $L = \mathcal{G} \sqcup Z \sqcup Z^{\text{op}} \sqcup \mathcal{H}$ [38]. In Sect. 4.1 we discuss instances of twisted groupoid Morita equivalence.

3 Frames of Translates and Wannier Bases from Groupoid Equivalences

We let $\mathcal{G} \xleftarrow{\rho} Z \xrightarrow{\phi} \mathcal{H}$ be a \mathcal{G} - \mathcal{H} equivalence of locally compact, second countable and Hausdorff groupoids such that $\mathcal{H}^{(0)}$ is compact and \mathcal{H} is étale. In particular, this implies that $C_r^*(\mathcal{H})$ is unital. We assume that \mathcal{G} has a Haar system $\{\nu^x\}_{x \in \mathcal{G}^{(0)}}$ and so ${}_{C_c(\mathcal{G})}C_c(Z)_{C_c(\mathcal{H})}$ is a pre-Morita equivalence bimodule described by Eqs. (6) and (7). Because \mathcal{H} is étale, the right-action in Eq. (6) reduces to a sum over the discrete set $r^{-1}(\phi(z))$.

3.1 C^* -Module Frame

Because all C^* -algebras are separable, there exists a countable right module frame for ${}_G L^2(Z)_{\mathcal{H}}$. Furthermore, since $C_r^*(\mathcal{H})$ is unital, ${}_G L^2(Z)_{\mathcal{H}}$ is finitely generated and projective as a left $C_r^*(\mathcal{G})$ -module and so has a finite left C^* -module frame.

A right module frame for the submodule $C_c(Z)$ is constructed in [34, Proposition 2.10]. We briefly review this construction. We say that a subset $L \subset \mathcal{G}$ is r -relatively compact if $L \cap r^{-1}(K)$ is relatively compact for every compact $K \subset \mathcal{G}$. We consider

a triple (K, U, ε) with $K \subset \mathcal{G}^{(0)}$ compact with $U \subset \mathcal{G}$ an r -relatively compact neighbourhood of $\mathcal{G}^{(0)}$ and $\varepsilon > 0$. Because $\mathcal{G}^{(0)}$ is paracompact and \mathcal{G} acts properly on Z , there are open, relatively compact sets $\{V_j\}_{j=1}^n \subset Z$ such that $\{\rho(V_j)\}_{j=1}^n$ cover K and are such that $(\gamma z, z) \in V_j \times V_j$ implies that $\gamma \in U$. We take a partition of unity $\{\psi_j\}_{j=1}^n$ subordinate to the cover $\{\rho(V_j)\}_{j=1}^n$. We can then find functions $\{\tilde{\psi}_j\}_{j=1}^n$ such that $\text{supp}(\tilde{\psi}_j) \subset V_j$ and

$$\sum_{\eta \in r^{-1}(\phi(z))} \tilde{\psi}_j(z\eta) = \psi_j(\rho(z)).$$

Finally, we can approximate $\tilde{\psi}_j$ with $\{\varphi_j\}_{j=1}^n$ such that $\text{supp}(\varphi_j) \subset V_j$ and

$$\left| \tilde{\psi}_j(z) - \varphi_j(z) \int_{\mathcal{G}} \varphi_j(\gamma^{-1}z) \, d\nu^{\rho(z)}(\gamma) \right| \leq \frac{\varepsilon}{M} \quad M = \sup_z \sum_{j=1}^n \sum_{\eta \in r^{-1}(\phi(z))} \chi_{V_j}(z\eta).$$

The functions $e_n = \sum_{j=1}^n \mathcal{G}(\varphi_j \mid \varphi_j)$ are an approximate identity for the left action of $C_c(\mathcal{G})$ on $C_c(Z)$. The functions $\{\varphi_j\} \subset C_c(Z)$ can then be used to construct a right C^* -module frame in the Morita equivalence bimodule ${}_G L^2(Z)_{\mathcal{H}}$ [38].

3.2 Localisation and Hilbert Space Frame

For any $x \in \mathcal{H}^{(0)}$ we have a state ω_x given by the restriction of $f \in C_c(\mathcal{H})$ to $\mathcal{H}^{(0)}$ and then evaluation at $x \in \mathcal{H}^{(0)}$. We see that

$$\omega_x(g^*g) = \sum_{\eta \in r^{-1}(x)} |g(\eta)|^2,$$

which shows that ω_x is positive. There are other possible states on $C_r^*(\mathcal{H})$ that one may also consider such as integration with respect to a quasi-invariant measure [36]. The reason we choose the evaluation state is because we would like to construct Parseval frames that have a discrete labeling. By choosing $x \in \mathcal{H}^{(0)}$, the discrete set $r^{-1}(x)$ provides us with such a labeling.

Lemma 3.1 *Fix an element $x \in \mathcal{H}^{(0)}$. Then there are real-valued functions $\{\delta_\alpha\}_{\alpha \in r^{-1}(x)} \subset C_c(\mathcal{H})$ such that $\omega_x(\delta_{\alpha_1} * \delta_{\alpha_2}^*) = \delta_{\alpha_1, \alpha_2}$.*

Proof Because $\mathcal{H}^{(0)}$ is compact and $r^{-1}(x)$ is discrete, for each $\alpha \in r^{-1}(x)$ we take δ_α a bump function supported on U_α such that $U_\alpha \cap r^{-1}(x) = \{\alpha\}$ and $\delta_\alpha(\alpha) = 1$. Then we compute that

$$\omega_x(\delta_{\alpha_1} * \delta_{\alpha_2}^*) = \sum_{\beta \in r^{-1}(x)} \delta_{\alpha_1}(\beta) \delta_{\alpha_2}(\beta) = \delta_{\alpha_2}(\alpha_1) = \delta_{\alpha_1, \alpha_2}$$

as the sum vanishes everywhere except for at most one term. □

Lemma 3.2 Let $Z_x = \phi^{-1}(x)$ and for $z \in Z_x$ define the measure $\nu^{\rho(z)}$ on Z_x by

$$\int_{Z_x} f(w) d\nu^{\rho(z)}(w) := \int_G f(\gamma^{-1} \cdot z) d\nu^{\rho(z)}(\gamma).$$

The Hilbert space localisation \mathfrak{h}_x of ${}_{\mathcal{G}}L^2(Z)_{\mathcal{H}}$ in ω_x is $L^2(Z_x, d\nu^{\rho(z)})$ with $z \in Z_x$ chosen arbitrarily.

Proof We consider the inner-product, where

$$\begin{aligned} \langle e_1, e_2 \rangle_x &= \omega_x((e_1 \mid e_2)_{\mathcal{H}}) = (e_1 \mid e_2)_{\mathcal{H}}(x) = \int_G \overline{e_1(\gamma^{-1}z)} e_2(\gamma^{-1}zx) d\nu^{\rho(z)}(\gamma) \\ &= \int_G \overline{e_1(\gamma^{-1}z)} e_2(\gamma^{-1}z\phi(z)) d\nu^{\rho(z)}(\gamma) = \int_G \overline{e_1(\gamma^{-1}z)} e_2(\gamma^{-1}z) d\nu^{\rho(z)}(\gamma) \end{aligned}$$

Hence the Hilbert space is the L^2 -completion of $C_c(Z_x)$ with respect to the measure $\nu^{\rho(z)}$. Since Z is an equivalence, for every $z, w \in Z_x$ there exists $\gamma \in r^{-1}(\rho(z)) \subset \mathcal{G}$ such that $w = \gamma^{-1}z$. The measure $\nu^{\rho(z)}$ is thus independent of the choice of $z \in Z_x$. \square

Given $x \in \mathcal{H}^{(0)}$ and $e \in {}_{\mathcal{G}}L^2(Z)_{\mathcal{H}}$, we let e_x be the corresponding element in the localisation $L^2(Z_x)$. Given any $\alpha \in r^{-1}(x)$, define a function $e_x^\alpha \in L^2(Z_x)$ by $(e_x^\alpha)(y) = e(y\alpha)$, $y \in Z_x$.

Lemma 3.3 Let $e \in {}_{\mathcal{G}}L^2(Z)_{\mathcal{H}}$, $a \in \mathcal{H}^{(0)}$ and $\alpha \in r^{-1}(x)$.

- (i) There is an equality $e_x^\alpha = (e \cdot \delta_\alpha^*)_x$ with δ_α be the bump functions from Lemma 3.1.
- (ii) If $e \in {}_{\mathcal{G}}L^2(Z)_{\mathcal{H}}$ is such that $(e \mid e)_{\mathcal{H}} = 1_{\mathcal{H}}$, then $\{e_x^\alpha\}_{\alpha \in r^{-1}(x)}$ is an orthonormal system in $L^2(Z_x)$;
- (iii) Given $e_1, e_2, \xi \in {}_{\mathcal{G}}L^2(Z)_{\mathcal{H}}$,

$$(\Theta_{e_1, e_2}^R(\xi))_x = \sum_{\alpha \in r^{-1}(x)} (e_1)_x^\alpha \langle (e_2)_x^\alpha, \xi_x \rangle_x.$$

Proof. We first note that $e(y\alpha)$ is well-defined as $\rho(y) = x = r(\alpha)$. For part (i) we compute for $y \in Z_x$,

$$(e \cdot \delta_\alpha^*)(y) = \sum_{\beta \in r^{-1}(x)} e(y\beta) \delta_\alpha^*(\beta^{-1}) = \sum_{\beta \in r^{-1}(x)} e(y\beta) \delta_\alpha(\beta) = e(y\alpha).$$

Using part (i) and Lemma 3.1, we see that for e such that $(e \mid e)_{\mathcal{H}} = 1$,

$$\langle e_x^{\alpha_1}, e_x^{\alpha_2} \rangle_x = \omega_x((e \cdot \delta_{\alpha_1}^* \mid e \cdot \delta_{\alpha_2}^*)_{\mathcal{H}}) = \omega_x(\delta_{\alpha_1} (e \mid e)_{\mathcal{H}} \delta_{\alpha_2}^*) = \delta_{\alpha_1, \alpha_2}$$

and so $\{e_x^\alpha\}_{\alpha \in r^{-1}(x)}$ is an orthonormal system, which proves part (ii).

For part (iii), we again compute for $y \in \phi^{-1}(x)$

$$\begin{aligned} (\Theta_{e_1, e_2}^R(\xi))(y) &= \sum_{\alpha \in r^{-1}(x)} e_1(y\alpha)(e_2 \mid \xi)_{\mathcal{H}}(\alpha^{-1}) \\ &= \sum_{\alpha \in r^{-1}(x)} e_1(y\alpha) \int_{\mathcal{G}} \overline{e_2(\gamma^{-1}z)} \xi(\gamma^{-1}z\alpha^{-1}) \, d\nu^{\rho(z)}(\gamma). \end{aligned}$$

We now let $u = z\alpha^{-1}$, where $u\alpha = z\phi(\alpha) = z\phi(z) = z$ and $\rho(u) = \rho(z\alpha^{-1}) = \rho(z)$ as Z is a groupoid equivalence. Hence

$$\begin{aligned} (\Theta_{e_1, e_2}^R(\xi))(y) &= \sum_{\alpha \in r^{-1}(x)} e_1(y\alpha) \int_{\mathcal{G}} \overline{e_2(\gamma^{-1}u\alpha)} \xi(\gamma^{-1}u) \, d\nu^{\rho(u)}(\gamma) \\ &= \sum_{\alpha \in r^{-1}(x)} (e_1)_x^\alpha(y) \langle (e_2)_x^\alpha, \xi_x \rangle_x. \end{aligned}$$

This completes the proof. □

For a countable set J and a C^* -algebra B we denote by $\ell^2(J, B)$ the standard Hilbert C^* -module of sequences indexed by J . That is

$$\ell^2(J, B) := \left\{ f : J \rightarrow B : \sum_{j \in J} f(j)^* f(j) < \infty \right\},$$

where the series converges in B . We now come to our main result relating C^* -module frames to localised frames of translates.

Theorem 3.4 *Let $\{e_j\}_{j \in J} \subset \mathcal{G}L^2(Z)\mathcal{H}$ be a countable subset and $E := \overline{\text{span}_{\mathcal{H}}\{e_j : j \in J\}}$ the closed $C_r^*(\mathcal{H})$ submodule generated by $\{e_j\}_{j \in J}$. The following are equivalent:*

- (1) *The sequence $\{e_j\}_{j \in J}$ is a right C^* -module frame of $E \subset \mathcal{G}L^2(Z)\mathcal{H}$;*
- (2) *For $\xi \in E$ the map $j \mapsto (e_j \mid \xi)_{\mathcal{H}}$ takes values in $\ell^2(J, C_r^*(\mathcal{H}))$ and for all $x \in \mathcal{H}^{(0)}$ the set $\{(e_j)_x^\alpha\}_{j \in J, \alpha \in r^{-1}(x)}$ is a normalised tight frame for $\overline{E_x} \subset L^2(Z_x)$;*

Proof (1) \Rightarrow (2): Using part (iii) of Lemma 3.3, we see that

$$\sum_{j \in J} \sum_{\alpha \in r^{-1}(x)} (e_j)_x^\alpha \langle (e_j)_x^\alpha, \xi_x \rangle_x = \sum_{j \in J} (\Theta_{e_j, e_j}^R(\xi))_x = \xi_x$$

as $(e_j)_{j \in J}$ is a right C^* -module frame.

(2) \Rightarrow (1): In order to prove that $\{e_j\}_{j \in J}$ is a Hilbert C^* -module frame for E , we need to show that the map

$$v : E \rightarrow \ell^2(J, C_r^*(\mathcal{H})), \quad \xi \mapsto (e_j \mid \xi)_{\mathcal{H}},$$

satisfies $v^*v = 1$. Note that v is well-defined since we assume that $j \mapsto (e_j \mid \xi)_{\mathcal{H}}$ is an element of $\ell^2(J, C_r^*(\mathcal{H}))$, and v is automatically adjointable with $v^*(b_j) = \sum_j e_j \cdot b_j$.

Since $\{(e_j)_x^\alpha\}_{j \in J, \alpha \in r^{-1}(x)}$ is a normalised tight frame for $\overline{E_x} \subset L^2(Z_x)$, the map

$$v_x : L^2(Z_x) \rightarrow \ell^2(J \times r^{-1}(x)), \quad \psi \mapsto \langle (e_j)_x^\alpha, \psi \rangle_x,$$

is an isometry, for

$$\begin{aligned} \langle \psi, \psi \rangle_x &= \|\psi\|_x^2 = \sum_{(j, \alpha) \in J \times r^{-1}(x)} |\langle (e_j)_x^\alpha, \psi \rangle_x|^2 = \|v_x(\psi)\|^2 \\ &= \langle v_x \psi, v_x \psi \rangle_{\ell^2(J \times r^{-1}(x))}, \end{aligned}$$

and it follows that $v_x^*v_x = 1$. The map v_x^* is given by

$$v_x^* : (\lambda_j^\alpha) \mapsto \sum_{j, \alpha} (e_j)_x^\alpha \lambda_j^\alpha,$$

and we find from Lemma 3.3

$$\xi_x = v_x^*v_x(\psi) = \sum_{\alpha, j} (e_j)_x^\alpha \langle (e_j)_x^\alpha, \xi_x \rangle_x = \left(\sum_j \Theta_{e_j, e_j}(\xi) \right)_x = (v^*v(\xi))_x.$$

Therefore we can conclude that

$$\|\xi - v^*v(\xi)\| = \sup_x \|\xi - v^*v(\xi)\|_x = \sup_x \|\xi_x - (v^*v(\xi))_x\|_x = 0,$$

and hence $v^*v = 1$. □

Remark 3.5 Well-definedness of the map $v : E \rightarrow \ell^2(J, C_r^*(\mathcal{H}))$ entails that for all $e \in \mathcal{E}$ the series

$$\sum_j (e \mid e_j)_{\mathcal{H}} (e_j \mid e)_{\mathcal{H}},$$

is norm convergent in $C_r^*(\mathcal{H})$. This is automatic when the index set J is finite but poses a non-trivial restriction for infinite J .

3.3 Finitely Generated and Projective Modules

Using the left-action of $C_r^*(\mathcal{G})$ on $L^2(Z)$, we can define a representation of $\pi_x : C_r^*(\mathcal{G}) \rightarrow \mathcal{B}(\mathfrak{h}_x)$, where

$$\pi_x(f)\xi_x = (f \cdot \xi)_x, \quad f \in C_r^*(\mathcal{G}), \quad \xi \in L^2(Z).$$

Concretely,

$$(\pi_x(f)\xi_x)(y) = \int_{\mathcal{G}} f(\gamma)\xi(\gamma^{-1}y) \, d\nu^{\rho \circ \phi^{-1}(x)}(\gamma), \quad f \in C_c(\mathcal{G}), \quad \xi \in C_c(\mathcal{Z}).$$

We consider projections, $p = p^* = p^2 \in M_n(C_r^*(\mathcal{G}))$, which act compactly on $L^2(\mathcal{Z})^{\oplus n}$.

Proposition 3.6 *Let $p = p^* = p^2 \in M_n(C_r^*(\mathcal{G}))$. There is a finite set $\{v_j\}_{j=1}^n \subset pL^2(\mathcal{Z})^{\oplus n}$ such that for any $x \in \mathcal{H}^{(0)}$, $\{v_1^\alpha, \dots, v_n^\alpha\}_{\alpha \in r^{-1}(x)}$ is a normalised tight frame of $\pi_x(p)\mathfrak{h}_x^{\oplus n}$.*

Proof By Lemma 2.9 (2), there is a finite frame $\{v_j\}_{j=1}^n$ of $pL^2(\mathcal{Z})^{\oplus n}$. It is immediate that the localisation of $pL^2(\mathcal{Z})^{\oplus n}$ in ω_x is $\pi_x(p)\mathfrak{h}_x^{\oplus n}$. The result then follows by the same proof as Theorem 3.4. \square

Let us now consider the converse, i.e. given the Hilbert space frames $\{w_1^\alpha, \dots, w_m^\alpha\}_{\alpha \in r^{-1}(x)}$ for $x \in \mathcal{H}^{(0)}$, we construct a finitely generated and projective module.

Proposition 3.7 *Let $W \subset \mathcal{G}L^2(\mathcal{Z})_{\mathcal{H}}$ be a closed submodule and $\{w_1, \dots, w_n\}$ a finite subset of $\mathcal{G}L^2(\mathcal{Z})_{\mathcal{H}}$ such that $W := \overline{\text{span}_{C_r^*(\mathcal{H})}\{w_1, \dots, w_n\}}$. Suppose that for all $x \in \mathcal{H}^{(0)}$, $\{w_1^\alpha, \dots, w_m^\alpha\}_{\alpha \in r^{-1}(x)}$ is a normalised tight frame of $\overline{W}_x \subset L^2(\mathcal{Z}_x, d\nu^{\rho(x)})$. Then W is a finitely generated and projective module over $C_r^*(\mathcal{H})$. If, for each $x \in \mathcal{H}^{(0)}$, $\{w_1^\alpha, \dots, w_m^\alpha\}_{\alpha \in r^{-1}(x)}$ is an orthonormal basis, then $W \cong C_r^*(\mathcal{H})^{\oplus m}$.*

Proof The first part of the Proposition follows immediately from Theorem 3.4 and the fact that any $C_r^*(\mathcal{H})$ -module with a finite C^* -module frame is finitely generated and projective. Now suppose that $\{w_1^\alpha, \dots, w_m^\alpha\}_{\alpha \in r^{-1}(x)}$ is an orthonormal basis and let $p_{jk} = (w_j \mid w_k)_{\mathcal{H}} \in C_r^*(\mathcal{H})$. For any $x \in \mathcal{H}^{(0)}$, we have that

$$\begin{aligned} \delta_{j,k} \delta_{\alpha_1, \alpha_2} &= \langle w_j^{\alpha_1}, w_k^{\alpha_2} \rangle_x = \omega_x(\delta_{\alpha_1}(w_j \mid w_k)_{\mathcal{H}} \delta_{\alpha_2}^*) = \sum_{\beta \in r^{-1}(x)} (\delta_{\alpha_1} * p_{jk})(\beta) \delta_{\alpha_2}(\beta) \\ &= (\delta_{\alpha_1} * p_{jk})(\alpha_2) = \sum_{\eta \in r^{-1}(x)} \delta_{\alpha_1}(\eta) p_{jk}(\eta^{-1} \alpha_2) = p_{jk}(\alpha_1^{-1} \alpha_2) \end{aligned}$$

for all $\alpha_1, \alpha_2 \in r^{-1}(x)$ and all $j, k \in \{1, \dots, m\}$. Now, for any $\eta \in \mathcal{H}$, we can find some $x \in \mathcal{H}^{(0)}$ and $\alpha, \beta \in r^{-1}(x)$ such that $\eta = \alpha^{-1}\beta$. Hence, $p_{jk}(\eta) = p_{jk}(\alpha^{-1}\beta) = \delta_{j,k} \delta_{\alpha, \beta}$. This implies that the matrix $p \in M_m(C_r^*(\mathcal{H}))$ is the identity matrix. Hence $W \xrightarrow{\cong} C_r^*(\mathcal{H})^{\oplus m}$. \square

Theorem 3.8 *Let $p = p^* = p^2 \in M_n(C_r^*(\mathcal{G}))$. The finitely generated and projective module $pL^2(\mathcal{Z})^{\oplus n}$ with frame $\{v_j\}_{j=1}^n$ is isomorphic to the free module $C_r^*(\mathcal{H})^{\oplus m}$ if and only if for all $x \in \mathcal{H}^{(0)}$, $\{v_1^\alpha, \dots, v_m^\alpha\}_{\alpha \in r^{-1}(x)}$ is an orthonormal basis of $\pi_x(p)\mathfrak{h}_x^{\oplus n}$.*

Proof Suppose there is a unitary isomorphism of C^* -modules $\varphi : pL^2(Z)^{\oplus n} \xrightarrow{\cong} C_r^*(\mathcal{H})^{\oplus m}$. It is clear that $\{1_j\}_{j=1}^m$ is right-frame of $C_r^*(\mathcal{H})^{\oplus m}$ and because φ respects the inner-product structure $\{v_j\}_{j=1}^m$ is a frame of $pL^2(Z)^{\oplus n}$ with $v_j = \varphi^{-1}(1_j)$. We know that $\{v_j^\alpha\}$ is a normalised tight frame by Proposition 3.6 and we see that

$$\begin{aligned} \langle (v_j)_x^{\alpha_1}, (v_k)_x^{\alpha_2} \rangle_x &= \omega_x((v_j \cdot \delta_{\alpha_1}^* \mid v_k \cdot \delta_{\alpha_2}^*)_{\mathcal{H}}) = \omega_x(\delta_{\alpha_1}(\varphi^{-1}(1_k) \mid \varphi^{-1}(1_j))_{\mathcal{H}} \delta_{\alpha_2}^*) \\ &= \delta_{j,k} \omega_x(\delta_{\alpha_1} * \delta_{\alpha_2}^*) = \delta_{j,k} \delta_{\alpha_1, \alpha_2}. \end{aligned}$$

Hence the tight frame is orthonormal and so is an orthonormal basis. The converse statement follows from Proposition 3.7. \square

Remark 3.9 (*K*-theoretic interpretation) Theorem 3.8 has an interpretation via the *K*-theory of the groupoid C^* -algebras. Using the $*$ -isomorphism $C_r^*(\mathcal{G}) \cong \mathbb{K}_{C_r^*(\mathcal{H})}(L^2(Z))$, we can naturally consider any projection $p \in M_n(C_r^*(\mathcal{G}))$ as a finite-rank operator on $L^2(Z)^{\oplus n}_{\mathcal{H}}$. Therefore we can also consider p as a projection in $M_m(C_r^*(\mathcal{H}))$ for some m , with corresponding *K*-theory class $[p] \in K_0(C_r^*(\mathcal{H}))$. If $pL^2(Z)_{\mathcal{H}} \cong C_r^*(\mathcal{H})^{\oplus m}$, then $[p] = m[1] \in K_0(C_r^*(\mathcal{H}))$ and the projection p is trivial in reduced *K*-theory.

Let us now consider the case where there are dense pre- C^* -algebras $\mathcal{A} \subset C_r^*(\mathcal{G})$ and $\mathcal{B} \subset C_r^*(\mathcal{H})$, which will allow us to consider C^* -modules and Hilbert space frames with additional regularity as in Proposition 2.13. Thus suppose there are families of commuting $*$ -derivations

$$\{\partial_j^{\mathcal{H}}\}_{j=1}^d : C_c(\mathcal{H}) \rightarrow C_c(\mathcal{H}), \quad \{\partial_j^{\mathcal{G}}\}_{j=1}^d : C_c(\mathcal{G}) \rightarrow C_c(\mathcal{G}),$$

as well as a family of maps $\nabla_j : C_c(Z) \rightarrow C_c(Z)$ such that for each j and all $a \in C_c(\mathcal{G})$, $b \in C_c(\mathcal{H})$, $\xi, \eta \in C_c(Z)$, Eqs. (3) and (4) hold. Denote by $\mathcal{S}_k(\mathcal{G})$ and $\mathcal{S}_k(\mathcal{H})$ the degree k Fréchet completions of $C_c(\mathcal{G})$ and $C_c(\mathcal{H})$ in these seminorms (Proposition 2.10), and by $\mathcal{S}_k(Z)$ the degree k Fréchet completion of $C_c(Z)$. We often write \mathcal{S} for \mathcal{S}_∞ in each of these cases. By Proposition 2.13, $\mathcal{S}_k(\mathcal{G})\mathcal{S}_k(Z)\mathcal{S}_k(\mathcal{H})$ is a pre-Morita equivalence bimodule for all $k = 0, \dots, \infty$.

Theorem 3.10 *Let $k = 0, \dots, \infty$ and $p = p^* = p^2 \in M_n(\mathcal{S}_k(\mathcal{G}))$. Then there is a finite frame $\{v_j\}_{j=1}^m$ of $p\mathcal{S}_k(Z)^{\oplus n}_{\mathcal{S}_k(\mathcal{H})}$ such that for all $x \in \mathcal{H}^{(0)}$ the normalised tight frame $\{v_1^\beta, \dots, v_m^\beta\}_{\beta \in r^{-1}(x)}$ of $\pi_x(p)\mathfrak{h}_x^{\oplus n}$ is finite with respect to the seminorms $\|\cdot\|_{l,x}$ on \mathfrak{h}_x for $0 \leq l \leq k$. There is an isomorphism $p\mathcal{S}_k(Z)^{\oplus n}_{\mathcal{S}_k(\mathcal{H})} \cong \mathcal{S}_k(\mathcal{H})^{\oplus m}$ if and only if for all $x \in \mathcal{H}^{(0)}$, $\{v_1^\beta, \dots, v_m^\beta\}_{\beta \in r^{-1}(x)}$ is an orthonormal basis.*

Proof Because $v_j \cdot \delta_\beta^* \subset \mathcal{S}_k(Z)$ for any $x \in \mathcal{H}^{(0)}$ and $\beta \in r^{-1}(x)$, the first statement then follows from Lemma 2.9 and Proposition 2.13. Lemma 2.9 and the fact that $\mathcal{S}_k(\mathcal{H})$ is a pre- C^* -algebra give that $p\mathcal{S}_k(\mathcal{H})^{\oplus n}$ is a free $\mathcal{S}_k(\mathcal{H})$ -module if and only if its C^* -completion is a free $C_r^*(\mathcal{H})$ -module.

If $p\mathcal{S}_k(Z)^{\oplus n}_{\mathcal{S}_k(\mathcal{H})} \cong \mathcal{S}_k(\mathcal{H})^{\oplus m}$, then there is a finite frame $\{v_1, \dots, v_m\} \subset p\mathcal{S}_k(Z)^{\oplus n}$ such that $(v_i \mid v_j)_{\mathcal{B}} = \delta_{i,j} 1_{\mathcal{S}_k(\mathcal{H})}$. By part (ii) of Lemma 3.3 we therefore see that $\{v_1, \dots, v_m^\alpha\}_{\alpha \in r^{-1}(x)}$ is an orthonormal system and hence must be an orthonormal

basis. For the converse, we can use Proposition 3.7 and the fact that the finite $C_r^*(\mathcal{H})$ -module frame can be approximated arbitrarily well by a finite $S_k(\mathcal{H})$ -module frame of the same size. □

4 Frames of Translates and Wannier Bases from Twisted Transversals

Here we consider our framework in the case of groupoid equivalences that come from abstract transversals with an additional twist by a normalised groupoid 2-cocycle. In Sect. 5 we apply these results to the Delone transversal groupoid with twist coming from a magnetic field.

4.1 Twisted Morita Equivalence

Definition 4.1 A topological groupoid \mathcal{G} admits an abstract transversal if there is a closed subset $X \subset \mathcal{G}^{(0)}$ such that

- (i) X meets every orbit of the \mathcal{G} -action on $\mathcal{G}^{(0)}$;
- (ii) for the relative topologies on X and

$$\mathcal{G}_X := \{\gamma \in \mathcal{G} : s(\gamma) \in X\} \subset \mathcal{G},$$

the restrictions $r : \mathcal{G}_X \rightarrow \mathcal{G}^{(0)}$ and $s : \mathcal{G}_X \rightarrow X$ are open maps.

Given an abstract transversal $X \subset \mathcal{G}^{(0)}$, $\mathcal{G} \xleftarrow{r} \mathcal{G}_X \xrightarrow{s} \mathcal{H}$ is a \mathcal{G} - \mathcal{H} groupoid equivalence for $\mathcal{H} = \{\gamma \in \mathcal{G}_X : r(\gamma) \in X\}$, see [34, Example 2.7]. Examples of abstract transversals include transitive groupoids and groupoids from foliations.

We now fix a locally compact, second countable and Hausdorff groupoid \mathcal{G} such that $X \subset \mathcal{G}^{(0)}$ is compact and admits an abstract transversal \mathcal{G}_X with \mathcal{H} étale. We also fix a normalised groupoid 2-cocycle $\sigma_{\mathcal{G}}$ on \mathcal{G} , i.e. $\sigma_{\mathcal{G}}(\gamma, \gamma^{-1}) = 1$ for all $\gamma \in \mathcal{G}$. The restriction of $\sigma_{\mathcal{G}}$ then gives a groupoid 2-cocycle $\sigma_{\mathcal{H}}$ for \mathcal{H} . The 2-cocycle twists the module structure

$$\begin{aligned} (f \cdot e)(z) &= \int_{\mathcal{G}} f(\gamma)e(\gamma^{-1}z) \sigma_{\mathcal{G}}(\gamma, \gamma^{-1}z) \, d\nu^{r(z)}(\gamma), \quad e \in C_c(\mathcal{G}_X), f \in C_c(\mathcal{G}, \sigma_{\mathcal{G}}), \\ (e \cdot g)(z) &= \sum_{\eta \in r^{-1}(s(z))} e(z\eta)g(\eta^{-1}) \sigma_{\mathcal{G}}(z\eta, \eta^{-1}), \quad e \in C_c(\mathcal{G}_X), g \in C_c(\mathcal{H}, \sigma_{\mathcal{H}}), \\ (e_1 | e_2)_{\mathcal{H}}(\eta) &= \int_{\mathcal{G}} \overline{e_1(\gamma^{-1}z)} e_2(\gamma^{-1}z\eta) \sigma_{\mathcal{G}}(z^{-1}\gamma, \gamma^{-1}z\eta) \, d\nu^{r(z)}(\gamma), \quad r(\eta) = s(z). \end{aligned}$$

Proposition 2.23 can be extended to the case of such simple twists. The case of general groupoid twists arising from S^1 -extensions is handled via equivalence of Fell bundles, see [33,39].

Proposition 4.2 *The module $C_c(\mathcal{G}_X)$ is a pre-Morita equivalence bimodule for $C_c(\mathcal{G}, \sigma_{\mathcal{G}})$ and $C_c(\mathcal{H}, \sigma_{\mathcal{H}})$. Consequently, $C_r^*(\mathcal{G}, \sigma_{\mathcal{G}})$ and $C_r^*(\mathcal{H}, \sigma_{\mathcal{H}})$ are Morita equivalent.*

4.2 Twisted Frames and Wannier Bases

We now outline the minor changes required to recover the results of Sect. 3 to the case of twisted algebras. One advantage of restricting to normalised cocycles is that in the Hilbert space localisation, the inner-product simplifies. Namely, for $x \in X = \mathcal{H}^{(0)}$,

$$\begin{aligned} \langle (e_1)_x, (e_2)_x \rangle_x &= (e_2 | e_1)_{\mathcal{H}}(x) = \int_{\mathcal{G}} \overline{e_2(\gamma^{-1}z)} e_1(\gamma^{-1}z) \sigma_{\mathcal{G}}(z^{-1}\gamma, \gamma^{-1}z) \, d\nu^{r \circ s^{-1}(x)}(\gamma) \\ &= \int_{\mathcal{G}} \overline{e_2(\gamma^{-1}z)} e_1(\gamma^{-1}z) \, d\nu^{r \circ s^{-1}(x)}(\gamma). \end{aligned}$$

Lemma 4.3 *Let $x \in X$ and $\alpha \in r^{-1}(x)$. For $e_x \in \mathfrak{h}_x$, define $e_x^\alpha(y) = e(y\alpha) \sigma_{\mathcal{G}}(y\alpha, \alpha^{-1})$.*

- (i) *If $e \in L^2(\mathcal{G}_X, \sigma)$ is such that $(e | e)_{\mathcal{H}} = 1_{\mathcal{H}}$, then $\{e_x^\alpha\}_{\alpha \in r^{-1}(x)}$ is an orthonormal system in \mathfrak{h}_x .*
- (ii) *Given $e_1, e_2, \xi \in \mathcal{G}L^2(\mathcal{G}_X, \sigma)_{\mathcal{H}}$,*

$$(\Theta_{e_1, e_2}^R(\xi))_x = \sum_{\alpha \in r^{-1}(x)} (e_1)_x^\alpha \langle \xi_x, (e_2)_x^\alpha \rangle_x$$

Proof. Like in the untwisted case, we see that for $y \in s^{-1}(x)$,

$$(e \cdot \delta_\alpha^*)(y) = \sum_{\beta \in r^{-1}(x)} e(y\beta) \delta_\alpha(\beta) \sigma_{\mathcal{G}}(y\beta, \beta^{-1}) = e(y\alpha) \sigma_{\mathcal{G}}(y\alpha, \alpha^{-1}).$$

Hence we can compute

$$\langle e_x^{\alpha_1}, e_x^{\alpha_2} \rangle_x = \omega_x(\delta_{\alpha_1} * \delta_{\alpha_2}^*) = \sum_{\beta \in r^{-1}(x)} \delta_{\alpha_1}(\beta) \delta_{\alpha_2}(\beta) \sigma_{\mathcal{H}}(\beta, \beta^{-1}) = \delta_{\alpha_1, \alpha_2}.$$

For part (ii), can follow the same argument as Lemma 3.3. For $y \in s^{-1}(x)$ and $u = z\alpha^{-1}$ with $u\alpha = zs(\alpha) = zs(z) = z$ and $r(u) = r(z\alpha^{-1}) = r(z)$,

$$\begin{aligned} &(\Theta_{e_1, e_2}^R(\xi))(y) \\ &= \sum_{\alpha \in r^{-1}(x)} e_1(y\alpha) \sigma_{\mathcal{G}}(y\alpha, \alpha^{-1}) \int_{\mathcal{G}} \overline{e_2(\gamma^{-1}u\alpha)} \xi(\gamma^{-1}u) \sigma_{\mathcal{G}}(\alpha^{-1}u^{-1}\gamma, \gamma^{-1}u) \, d\nu^{r(u)}(\gamma) \\ &= \sum_{\alpha \in r^{-1}(x)} e_1(y\alpha) \sigma_{\mathcal{G}}(y\alpha, \alpha^{-1}) \int_{\mathcal{G}} \overline{e_2(\gamma^{-1}u\alpha)} \sigma_{\mathcal{G}}(\gamma^{-1}u\alpha, \alpha^{-1}) \xi(\gamma^{-1}u) \, d\nu^{r(u)}(\gamma) \\ &= \sum_{\alpha \in r^{-1}(x)} (e_1)_x^\alpha(y) \langle (e_2)_x^\alpha, \xi_x \rangle_x, \end{aligned}$$

where we used the 2-cocycle identity

$$\sigma_{\mathcal{G}}(\alpha^{-1}, u^{-1}\gamma) \sigma_{\mathcal{G}}(\alpha^{-1}u^{-1}\gamma, \gamma^{-1}u) = \sigma_{\mathcal{G}}(\alpha^{-1}, s(\gamma^{-1}u)) \sigma_{\mathcal{G}}(u^{-1}\gamma, \gamma^{-1}u) = 1$$

which implies that $\sigma_{\mathcal{G}}(\alpha^{-1}u^{-1}\gamma, \gamma^{-1}u) = \overline{\sigma_{\mathcal{G}}(\alpha^{-1}, u^{-1}\gamma)}$. Then using Lemma 2.18

$$\overline{\sigma_{\mathcal{G}}(\alpha^{-1}, u^{-1}\gamma)} = \sigma_{\mathcal{G}}(\gamma^{-1}u, \alpha) = \overline{\sigma_{\mathcal{G}}(\gamma^{-1}u\alpha, \alpha^{-1})}.$$

This completes the proof. □

Given elements $w_j^{\alpha_1}, w_k^{\alpha_2} \in \mathfrak{h}_x = L^2(\mathcal{G}_X, dv^{r\circ s^{-1}(x)})$, we can compute that

$$\begin{aligned} \langle w_j^{\alpha_1}, w_k^{\alpha_2} \rangle_x &= \omega_x(\delta_{\alpha_1}(w_j | w_k)\mathcal{H}\delta_{\alpha_2}^*) = (\delta_{\alpha_1} * (w_j | w_k)\mathcal{H})(\alpha_2) \\ &= (w_j | w_k)\mathcal{H}(\alpha_1^{-1}\alpha_2) \sigma_{\mathcal{H}}(\alpha_1, \alpha_1^{-1}\alpha_2). \end{aligned}$$

Hence, if $\langle w_j^{\alpha_1}, w_k^{\alpha_2} \rangle_x = \delta_{j,k} \delta_{\alpha_1, \alpha_2}$ for some $\alpha_1, \alpha_2 \in r^{-1}(x)$, the 2-cocycle term will be 1.

At this point we can follow the same arguments as those in Theorem 3.4 and Sect. 3.3, so we summarise our results.

Proposition 4.4 *If $(e_j)_{j \in J}$ is a right C^* -module frame of $\mathcal{G}L^2(\mathcal{G}_X, \sigma)_{\mathcal{H}}$, then for all $x \in X$ the set $\{(e_j)_x^\alpha\}$ for $j \in J$ and $\alpha \in r^{-1}(x)$ is a normalised tight frame of \mathfrak{h}_x .*

We can again define a representation of $\pi_x : C_r^*(\mathcal{G}, \sigma_{\mathcal{G}}) \rightarrow \mathcal{B}(\mathfrak{h}_x)$, where

$$\begin{aligned} (\pi_x(f)\xi_x)(y) &= \int_{\mathcal{G}} f(\gamma)\xi(\gamma^{-1}y) \sigma_{\mathcal{G}}(\gamma, \gamma^{-1}y) dv^{r\circ s^{-1}(x)}(\gamma), \\ f &\in C_c(\mathcal{G}), \xi \in C_c(\mathcal{G}_X). \end{aligned}$$

We consider the case of pre- C^* -algebras $\mathcal{S}_k(\mathcal{G}, \sigma) \subset C_r^*(\mathcal{G}, \sigma_{\mathcal{G}})$ and $\mathcal{S}_k(\mathcal{H}, \sigma) \subset C_r^*(\mathcal{H}, \sigma_{\mathcal{H}})$ defined from families of derivations with a pre-Morita equivalence bimodule $\mathcal{S}_k(\mathcal{G}, \sigma)\mathcal{S}_k(\mathcal{G}_X, \sigma)\mathcal{S}_k(\mathcal{H}, \sigma)$ defined from a family of maps $\nabla_j : C_c(Z) \rightarrow C_c(Z)$ using the construction in Proposition 2.13.

Proposition 4.5 *Let $k = 0, \dots, \infty$ and $p = p^* = p^2 \in M_n(\mathcal{S}_k(\mathcal{G}, \sigma))$. There are elements $\{v_j\}_{j=1}^m \subset p\mathcal{S}_k(\mathcal{G}_X, \sigma)_{\mathcal{S}_k(\mathcal{H}, \sigma)}^{\oplus n}$ such that for all $x \in X$, $\{v_1^\beta, \dots, v_m^\beta\}_{\beta \in r^{-1}(x)}$ is a normalised tight frame of $\pi_x(p)\mathfrak{h}_x^{\oplus n}$ that is finite under the seminorms $\|\cdot\|_{l,x}$ on \mathfrak{h}_x for $0 \leq l \leq k$. This tight frame is an orthonormal basis for all $x \in \mathcal{H}^{(0)}$ if and only if $p\mathcal{S}_k(\mathcal{G}_X, \sigma)_{\mathcal{S}_k(\mathcal{H}, \sigma)}^{\oplus n} \cong \mathcal{S}_k(\mathcal{H}, \sigma)^{\oplus m}$.*

5 Frames of Translates and Wannier Bases for the Delone Groupoid

5.1 Delone Sets and the Transversal Groupoid

We review some of the material from [6] as outlined in [11]. We denote by $B(x; K) \subset \mathbb{R}^d$ the open ball centered at x with radius $K > 0$.

Definition 5.1 Let $\mathcal{L} \subset \mathbb{R}^d$ be discrete and infinite and fix $0 < r < R$.

(1) \mathcal{L} is r -uniformly discrete if $|B(x; r) \cap \mathcal{L}| \leq 1$ for all $x \in \mathbb{R}^d$.

(2) \mathcal{L} is R -relatively dense if $|B(x; R) \cap \mathcal{L}| \geq 1$ for all $x \in \mathbb{R}^d$.

An r -uniformly discrete and R -relatively dense set \mathcal{L} is called an (r, R) -Delone set.

Proposition 5.2 [4, Sect. 3.2] *The set of (r, R) -Delone sets is a compact and metrizable space. Let d_H denote the Hausdorff distance between sets. A neighbourhood base at $\mathcal{L} \in \text{Del}_{(r,R)}$ is given by the sets*

$$U_{\varepsilon, M}(\mathcal{L}) = \{ \mathcal{L}' \in \text{Del}_{(r,R)} : d_H(\mathcal{L} \cap B(0; M), \mathcal{L}' \cap B(0; M)) < \varepsilon \}, \quad M, \varepsilon > 0.$$

The set of Delone sets $\text{Del}_{(r,R)}$ is clearly invariant under translations and rotations.

Definition 5.3 Let Λ be an (r, R) -Delone subset of \mathbb{R}^d . The *continuous hull* of Λ is the dynamical system $(\Omega_\Lambda, \mathbb{R}^d, T)$, where $\Omega_\Lambda \subset \text{Del}_{(r,R)}$ is the closure of the orbit of Λ under the translation action.

The continuous hull of Λ therefore gives a locally compact Hausdorff groupoid $\Omega_\Lambda \rtimes \mathbb{R}^d$. This groupoid admits a transversal in the sense of Definition 4.1.

Definition 5.4 The transversal of Λ is given by the set

$$\Omega_0 = \{ \mathcal{L} \in \Omega_\Lambda : 0 \in \mathcal{L} \},$$

We see that Ω_0 is a closed subset of Ω_Λ and so is compact.

Proposition 5.5 [21, Lemma 2] *Given a Delone set Λ with transversal Ω_0 , define the set*

$$\mathcal{G}_{\text{Del}} := \{ (\mathcal{L}, x) \in \Omega_0 \times \mathbb{R}^d : x \in \mathcal{L} \},$$

with maps

$$\begin{aligned} (\mathcal{L}, x)^{-1} &= (\mathcal{L} - x, -x), & (\mathcal{L}, x) \cdot (\mathcal{L} - x, y) &= (\mathcal{L}, x + y), \\ s(\mathcal{L}, x) &= \mathcal{L} - x, & r(\mathcal{L}, x) &= \mathcal{L} \end{aligned}$$

and unit space $\mathcal{G}^{(0)} = \Omega_0$. Then \mathcal{G}_{Del} is a Hausdorff étale groupoid in the relative topology inherited from $\Omega_0 \times \mathbb{R}^d$.

Notation Following the previous proposition, we will let \mathcal{G}_{Del} denote the étale groupoid from a Delone set. We let $\mathcal{F} = \Omega_\Lambda \rtimes \mathbb{R}^d$ be the crossed product groupoid.

Proposition 5.6 [11, Proposition 2.16] *Let $\mathcal{L} \subset \mathbb{R}^d$ be an (r, R) -Delone set with transversal Ω_0 and associated groupoid \mathcal{G}_{Del} . For $U \subset \Omega_0$ an open set, the sets*

$$V_{(U, y, \varepsilon)} := (U \times B(y; \varepsilon)) \cap \mathcal{G}_{\text{Del}} = \{ (\mathcal{L}, x) \in \Omega_0 \times \mathbb{R}^d : \mathcal{L} \in U, x \in \mathcal{L} \cap B(y; \varepsilon) \},$$

form a base for the topology on \mathcal{G}_{Del} . For $0 < \varepsilon < r/2$, the restriction $s : V_{(U, y, \varepsilon)} \rightarrow \Omega_0$ is a homeomorphism onto its image. Moreover, the set $\Omega_0 \subset \Omega_\Lambda$ is an abstract transversal and the groupoid $\mathcal{G}_{\text{Del}} \subset \Omega_\Lambda \rtimes \mathbb{R}^d$, with the subspace topology, is equivalent to $\Omega_\Lambda \rtimes \mathbb{R}^d$.

Let us now fix a normalised 2-cocycle $\sigma : (\Omega_\Lambda \rtimes \mathbb{R}^d)^{(2)} \rightarrow \mathbb{T}$, $\sigma(\gamma, \gamma^{-1}) = 1$ for all $\gamma \in \Omega_\Lambda \rtimes \mathbb{R}^d$, which also restricts to a 2-cocycle on \mathcal{G}_{Del} . Our main motivation to consider such twists comes from the following example.

Example 5.7 (Magnetic twists, Sect. 2.2 of [7]) Working with the continuous hull $\Omega_\Lambda \rtimes \mathbb{R}^d$, we can define a 2-cocycle, $\sigma : \mathcal{F}^{(2)} \rightarrow \mathbb{T}$ as follows. We first define a parametrised magnetic field as a continuous map $B : \Omega_\Lambda \rightarrow \wedge^2 \mathbb{R}^d$. Then we define

$$\sigma((\mathcal{L}, x), (\mathcal{L} - x, y)) = \exp(-i\Gamma_{\mathcal{L}}\langle 0, x, x + y \rangle), \quad \Gamma_{\mathcal{L}}\langle x, y, z \rangle = \int_{\langle x, y, z \rangle} B(\mathcal{L})$$

and $\langle x, y, z \rangle \subset \mathbb{R}^{2d}$ is the triangle with corners $x, y, z \in \mathbb{R}^d$. Hence $\Gamma_{\mathcal{L}}\langle 0, x, x + y \rangle$ measures the magnetic flux through the triangle defined by the points $0, x, x + y \in \mathbb{R}^d$. It is shown in [7] that σ is a well-defined 2-cocycle. We remark that σ will always be trivial for $d = 1$ and is normalised because

$$\sigma((\mathcal{L}, x), (\mathcal{L} - x, -x)) = \exp(-i\Gamma\langle 0, x, 0 \rangle) = 1.$$

If the magnetic field is constant over Ω_Λ , then our general flux equation can be described using a real-valued and skew-symmetric matrix B with

$$\sigma((\mathcal{L}, x), (\mathcal{L} - x, y)) = \exp(-i\langle x, B(x + y) \rangle) = \exp(-i\langle x, By \rangle).$$

The 2-cocycle σ on $\Omega_\Lambda \rtimes \mathbb{R}^d$ also restricts to a normalised 2-cocycle on \mathcal{G}_{Del} , where we note that if $((\mathcal{L}, x), (\mathcal{L} - x, y)) \in \mathcal{G}_{\text{Del}}^{(2)}$, the points $0, x, x + y \in \mathcal{L}$ and so $\Gamma_{\mathcal{L}}\langle 0, x, x + y \rangle$ gives a flux through the triangle with points in \mathcal{L} .

Remark 5.8 Given a groupoid 2-cocycle $\sigma : (\Omega_\Lambda \rtimes \mathbb{R}^d)^{(2)} \rightarrow \mathbb{T}$, the twisted groupoid C^* -algebra $C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma)$ is canonically isomorphic to the twisted crossed product $C(\Omega_\Lambda) \rtimes_{\sigma'} \mathbb{R}^d$, where

$$\sigma' : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathcal{U}(C(\Omega_\Lambda)), \quad \sigma'(x, y) = \sigma((\mathcal{L}, x), (\mathcal{L} - x, y)).$$

Regularity, smoothness and decay properties of functions on $\Omega_\Lambda \rtimes \mathbb{R}^d$ and \mathcal{G}_{Del} are encoded via the groupoid 1-cocycles

$$c_k : \Omega_\Lambda \rtimes \mathbb{R}^d \rightarrow \mathbb{R}, \quad c(\mathcal{L}, x) = x_k, \quad k \in \{1, \dots, d\}.$$

and their restrictions to \mathcal{G}_{Del} . It is shown in [11, Proposition 2.17] that the groupoid cycles are *exact*, in that $c_k^{-1}(0)$ has a Haar system and c_k is a quotient map onto its image.

Given the cocycles $c_j : \Omega_\Lambda \rtimes \mathbb{R}^d \rightarrow \mathbb{R}^d$, we obtain families of commuting derivations $\{\partial_j\}_{j=1}^d$ on both $C_c(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma)$ and $C_c(\mathcal{G}_{\text{Del}}, \sigma)$ given by $(\partial_j f)(\mathcal{L}, x) = x_j f(\mathcal{L}, x)$. For $k = 0, \dots, \infty$, we obtain pre- C^* -algebra completions \mathcal{A}_k of $C_c(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma)$ and \mathcal{B}_k of $C_c(\mathcal{G}_{\text{Del}}, \sigma)$ using Proposition 2.10.

5.2 The Transversal \mathcal{G}_{Del} -Space and Its Localisation

Following Sect. 4, we consider the space

$$\mathcal{F}_{\Omega_0} := \{(\mathcal{L}, x) \in \Omega_\Lambda \rtimes \mathbb{R}^d : x \in \mathcal{L}\}, \quad s : \mathcal{F}_{\Omega_0} \rightarrow \Omega_0, \quad s(\mathcal{L}, x) = \mathcal{L} - x,$$

which implements a groupoid equivalence between $\mathcal{F} = \Omega_\Lambda \rtimes \mathbb{R}^d$ and \mathcal{G}_{Del} . Thus $C_c(\mathcal{F}_{\Omega_0})$ is a pre-Morita equivalence bimodule for $C_c(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma)$ and $C_c(\mathcal{G}_{\text{Del}}, \sigma)$ and can be completed into the Morita equivalence bimodule ${}_{\mathcal{F}}L^2(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{G}_{\text{Del}}}$ between $C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma)$ and $C_r^*(\mathcal{G}_{\text{Del}}, \sigma)$.

Lemma 5.9 *The restrictions of the cocycles $c_j : \Omega_\Lambda \rtimes \mathbb{R}^d \rightarrow \mathbb{R}$ to \mathcal{F}_{Ω_0} define maps*

$$\nabla_j : C_c(\mathcal{F}_{\Omega_0}) \rightarrow C_c(\mathcal{F}_{\Omega_0}), \quad \nabla_j(f)(\mathcal{L}, x) := x_j f(\mathcal{L}, x).$$

For all $a \in C_c(\mathcal{F}, \sigma)$, $b \in C_c(\mathcal{G}_{\text{Del}}, \sigma)$ and $\xi, \eta \in C_c(\mathcal{F}_{\Omega_0})$ we have

$$\begin{aligned} \nabla_j(a \cdot \xi \cdot b) &= \partial_j(a) \cdot \xi \cdot b + a \cdot \nabla_j(\xi) \cdot b + a \cdot \xi \cdot \partial_j(b), \\ \partial_j(\xi \mid \eta)_{\mathcal{G}_{\text{Del}}} &= (\xi \mid \nabla_j(\eta))_{\mathcal{G}_{\text{Del}}} - (\nabla_j(\xi) \mid \eta)_{\mathcal{G}_{\text{Del}}}. \end{aligned}$$

Consequently, for all $k = 0, \dots, \infty$, the space $C_c(\mathcal{F}_{\Omega_0})$ can be completed into a pre-Morita equivalence bimodule ${}_{\mathcal{A}_k}S_k(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{B}_k}$ for the pre- C^* -algebras $\mathcal{A}_k \subset C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma)$ and $\mathcal{B}_k \subset C_r^*(\mathcal{G}_{\text{Del}}, \sigma)$.

Proof As \mathcal{F}_{Ω_0} and \mathcal{G}_{Del} are subspaces of $\Omega_\Lambda \rtimes \mathbb{R}^d$ and the bimodule structure and inner product are induced by the convolution product in $C_c(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma)$, the required identities follow from the fact that multiplication by x_j is a derivation of $C_c(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma)$. The result now follows from Propositions 2.10 and 2.13. \square

For every $\mathcal{L} \in \Omega_0$, there is a state $\omega_{\mathcal{L}}$ on $C_r^*(\mathcal{G}_{\text{Del}}, \sigma)$ such that $\omega_{\mathcal{L}}(f) = f(\mathcal{L}, 0)$ for $f \in C_c(\mathcal{G}_{\text{Del}}, \sigma)$. Note that

$$\omega_{\mathcal{L}}(f^*f) = \sum_{y \in \mathcal{L}} |f(\mathcal{L} - y, -y)|^2, \quad \omega_{\mathcal{L}}(1_{C_r^*(\mathcal{G}_{\text{Del}})}) = 1$$

and so $\omega_{\mathcal{L}}$ is faithful. From this point, all results from Sect. 4 apply to the Delone groupoid setting. Though for concreteness, we highlight some key aspects of this example.

Lemma 5.10 *For every $\mathcal{L} \in \Omega_0$, the localised Hilbert space $\mathfrak{h}_{\mathcal{L}} \cong L^2(\mathbb{R}^d)$.*

Proof We define a map $\beta_{\mathcal{L}} \rightarrow L^2(\mathbb{R}^d)$ that agrees with the localised Hilbert space inner product. Namely, we consider $\beta_{\mathcal{L}} : L^2(\mathcal{F}_{\Omega_0}) \rightarrow L^2(\mathbb{R}^d)$, given by $[\beta_{\mathcal{L}}(\xi)](x) = \xi(\mathcal{L} - x, -x)$ for almost all x . To see why this is true, we first note that for any $\mathcal{L} \in \Omega_0$, $s^{-1}(\mathcal{L}) = \{(\mathcal{L} - x, -x)\}_{x \in \mathbb{R}^d}$ and the measure on $s^{-1}(\mathcal{L})$ is just the Lebesgue measure on \mathbb{R}^d . We also see that

$$\langle \xi_1, \xi_2 \rangle_{\mathcal{L}} = (\xi_1 \mid \xi_2)_{C_r^*(\mathcal{G}_{\text{Del}})}(\mathcal{L}, 0) = \int_{\mathbb{R}^d} \overline{\xi_1(\mathcal{L} - y, -y)} \xi_2(\mathcal{L} - y, -y) \, dy.$$

Hence we see there is a canonical identification of $\mathfrak{h}_{\mathcal{L}}$ with $\beta_{\mathcal{L}}[L^2(\mathcal{F}_{\Omega_0})] \cong L^2(\mathbb{R}^d)$. □

For any $\xi \in \mathcal{F}L^2(\mathcal{F}_{\Omega_0})_{\mathcal{G}_{\text{Del}}}$, let $\xi_{\mathcal{L}} \in \mathfrak{h}_{\mathcal{L}}$ be its localisation. Because $\nabla^\alpha \xi(\mathcal{L}, x) = x^\alpha \xi(\mathcal{L}, x)$ for $\alpha \in \mathbb{N}^d$ and $\xi \in C_c(\mathcal{F}_{\Omega_0})$, Proposition 2.13 gives the following.

Lemma 5.11 *For any $k = 0, \dots, \infty$ and $\mathcal{L} \in \Omega_0$, every element in dense subspace $\beta_{\mathcal{L}}[\mathcal{S}_k(\mathcal{F}_{\Omega_0})] \subset \mathfrak{h}_{\mathcal{L}}$ has polynomial decay of at least degree k ,*

$$\|x^\alpha \xi_{\mathcal{L}}\|_{\mathfrak{h}_{\mathcal{L}}} \leq C, \quad \xi \in \mathcal{S}_k(\mathcal{F}_{\Omega_0}), \quad \alpha \in \mathbb{N}^d, \quad |\alpha| \leq k.$$

Lemma 5.12 *Let χ be a smooth and real-valued bump-function such that $\text{supp}(\chi) \subset B(0; r/2)$, $\chi(x) = \chi(-x)$, $\chi(0) = 1$ and $\|\chi\|_2 = 1$.*

- (i) *Extend χ to a function $\chi \in C_c(\mathcal{F}_{\Omega_0})$ such that $\chi(\mathcal{L}, x) = \chi(x)$. Then $(\chi \mid \chi)_{\mathcal{G}_{\text{Del}}} = 1_{\mathcal{G}_{\text{Del}}}$.*
- (ii) *Given $p \in \mathbb{R}^d$, define the function $\chi_p \in C_c(\mathcal{G}_{\text{Del}}, \sigma)$ by $\chi_p(\mathcal{L}, x) = \chi(x - p)$. Then for any $\mathcal{L} \in \Omega_0$ and $p, q \in \mathcal{L}$, $\omega_{\mathcal{L}}(\chi_p * \chi_q^*) = \delta_{p,q}$.*

Proof For part (i) we compute

$$(\chi \mid \chi)_{\mathcal{G}_{\text{Del}}}(\mathcal{L}, y) = \int_{\mathbb{R}^d} \chi(\mathcal{L} - z, -z) \chi(\mathcal{L} - z, y - z) \sigma((\mathcal{L}, z), (\mathcal{L} - z, y - z)) \, dz.$$

The integral will be zero unless $-z, y - z \in \mathcal{L} - z \cap B(0; r/2)$. But because $y \in \mathcal{L}$, $\mathcal{L} \in \Omega_0$ and \mathcal{L} is uniformly r -discrete, this will only happen when $y = 0$. Then, because the 2-cocycle is 1 when $y = 0$ and $\|\chi\|_2 = 1$, $(\chi \mid \chi)_{\mathcal{G}_{\text{Del}}}(\mathcal{L}, y) = \delta_{y,0} = 1_{\mathcal{G}_{\text{Del}}}(\mathcal{L}, y)$.

For part (ii) we compute that

$$\begin{aligned} (\chi_p * \chi_q^*)(\mathcal{L}, 0) &= \sum_{y \in \mathcal{L}} \chi_p(\mathcal{L}, y) \overline{\chi_q(\mathcal{L}, y)} \sigma((\mathcal{L}, y), (\mathcal{L} - y, -y)) \\ &= \sum_{y \in \mathcal{L}} \chi(y - p) \chi(y - q) = \chi(p - q) = \delta_{p,q} \end{aligned}$$

where we have used that $\text{supp}(\chi) \subset B(0, r/2)$ and \mathcal{L} is r -discrete. □

Given $(\mathcal{L}, y) \in \mathcal{G}_{\text{Del}}$, define the action

$$\xi^{(\mathcal{L}, y)}(x) = \xi(\mathcal{L} - x, y - x) \sigma((\mathcal{L} - x, y - x), (\mathcal{L} - y, -y)) \in \mathfrak{h}_{\mathcal{L}}, \quad \xi \in L^2(\mathcal{F}_{\Omega_0}).$$

In the case that σ is comes from a magnetic twist that is constant over the unit space Ω_Λ , we see that $\xi^{(\mathcal{L}, y)}(x) = e^{-i\langle x, By \rangle} \xi(\mathcal{L} - x, y - x)$ with B a real-valued skew-adjoint matrix describing the magnetic field. Lemmas 4.3 and 5.12 now give the following.

- Lemma 5.13** (i) *Recall the functions $\{\chi_p\}$ from Lemma 5.12. Then for any $\xi \in L^2(\mathcal{F}_{\Omega_0})$ and $(\mathcal{L}, p) \in \mathcal{G}_{\text{Del}}$, $\xi^{(\mathcal{L}, p)} = (\xi \cdot \chi_p^*)_{\mathcal{L}}$.*
- (ii) *Let $e \in \mathcal{F}L^2(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{G}_{\text{Del}}}$ be such that $(e \mid e)_{C^*_r(\mathcal{G}_{\text{Del}})} = 1_{C^*_r(\mathcal{G}_{\text{Del}})}$. Then for any $\mathcal{L} \in \Omega_0$ the set $\{e^{(\mathcal{L}, y)}\}_{y \in \mathcal{L}}$ is an orthonormal system in $\mathfrak{h}_{\mathcal{L}}$.*

Proposition 5.14 *Let $(e_j)_{j \in J}$ be a (countable) right frame of $\mathcal{F}L^2(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{G}_{\text{Del}}}$. Then for all $\mathcal{L} \in \Omega_0$, the set $\{e_j^{(\mathcal{L}, y)}\}_{j \in J, y \in \mathcal{L}}$ is a normalised tight frame for $\mathfrak{h}_{\mathcal{L}} \cong L^2(\mathbb{R}^d)$. If $(e_j)_{j \in J} \subset \mathcal{A}_k \mathcal{S}_k(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{B}_k}$ for $k = 0, \dots, \infty$, then the normalised tight frame $\{e_j^{(\mathcal{L}, y)}\}$ has polynomial decay of at least order k .*

Proof The first statement is a special case of Proposition 4.4. Lemma 5.11 ensures that the elements $e_j^{(\mathcal{L}, 0)}(x) = e_j(\mathcal{L} - x, -x)$ have polynomial decay of at least degree k . The translation $e_j^{(\mathcal{L}, y)}(x) = e_j(\mathcal{L} - x, y - x) \sigma((\mathcal{L} - x, y - x), (\mathcal{L} - y, -y))$ will have the same decay properties for all $y \in \mathcal{L}$. \square

We can define an action $\pi_{\mathcal{L}} : C(\Omega_{\Lambda}) \rtimes \mathbb{R}^d$ on the localisation Hilbert space $\mathfrak{h}_{\mathcal{L}} \cong L^2(\mathbb{R}^d)$ by

$$(\pi_{\mathcal{L}}(f)\xi_{\mathcal{L}})(x) = (f \cdot \xi)_{\mathcal{L}}(x) = (f \cdot \xi)(\mathcal{L} - x, -x).$$

Explicitly, we can compute that

$$(f \cdot \xi)(\mathcal{L} - x, -x) = \int_{\mathbb{R}^d} f(\mathcal{L} - x, u - x) \xi_{\mathcal{L}}(u) \sigma((\mathcal{L} - x, u - x), (\mathcal{L} - u, -u)) du.$$

Let us now consider the localisation $\pi_{\mathcal{L}}(p)\mathfrak{h}_{\mathcal{L}}^{\oplus n}$ for $p \in M_n(C_r^*(\Omega_{\Lambda} \rtimes \mathbb{R}^d, \sigma))$ a projection.

Theorem 5.15 *Let $k = 0, \dots, \infty$ and $p = p^* = p^2 \in M_n(\mathcal{A}_k)$. Then there are elements $\{e_j\}_{j=1}^m \subset p\mathcal{S}_k(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{B}_k}^{\oplus n}$ such that for all $\mathcal{L} \in \Omega_0$, $\{e_1^{(\mathcal{L}, y)}, \dots, e_m^{(\mathcal{L}, y)}\}_{y \in \mathcal{L}}$ is a normalised tight frame of $\pi_{\mathcal{L}}(p)L^2(\mathbb{R}^d, \mathbb{C}^n)$ with polynomial decay of at least degree k . The normalised tight frame $\{e_1^{(\mathcal{L}, y)}, \dots, e_m^{(\mathcal{L}, y)}\}_{y \in \mathcal{L}}$ is an orthonormal basis for all $\mathcal{L} \in \Omega_0$ if and only if $p\mathcal{S}_k(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{B}_k}^{\oplus n} \cong \mathcal{B}_k^{\oplus m}$.*

For concreteness, we note that the case $p \in M_n(\mathcal{A})$ with $\mathcal{A} = \mathcal{A}_{\infty}$ gives a normalised tight frame with faster than polynomial decay.

Remark 5.16 (Invariance under homotopies of 2-cocycles) Let us briefly consider the stability of Theorem 5.15 under deformations of the groupoid 2-cocycle σ using results from Gillaspay [17]. Given the groupoid $\mathcal{F} = \Omega_{\Lambda} \rtimes \mathbb{R}^d$, we can consider the trivial bundle of groupoids $\mathcal{F} \times [0, 1]$ equipped with the product topology so that groupoid operations preserve the fibres and such that $\mathcal{F} \times [0, 1]$ is a locally compact Hausdorff groupoid. A homotopy of groupoid 2-cocycles is a groupoid 2-cocycle $\omega : (\mathcal{F} \times [0, 1])^{(2)} \rightarrow \mathbb{T}$, which will give rise to a family of 2-cocycles $\{\omega_t\}_{t \in [0, 1]}$ on \mathcal{F} that is continuous in t .

Because $\Omega_{\Lambda} \rtimes \mathbb{R}^d$ satisfies the Baum–Connes conjecture with coefficients, [17, Theorem 5.1] applies, which says that the evaluation map

$$q_t : C_r^*((\Omega_{\Lambda} \rtimes \mathbb{R}^d) \times [0, 1], \omega) \rightarrow C_r^*(\Omega_{\Lambda} \rtimes \mathbb{R}^d, \omega_t)$$

induces an isomorphism of K -theory groups. Composing this isomorphism with the Morita equivalence of $\Omega_{\Lambda} \rtimes \mathbb{R}^d$ with \mathcal{G}_{Del} , given a homotopy of 2-cocycles σ_{\bullet} on

$\mathcal{F} \times [0, 1]$ (and so on $\mathcal{G}_{\text{Del}} \times [0, 1]$ by restriction), we can consider finitely generated and projective modules P_\bullet over $C_r^*(\mathcal{G}_{\text{Del}} \times [0, 1], \sigma_\bullet)$. Composing with the evaluation map, P_0 is a free $C_r^*(\mathcal{G}_{\text{Del}}, \sigma_0)$ -module if and only if P_1 is a free $C_r^*(\mathcal{G}_{\text{Del}}, \sigma_1)$ -module.

Considering the magnetic twists of Example 5.7, we can easily construct homotopies of 2-cocycles via a continuous map $B_\bullet : \Omega_\Lambda \times [0, 1] \rightarrow \bigwedge^2 \mathbb{R}^d$ which restricts to a continuous path $\{B_t\}_{t \in [0,1]}$ of magnetic fields.

5.3 The Localisation Dichotomy

Decay properties of the normalised tight frame in Theorem 5.15 come from the seminorms on the dense submodules $\mathcal{S}_k(\mathcal{F}_{\Omega_0}, \sigma)$ over the pre- C^* -algebras $\mathcal{A}_k \subset C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma)$ and $\mathcal{B}_k \subset C_r^*(\mathcal{G}_{\text{Del}}, \sigma)$ for $k = 0, \dots, \infty$. Because each \mathcal{A}_k is a pre- C^* -algebra, given $p = p^* = p^2 \in M_n(C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma))$ and $0 < \varepsilon < 1$, there is some $p_k = p_k^* = p_k^2 \in M_n(\mathcal{A}_k) \subset M_n(C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma))$ such that $\|p - p_k\| < \varepsilon$ in C^* -norm as well as a unitary u_k in $M_n(C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma))$ such that $p_k = u_k^* p u_k$ (see [9, Sect. 4]). Consequently the finitely generated and projective $C^*(\mathcal{G}_{\text{Del}}, \sigma)$ -modules $pL^2(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{G}_{\text{Del}}}^{\oplus n}$ and $p_kL^2(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{G}_{\text{Del}}}^{\oplus n}$ are isomorphic. The module $p_kL^2(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{G}_{\text{Del}}}^{\oplus n}$ contains the dense submodule $p_k\mathcal{S}_k(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{B}_k}^{\oplus n}$. Note that we can choose $p_k = p_\infty$ for all $k \geq 1$.

By comparing C^* -module frames for $\mathcal{S}_\infty(\mathcal{F}_{\Omega_0}, \sigma) \subset \mathcal{S}_1(\mathcal{F}_{\Omega_0}, \sigma) \subset L^2(\mathcal{F}_{\Omega_0}, \sigma)$, we can prove a weak version of the localisation dichotomy considered in [29, Sect. 5 (arXiv version)].

Proposition 5.17 (Weak localisation dichotomy) *Let $p = p^* = p^2 \in M_n(C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \sigma))$ and $p_k = p_k^* = p_k^2 \in M_n(\mathcal{A}_k)$ be equivalent projections as above. The following statements are equivalent.*

- (i) *There is a C^* -module isomorphism $pL^2(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{G}_{\text{Del}}}^{\oplus n} \cong C_r^*(\mathcal{G}_{\text{Del}}, \sigma)^{\oplus m}$.*
- (ii) *There are elements $\{w_j\}_{j=1}^m \subset p_1\mathcal{S}_1(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{B}_1}^{\oplus n}$ such that for all $\mathcal{L} \in \Omega_0$, the collection $\{w_1^{(\mathcal{L}, y)}, \dots, w_m^{(\mathcal{L}, y)}\}_{y \in \mathcal{L}}$ is an orthonormal basis of $\pi_{\mathcal{L}}(p)L^2(\mathbb{R}^d, \mathbb{C}^n)$ and for all $y \in \mathcal{L}$,*

$$\sum_{j=1}^m \int_{\mathbb{R}^d} (1 + |x - y|^2) |w_j^{(\mathcal{L}, y)}(x)|^2 dx < \infty. \tag{8}$$

- (iii) *There exists $\{e_j\}_{j=1}^m \subset p_\infty\mathcal{S}_\infty(\mathcal{F}_{\Omega_0}, \sigma)_{\mathcal{B}_\infty}^{\oplus n}$ where for all $\mathcal{L} \in \Omega_0$, $\{e_1^{(\mathcal{L}, y)}, \dots, e_m^{(\mathcal{L}, y)}\}_{y \in \mathcal{L}}$ is an orthonormal basis of $\pi_{\mathcal{L}}(p)L^2(\mathbb{R}^d, \mathbb{C}^n)$ with faster than polynomial decay.*

Proof All statements except for Eq. (8) immediately follow from Theorem 5.15. To see Eq. (8), we note that the frame elements are such that $\|w_j \cdot \chi_p^*\|_1 < \infty$ for $j \in \{1, \dots, m\}$, $p \in \mathbb{R}^d$ and $\|\cdot\|_1$ the seminorm on $\mathcal{S}_1(\mathcal{F}_{\Omega_0}, \sigma)$,

$$\|\xi\|_1 = \|\xi\| + \sum_{l=1}^d \|\nabla_l \xi\|.$$

Passing to the localisation, the functions $w_j^{(\mathcal{L},y)}$ and $[\nabla_l(w_j \cdot \chi_y^*)]_{\mathcal{L}}(x) = (x_l - y_l)w_j^{(\mathcal{L},y)}(x)$ are in $\pi_{\mathcal{L}}(p)L^2(\mathbb{R}^d, \mathbb{C}^n)$ for any $y \in \mathcal{L}$, $j \in \{1, \dots, m\}$ and $l \in \{1, \dots, d\}$. We can combine these cases to obtain Eq. (8). \square

Proposition 5.17 should be compared to the Localisation Dichotomy Conjecture in [29, Sect. 5 (arXiv version)]. We have shown that an s_* -localised Wannier basis for $s_* = 1$ is equivalent to a Wannier basis with faster than polynomial decay, which in turn is equivalent to a free finitely generated and projective module. To improve condition (iii) to an exponentially localised Wannier basis will require more analytic arguments that fall outside the framework of pre- C^* -algebras we have considered. In Sect. 5.5, we show that conditions (i)-(iii) of Proposition 5.17 imply that the (even) noncommutative Chern numbers vanish.

Since the submission of this manuscript, the preprints [25,30] have appeared that further develop the Localisation Dichotomy Conjecture for generalised Wannier bases in dimension 2. In particular, [25, Theorem 1] shows that for gapped spectral projections of a magnetic Schrödinger operator on $L^2(\mathbb{R}^2)$ (with mild regularity assumptions), an exponentially localised Wannier basis is equivalent to an s_* -localised Wannier basis with $s_* > 5/2$. Hence, for projections $p \in C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^2, \sigma)$ that fall into the framework of [25], we can improve condition (iii) of Proposition 5.17 to exponential decay. The magnetic Schrödinger operator with Delone atomic potential we consider below (see Equation (9)) satisfies the regularity assumptions of [25] when $d = 2$.

Remark 5.18 (Connections to the Balian–Low Theorem) Theorem 5.15 and Proposition 5.17 plays a similar role to the Balian–Low Theorem in time–frequency analysis. Briefly, the theorem states that if a Gabor system $\{e^{2\pi imt} g(t - n)\}_{m,n \in \mathbb{Z}}$ forms an orthonormal basis of $L^2(\mathbb{R})$, then either g or the Fourier transform \hat{g} is such that the sum in Equation (8) with $m = 1$ diverges. By the work of Luef [28], the Balian–Low Theorem can also be interpreted using finitely generated and projective modules over $C^*(\mathbb{Z}^2)$ and the fact that $C(\mathbb{T}^2)$ has no non-trivial projections. See [28] for more details and a generalisation to the rotation algebra $A_\theta \simeq C^*(\mathbb{Z}^2, \theta)$.

5.4 Wannier Bases for Hamiltonians on $L^2(\mathbb{R}^d, \mathbb{C}^n)$ with Delone Potentials

We model a particle in \mathbb{R}^d subject to a uniform magnetic field perpendicular to the sample. We take a magnetic potential $A = (A_1, \dots, A_d)$ such that $A_j \in L^2_{\text{loc}}(\mathbb{R}^d)$ and differentiable with

$$\frac{\partial}{\partial x_j} A_k - \frac{\partial}{\partial x_k} A_j = \text{const.}$$

for all $j, k \in \{1, \dots, d\}$. For simplicity, we consider constant magnetic field strength but more general fields are possible (cf. Example 5.7). The magnetic Schrödinger operator is given by

$$H_0 = \sum_{j=1}^d \left(-i \frac{\partial}{\partial x_j} - A_j \right)^2,$$

We choose the symmetric gauge and define $A_j = -\frac{1}{2} \sum_{k=1}^d \theta_{j,k} x_k$ for $j = 1, \dots, d$, where $\theta_{j,k}$ is antisymmetric and real. Our choice of gauge gives the magnetic translations $\{U_a\}_{a \in \mathbb{R}^d}$, where for any $a \in \mathbb{R}^d$,

$$[H_0, U_a] = 0, \quad (U_a \psi)(x) = e^{-i(x, \theta a)} \psi(x - a), \quad \psi \in L^2(\mathbb{R}^d).$$

Given a compact space Ω with action $T : \mathbb{R}^d \rightarrow \text{Homeo}(\Omega)$, we can define the Delone groupoid 2-cocycle

$$\theta : (\Omega \times \mathbb{R}^d)^{(2)} \rightarrow \mathbb{T}, \quad \theta((\omega, x), (T_{-x}\omega, y)) = e^{-i(x, \theta y)},$$

which is normalised, $\theta((\omega, x), (T_{-x}\omega, -x)) = 1$, and constant over the unit space.

We wish to relate spectral properties of aperiodic Schrödinger operators to the Delone groupoid. We do this by considering atomic potentials on point sets,

$$H_\Gamma = H_0 + \sum_{p \in \Gamma} v(\cdot - p), \quad H_0 = \sum_{j=1}^d \left(-i \frac{\partial}{\partial x_j} - A_j \right)^2, \quad (9)$$

where v an atomic potential function. Provided the potential $V_\Gamma = \sum_{p \in \Gamma} v(\cdot - p)$ is essentially bounded, real valued and measurable, H_Γ is essentially self-adjoint on the dense core $C_c^\infty(\mathbb{R}^d)$. We assume Γ is r -discrete and restrict our potentials to the K -subharmonic functions on \mathbb{R}^d ,

$$L_{K,r}^1(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d) : |f(x)| \leq Kr^{-d} \int_{|x-y| < r/2} |f(y)| dy \text{ for a.e. } x \right\}.$$

Theorem 5.19 [5] *Let Λ be an (r, R) -Delone set and $v \in L_{K,r}^1(\mathbb{R}^d)$ be a uniformly continuous and \mathbb{R} -valued atomic potential. Then the family of operators $h = \{H_\mathcal{L}\}_{\mathcal{L} \in \Omega_\Lambda}$ with $H_\mathcal{L}$ as in Eq. (9) is affiliated to the groupoid C^* -algebra $C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)$.*

Remark 5.20 Theorem 5.19 is proved by showing that $h = \{H_\mathcal{L}\}_{\mathcal{L} \in \Omega_\Lambda}$ is affiliated to a crossed product groupoid constructed from the continuous hull of the potential $V_\Lambda = \sum_{p \in \Lambda} v(\cdot - p)$. In general the continuous hull of a Delone set is topologically semi-conjugate to the continuous hull of a Delone potential via a surjective map $\Omega_\Lambda \rightarrow \Omega_{V_\Lambda}$. If $\text{supp}(v) \subset B(0, r_v)$ for some $r_v \leq r$, then the map is injective, see [6, Sect. 2.7].

Let us now fix an (r, R) -Delone set Λ and Hamiltonian of the form Eq. (9) such that the family of Schrödinger operators $h = \{H_\mathcal{L}\}_{\mathcal{L} \in \Omega_\Lambda}$ is affiliated to the twisted

groupoid C^* -algebra $C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)$. With the choice of twist θ , we see that for $\pi_{\mathcal{L}} : C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta) \rightarrow \mathcal{B}[L^2(\mathbb{R}^d)]$,

$$(\pi_{\mathcal{L}}(f)\psi)(x) = \int_{\mathbb{R}^d} e^{-i\langle x, \theta u \rangle} f(\mathcal{L} - x, u - x)\psi(u) \, du, \quad U_a \pi_{\mathcal{L}}(f) U_a^* = \pi_{\mathcal{L}-a}(f).$$

The twisted groupoid algebra $C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)$ is Morita equivalent to $C_r^*(\mathcal{G}_{\text{Del}}, \theta)$ and for every $(\mathcal{L}, y) \in \mathcal{G}_{\text{Del}}$, we have a twisted translation action

$$\psi^{(\mathcal{L}, y)}(x) = e^{-i\langle x, B y \rangle} \psi(x - y) = (U_y \psi)(x).$$

Hence, the discrete groupoid translations are just the magnetic translations restricted to $\mathcal{L} \in \Omega_0$.

Recall the pre- C^* -algebra $\mathcal{A} = \mathcal{A}_\infty \subset C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)$ that comes from the Fréchet completion of $C_c(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)$ in the seminorms defined from the derivations $\{\partial_j\}_{j=1}^d$ (Proposition 2.10).

Lemma 5.21 *Let h be a self-adjoint element affiliated to $M_n(C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta))$. Suppose that $\Delta \subset \sigma(h)$ is a bounded spectral region separated from $\sigma(h) \setminus \Delta$ with positive distance. Then $p_\Delta(h) = \chi_\Delta(h) \in M_n(\mathcal{A})$.*

Proof Because Δ is an isolated spectral region, $p_\Delta(h)$ can be approximated arbitrarily well by $\varphi(h)$ with $\varphi \in C_c^\infty(\mathbb{R})$ such that $\varphi(x) = 1$ for $x \in \Delta$ and $\varphi(x) = 0$ for $x \in \sigma(h) \setminus \Delta$. Hence $\varphi(h) \in M_n(\mathcal{A})$. \square

Hence, we can adapt Theorem 5.15 to the case of Schrödinger operators on $L^2(\mathbb{R}^d, \mathbb{C}^n)$ with Delone atomic potentials.

Theorem 5.22 *Let Λ be a (r, R) -Delone set and let H_Λ be a magnetic Schrödinger operator on $L^2(\mathbb{R}^d, \mathbb{C}^n)$ with Delone atomic potential as in Eq. (9) with $v \in L^1_{K,r}(\mathbb{R}^d)$ and uniformly continuous. Suppose that Δ is an isolated and bounded spectral region of $\sigma(H_{\mathcal{L}})$ for all $\mathcal{L} \in \Omega_\Lambda$. Then there are elements $w_1, \dots, w_m \in p_\Delta \mathcal{S}_\infty(\mathcal{F}_{\Omega_0}, \theta)^{\oplus n}$ such that for all $\mathcal{L} \in \Omega_0$ the magnetic translates $\{U_y w_1^{\mathcal{L}}, \dots, U_y w_m^{\mathcal{L}}\}_{y \in \mathcal{L}}$ give a normalised tight frame of $p_\Delta(H_{\mathcal{L}})L^2(\mathbb{R}^d, \mathbb{C}^n)$ with faster than polynomial decay.*

The frame $\{U_y w_1^{\mathcal{L}}, \dots, U_y w_m^{\mathcal{L}}\}_{y \in \mathcal{L}}$ is an orthonormal basis of $p_\Delta(H_{\mathcal{L}})L^2(\mathbb{R}^d, \mathbb{C}^n)$ for all $\mathcal{L} \in \Omega_0$ if and only if the finitely generated and projective C^ -module $p_\Delta L^2(\mathcal{F}_{\Omega_0}, \theta)^{\oplus n}_{\mathcal{G}_{\text{Del}}} \cong C_r^*(\mathcal{G}_{\text{Del}}, \theta)^{\oplus m}$.*

Proof By the spectral gap assumption, the family of spectral projections $\{p_\Delta(H_{\mathcal{L}})\}_{\mathcal{L} \in \Omega_\Lambda}$ give an element $p_\Delta(h) \in M_n(\mathcal{A})$. As such, we can apply Theorem 5.15 which gives the faster than polynomially decaying tight frame or orthonormal basis of the localisation $\pi_{\mathcal{L}}(p_\Delta)h_{\mathcal{L}}^{\oplus n} = p_\Delta(H_{\mathcal{L}})L^2(\mathbb{R}^d, \mathbb{C}^n)$. \square

Remarks 5.23 (i) The existence of an isomorphism $p_\Delta L^2(\mathcal{F}_{\Omega_0}, \theta)^{\oplus m}_{\mathcal{G}_{\text{Del}}} \cong C_r^*(\mathcal{G}_{\text{Del}}, \theta)^{\oplus m}$ is a K -theoretic statement and implies that $[p_\Delta] = m[1] \in K_0(C_r^*(\mathcal{G}_{\text{Del}}, \theta))$.

(ii) If we take a deformation of the magnetic field $\{\theta_t\}_{t \in [0,1]}$ such that the \mathbb{R} -valued 2-cocycle $\omega_t(x, y) = \langle x, \theta_t y \rangle$ is continuous in t , we obtain a homotopy of 2-cocycles in the sense of Remark 5.16. Therefore, Theorem 5.22 is stable under

deformations of the magnetic field provided that the spectral gap remains open throughout the deformation.

5.5 Obstruction to Localised Wannier Basis by (Noncommutative) Chern Numbers

Let us briefly recall the periodic setting. If the atomic potential $V_\Lambda = \sum_{p \in \Lambda} v(\cdot - p)$ is such that Λ is a periodic and co-compact group G , then H_Λ is affiliated to the algebra

$$C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta) \cong C_r^*((\mathbb{R}^d/G) \rtimes \mathbb{R}^d, \theta) \cong C_r^*(G, \theta) \otimes \mathbb{K}.$$

In the case $G = \mathbb{Z}^d$, the non-triviality of finitely generated and projective $C_r^*(\mathbb{Z}^d, \theta)$ -modules with θ rational can be examined by studying the Chern classes of the spectral subspaces of the Hamiltonian viewed as a complex vector bundle over the Brillouin torus, $\widehat{\mathbb{Z}^d}$. In the aperiodic setting, we can use tools from noncommutative geometry to carry out an analogous argument. Indeed, noncommutative Chern numbers for Hamiltonians affiliated to $C(\Omega) \rtimes_\theta \mathbb{R}^d$ and $C_r^*(\mathcal{G}_{\text{Del}}, \theta)$ have already been studied [11–13].

Throughout this section, we will regularly take advantage of the equivalence between the continuous hull $\Omega_\Lambda \rtimes \mathbb{R}^d$ and \mathcal{G}_{Del} , which gives an isomorphism $K_0(C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)) \cong K_0(C_r^*(\mathcal{G}_{\text{Del}}, \theta))$.

We first recall the top degree noncommutative Chern numbers for aperiodic or disordered magnetic Schrödinger Hamiltonians with a spectral gap. To do this, we recall the trace per unit volume on $L^2(\mathbb{R}^d)$. Let Λ_j be an increasing sequence of sets that converge to \mathbb{R}^d in an appropriate sense, e.g. $\Lambda_j = [-j, j]^d$. Then for any $a \in \mathcal{B}(L^2(\mathbb{R}^d))$,

$$\text{Tr}_{\text{Vol}}(a) = \lim_{j \rightarrow \infty} \frac{1}{\text{Vol}(\Lambda_j)} \text{Tr}(\Pi_{\Lambda_j} a), \quad \Pi_{\Lambda_j} : L^2(\mathbb{R}^d) \rightarrow L^2(\Lambda_j).$$

Proposition 5.24 [13] *Fix a probability measure \mathbf{P} on Ω_Λ that is invariant and ergodic under the \mathbb{R}^d -action and let S_d denote the permutation group of $\{1, \dots, d\}$. If $d > 0$ is even and $p = p^* = p^2 \in M_n(\mathcal{A})$, then for almost all $\mathcal{L} \in \Omega_\Lambda$ the functional*

$$C_d(p) = \frac{(-2\pi i)^{d/2}}{(d/2)!} \sum_{\tau \in S_d} (-1)^\tau (\text{Tr}_{C^n} \otimes \text{Tr}_{\text{Vol}}) \left(\pi_{\mathcal{L}}(p) \prod_{j=1}^d [X_{\tau(j)}, \pi_{\mathcal{L}}(p)] \right) \quad (10)$$

is integer valued and almost surely constant in Ω_Λ .

The number C_d almost surely defines a homomorphism $K_0(C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)) \rightarrow \mathbb{Z}$, which we can also consider as a homomorphism $K_0(C_r^*(\mathcal{G}_{\text{Del}}, \theta)) \rightarrow \mathbb{Z}$. In particular $C_d(p) = C_d(p')$ if $[p] = [p'] \in K_0(C_r^*(\mathcal{G}_{\text{Del}}, \theta))$ and $C_d(p) = 0$ if $[p] = m[1] \in K_0(C_r^*(\mathcal{G}_{\text{Del}}, \theta))$.

For systems with $d \geq 3$, we may also wish to consider lower-dimensional invariants. These invariants are not integer-valued in general, but can still be used to study the topology of gapped spectral projections. We fix a probability measure \mathbf{P} on Ω_Λ that

is invariant and under the \mathbb{R}^d -action. Then we recall the noncommutative calculus for $\mathcal{A} \subset C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)$,

$$\mathcal{T}(f) = \int_{\Omega_\Lambda} f(\mathcal{L}, 0) \, d\mathbf{P}, \quad (\partial_j f)(\mathcal{L}, x) = x_j f(\mathcal{L}, x), \quad f \in \mathcal{A}, \quad j \in \{1, \dots, d\}.$$

Proposition 5.25 [13] *Let $p = p^* = p^2 \in M_n(\mathcal{A})$ and \mathbf{P} a probability measure on Ω_Λ that is invariant and under the \mathbb{R}^d -action. Then for any $k \leq d$ an even integer, the functional*

$$C_k(p) = \frac{(-2\pi i)^{k/2}}{(k/2)!} \sum_{\tau \in S_k} (-1)^\tau (\text{Tr}_{\mathbb{C}^n} \otimes \mathcal{T}) \left(p \prod_{j=1}^k \partial_{\tau(j)} p \right), \tag{11}$$

defines a homomorphism $K_0(C_r^*(\Omega_\Lambda \rtimes \mathbb{R}^d, \theta)) \rightarrow \mathbb{R}$. If \mathbf{P} is ergodic under the \mathbb{R}^d -action and $k = d$, then $C_k(p) = C_d(p) \in \mathbb{Z}$ from Eq. (10) almost surely.

We again note that if $[p] = m[1] \in K_0(C_r^*(\mathcal{G}_{\text{Del}}, \theta))$, then $C_k(p) = 0$ for any $k \geq 2$. Combining our results on the noncommutative Chern numbers with Theorem 5.15 and the weak localisation dichotomy (Proposition 5.17), we have the following.

Corollary 5.26 *Let $p = p^* = p^2 \in M_n(\mathcal{A})$ and \mathbf{P} a probability measure on Ω_Λ that is invariant under the \mathbb{R}^d -action. If $C_k(p)$ from Eq. (11) is non-zero for some $k \geq 2$, then for any $\mathcal{L} \in \Omega_0$ there can not be Wannier basis of $\pi_{\mathcal{L}}(p)L^2(\mathbb{R}^d, \mathbb{C}^n)$ constructed from magnetic translations in \mathcal{L} of elements $\{w_j\}_{j=1}^m \subset p\mathcal{S}_1(\mathcal{F}_{\Omega_0}, \theta)_{\mathcal{B}_1}^{\oplus n}$ such that for any $y \in \mathcal{L}$*

$$\sum_{j=1}^m \int_{\mathbb{R}^d} (1 + |x - y|^2) |(U_y w_j^{\mathcal{L}})(x)|^2 \, dx < \infty.$$

5.6 Deformation of the Delone Atomic Potential

We would like to consider the stability of our results on aperiodic Schrödinger operators H_Λ when the Delone set Λ is deformed (e.g. from an aperiodic set to a periodic lattice). Deforming a Delone set Λ will change the crossed product groupoid and the K -theory may change substantially as the Delone set changes. However, we will show the pairings in cyclic cohomology considered in Sect. 5.5 are unaffected by such deformations.

Lemma 5.27 *Let $v \in C_c(\mathbb{R}^d)$ be a continuous atomic potential with compact support. If $\{\Lambda_t\}_{t \in [0,1]}$ is continuous path of (r, R) -Delone sets in $\text{Del}_{(r,R)}$, then the path of Schrödinger operators $\{H_{\Lambda_t}\}_{t \in [0,1]}$ is norm-continuous in the resolvent topology.*

Proof It is straightforward to see that for any $v \in C_c(\mathbb{R}^d)$, $v \in L^1_{K,r}(\mathbb{R}^d)$ and the function $v_\Lambda(x) = \sum_{p \in \Lambda} v(p-x)$ is real-valued, measurable and essentially bounded for any (r, R) -Delone set Λ . Because $\text{Dom}(H_\Lambda)$ is constant for any $\Lambda \in \text{Del}_{(r,R)}$, we can use the resolvent identity to bound

$$\begin{aligned} \|(z - H_{\Lambda_1})^{-1} - (z - H_{\Lambda_2})^{-1}\| &= \|(z - H_{\Lambda_1})^{-1}(H_{\Lambda_1} - H_{\Lambda_2})(z - H_{\Lambda_2})^{-1}\| \\ &\leq \text{ess. sup } |v_{\Lambda_1} - v_{\Lambda_2}| \|(z - H_{\Lambda_1})^{-1}\| \|(z - H_{\Lambda_2})^{-1}\| \end{aligned}$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$. The result will therefore follow if we can show that the essential supremum is controlled by the topology on $\text{Del}_{(r,R)}$. Suppose that $\text{supp}(v) \subset B(0; M)$ for some $M > 0$. Recalling the topology of $\text{Del}_{(r,R)}$ (Proposition 5.2) with d_H the Hausdorff metric, we take $x \in \mathbb{R}^d$ and suppose that $d_H(\Lambda_1 - x \cap B(0; M), \Lambda_2 - x \cap B(0; M)) \leq \delta < r/2$. Taking δ small enough, we can ensure that $|\Lambda_1 - x \cap B(0; M + r/2)| = |\Lambda_2 - x \cap B(0; M + r/2)|$ and, furthermore, we can decompose the sets $\Lambda_1 - x \cap B(0; M + r/2)$ and $\Lambda_2 - x \cap B(0; M + r/2)$ as pairs $(p, q) \in \Lambda_M := \Lambda_1 \cap B(x; M + r/2) \times \Lambda_2 \cap B(x; M + r/2)$ such that $\|p - q\| \leq \delta$. We can therefore estimate

$$\begin{aligned} |(v_{\Lambda_1} - v_{\Lambda_2})(x)| &= \left| \sum_{p \in \Lambda_1} v(p-x) - \sum_{q \in \Lambda_2} v(q-x) \right| \\ &= \left| \sum_{p \in \Lambda_1 \cap B(x; M)} v(p-x) - \sum_{q \in \Lambda_2 \cap B(x; M)} v(q-x) \right| \\ &= \left| \sum_{\substack{(p,q) \in \Lambda_M \\ \|p-q\| \leq \delta}} v(p-x) - v(q-x) \right| \\ &\leq \sum_{\substack{(p,q) \in \Lambda_M \\ \|p-q\| \leq \delta}} |v(p-x) - v(q-x)|. \end{aligned}$$

By continuity of v , given any $\varepsilon > 0$ we can choose a small enough δ so that $|v(p-x) - v(q-x)| < \frac{\varepsilon}{|\Lambda_1 \cap B(x; M+r/2)|}$ for all $\|p - q\| \leq \delta$. Hence $|(v_{\Lambda_1} - v_{\Lambda_2})(x)| < \varepsilon$ and the essential supremum is also bounded by ε . \square

Definition 5.28 Let $\{\Lambda_t\}_{t \in [0,1]}$ be a continuous path in $\text{Del}_{(r,R)}$. We say that $\{H_{\Lambda_t}\}_{t \in [0,1]}$ is a gapped path if there exists a bounded interval $\Delta \subset \mathbb{R}$ such that for all $t \in [0, 1]$ and $\mathcal{L}_t \in \Omega_{\Lambda_t}$, $\Delta \cap \sigma(H_{\mathcal{L}_t})$ is non-empty and Δ is separated from the rest of the spectrum of $H_{\mathcal{L}_t}$ by a positive distance.

The conditions to obtain a gapped path are quite strong, but if satisfied give a path of operators $\{h_t\}_{t \in [0,1]}$ such that h_t is affiliated to $C_r^*(\Omega_{\Lambda_t} \rtimes \mathbb{R}^d, \theta)$ and $p_t = \chi_\Delta(h_t) \in \mathcal{A}_t \subset C_r^*(\Omega_{\Lambda_t} \rtimes \mathbb{R}^d, \theta)$, a dense pre- C^* -algebra.

Proposition 5.29 Let $\{\Lambda_t\}_{t \in [0,1]}$ be a continuous path in $\text{Del}_{(r,R)}$ and fix an atomic potential $v \in C_c(\mathbb{R}^d)$. Suppose that $\{H_{\Lambda_t}\}_{t \in [0,1]}$ is a gapped path with isolated spectral region $\Delta \subset \mathbb{R}$. Then for $p_t = \{\chi_\Delta(H_{\mathcal{L}_t})\}_{\mathcal{L}_t \in \Omega_{\Lambda_t}} \in \mathcal{A}_t$ and any even integer $k \leq d$,

the function

$$[0, 1] \ni t \mapsto C_k(p_t) = \frac{(-2\pi i)^{k/2}}{(k/2)!} \sum_{\tau \in S_k} (-1)^\tau (\text{Tr}_{\mathbb{C}^n} \otimes \mathcal{T}) \left(p_t \prod_{j=1}^k \partial_{\tau(j)} p_t \right) \in \mathbb{R}$$

is continuous, where $C_k(p)$ is the weak Chern number from Eq. (11).

Proof The assumption on the spectral gap implies that $C_k(p_t)$ is well-defined for all t . Because we have a uniform isolated spectral region Δ , we can write for all $t \in [0, 1]$,

$$p_t = \frac{1}{2\pi i} \oint_{\mathcal{C}} (z - h_t)^{-1} dz, \quad h_t = \{(z - H_{\mathcal{L}_t})^{-1}\}_{\mathcal{L}_t \in \Omega_{\Lambda_t}}$$

with \mathcal{C} a contour enclosing Δ and not intersecting any other part of the spectrum. If $\{\Lambda_t\}_{t \in [0,1]}$ is a continuous path in $\text{Del}_{(r,R)}$, then there is a corresponding continuous path $\{\mathcal{L}_t\}_{t \in [0,1]}$ with \mathcal{L}_t in the orbit space of Λ_t . Because $t \mapsto (z - H_{\mathcal{L}_t})^{-1}$ is norm-continuous by Lemma 5.27, so is $t \mapsto \|(z - h_t)^{-1}\|$. By the integral formula for the spectral projections, $t \mapsto \|p_t\|$ is continuous. Because the functional C_k induces a weaker topology than the norm topology, $t \mapsto C_k(p_t)$ is also continuous. \square

Continuity of the function $[0, 1] \ni t \mapsto \|(z - h_t)^{-1}\| \in \mathbb{R}$ for all z in the resolvent set implies that the spectral edges of $\sigma(h_t)$ are continuous away from gap closing points, see [3].

Continuity of the Chern numbers under deformations of Delone sets means that if the range of the pairing is quantised, then it is constant under deformations.

Corollary 5.30 *Let $\{\Lambda_t\}_{t \in [0,1]}$ be a continuous path in $\text{Del}_{(r,R)}$ and fix an atomic potential $v \in C_c(\mathbb{R}^d)$. Suppose that $\{H_{\Lambda_t}\}_{t \in [0,1]}$ is a gapped path with isolated spectral region $\Delta \subset \mathbb{R}$. Let \mathbf{P}_0 be an invariant and ergodic probability measure on Ω_{Λ_0} . Then for almost all $\mathcal{L}_0 \in \Omega_{\Lambda_0}$,*

$$\begin{aligned} C_d(p_t) &= C_d(p_{\mathcal{L}_0}) \\ &:= \frac{(-2\pi i)^{d/2}}{(d/2)!} \sum_{\tau \in S_d} (-1)^\tau (\text{Tr}_{\mathbb{C}^n} \otimes \text{Tr}_{\text{Vol}}) \left(\chi_\Delta(H_{\mathcal{L}_0}) \prod_{j=1}^d [X_{\tau(j)}, \chi_\Delta(H_{\mathcal{L}_0})] \right) \end{aligned}$$

is integer valued and constant for all $t \in [0, 1]$.

Corollaries 5.30 and 5.26 then give us the following stability result on delocalised Wannier bases.

Corollary 5.31 (Stability of delocalised Wannier basis under atomic deformations) *Let Λ_0 be an (r, R) -Delone set and consider H_{Λ_0} with atomic potential $v \in C_c(\mathbb{R}^d)$. Fix an invariant and ergodic probability measure on Ω_{Λ_0} and suppose that $C_d(p_{\mathcal{L}_0}) \neq 0$ for almost all $\mathcal{L}_0 \in \Omega_{\Lambda_0}$. Then for any gapped path $\{H_{\Lambda_t}\}_{t \in [0,1]}$ with isolated spectral region $\Delta \subset \mathbb{R}$ and any \mathcal{L}_t in the transversal $\Omega_{0,t}$, there can not be a faster than polynomially decaying Wannier basis of $\chi_\Delta(H_{\mathcal{L}_t})L^2(\mathbb{R}^d, \mathbb{C}^n)$ built from magnetic translates in \mathcal{L}_t .*

We again note that by Proposition 5.17 the non-existence of a faster than polynomially decaying Wannier basis also implies that the weaker localisation bound of Eq. (8) diverges,

$$\sum_{j=1}^m \int_{\mathbb{R}^d} (1 + |x|^2) |w_j^{\mathcal{L}}(x)|^2 dx = \infty.$$

Acknowledgements The authors thank Franz Luef, Domenico Monaco and Guo Chuan Thiang for valuable feedback on an earlier version of this manuscript. We also thank Giovanna Marcelli, Massimo Moscolari and Gianluca Panati for sharing the results of [29,30] with us. CB is supported by a JSPS Grant-in-Aid for Early-Career Scientists (No. 19K14548) and thanks the Mathematical Institute, Universiteit Leiden, for hospitality during the conference *Noncommutative Geometry, Analysis, and Topological Insulators* in February 2020, where this work took shape. Both authors thank the Casa Matematica Oaxaca for hospitality and support during the workshop *Topological Phases of Interacting Quantum Systems* in June 2019.

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