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Weak and Strong Type Estimates for the Multilinear Littlewood–Paley Operators

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Abstract

Let S_α be the multilinear square function defined on the cone with aperture $\alpha \geq 1$. In this paper, we investigate several kinds of weighted norm inequalities for S_α. We first obtain a sharp weighted estimate in terms of aperture α and $\vec{w} \in A_{\vec{p}}$. By means of some pointwise estimates, we also establish two-weight inequalities including bump and entropy bump estimates, and Fefferman–Stein inequalities with arbitrary weights. Beyond that, we consider the mixed weak type estimates corresponding Sawyer's conjecture, for which a Coifman–Fefferman inequality with the precise A_∞ norm is proved. Finally, we present the local decay estimates using the extrapolation techniques and dyadic analysis respectively. All the conclusions aforementioned hold for the Littlewood–Paley g_λ^* function. Some results are new even in the linear case.

Keywords Multilinear square functions · Bump conjectures · Mixed weak type estimates · Local decay estimates · Sharp aperture dependence

Mathematics Subject Classification 42B20 · 42B25

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1 Introduction

Given $\alpha > 0$, let S_α be the square function defined by

$$S_\alpha(f)(x) = \left(\iint_{\Gamma_\alpha(x)} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},$$

where $\psi_t(x) = t^{-n}\psi(x/t)$ and $\Gamma_\alpha(x)$ is the cone at vertex x with aperture α. Lerner [30], by applying the intrinsic square function introduced in [46], proved sharp weighted norm inequalities for $S_\alpha(f)$. Later on, he improved the result in the sense of determination of sharp dependence on α in [32] by using the local mean oscillation formula. More precisely,

$$\|S_\alpha\|_{L^p(w)\to L^p(w)} \lesssim \alpha^n [w]_{A_p}^{\max\{\frac{1}{2},\frac{1}{p-1}\}}, \quad 1 < p < \infty. \tag{1.1}$$

The preceding result is among the plenty important results in the fruitful realm of weighted inequalities concerning the precise determination of the optimal bounds of the weighted operator norm of different singular integral operators. We refer the interested reader to [24,25,27,31] and the references therein for a survey on the advances on the topic.

Let us recall the definition of multilinear square functions considered in this paper. The standard kernel for multilinear square functions was introduced in [45]. Let $\psi(x, \vec{y}) := \psi(x, y_1, \ldots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. We assume that there are positive constants δ and A so that the following conditions hold:

- Size condition:

$$|\psi(x, \vec{y})| \leq \frac{A}{\left(1 + \sum_{i=1}^m |x - y_i|\right)^{mn+\delta}}.$$

- Smoothness condition: There exists $\gamma > 0$ so that

$$|\psi(x, \vec{y}) - \psi(x', \vec{y})| \leq \frac{A|x - x'|^\gamma}{\left(1 + \sum_{i=1}^m |x - y_i|\right)^{mn+\delta+\gamma}},$$

whenever $|x - x'| < \frac{1}{2} \max_j |x - y_j|$, and

$$\left|\psi(x, \vec{y}) - \psi(x, y_1, \ldots, y_i', \ldots, y_m)\right| \leq \frac{A|y_i - y_i'|^\gamma}{\left(1 + \sum_{i=1}^m |x - y_i|\right)^{mn+\delta+\gamma}},$$

whenever $|y_i - y_i'| < \frac{1}{2} \max_j |x - y_j|$ for $i = 1, 2, \ldots, m$.

For $t > 0$, denote ψ_t

$$\psi_t(\vec{f})(x) := \frac{1}{t^{mn}} \int_{(\mathbb{R}^n)^m} \psi\Big(\frac{x}{t}, \frac{y_1}{t}, \cdots, \frac{y_m}{t}\Big) \prod_{j=1}^m f_j(y_j)dy_j,$$

for all $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$ and $\vec{f} = (f_1, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)$.

Given $\alpha > 0$ and $\lambda > 2m$, the multilinear square functions S_α and g_λ^* are defined by

$$S_\alpha(\vec{f})(x) := \Big(\iint_{\Gamma_\alpha(x)} |\psi_t(\vec{f})(y)|^2 \frac{dydt}{t^{n+1}} \Big)^{1/2},$$

where $\Gamma_\alpha(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < \alpha t\}$, and

$$g_\lambda^*(\vec{f})(x) := \Big(\iint_{\mathbb{R}_+^{n+1}} \Big(\frac{t}{t + |x - y|}\Big)^{n\lambda} |\psi_t(\vec{f})(y)|^2 \frac{dydt}{t^{n+1}} \Big)^{1/2}.$$

Hereafter, we assume that for $\lambda > 2m$ there exist some $1 \le p_1, \ldots, p_m \le \infty$ and some $0 < p < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, such that g_λ^* maps continuously $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Under this condition, it was proved in [45] that g_λ^* maps continuously $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n) \to L^{1/m,\infty}(\mathbb{R}^n)$ provided $\lambda > 2m$. Moreover, since S_α is dominated by g_λ^*, we also get that S_α maps continuously $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n) \to L^{1/m,\infty}(\mathbb{R}^n)$.

These two mutilinear square functions were introduced and investigated in [45,47]. Indeed, the theory of multilinear Littlewood–Paley operators originated in the works of Coifman and Meyer [14]. The multilinear square functions has important applications in PDEs and other fields. In particular, Fabes, Jerison, and Kenig brought very important applications of multilinear square functions in PDEs to the attention. In [21], they studied the solutions of Cauchy problem for non-divergence form parabolic equations by obtaining some multilinear Littlewood–Paley type estimates for the square root of an elliptic operator in divergence form. Also, the necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure were achieved relying upon a multilinear Littlewood–Paley estimate, in [22]. Moreover, in [23], they applied a class of multilinear square functions to Kato's problem. For further details on the theory of multilinear square functions and their applications, we refer to [8,12–14,21,23] and the references therein.

In this paper, we investigate some weak and strong type estimates for multilinear Littlewood–Paley operators. This kind of inequalities has its origin in classical potential theory. A big breakthrough in understanding Poisson's equation, made by Lichtenstein [37] in 1916, raised problems that have been central to analysis over the past decades. The theory of singular integral operators owes its impetus to the change of point of view of potential theory generated by this work. The action of singular integral operators on the standard Lebesgue spaces $L^p(\mathbb{R}^n)$ was for a long time the main object of study. But these operators have natural analogs in which $\mathbb{R}^n$ is replaced by a Lie group or Lebesgue measure on $\mathbb{R}^n$ is replaced by a weighted measure. It is in the setting that our work is focused on.

The contributions of this paper are as follows. Based on the ideas from Fefferman's celebrated paper [19], in this work, we first prove the upper bound for S_α is sharp in the aperture α on all class $A_{\vec{p}}$ which proves a conjecture given in [2]. Secondly, we focus on bump and entropy bump estimates, mixed weak type estimates, local decay estimates, and multilinear version of Fefferman–Stein inequality with arbitrary weights for multilinear square functions respectively. These interesting estimates have aroused the attention of many researchers. For example, A_p bump conditions may be thought of as the classical two-weight A_p condition with the localized L^p and $L^{p'}$ norms "bumped up" in the scale of Orlicz spaces. These conditions have a long history, we refer to [20,43]. Muckenhoupt and Wheeden [38] first formulated the mixed weak type estimates for Hardy–Littlewood maximal function and the Hilbert transform on the real line although Sawyer [44] considered a more singular case, namely he showed that if $\mu \in A_1$ and $\nu \in A_\infty$, then

$$\left\| \frac{M(f\nu)}{\nu} \right\|_{L^{1,\infty}(\mu\nu)} \lesssim \|f\|_{L^1(\mu\nu)} \tag{1.2}$$

and conjectured that such an inequality should hold with M replaced by the Hilbert transform. Later on Cruz-Uribe et al. [15] extended Sawyer's result to higher dimensions and also settled Sawyer's conjecture and extended that result for general Calderón–Zygmund operators reducing it to the case of maximal functions via an extrapolation argument. That extrapolation argument allowed them to take $\mu \in A_1$ and $\nu \in A_\infty$. That led them to conjecture that (1.2) should hold $\mu \in A_1$ and $\nu \in A_\infty$. Recently, that conjecture was settled by Li, Ombrosi and Pérez [36]. That result was extended to maximal operators with Young functions [3]. Analogous results were obtained for commutators [4], fractional operators [5] or in the multilinear setting [35]. Also quantitative estimates have been studied in [6,39]. Local exponential decay estimates for CZOs and square functions, multilinear pseudo-differential operators and its commutator were studied in [7,40] respectively.

The main results of this paper can be stated as follows. We begin with a sharp weighted inequality in terms of both α and $[\vec{w}]_{A_{\vec{p}}}$.

Theorem 1.1 *Let $\alpha \geq 1$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \ldots, p_m < \infty$. If $\vec{w} \in A_{\vec{p}}$, then*

$$\|S_\alpha(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim \alpha^{mn} [\vec{w}]_{A_{\vec{p}}}^{\max\{\frac{1}{2}, \frac{p_1'}{p}, \cdots, \frac{p_m'}{p}\}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}, \tag{1.3}$$

where the implicit constant is independent of α and $\vec{w}$. Moreover, (1.3) is sharp in α on all class $A_{\vec{p}}$.

In order to present two-weight inequalities for square functions, we give the definition of bump conditions. Given Young functions A and $\vec{B} = (B_1, \ldots, B_m)$, we denote

$$\|(u,\vec{v})\|_{A,\vec{B},\vec{p}} := \begin{cases} \sup_Q \|u^{\frac{1}{p}}\|_{p,Q} \prod_{j=1}^m \|v_j^{-\frac{1}{p_j}}\|_{B_j,Q}, & \text{if } 1 < p \le 2, \\ \sup_Q \|u^{\frac{2}{p}}\|_{A,Q}^{\frac{1}{2}} \prod_{j=1}^m \|v_j^{-\frac{1}{p_j}}\|_{B_j,Q}, & \text{if } 2 < p < \infty. \end{cases}$$

Theorem 1.2 *Let $\alpha \ge 1$, $\lambda > 2m$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \ldots, p_m < \infty$. If the pair $(u, \vec{v})$ satisfies $\|(u,\vec{v})\|_{A,\vec{B},\vec{p}} < \infty$ with $\bar{A} \in B_{(p/2)'}$ $(2 < p < \infty)$ and $\bar{B}_j \in B_{p_j}$, then*

$$\|S_\alpha(\vec{f})\|_{L^p(u)} \lesssim \alpha^{mn} \mathscr{N}_{\vec{p}} \prod_{j=1}^m \|f_j\|_{L^{p_j}(v_j)}, \tag{1.4}$$

$$\|g_\lambda^*(\vec{f})\|_{L^p(u)} \lesssim \frac{\mathscr{N}_{\vec{p}}}{2^{n(\lambda-2m)}-1} \prod_{j=1}^m \|f_j\|_{L^{p_j}(v_j)}, \tag{1.5}$$

where

$$\mathscr{N}_{\vec{p}} := \begin{cases} \|(u,\vec{v})\|_{A,\vec{B},\vec{p}} \prod_{j=1}^m [\bar{B}_j]_{(B_j)_{p_j}}^{\frac{1}{p_j}}, & \text{if } 1 < p \le 2, \\ \|(u,\vec{v})\|_{A,\vec{B},\vec{p}} [\bar{A}]_{B_{(p/2)'}}^{\frac{1}{2}-\frac{1}{p}} \prod_{j=1}^m [\bar{B}_j]_{(B_j)_{p_j}}^{\frac{1}{p_j}}, & \text{if } 2 < p < \infty. \end{cases}$$

For arbitrary weights, we have the following Fefferman–Stein inequalities.

Theorem 1.3 *Let $\alpha \ge 1$ and $\lambda > 2m$. Then for all exponents $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $0 < p \le 2$ and $1 < p_1, \ldots, p_m < \infty$, and for all weights $\vec{w} = (w_1, \ldots, w_m)$,*

$$\|S_\alpha(\vec{f})\|_{L^p(v_{\vec{w}})} \lesssim \alpha^{mn} \prod_{i=1}^m \|f_i\|_{L^{p_i}(Mw_i)}, \tag{1.6}$$

$$\|g_\lambda^*(\vec{f})\|_{L^p(v_{\vec{w}})} \lesssim \frac{1}{2^{n(\lambda-2m)}-1} \prod_{i=1}^m \|f_i\|_{L^{p_i}(Mw_i)}, \tag{1.7}$$

where $v_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

We are going to establish entropy bump estimates. See Sect. 5 for the entropy bump conditions $\lfloor \vec{\sigma}, \nu \rfloor_{\frac{2}{\vec{p}'},\vec{p},\varepsilon,\frac{2}{p},m+1}$ and $\lfloor \vec{\sigma}, \nu \rfloor_{\vec{p},2,\varepsilon}$.

Theorem 1.4 *Let $\alpha \ge 1$, $\lambda > 2m$, and let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \ldots, p_m < \infty$. Let ν and $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$ weights. Assume that ε is a monotonic increasing* *function on* $(1, \infty)$ *satisfying* $\int_1^\infty \frac{dt}{\varepsilon(t)t} < \infty$. *Then,*

$$\|S_\alpha(\vec{f}\sigma)\|_{L^p(\nu)} \lesssim \alpha^{mn} \mathcal{N}_{\vec{p},\varepsilon} \prod_{i=1}^m \|f\|_{L^{p_i}(\sigma_i)}, \tag{1.8}$$

$$\|g_\lambda^*(\vec{f}\sigma)\|_{L^p(\nu)} \lesssim \frac{\mathcal{N}_{\vec{p},\varepsilon}}{2^{n(\lambda-2m)}-1} \prod_{i=1}^m \|f\|_{L^{p_i}(\sigma_i)}, \tag{1.9}$$

where

$$\mathcal{N}_{\vec{p},\varepsilon} := \begin{cases} \lfloor \vec{\sigma}, \nu \rfloor^{\frac{1}{2}}_{\frac{2}{\vec{p}'}, \vec{p}, \varepsilon, \frac{2}{p}, m+1}, & \text{if } 0 < p \le 2, \\ \lfloor \vec{\sigma}, \nu \rfloor^{\frac{1}{p}}_{\vec{p}, 2, \varepsilon}, & \text{if } 2 < p < \infty. \end{cases}$$

with $\vec{p} = (p_1, \ldots, p_m, p')$ *and* $\frac{2}{\vec{p}'} = \big(\frac{2}{p_1'}, \ldots, \frac{2}{p_m'}, \frac{2}{p}\big)$.

Next, we turn to the weak type estimates for Littlewood–Paley operators.

Theorem 1.5 *Let* $\alpha \ge 1$ *and* $\lambda > 2m$. *Let* $\vec{w} = (w_1, \ldots, w_m)$ *and* $u = \prod_{i=1}^m w_i^{1/m}$. *If* $\vec{w}$ *and* v *satisfy*

$$(1) \quad \vec{w} \in A_{\vec{1}} \text{ and } uv^{1/m} \in A_\infty, \quad \text{or} \quad (2) \quad w_1, \ldots, w_m \in A_1 \text{ and } v \in A_\infty,$$

then we have

$$\left\| \frac{S_\alpha(\vec{f})}{v} \right\|_{L^{1/m,\infty}(uv^{1/m})} \lesssim \prod_{i=1}^m \|f_i\|_{L^1(w_i)}, \tag{1.10}$$

$$\left\| \frac{g_\lambda^*(\vec{f})}{v} \right\|_{L^{1/m,\infty}(uv^{1/m})} \lesssim \prod_{i=1}^m \|f_i\|_{L^1(w_i)}. \tag{1.11}$$

In particular, both S_α *and* g_λ^* *are bounded from* $L^1(w_1) \times \cdots \times L^1(w_m)$ *to* $L^{1/m,\infty}(\nu_{\vec{w}})$ *for every* $\vec{w} \in A_{\vec{1}}$.

Theorem 1.6 *Let* $\alpha \ge 1$ *and* $\lambda > 2m$. *Let* Q *be a cube and every function* $f_j \in L_c^\infty(\mathbb{R}^n)$ *with* $\operatorname{supp}(f_j) \subset Q$, $j = 1, \ldots, m$. *Then there exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\big|\{x \in Q : S_\alpha(\vec{f})(x) > t\mathcal{M}(\vec{f})(x)\}\big| \le c_1 e^{-c_2\beta_1 t^2}|Q|, \tag{1.12}$$

$$\big|\{x \in Q : g_\lambda^*(\vec{f})(x) > t\mathcal{M}(\vec{f})(x)\}\big| \le c_1 e^{-c_2\beta_2 t^2}|Q|, \tag{1.13}$$

for all $t > 0$, *where* $\beta_1 = \alpha^{-2mn}$ *and* $\beta_2 = (1 - 2^{-n(\lambda-2m)/2})^2$.

2 Preliminaries

2.1 Multiple Weights

The multilinear maximal operators $\mathcal{M}$ are defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \fint_Q |f_j(y_j)| dy_j,$$

where the supremum is taken over all the cubes containing x. The corresponding theory of weights for this new maximal function gives the right class of multiple weights for multilinear Calderón-Zygmund operators.

Definition 2.1 Let $1 \leq p_1, \ldots, p_m < \infty$. Given a vector of weights $\vec{w} = (w_1, \cdots, w_m)$, we say that $\vec{w} \in A_{\vec{p}}$ if

$$[\vec{w}]_{A_{\vec{p}}} := \sup_Q \left(\fint_Q \nu_{\vec{w}}\, dx \right)^{\frac{1}{p}} \prod_{i=1}^{m} \left(\fint_Q w_i^{1-p_i'} dx \right)^{\frac{1}{p_i'}} < \infty,$$

where $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$. When $p_i = 1$, $\left(\fint_Q w_i^{1-p_i'} dx \right)^{1/p_i'}$ is understood as $(\inf_Q w_i)^{-1}$.

The characterizations of multiple weights were given in [10,33] .

Lemma 2.2 *Let* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *with* $1 \leq p_1, \ldots, p_m < \infty$, *and* $p_0 = \min\{p_i\}_i$. *Then the following statements hold :*

(1) $A_{r_1 \vec{p}} \subsetneq A_{r_2 \vec{p}}$, *for any* $1/p_0 \leq r_1 < r_2 < \infty$.
(2) $A_{\vec{p}} = \bigcup_{1/p_0 \leq r < 1} A_{r\vec{p}}$.
(3) $\vec{w} \in A_{\vec{p}}$ *if and only if* $\nu_{\vec{w}} \in A_{mp}$ *and* $w_i^{1-p_i'} \in A_{mp_i'}$, $i = 1, \ldots, m$. *Here, if* $p_i = 1$, $w_i^{1-p_i'} \in A_{mp_i'}$ *is understood as* $w_i^{1/m} \in A_1$.

2.2 Dyadic Cubes

Denote by $\ell(Q)$ the sidelength of the cube Q. Given a cube $Q_0 \subset \mathbb{R}^n$, let $\mathcal{D}(Q_0)$ denote the set of all dyadic cubes with respect to Q_0, that is, the cubes obtained by repeated subdivision of Q_0 and each of its descendants into 2^n congruent subcubes.

Definition 2.3 A collection $\mathcal{D}$ of cubes is said to be a dyadic grid if it satisfies

(1) For any $Q \in \mathcal{D}$, $\ell(Q) = 2^k$ for some $k \in \mathbb{Z}$.
(2) For any $Q, Q' \in \mathcal{D}$, $Q \cap Q' = \{Q, Q', \emptyset\}$.
(3) The family $\mathcal{D}_k = \{Q \in \mathcal{D}; \ell(Q) = 2^k\}$ forms a partition of $\mathbb{R}^n$ for any $k \in \mathbb{Z}$.

Definition 2.4 A subset $\mathcal{S}$ of a dyadic grid is said to be η-sparse, $0 < \eta < 1$, if for every $Q \in \mathcal{S}$, there exists a measurable set $E_Q \subset Q$ such that $|E_Q| \ge \eta|Q|$, and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint.

By a median value of a measurable function f on a cube Q we mean a possibly non-unique, real number $m_f(Q)$ such that

$$\max\Big\{|\{x \in Q : f(x) > m_f(Q)\}|, |\{x \in Q : f(x) < m_f(Q)\}|\Big\} \le |Q|/2.$$

The decreasing rearrangement of a measurable function f on $\mathbb{R}^n$ is defined by

$$f^*(t) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| < t\}, \quad 0 < t < \infty.$$

The local mean oscillation of f is

$$\omega_\lambda(f; Q) = \inf_{c\in\mathbb{R}} \big((f - c)\mathbf{1}_Q\big)^*(\lambda|Q|), \quad 0 < \lambda < 1.$$

Given a cube Q_0, the local sharp maximal function is defined by

$$M^\sharp_{\lambda;Q_0} f(x) = \sup_{x\in Q\subset Q_0} \omega_\lambda(f; Q).$$

Observe that for any $\delta > 0$ and $0 < \lambda < 1$

$$|m_f(Q)| \le (f\mathbf{1}_Q)^*(|Q|/2) \ \text{ and } \ (f\mathbf{1}_Q)^*(\lambda|Q|) \le \left(\frac{1}{\lambda|Q|}\int_Q |f|^\delta dx\right)^{1/\delta}. \tag{2.1}$$

The following theorem was proved by Hytönen [25, Theorem 2.3] in order to improve Lerner's formula given in [30] by getting rid of the local sharp maximal function.

Lemma 2.5 *Let f be a measurable function on $\mathbb{R}^n$ and let Q_0 be a fixed cube. Then there exists a (possibly empty) sparse family $\mathcal{S}(Q_0) \subset \mathcal{D}(Q_0)$ such that*

$$|f(x) - m_f(Q_0)| \le 2 \sum_{Q\in\mathcal{S}(Q_0)} \omega_{2^{-n-2}}(f; Q)\mathbf{1}_Q(x), \quad a.e.\ x \in Q_0. \tag{2.2}$$

2.3 Orlicz Maximal Operators

A function $\Phi : [0,\infty) \to [0,\infty)$ is called a Young function if it is continuous, convex, strictly increasing, and satisfies

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty.$$

Given $p \in [1,\infty)$, we say that a Young function Φ is a p-Young function, if $\Psi(t) = \Phi(t^{1/p})$ is a Young function.

If A and B are Young functions, we write $A(t) \simeq B(t)$ if there are constants $c_1, c_2 > 0$ such that $c_1 A(t) \leq B(t) \leq c_2 A(t)$ for all $t \geq t_0 > 0$. Also, we denote $A(t) \preceq B(t)$ if there exists $c > 0$ such that $A(t) \leq B(ct)$ for all $t \geq t_0 > 0$. Note that for all Young functions ϕ, $t \preceq \phi(t)$. Further, if $A(t) \leq cB(t)$ for some $c > 1$, then by convexity, $A(t) \leq B(ct)$.

A function Φ is said to be doubling, or $\Phi \in \Delta_2$, if there is a constant $C > 0$ such that $\Phi(2t) \leq C\Phi(t)$ for any $t > 0$. Given a Young function Φ, its complementary function $\bar{\Phi} : [0, \infty) \to [0, \infty)$ is defined by

$$\bar{\Phi}(t) := \sup_{s>0}\{st - \Phi(s)\}, \quad t > 0,$$

which clearly implies that

$$st \leq \Phi(s) + \bar{\Phi}(t), \quad s, t > 0. \tag{2.3}$$

Moreover, one can check that $\bar{\Phi}$ is also a Young function and

$$t \leq \Phi^{-1}(t)\bar{\Phi}^{-1}(t) \leq 2t, \qquad t > 0. \tag{2.4}$$

In turn, by replacing t by $\Phi(t)$ in first inequality of (2.4), we obtain

$$\bar{\Phi}\Big(\frac{\Phi(t)}{t}\Big) \leq \Phi(t), \qquad t > 0. \tag{2.5}$$

Given a Young function Φ, we define the Orlicz space $L^{\Phi}(\Omega, \mu)$ to be the function space with Luxemburg norm

$$\|f\|_{L^{\Phi}(\Omega,\mu)} := \inf\left\{\lambda > 0 : \int_{\Omega} \Phi\Big(\frac{|f(x)|}{\lambda}\Big) d\mu(x) \leq 1\right\}. \tag{2.6}$$

Now we define the Orlicz maximal operator

$$M_{\Phi} f(x) := \sup_{Q \ni x} \|f\|_{\Phi, Q} := \sup_{Q \ni x} \|f\|_{L^{\Phi}(Q, \frac{dx}{|Q|})},$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$. When $\Phi(t) = t^p$, $1 \leq p < \infty$,

$$\|f\|_{\Phi, Q} = \left(\fint_Q |f(x)|^p dx\right)^{\frac{1}{p}} =: \|f\|_{p, Q}.$$

In this case, if $p = 1$, M_{Φ} agrees with the classical Hardy–Littlewood maximal operator M; if $p > 1$, $M_{\Phi} f = M_p f := M(|f|^p)^{1/p}$. If $\Phi(t) \preceq \Psi(t)$, then $M_{\Phi} f(x) \leq c M_{\Psi} f(x)$ for all $x \in \mathbb{R}^n$.

The Hölder inequality can be generalized to the scale of Orlicz spaces [16, Lemma 5.2].

Lemma 2.6 *Given a Young function A, then for all cubes Q,*

$$\fint_Q |fg|dx \leq 2\|f\|_{A,Q}\|g\|_{\bar{A},Q}. \tag{2.7}$$

More generally, if A, B and C are Young functions such that $A^{-1}(t)B^{-1}(t) \leq c_1 C^{-1}(t)$, *for all* $t \geq t_0 > 0$, *then*

$$\|fg\|_{C,Q} \leq c_2\|f\|_{A,Q}\|g\|_{B,Q}. \tag{2.8}$$

The following result is an extension of the well-known Coifman–Rochberg theorem. The proof can be found in [26, Lemma 4.2].

Lemma 2.7 *Let* Φ *be a Young function and* w *be a nonnegative function such that* $M_\Phi w(x) < \infty$ *a.e.. Then*

$$[(M_\Phi w)^\delta]_{A_1} \leq c_{n,\delta}, \quad \forall\delta \in (0, 1), \tag{2.9}$$

$$[(M_\Phi w)^{-\lambda}]_{RH_\infty} \leq c_{n,\lambda}, \quad \forall\lambda > 0. \tag{2.10}$$

Given $p \in (1, \infty)$, a Young function Φ is said to satisfy the B_p condition (or, $\Phi \in B_p$) if for some $c > 0$,

$$\int_c^\infty \frac{\Phi(t)}{t^p}\frac{dt}{t} < \infty. \tag{2.11}$$

Observe that if (2.11) is finite for some $c > 0$, then it is finite for every $c > 0$. Let $[\Phi]_{B_p}$ denote the value if $c = 1$ in (2.11). It was shown in [16, Proposition 5.10] that if Φ and $\bar{\Phi}$ are doubling Young functions, then $\Phi \in B_p$ if and only if

$$\int_c^\infty \left(\frac{t^{p'}}{\bar{\Phi}(t)}\right)^{p-1}\frac{dt}{t} < \infty.$$

Let us present two types of B_p bumps. An important special case is the “log-bumps" of the form

$$A(t) = t^p \log(e+t)^{p-1+\delta}, \quad B(t) = t^{p'} \log(e+t)^{p'-1+\delta}, \quad \delta > 0. \tag{2.12}$$

Another interesting example is the “loglog-bumps" as follows:

$$A(t) = t^p \log(e+t)^{p-1} \log\log(e^e+t)^{p-1+\delta}, \quad \delta > 0 \tag{2.13}$$

$$B(t) = t^{p'} \log(e+t)^{p'-1} \log\log(e^e+t)^{p'-1+\delta}, \quad \delta > 0. \tag{2.14}$$

Then one can verify that in both cases above, $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$ for any $1 < p < \infty$.

The B_p condition can be also characterized by the boundedness of the Orlicz maximal operator M_Φ. Indeed, the following result was given in [16, Theorem 5.13] and [26, eq. (25)].

Lemma 2.8 *Let $1 < p < \infty$. Then M_Φ is bounded on $L^p(\mathbb{R}^n)$ if and only if $\Phi \in B_p$. Moreover, $\|M_\Phi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \le C_{n,p}[\Phi]_{B_p}^{\frac{1}{p}}$. In particular, if the Young function A is the same as the first one in (2.12) or (2.13), then*

$$\|M_{\bar{A}}\|_{L^{p'}(\mathbb{R}^n)\to L^{p'}(\mathbb{R}^n)} \le c_n p^2 \delta^{-\frac{1}{p'}}, \quad \forall \delta \in (0, 1]. \tag{2.15}$$

Definition 2.9 Given $p \in (1, \infty)$, let A and B be Young functions such that $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$. We say that the pair of weights (u, v) satisfies the `double bump condition` with respect to A and B if

$$[u, v]_{A,B,p} := \sup_Q \|u^{\frac{1}{p}}\|_{A,Q}\|v^{-\frac{1}{p}}\|_{B,Q} < \infty. \tag{2.16}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$. Also, (u, v) is said to satisfy the separated bump condition if

$$[u, v]_{A,p'} := \sup_Q \|u^{\frac{1}{p}}\|_{A,Q}\|v^{-\frac{1}{p}}\|_{p',Q} < \infty, \tag{2.17}$$

$$[u, v]_{p,B} := \sup_Q \|u^{\frac{1}{p}}\|_{p,Q}\|v^{-\frac{1}{p}}\|_{B,Q} < \infty. \tag{2.18}$$

Note that if $A(t) = t^p$ in (2.17) or $B(t) = t^p$ in (2.18), each of them actually is two-weight A_p condition and we denote them by $[u, v]_{A_p} := [u, v]_{p,p'}$. Also, the separated bump condition is weaker than the double bump condition. Indeed, (2.16) implies (2.17) and (2.18), but the reverse direction is incorrect. The first fact holds since $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$ respectively indicate A is a p-Young function and B is a p'-Young function. The second fact was shown in [1, Section 7] by constructing log-bumps.

Lemma 2.10 *Let $1 < p < \infty$, let A, B and Φ be Young functions such that $A \in B_p$ and $A^{-1}(t)B^{-1}(t) \lesssim \Phi^{-1}(t)$ for any $t > t_0 > 0$. If a pair of weights (u, v) satisfies $[u, v]_{p,B} < \infty$, then*

$$\|M_\Phi f\|_{L^p(u)} \le C[u, v]_{p,B}[A]_{B_p}^{\frac{1}{p}}\|f\|_{L^p(v)}. \tag{2.19}$$

Moreover, (2.19) holds for $\Phi(t) = t$ and $B = \bar{A}$ satisfying the same hypotheses. In this case, $\bar{A} \in B_p$ is necessary.

The two-weight inequality above was established in [16, Theorem 5.14] and [17, Theorem 3.1]. The weak type inequality for M_Φ was also obtained in [16, Proposition 5.16] as follows.

Lemma 2.11 *Let $1 < p < \infty$, let B and Φ be Young functions such that $t^{\frac{1}{p}}B^{-1}(t) \lesssim \Phi^{-1}(t)$ for any $t > t_0 > 0$. If a pair of weights (u, v) satisfies $[u, v]_{p,B} < \infty$, then*

$$\|M_\Phi f\|_{L^{p,\infty}(u)} \leq C\|f\|_{L^p(v)}. \tag{2.20}$$

Moreover, (2.20) *holds for* M *if and only if* $[u, v]_{A_p} < \infty$.

3 Sharpness in Aperture α

The goal of this section is to give the proof of Theorem 1.1. To this end, we establish some fundamental estimates.

Lemma 3.1 *$\psi(x, \vec{y})$ is continuous at $(x_0, y_{1,0}, \ldots, y_{m,0})$ with $x_0 \neq y_{j,0}$, $j = 1, 2, \ldots, m$.*

Proof Let $x_0 \neq y_{j,0}$ for $j = 1, 2, \ldots, m$, and let

$$|x - x_0| < \frac{1}{4} \min_{1\leq i\leq m} \{|x_0 - y_{i,0}|\}, \qquad |y_j - y_{j,0}| < \frac{1}{2} \min_{1\leq i\leq m} \{|x_0 - y_{i,0}|\}.$$

Then we get

$$|y_j - y_{j,0}| < \frac{1}{2}|x_0 - y_{j,0}|$$

and

$$|x_0 - y_{j,0}| \leq |x_0 - y_j| + |y_j - y_{j,0}| < |x_0 - y_j| + \frac{1}{2}|x_0 - y_{j,0}|$$

and so

$$|x_0 - y_{j,0}| < 2|x_0 - y_j|, \quad j = 1, \ldots, m,$$

which implies

$$|x - x_0| < \frac{1}{4}|x_0 - y_{j,0}| < \frac{1}{2}|x_0 - y_j|, \quad j = 1, \ldots, m,$$

Therefore, we have

$$\begin{aligned}
&|\psi(x, y_1, \ldots, y_m) - \psi(x_0, y_{1,0}, \ldots, y_{m,0})| \\
&\quad \leq |\psi(x, y_1, \ldots, y_m) - \psi(x_0, y_1, \ldots, y_m)| \\
&\quad\quad + \sum_{j=1}^{m} |\psi(x_0, y_{1,0}, \ldots, y_{j-1,0}, y_j, y_{j+1}, \ldots, y_m) \\
&\quad\quad - \psi(x_0, y_{1,0}, \ldots, y_{j-1,0}, y_{j,0}, y_{j+1}, \ldots, y_m)| \\
&\quad \leq \frac{A|x - x_0|^\gamma}{(1 + \sum_{i=1}^m |x - y_i|)^{mn+\delta+\gamma}} + \sum_{j=1}^{m} \frac{A|y_j - y_{j,0}|^\gamma}{(1 + \sum_{i=1}^m |x_0 - y_i|)^{mn+\delta+\gamma}}.
\end{aligned}$$

This shows $\psi(x, \vec{y})$ is continuous at $(x_0, y_{1,0}, \ldots, y_{m,0}) \in \mathbb{R}^{n(m+1)}$ with $x_0 \neq y_{j,0}$, $j = 1, 2, \ldots, m$. □

Lemma 3.2 *There exist* $x_0 \in \mathbb{R}^n$, $r_0 > 0$, $t_0 > 1$ *and* $f_j \in \mathcal{S}(\mathbb{R}^n)$, $j = 1, \ldots, m$, *such that*

$$A_0 := \iint_{\Omega_0} |\psi_t(\vec{f})(y)|^2 dydt \in (0, \infty), \tag{3.1}$$

where $\Omega_0 := B(0, |x_0| + r_0) \times [1, t_0]$.

Proof Since ψ is a non-zero function in $\mathbb{R}^{n(m+1)}$, there exist $x_0, y_{1,0}, \ldots, y_{m,0} \in \mathbb{R}^n$ such that $x_0 \neq y_{i,0} (i = 1, \ldots, m)$ and $\psi(x_0, y_{1,0}, \ldots, y_{m,0}) \neq 0$. By Lemma 3.1, there exists $r_0 > 0$ such that $\psi(x, \vec{y}) > 0$ or $\psi(x, \vec{y}) < 0$ for all $x \in B(x_0, r_0)$ and $y_j \in B(y_{j,0}, r_0)$, $j = 1, \ldots, m$. Without loss of generality, we assume the case $\psi(x, \vec{y}) > 0$. Keeping these notations in mind, we set

$$t_0 = \begin{cases} \left(1 - \frac{r_0}{2\max\{|x_0|, |y_{1,0}|, \ldots, |y_{m,0}|\}}\right)^{-1}, & \max\{|x_0|, |y_{1,0}|, \ldots, |y_{m,0}|\} \geq r_0; \\ 2, & \text{otherwise.} \end{cases}$$

We claim that

$$\left|\frac{x}{t} - x_0\right| < r_0 \quad \text{and} \quad \left|\frac{y_i}{t} - y_{i,0}\right| < r_0, \quad i = 1, \ldots, m, \tag{3.2}$$

for all $1 < t < t_0$, $|x - x_0| < \frac{r_0}{2}$ and $|y_i - y_{i,0}| < \frac{r_0}{2}$, $i = 1, \ldots, m$. Indeed, if $\max\{|x_0|, |y_{1,0}|, \ldots, |y_{m,0}|\} < r_0$, it follows

$$|\frac{x}{t} - x_0| < \frac{|x - x_0|}{t} + (1 - \frac{1}{t})|x_0| < |x - x_0| + \frac{|x_0|}{2} < r_0,$$

and similarly we get $|\frac{y_i}{t} - y_{i,0}| < r_0$, $i = 1, \ldots, m$. In the case

$$y_{j_0,0} := \max\{|x_0|, |y_{1,0}|, \ldots, |y_{m,0}|\} \geq r_0,$$

we have

$$\frac{|y_i - y_{i,0}|}{t} < |y_i - y_{i,0}| < \frac{r_0}{2}$$

and

$$\left(1 - \frac{1}{t}\right)|y_{j_0,0}| < \left(1 - \frac{1}{t_0}\right)|y_{j_0,0}| = \left(1 - \left(1 - \frac{r_0}{2|y_{j_0,0}|}\right)\right)|y_{j_0,0}| = \frac{r_0}{2}.$$

As a consequence,

$$\left|\frac{y_i}{t} - y_{i,0}\right| < r_0, \quad i = 1, \ldots, m.$$

Similarly, we get $|\frac{x}{t} - x_0| < r_0$. This shows (3.2).

Thus, we have $\psi(\frac{x}{t}, \frac{\vec{y}}{t}) > 0$ for $1 < t < t_0$ and $|x - x_0| < \frac{r_0}{2}$, $|y_i - y_{i,0}| < \frac{r_0}{2}$, $i = 1, \ldots, m$. Pick non-negative valued $f_j \in \mathcal{S}(\mathbb{R}^n)$, $j = 1, \ldots, m$ such that $\operatorname{supp} f_j \subset B(y_{j,0}, \frac{r_0}{2})$ and $f_j(y_j) > 0$ for $|y_j - y_{j,0}| < \frac{r_0}{2}$, $j = 1, 2, \ldots, m$. Then it follows from (3.2) that

$$\psi_t(\vec{f})(x) = \frac{1}{t^{mn}} \int_{B(y_{1,0}, \frac{r_0}{2}) \times \cdots \times B(y_{m,0}, \frac{r_0}{2})} \psi(\frac{x}{t}, \frac{y_1}{t}, \ldots, \frac{y_m}{t}) \prod_{j=1}^{m} f_j(y_j) d\vec{y} > 0$$

for all $1 < t < t_0$ and $|x - x_0| < \frac{r_0}{2}$. Therefore,

$$\iint_{B(x_0, \frac{r_0}{2}) \times [1, t_0]} \left|\psi_t(\vec{f})(y)\right|^2 dy dt > 0.$$

In particular, since $B(x_0, \frac{r_0}{2}) \subset B(0, |x_0| + r_0)$, we have

$$A_0 = \iint_{B(0, |x_0| + r_0) \times [1, t_0]} \left|\psi_t(\vec{f})(y)\right|^2 dy dt > 0. \tag{3.3}$$

On the other hand, by using the size condition of ψ, we obtain for every $(y, t) \in \Omega_0$,

$$\begin{aligned} |\psi_t(\vec{f})(y)| &\le \frac{1}{t^{mn}} \int_{\mathbb{R}^{mn}} \left|\psi\left(\frac{y}{t}, \frac{y_1}{t}, \cdots, \frac{y_m}{t}\right)\right| \prod_{j=1}^{m} |f_j(y_j)| dy_j \\ &\le \frac{1}{t^{mn}} \int_{\mathbb{R}^{mn}} \frac{\prod_{j=1}^{m} |f_j(y_j)| dy_j}{\left(1 + \frac{|y - y_1|}{t} + \cdots + \frac{|y - y_m|}{t}\right)^{mn+\delta}} \le \frac{\prod_{j=1}^{m} \|f_j\|_{L^1}}{t^{mn}}. \end{aligned}$$

This immediately yields that

$$A_0 \le \iint_{\Omega_0} \frac{\prod_{j=1}^{m} \|f_j\|_{L^1}^2}{t^{2mn}} dy dt \lesssim \prod_{j=1}^{m} \|f_j\|_{L^1}^2 < \infty. \tag{3.4}$$

Consequently, the desired result follows from (3.3) and (3.4). □

Lemma 3.3 *Let* $0 < \lambda < 2m$ *and* $\frac{1}{m} < p < \frac{2}{\lambda}$. *Then* g_λ^* *is not bounded from* $L^{p_1} \times \cdots \times L^{p_m}$ *to* L^p, *where* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *with* $1 \le p_1, \ldots, p_m < \infty$.

Proof By Lemma 3.2, there exist $x_0 \in \mathbb{R}^n$, $r_0 > 0$, $t_0 > 1$ and $f_j \in \mathcal{S}(\mathbb{R}^n)$, $j = 1, \ldots, m$, such that $0 < A_0 < \infty$, where A_0 is defined in (3.1). Write $R_0 := 2(|x_0| + r_0 + t_0)$. Then for all $|x| > R_0$ and $(y, t) \in \Omega_0$,

$$\frac{|x|}{2} < |x| - |y| \le t + |x - y| \le |x| + |y| + t_0 \le 2|x|.$$

Thus, $t+|x-y| \simeq |x|$. This gives that for all $|x|>R_0$,

$$g_\lambda^*(\vec{f})(x)^2 \geq \iint_{\Omega_0} \frac{t^{n\lambda-n-1}}{|x|^{n\lambda}} |\psi_t(\vec{f})(y)|^2 dydt$$
$$\gtrsim \frac{1}{|x|^{n\lambda}} \iint_{\Omega_0} |\psi_t(\vec{f})(y)|^2 dydt = \frac{A_0}{|x|^{n\lambda}}.$$

Therefore, for any $\lambda \leq \frac{2}{p}$,

$$\|g_\lambda^*(\vec{f})\|_{L^p}^p \gtrsim A_0^{\frac{p}{2}} \int_{|x|>R_0} \frac{dx}{|x|^{\frac{n\lambda p}{2}}} = \infty.$$

On the other hand, for $\vec{f} \in \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)$, we have $\prod_{j=1}^m \|f_j\|_{L^{p_j}} < \infty$. As a consequence, g_λ^* is not bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p whenever $\lambda \leq \frac{2}{p}$.

In particular, for $0<\lambda<2m$ (equivalently $\frac{1}{m}<\frac{2}{\lambda}$), and $p \in (\frac{1}{m}, \frac{2}{\lambda})$, $1<p_1, \ldots, p_m<\infty$ with $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, g_λ^* is not bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p. □

Proof of Theorem 1.1 It follows from [2] that

$$\|S_\alpha(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim \alpha^{mn} [\vec{w}]_{A_{\vec{p}}}^{\max\{\frac{1}{2}, \frac{p_1'}{p}, \cdots, \frac{p_m'}{p}\}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}, \tag{3.5}$$

for all $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ with $1<p_1, \ldots, p_m<\infty$, and for all $\vec{w} \in A_{\vec{p}}$, where the implicit constant is independent of α and $\vec{w}$. Now, we seek for $\gamma(\alpha)=\alpha^r$ such that

$$\|S_\alpha(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim \gamma(\alpha) [\vec{w}]_{A_{\vec{p}}}^{\max\{\frac{1}{2}, \frac{p_1'}{p}, \cdots, \frac{p_m'}{p}\}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

We follow Lerner's idea to show $r \geq mn$ for any $1/m<p<\infty$. In fact, for the case $r<mn$ we can reach a contradiction as follows. This means that the power growth $\gamma(\alpha)=\alpha^{mn}$ in (3.5) is sharp.

Using the standard estimate

$$g_\lambda^*(\vec{f})(x) \leq S_1(\vec{f})(x) + \sum_{k=0}^\infty 2^{-\frac{k\lambda n}{2}} S_{2^{k+1}}(\vec{f})(x), \tag{3.6}$$

we get for some fixed $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$ with $1<q_1, \ldots, q_m<\infty$, and $\gamma(\alpha)=\alpha^{r_0}$

$$\|g_\lambda^*(\vec{f})\|_{L^q(\nu_{\vec{w}})} \lesssim \left(\sum_{k=0}^\infty 2^{-\frac{k\lambda n}{2}} 2^{kr_0}\right) [\vec{w}]_{A_{\vec{q}}}^{\max\{\frac{1}{2}, \frac{q_1'}{q}, \cdots, \frac{q_m'}{q}\}} \prod_{i=1}^m \|f_i\|_{L^{q_i}(w_i)}.$$

This means that if $\lambda > \frac{2r_0}{n}$, g_λ^* is bounded from $L^{q_1}(w_1) \times \cdots \times L^{q_m}(w_m)$ to $L^q(\nu_{\vec{w}})$. From this, by extrapolation(see [34]), we get g_λ^* is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p for any $p > 1/m$, whenever $\lambda > \frac{2r_0}{n}$. But by Lemma 3.3, we know g_λ^* is not bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p for $\lambda < 2m$ and $\frac{1}{m} < p < \frac{2}{\lambda}$. If $r_0 < mn$, we would obtain a contradiction to the latter fact for p sufficiently close to $1/m$. □

4 Bump and Fefferman–Stein Inequalities

In this section, we will prove bump inequalities (Theorem 1.2) and Fefferman–Stein inequalities (Theorem 1.3). Our strategy is to use the sparse domination for the multilinear Littlewood–Paley operators.

Proof of Theorem 1.2 Given $r \geq 1$ and a sparse family $\mathcal{S}$, we denote

$$\mathcal{A}_{\mathcal{S}}^r(\vec{f})(x) := \left(\sum_{Q \in \mathcal{S}} \prod_{i=1}^m \langle f_i \rangle_Q^r \mathbf{1}_Q(x) \right)^{\frac{1}{r}}.$$

The sparse domination below will provide us great convenience:

$$S_\alpha \vec{f}(x) \leq c_n \alpha^{mn} \sum_{j=1}^{3^n} \mathcal{A}_{\mathcal{S}_j}^2(|\vec{f}|)(x), \quad \text{a.e. } x \in \mathbb{R}^n, \tag{4.1}$$

$$g_\lambda^* \vec{f}(x) \leq \frac{c_n}{2^{n(\lambda-2m)} - 1} \sum_{j=1}^{3^n} \mathcal{A}_{\mathcal{S}_j}^2(|\vec{f}|)(x), \quad \text{a.e. } x \in \mathbb{R}^n, \tag{4.2}$$

where $\mathcal{S}_j$ is a sparse family for each $j = 1, \ldots, 3^n$. These results are explicitly proved in [2]. By (4.1) and (4.2), the inequalities (1.4) and (1.5) follow from the following

$$\|\mathcal{A}_{\mathcal{S}}^2(\vec{f})\|_{L^p(u)} \lesssim \mathcal{N}_p \prod_{j=1}^m \|f_j\|_{L^{p_j}(v_j)}, \tag{4.3}$$

for every sparse family $\mathcal{S}$, where the implicit constant does not depend on $\mathcal{S}$.

To show (4.3), we begin with the case $1 < p \leq 2$. Actually, the Hölder inequality (2.7) gives that

$$\begin{aligned}
\|\mathcal{A}_{\mathcal{S}}^2(\vec{f})\|_{L^p(u)}^p &= \int_{\mathbb{R}^n} \left(\sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle f_j \rangle_Q^2 \mathbf{1}_Q(x) \right)^{\frac{p}{2}} u(x)dx \leq \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle |f_j| \rangle_Q^p u(Q) \\
&\lesssim \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \|f_j v_j^{\frac{1}{p_j}}\|_{\bar{B}_j,Q}^p \|v_j^{-\frac{1}{p_j}}\|_{B_j,Q}^p \|u^{\frac{1}{p}}\|_{p,Q}^p |Q| \\
&\lesssim \|(u, \vec{v})\|_{A,\vec{B},\vec{p}}^p \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \left(\inf_Q M_{\bar{B}_j}(f_j v_j^{\frac{1}{p_j}}) \right)^p |E_Q|
\end{aligned}$$

$$\leq \|(u,\vec{v})\|^p_{A,\vec{B},\vec{p}} \prod_{j=1}^m \left(\int_{\mathbb{R}^n} M_{\bar{B}_j}(f_j v_j^{\frac{1}{p_j}})(x)^{p_j} dx \right)^{p/p_j}$$

$$\leq \|(u,\vec{v})\|^p_{A,\vec{B},\vec{p}} \prod_{j=1}^m \|M_{\bar{B}_j}\|^p_{L^{p_j}(\mathbb{R}^n)} \|f_j\|^p_{L^{p_j}(v_j)}, \tag{4.4}$$

where Lemma 2.8 is used in the last step.

Next let us deal with the case $2 < p < \infty$. By duality, one has

$$\|\mathcal{A}^2_{\mathcal{S}}(\vec{f})\|^2_{L^p(u)} = \|\mathcal{A}^2_{\mathcal{S}}(\vec{f})^2\|_{L^{p/2}(u)} = \sup_{\substack{0 \leq h \in L^{(p/2)'}(u) \\ \|h\|_{L^{(p/2)'}(u)=1}}} \int_{\mathbb{R}^n} \mathcal{A}^2_{\mathcal{S}}(\vec{f})(x)^2 h(x) u(x) dx. \tag{4.5}$$

Fix a nonnegative function $h \in L^{(p/2)'}(u)$ with $\|h\|_{L^{(p/2)'}(u)} = 1$. Then using Hölder's inequality (2.7) and Lemma 2.8, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{A}^2_{\mathcal{S}}(\vec{f})(x)^2 h(x) u(x) dx \\
&\lesssim \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \langle |f_j| \rangle_Q^2 \langle hu \rangle_Q |Q| \\
&\lesssim \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \|f_j v_j^{\frac{1}{p_j}}\|^2_{\bar{B}_j,Q} \|v^{-\frac{1}{p_j}}\|^2_{B_j,Q} \|hu^{1-\frac{2}{p}}\|_{\bar{A},Q} \|u^{\frac{2}{p}}\|_{A,Q} |Q| \\
&\lesssim \|(u,\vec{v})\|^2_{A,\vec{B},\vec{p}} \sum_{Q \in \mathcal{S}} \prod_{j=1}^m \left(\inf_Q M_{\bar{B}_j}(f_j v_j^{\frac{1}{p_j}}) \right)^2 \left(\inf_Q M_{\bar{A}}(hu^{1-\frac{2}{p}}) \right) |E_Q| \\
&\leq \|(u,\vec{v})\|^2_{A,\vec{B},\vec{p}} \int_{\mathbb{R}^n} \prod_{j=1}^m M_{\bar{B}_j}(f_j v_j^{\frac{1}{p_j}})(x)^2 M_{\bar{A}}(hu^{1-\frac{2}{p}})(x) dx \\
&\leq \|(u,\vec{v})\|^2_{A,\vec{B},\vec{p}} \Big\| \prod_{j=1}^m M_{\bar{B}_j}(f_j v_j^{\frac{1}{p_j}})^2 \Big\|_{L^{p/2}(\mathbb{R}^n)} \|M_{\bar{A}}(hu^{1-\frac{2}{p}})\|_{L^{(p/2)'}(\mathbb{R}^n)} \\
&\leq \|(u,\vec{v})\|^2_{A,\vec{B},\vec{p}} \prod_{j=1}^m \|M_{\bar{B}_j}(f_j v_j^{\frac{1}{p_j}})\|^2_{L^{p_j}(\mathbb{R}^n)} \|M_{\bar{A}}(hu^{1-\frac{2}{p}})\|_{L^{(p/2)'}(\mathbb{R}^n)} \\
&\leq \|(u,\vec{v})\|^2_{A,\vec{B},\vec{p}} \prod_{j=1}^m \|M_{\bar{B}_j}\|^2_{\mathcal{L}(L^{p_j}(\mathbb{R}^n))} \|f_j\|^2_{L^{p_j}(v_j)} \|M_{\bar{A}}\|_{\mathcal{L}(L^{(p/2)'}(\mathbb{R}^n))} \|h\|_{L^{(p/2)'}(u)},
\end{aligned} \tag{4.6}$$

where

$$\|M_{\bar{B}_j}\|_{\mathcal{L}(L^{p_j}(\mathbb{R}^n))} = \|M_{\bar{B}_j}\|_{L^{p_j}(\mathbb{R}^n) \to L^{p_j}(\mathbb{R}^n)}$$

and

$$\|M_{\bar{A}}\|_{\mathcal{L}(L^{(p/2)'}(\mathbb{R}^n))} = \|M_{\bar{A}}\|_{L^{(p/2)'}(\mathbb{R}^n)\to L^{(p/2)'}(\mathbb{R}^n)}.$$

Therefore, (4.3) immediately follows from (4.4), (4.5) and (4.6). □

Proof of Theorem 1.3 Fix exponents $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \ldots, p_m < \infty$, $0 < p \le 2$ and weights $\vec{w} = (w_1, \ldots, w_m)$. Note that $v_i(x) := Mw_i(x) \ge \langle w_i\rangle_Q$ for any dyadic cube $Q \in \mathcal{S}$ containing x. For each i, let A_i be a Young function such that $\bar{A}_i \in B_{p_i}$. By Lemma 2.8, we have

$$\|M_{\bar{A}_i}(f_i v_i^{\frac{1}{p_i}})\|_{L^{p_i}(\mathbb{R}^n)} \lesssim \|f_i\|_{L^{p_i}(v_i)}, \quad i = 1, \ldots, m. \tag{4.7}$$

Thus, using sparse domination (4.1), Hölder's inequality and (4.7), we deduce that

$$\begin{aligned}
\|S_\alpha(\vec{f})\|_{L^p(\nu_{\vec{w}})}^p &\lesssim \alpha^{pmn} \sum_{j=1}^{3^n} \sum_{Q\in\mathcal{S}_j} \prod_{i=1}^m \langle |f_i|\rangle_Q^p \nu_{\vec{w}}(Q) \\
&\le \alpha^{pmn} \sum_{j=1}^{3^n} \sum_{Q\in\mathcal{S}_j} \prod_{i=1}^m \|f_i v_i^{\frac{1}{p_i}}\|_{\bar{A}_i,Q}^p \|v_i^{-\frac{1}{p_i}}\|_{A_i,Q}^p \nu_{\vec{w}}(Q) \\
&\le \alpha^{pmn} \sum_{j=1}^{3^n} \sum_{Q\in\mathcal{S}_j} \prod_{i=1}^m \|f_i v_i^{\frac{1}{p_i}}\|_{\bar{A}_i,Q}^p \langle w_i\rangle_Q^{-\frac{p}{p_i}} \langle \nu_{\vec{w}}\rangle_Q |Q| \\
&\lesssim \alpha^{pmn} \sum_{j=1}^{3^n} \sum_{Q\in\mathcal{S}_j} \prod_{i=1}^m \Big(\inf_Q M_{\bar{A}_i}(f_i v_i^{\frac{1}{p_i}})\Big)^p |E_Q| \\
&\lesssim \alpha^{pmn} \int_{\mathbb{R}^n} \prod_{i=1}^m M_{\bar{A}_i}(f_i v_i^{\frac{1}{p_i}})(x)^p dx \lesssim \alpha^{pmn} \prod_{i=1}^m \|M_{\bar{A}_i}(f_i v_i^{\frac{1}{p_i}})\|_{L^{p_i}(\mathbb{R}^n)}^p \\
&\lesssim \alpha^{pmn} \prod_{i=1}^m \|f_i\|_{L^{p_i}(v_i)}^p = \alpha^{pmn} \prod_{i=1}^m \|f_i\|_{L^{p_i}(Mw_i)}^p.
\end{aligned}$$

This shows (1.6). Likewise, one can obtain (1.7). □

5 Entropy Bumps

In this section, we will prove entropy bump inequalities (Theorem 1.4). By the sparse domination for Littlewood–Paley operators, see (4.1) and (4.2), it suffices to prove the results for $\mathcal{A}_{\mathcal{S}}^r$, $r \ge 1$.

Let us call $(\alpha_i) = (\alpha_1, \alpha_2, \ldots, \alpha_m)$. We will denote $(\alpha_i)_{i\neq j} = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_m)$. Having that notation at our disposal we define the following submultilinear maximal function.

$$\mathcal{M}^{(\alpha_i)_{i\neq j}}(\vec{\sigma})(x) := \sup_{x\in Q} \prod_{i\in\{1,\ldots,m\},i\neq j} \langle\sigma_i\rangle_Q^{\alpha_i}$$

and given $\vec{p} = (p_1, \ldots, p_m)$

$$\mathcal{M}^{\frac{1}{\vec{p}}}(\vec{\sigma})(x) := \sup_{x\in Q} \prod_{i=1}^{m} \langle\sigma_i\rangle_Q^{\frac{1}{p_i}}$$

Let $1 < p_1, \ldots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. We define

$$\rho_{\vec{\sigma},\vec{p}}(Q) = \left(\int_Q \mathcal{M}^{\frac{p}{\vec{p}}}(\sigma_i\chi_Q)(x)dx\right)\left(\int_Q \prod_{i=1}^{m} \sigma_i(x)^{\frac{p}{p_i}}dx\right)^{-1}.$$

In the scalar case we shall denote just

$$\rho_\nu(Q) = \frac{1}{\nu(Q)}\int_Q M(\nu\chi_Q)(x)dx.$$

Given an increasing function $\varepsilon : [1, +\infty) \to (0, +\infty)$ let us denote

$$\rho_{\vec{\sigma},\vec{p},\varepsilon}(Q) := \rho_{\vec{\sigma},\vec{p}}(Q)\varepsilon(\rho_{\vec{\sigma},\vec{p}}(Q)) \quad \text{and} \quad \rho_{\nu,\varepsilon}(Q) := \rho_\nu(Q)\varepsilon(\rho_\nu(Q)).$$

With the notation we have just fixed, we are in the position to introduce the entropy bump conditions. For weights $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$ and ν, we define

$$\lfloor\vec{\sigma},\nu\rfloor_{\vec{p},r,\varepsilon} = \sup_Q \left(\prod_{i=1}^{m}\langle\sigma_i\rangle_Q^{\frac{p}{p_i'}}\right)\langle\nu\rangle_Q \rho_{\vec{\sigma},\vec{p},\varepsilon}(Q)\rho_{\nu,\varepsilon}(Q)^{\frac{p}{r}-1}. \tag{5.1}$$

Also, if $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$, we denote

$$\lfloor\vec{\sigma}\rfloor_{\vec{q},\vec{p},\rho,\theta,j} := \sup_Q \prod_{i=1}^{m}\langle\sigma_i\rangle_Q^{q_i}\left(\frac{\int_Q \mathcal{M}^{(1/(\theta p_i))_{i\neq j}}(\vec{\sigma}\chi_Q)}{\int_Q \prod_{i\neq j}\sigma_i^{1/(\theta p_i)}}\right)^{\theta}\rho\left(\left(\frac{\int_Q \mathcal{M}^{(1/(\theta p_i))_{i\neq j}}(\vec{\sigma}\chi_Q)}{\int_Q \prod_{i\neq j}\sigma_i^{1/(\theta p_i)}}\right)^{\theta}\right).$$

Denote $\overrightarrow{f\sigma} := (f_1\sigma_1, \ldots, f_m\sigma_m)$. Armed with the notation and the definitions of the entropy bumps just introduced, we can finally state and prove the main theorems of this section.

Theorem 5.1 *Let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $p > r$ and $1 < p_1, \ldots, p_m < \infty$. Let $\sigma_1, \ldots, \sigma_m$ and ν be weights. Assume that ε is a monotonic increasing function on*

$(1,\infty)$ satisfying $\int_1^\infty \frac{dt}{\varepsilon(t)t} < \infty$. Then

$$\|\mathcal{A}^r_{\mathcal{S}}(\overrightarrow{f\sigma})\|_{L^p(\nu)} \lesssim \lfloor\vec{\sigma},\nu\rfloor^{\frac{1}{p}}_{\vec{p},r,\varepsilon} \prod_{i=1}^m \|f\|_{L^{p_i}(\sigma_i)}. \tag{5.2}$$

Note that the theorem above extends to the multilinear setting [28, Theorem 3.2].

Theorem 5.2 *Let $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $p \le r$ and $1 < p_1, \ldots, p_m < \infty$. Let $\sigma_1, \ldots, \sigma_m$ and ν be weights. Assume that ρ is a monotonic increasing function on $(1,\infty)$ satisfying $\int_1^\infty \frac{dt}{\rho^{\frac{p}{r}}(t)t} < \infty$ and $\rho(2t) \le C\rho(t)$ for $t \ge 1$. Then*

$$\|\mathcal{A}^r_{\mathcal{S}}(\overrightarrow{f\sigma})\|_{L^p(\nu)} \lesssim \lfloor\vec{\sigma},\nu\rfloor^{\frac{1}{r}}_{\frac{r}{\vec{p}'},\vec{p},\rho,\frac{r}{p},m+1} \prod_{i=1}^m \|f\|_{L^{p_i}(\sigma_i)}, \tag{5.3}$$

where $\vec{p} = (p_1, \ldots, p_m, p')$ and $\frac{r}{\vec{p}'} = \big(\frac{r}{p_1'}, \ldots, \frac{r}{p_m'}, \frac{r}{p}\big)$.

Note that in this case the linear version of the estimate obtained is slightly different from [28, Theorem 3.3] since the entropy bump constant involved in that case is the following

$$\lfloor\sigma,\nu\rfloor_{\left(\frac{r}{p'},\frac{r}{p}\right),(p,p'),\rho,\frac{r}{p},2} := \sup_Q \langle\sigma\rangle_Q^{\frac{r}{p'}} \langle\nu\rangle_Q^{\frac{r}{p}} \left(\frac{\int_Q M(\sigma\chi_Q)^{\frac{1}{r}}}{\int_Q \sigma^{\frac{1}{r}}}\right)^{\frac{r}{p}} \rho\left(\left(\frac{\int_Q M(\sigma\chi_Q)^{\frac{1}{r}}}{\int_Q \sigma^{\frac{1}{r}}}\right)^{\frac{r}{p}}\right).$$

Also the integrability condition imposed on ρ does not match the one in [28, Theorem 3.3].

5.1 Proof of Theorem 5.1

We need a multilinear version of Carleson embedding theorem from [11].

Lemma 5.3 *Let $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$ be weights. Let $1 < p_i < \infty$ and $p \in (1,\infty)$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Assume that $\{a_Q\}_{Q\in\mathcal{D}}$ is a sequence of non-negative numbers for which the following condition holds*

$$\sum_{Q'\subset Q} a_{Q'} \le A \int_Q \prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}\,dx, \quad \forall Q \in \mathcal{D}. \tag{5.4}$$

Then for all $f_i \in L^{p_i}(\sigma_i)$,

$$\left(\sum_{Q\in\mathcal{D}} a_Q\Big(\prod_{i=1}^m \fint_Q f_i\,d\sigma_i\Big)^p\right)^{\frac{1}{p}} \le A \prod_{i=1}^m p_i'\|f_i\|_{L^{p_i}(\sigma_i)}. \tag{5.5}$$

With this result in hand, we are in the position to settle Theorem 5.1 following ideas in [29].

Proof of Theorem 5.1 First we split the sparse family as follows. We say that $Q \in \mathcal{S}_a$ if and only if

$$\left(\prod_{i=1}^{m}\langle\sigma_i\rangle_Q^{\frac{r}{p_i'}}\right)\langle\nu\rangle_Q^{\frac{r}{p}}\rho_{\vec{\sigma},\vec{p},\varepsilon}(Q)^{\frac{r}{p}}\rho_{\nu,\varepsilon}(Q)^{1/(p/r)'}\simeq 2^a.$$

Let us begin providing a suitable estimate for each of those pieces of the sparse family. Given a weight γ let us denote $\langle h\rangle_Q^{\gamma}:=\frac{1}{\gamma(Q)}\int_Q|h(x)|\gamma(x)dx$. Assume that $g\in L^{(p/r)'}(\nu)$. By duality we can write

$$\begin{aligned}
&\Big\langle\sum_{Q\in\mathcal{S}_a}\Big(\prod_{i=1}^{m}\langle f_i\sigma_i\rangle_Q\Big)^r\mathbf{1}_Q, g\nu\Big\rangle\\
&=\sum_{Q\in\mathcal{S}_a}\Big(\prod_{i=1}^{m}\langle f_i\rangle_Q^{\sigma_i}\Big)^r\Big(\prod_{i=1}^{m}\langle\sigma_i\rangle_Q\Big)^r\langle g\rangle_Q^{\nu}\langle\nu\rangle_Q|Q|\\
&=\sum_{Q\in\mathcal{S}_a}\Big(\prod_{i=1}^{m}\langle f_i\rangle_Q^{\sigma_i}\Big)^r\Big(\prod_{i=1}^{m}\langle\sigma_i\rangle_Q^{\frac{r}{p_i}}\Big)\Big\{\Big(\prod_{i=1}^{m}\langle\sigma_i\rangle_Q^{\frac{r}{p_i'}}\Big)\langle\nu\rangle_Q^{\frac{r}{p}}\Big\}\langle\nu\rangle_Q^{1/(p/r)'}\langle g\rangle_Q^{\nu}\cdot|Q|\\
&=\sum_{Q\in\mathcal{S}_a}\Big(\prod_{i=1}^{m}\langle f_i\rangle_Q^{\sigma_i}\Big)^r\frac{\prod_{i=1}^{m}\langle\sigma_i\rangle_Q^{\frac{r}{p_i}}}{\rho_{\vec{\sigma},\vec{p},\varepsilon}(Q)^{\frac{r}{p}}}\frac{\langle\nu\rangle_Q^{1/(p/r)'}}{\rho_{\nu,\varepsilon}(Q)^{1/(p/r)'}}\langle g\rangle_Q^{\nu}\cdot|Q|\\
&\quad\times\Big\{\Big(\prod_{i=1}^{m}\langle\sigma_i\rangle_Q^{\frac{r}{p_i'}}\Big)\langle\nu\rangle_Q^{\frac{r}{p}}\rho_{\vec{\sigma},\vec{p},\varepsilon}(Q)^{\frac{r}{p}}\rho_{\nu,\varepsilon}(Q)^{1/(p/r)'}\Big\}\\
&\lesssim 2^a\sum_{Q\in\mathcal{S}_a}\Big(\prod_{i=1}^{m}\langle f_i\rangle_Q^{\sigma_i}\Big)^r\frac{\prod_{i=1}^{m}\sigma_i(Q)^{\frac{r}{p_i}}}{\rho_{\vec{\sigma},\vec{p},\varepsilon}(Q)^{\frac{r}{p}}}\frac{\nu(Q)^{1/(p/r)'}}{\rho_{\nu,\varepsilon}(Q)^{1/(p/r)'}}\langle g\rangle_Q^{\nu}\\
&\le 2^a\Big(\sum_{Q\in\mathcal{S}_a}\Big(\prod_{i=1}^{m}\langle f_i\rangle_Q^{\sigma_i}\Big)^p\frac{\prod_{i=1}^{m}\sigma_i(Q)^{\frac{p}{p_i}}}{\rho_{\vec{\sigma},\vec{p},\varepsilon}(Q)}\Big)^{\frac{r}{p}}\Big(\sum_{Q\in\mathcal{S}_a}(\langle g\rangle_Q^{\nu})^{(p/r)'}\frac{\nu(Q)}{\rho_{\nu,\varepsilon}(Q)}\Big)^{\frac{1}{(p/r)'}}.
\end{aligned}$$

For the second term, we would like to get that

$$\sum_{Q\in\mathcal{S}_a}(\langle g\rangle_Q^{\nu})^{(p/r)'}\frac{\nu(Q)}{\rho_{\nu,\varepsilon}(Q)}\lesssim\|g\|_{L^{(p/r)'}(\nu)}^{(p/r)'}. \tag{5.6}$$

We omit the proof of (5.6) and focus on the first term above, since the argument that we are going provide, essentially contains the linear case. For the first term, it needs to show

$$\sum_{Q\in\mathcal{S}_a}\Big(\prod_{i=1}^m \langle f_i\rangle_Q^{\sigma_i}\Big)^p \frac{\prod_{i=1}^m \sigma_i(Q)^{\frac{p}{p_i}}}{\rho_{\vec{\sigma},\vec{p},\varepsilon}(Q)} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}^p. \tag{5.7}$$

Taking into account Lemma 5.3, it suffices to verify that (5.4) holds with

$$a_Q = \begin{cases} \frac{\prod_{i=1}^m \sigma_i(Q)^{\frac{p}{p_i}}}{\rho_{\vec{\sigma},\vec{p},\varepsilon}(Q)} & Q\in\mathcal{S}_a, \\ 0 & \text{otherwise.}\end{cases}$$

Indeed, let us call $\mathcal{S}_a(R)$ the set of cubes of $\mathcal{S}_a$ that are contained in $R\in\mathcal{D}$. Then

$$\begin{aligned}
\sum_{Q\in\mathcal{S}_a(R)} \frac{\prod_{i=1}^m \sigma_i(Q)^{\frac{p}{p_i}}}{\rho_{\vec{\sigma},\vec{p},\varepsilon}(Q)} &\lesssim \sum_{j=1}^\infty \sum_{\substack{\rho_{\vec{\sigma},\vec{p}}(Q)\sim 2^j\\ Q\in\mathcal{S}_a(R)}} \frac{\prod_{i=1}^m \sigma_i(Q)^{\frac{p}{p_i}}}{\rho_{\vec{\sigma},\vec{p},\varepsilon}(Q)} \\
&\lesssim \sum_{j=1}^\infty \sum_{\substack{\text{maximal } Q\in\mathcal{S}_a(R)\\ \rho_{\vec{\sigma},\vec{p}}(Q)\simeq 2^j}} \sum_{\substack{P\subset Q\\ P\in\mathcal{S}_a(R)}} \frac{\prod_{i=1}^m \sigma_i(P)^{\frac{p}{p_i}}}{\rho_{\vec{\sigma},\vec{p},\varepsilon}(P)} \\
&\le \sum_{j=1}^\infty \sum_{\substack{\text{maximal } Q\in\mathcal{S}_a(R)\\ \rho_{\vec{\sigma},\vec{p}}(Q)\simeq 2^j}} \sum_{\substack{P\subset Q\\ P\in\mathcal{S}_a(R)}} \frac{2^{-j}}{\varepsilon(2^j)} \int_{E_P} \mathcal{M}^{\frac{p}{\vec{p}}}(\sigma_i\mathbf{1}_Q)(x)dx \\
&\lesssim \sum_{j=1}^\infty \sum_{\substack{\text{maximal } Q\in\mathcal{S}_a(R)\\ \rho_{\vec{\sigma},\vec{p}}(Q)\simeq 2^j}} \frac{2^{-j}}{\varepsilon(2^j)} \int_Q \mathcal{M}^{\frac{p}{\vec{p}}}(\sigma_i\mathbf{1}_Q)(x)dx \\
&\lesssim \Big(\prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}\Big)(R) \sum_{j=0}^\infty \frac{1}{\varepsilon(2^j)} \lesssim \Big(\prod_{i=1}^m \sigma_i^{\frac{p}{p_i}}\Big)(R)\int_1^\infty \frac{dt}{t\varepsilon(t)}.
\end{aligned}$$

This provides the desired bound.

Collecting (5.6) and (5.7), we have shown that

$$\Big\langle \sum_{Q\in\mathcal{S}_a}\Big(\prod_{i=1}^m \langle f_i\sigma_i\rangle_Q\Big)^r \mathbf{1}_Q, g\nu\Big\rangle \lesssim 2^a \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}^r \cdot \|g\|_{L^{(p/r)'}(\nu)}.$$

Since for the largest a for which $\mathcal{S}_a$ is not empty we have that $\lfloor\vec{\sigma},\nu\rfloor_{\vec{p},r,\varepsilon}^{\frac{r}{p}} \simeq 2^a$, summing in a yields

$$\Big\langle \sum_{Q\in\mathcal{S}}\Big(\prod_{i=1}^m \langle f_i\sigma_i\rangle_Q\Big)^r \mathbf{1}_Q, g\nu\Big\rangle \lesssim \lfloor\vec{\sigma},\nu\rfloor_{\vec{p},r,\varepsilon}^{\frac{r}{p}} \prod_{i=1}^m \|f_i\|_{L^p(\sigma_i)}^r \|g\|_{L^{(p/r)'}(\nu)}.$$

Consequently,

$$\|\mathcal{A}^r_{\mathcal{S}}(\overrightarrow{f\sigma})\|_{L^p(\nu)} \lesssim \lfloor\vec{\sigma},\nu\rfloor_{\vec{p},r,\varepsilon}^{\frac{1}{p}} \prod_{i=1}^m \|f_i\|_{L^p(\sigma_i)}.$$

This shows Theorem 5.1. □

5.2 Proof of Theorem 5.2

To settle Theorem 5.2 we are going to follow the scheme in [48]. First we borrow a result from [9].

Lemma 5.4 *For every $1 < s < \infty$ we have that for every positive locally finite measure σ on $\mathbb{R}^n$ and any positive numbers λ_Q, $Q \in \mathcal{D}$, we have*

$$\int_{\mathbb{R}^n} \Big(\sum_{Q\in\mathcal{D}} \frac{\lambda_Q}{\sigma(Q)}\mathbf{1}_Q(x)\Big)^s d\sigma(x) \lesssim_s \sum_{Q\in\mathcal{D}} \lambda_Q \Big(\sigma(Q)^{-1}\sum_{Q'\subseteq Q}\lambda_{Q'}\Big)^{s-1}.$$

Given a sparse family $\mathcal{S}$ contained in a dyadic grid $\mathcal{D}$, for every $Q \in \mathcal{S}$ we will denote $\mathcal{S}(Q)$ the family of cubes of $\mathcal{S}$ that are contained in Q. For $\mathcal{S}$ and $\vec{\omega} = (\omega_1,\dots,\omega_m)$, we denote

$$\lfloor\vec{\omega}\rfloor_{\vec{q},\vec{p},\rho,\theta,j,\mathcal{S}} := \sup_{Q\in\mathcal{S}} \prod_{i=1}^m \langle\omega_i\rangle_Q^{q_i} \left(\frac{\int_Q \mathcal{M}^{(1/(\theta p_i))_{i\neq j}}(\vec{\omega}\chi_Q)}{\int_Q \prod_{i\neq j}\omega_i^{1/(\theta p_i)}}\right)^{\theta} \rho\left(\left(\frac{\int_Q \mathcal{M}^{(1/(\theta p_i))_{i\neq j}}(\vec{\omega}\chi_Q)}{\int_Q \prod_{i\neq j}\omega_i^{1/(\theta p_i)}}\right)^{\theta}\right).$$

The following lemma is a particular case of [48, Lemma 2.3]. The proof is also essentially contained in the earlier work [18, Proposition 4.8].

Lemma 5.5 *Let $\beta_1,\dots,\beta_m \geq 0$ be such that $\beta := \sum_{i=1}^m \beta_i < 1$. Let $\mathcal{S} \subset \mathcal{D}$ be a sparse family. Then for every cube $Q \in \mathcal{S}$ and all functions $w_1,\dots,w_m$,*

$$\sum_{Q'\in\mathcal{S}(Q)} |Q'| \prod_{i=1}^m \langle w_i\rangle_{Q'}^{\beta_i} \lesssim |Q| \prod_{i=1}^m \langle w_i\rangle_Q^{\beta_i}.$$

The following lemma will be one of the fundamental pieces to settle Theorem 5.2.

Lemma 5.6 *Let $j \in \{1,\dots,m\}$, $s_1,\dots,s_m \in \mathbb{R}$ with $s_i > 0$ for each $i \in \{1,\dots,m\}$ with $i \neq j$, and $q_1,\dots,q_m > 0$ with $q_j = 1 + s_j$ be such that*

$$\sum_i s_i \leq \sum_i q_i, \quad \frac{\sum_i s_i}{\sum_i q_i} < \min_{i\neq j} \frac{s_i}{q_i}.$$

Let $\mathcal{S}$ be a sparse family such that for every $Q \in \mathcal{S}$ and some $\theta > 0$

$$2^r \le \left(\frac{\int_Q \mathcal{M}^{(1/(\theta p_i))_{i\neq j}}(\vec{w}\chi_Q)}{\int_Q \prod_{i\neq j} w_i^{1/(\theta p_i)}} \right)^{\theta} \le 2^{r+1}, \tag{5.8}$$

where $p_i \in (0, +\infty)$. Then, if ρ is a monotonic increasing function on $(1, \infty)$, for every $0 < \alpha < \infty$ we have that

$$\mathscr{A} := \int_{\mathbb{R}^n} \Big(\sum_{Q\in\mathcal{S}} \prod_{i=1}^{m} \langle w_i \rangle_Q^{s_i\alpha} \mathbf{1}_Q \Big)^{\frac{1}{\alpha}} dw_j \lesssim \frac{\lfloor \vec{\omega} \rfloor_{\vec{q},\vec{p},\rho,\theta,j,\mathcal{S}}}{2^r \rho(2^r)} \sum_{Q\in\mathcal{S}} |Q| \prod_{i\neq j} \langle w_i \rangle_Q^{s_i - q_i}.$$

Proof The left-hand side of the conclusion is monotonically decreasing in α and the right-hand side does not depend on α, so it suffices to consider small α, in particular we may assume $\alpha < 1$.

It follows from the hypothesis that for sufficiently small α there exists an ϵ such that

$$\frac{\alpha \sum_i s_i}{\sum_i q_i} < \epsilon \le \min \left\{ \frac{1}{1/\alpha - 1}, \min_i \frac{\alpha s_i + \delta_{ij}}{q_i} \right\}.$$

where as usual $\delta_{ij} = 1$ if $i = j$ or 0 otherwise. By the assumption $\alpha < 1$ and Lemma 5.4,

$$\mathscr{A} \lesssim \sum_{Q\in\mathcal{S}} |Q| \prod_{i=1}^{m} \langle w_i \rangle_Q^{\alpha s_i + \delta_{ij}} \Big(\sum_{Q'\in\mathcal{S}(Q)} \frac{|Q'|}{w_j(Q)} \prod_{i=1}^{m} \langle w_i \rangle_{Q'}^{\alpha s_i + \delta_{ij}} \Big)^{\frac{1}{\alpha} - 1}.$$

Taking into account the definition of $\lfloor \vec{\omega} \rfloor_{\vec{q},\vec{p},\rho,\theta,j,\mathcal{S}}$ and (5.8), we get

$$\mathscr{A} \lesssim \left(\frac{\lfloor \vec{\omega} \rfloor_{\vec{q},\vec{p},\rho,\theta,j,\mathcal{S}}}{2^r \rho(2^r)} \right)^{\epsilon(\frac{1}{\alpha}-1)} \sum_{Q\in\mathcal{S}} |Q| \prod_{i=1}^{m} \langle w_i \rangle_Q^{\alpha s_i + \delta_{ij}} \Big(\sum_{Q'\in\mathcal{S}(Q)} \frac{|Q'|}{w_j(Q)} \prod_{i=1}^{m} \langle w_i \rangle_{Q'}^{\alpha s_i + \delta_{ij} - \epsilon q_i} \Big)^{\frac{1}{\alpha} - 1}.$$

Observe that $\alpha s_i + \delta_{ij} - \epsilon q_i \ge 0$ and $\sum_i (\alpha s_i + \delta_{ij} - \epsilon q_i) < 1$. Hence, Lemma 5.5 implies that

$$\begin{aligned}
\mathscr{A} &\lesssim \left(\frac{\lfloor \vec{\omega} \rfloor_{\vec{q},\vec{p},\rho,\theta,j,\mathcal{S}}}{2^r \rho(2^r)} \right)^{\epsilon(\frac{1}{\alpha}-1)} \sum_{Q\in\mathcal{S}} |Q| \prod_{i=1}^{m} \langle w_i \rangle_Q^{\alpha s_i + \delta_{ij}} \Big(\frac{|Q|}{w_j(Q)} \prod_{i=1}^{m} \langle w_i \rangle_Q^{\alpha s_i + \delta_{ij} - \epsilon q_i} \Big)^{\frac{1}{\alpha} - 1} \\
&= \left(\frac{\lfloor \vec{\omega} \rfloor_{\vec{q},\vec{p},\rho,\theta,j,\mathcal{S}}}{2^r \rho(2^r)} \right)^{\epsilon(\frac{1}{\alpha}-1)} \sum_{Q\in\mathcal{S}} |Q| \prod_{i=1}^{m} \langle w_i \rangle_Q^{\delta_{ij} + s_i - \epsilon q_i (\frac{1}{\alpha} - 1)}.
\end{aligned}$$

By construction $1-\epsilon(1/\alpha-1)\geq 0$, and again by the definition of $\lfloor\vec{\omega}\rfloor_{\vec{q},\vec{p},\rho,\theta,j,\mathcal{S}}$ and (5.8), we conclude that

$$\mathscr{A}\lesssim\frac{\lfloor\vec{\omega}\rfloor_{\vec{q},\vec{p},\rho,\theta,j,\mathcal{S}}}{2^r\rho(2^r)}\sum_Q|Q|\prod_{i=1}^m\langle w_i\rangle_Q^{\delta_{ij}+s_i-q_i}.$$

and we are done, since $q_j=1+s_j$. □

Now we present a stopping time condition. Let $\mathcal{S}\subset\mathcal{D}$ be a finite sparse family and let $\lambda_i:\mathcal{S}\to[0,\infty)$, $Q\mapsto\lambda_{i,Q}$ be a function that takes a cube to a non-negative real number. Then we have that $\mathcal{F}_i$ is the minimal family of cubes such that the maximal members of $\mathcal{S}$ are contained in $\mathcal{F}_i$, and if $F\in\mathcal{F}_i$, then every maximal subcube $F'\subset F$ with $\lambda_{i,F'}\geq 2\lambda_{i,F}$ is also a member of $\mathcal{F}_i$.

For each cube Q, let $\pi_i(Q)$ (the parent of Q in the stopping family $\mathcal{F}_i$) be the smallest cube with $Q\subseteq\pi_i(Q)\in\mathcal{F}_i$. We write $\sum_{F_1,\ldots,F_m}$ for the sum running over $F_i\in\mathcal{F}_i$. We also write

$$M\lambda_i(x):=\sup_{x\in Q\in\mathcal{D}}\lambda_{i,Q}.$$

Lemma 5.7 *Let $m\geq 2$, $0<p_1,\ldots,p_{m-1}<\infty$. Define $\alpha:=\sum_{i=1}^{m-1}1/p_i$ and assume*

$$0<q_i:=s_i-\begin{cases}1/p_i, & i<m,\\ 1-\alpha, & i=m.\end{cases}$$

Assume that $\mathcal{S}$ is a sparse family such that for every $Q\in\mathcal{S}$,

$$2^r\leq\left(\frac{\int_Q\mathcal{M}^{(1/(\alpha p_i))_{i\neq m}}(\vec{w}\chi_Q)}{\int_Q\prod_{i=1}^{m-1}w_i^{1/(\alpha p_i)}}\right)^\alpha\leq 2^{r+1}. \tag{5.9}$$

Then, if ρ is a monotonic increasing function on $(1,\infty)$ and $\rho(2t)\leq C\rho(t)$ for $t\geq 1$, one has

$$\mathscr{B}\lesssim\lfloor\vec{w}\rfloor_{\vec{q},\vec{p},\rho,\alpha,m}\rho(2^r)^{-1}\prod_{i=1}^{m-1}\|M\lambda_i\|_{L^{p_i}(w_i)},$$

where

$$\mathscr{B}:=\Bigg(\sum_{F_1,\ldots,F_{m-1}}\prod_{i=1}^{m-1}\lambda_{i,F_i}^{\frac{1}{\alpha}}\int\Bigg(\sum_{Q:\forall j,\pi_j(Q)=F_j}\mathbf{1}_Q\prod_{i=1}^m\langle w_i\rangle_Q^{s_i-\delta_{im}}\Bigg)^{\frac{1}{\alpha}}dw_m\Bigg)^\alpha.$$

Proof We will estimate $\mathscr{B}$ by means of Lemma 5.6 letting $s_i \to \tilde{s}_i = (s_i - \delta_{im})/\alpha$, $i \le m$, $q_i \to \tilde{q}_i = q_i/\alpha$ and $\theta \to \alpha$. We can provide such an estimate since

$$\sum_{i\le m} \alpha\tilde{q}_i = \sum_{i\le m} s_i - (1-\alpha) - \sum_{i<m} 1/p_i = \sum_{i\le m} s_i - 1 = \sum_{i\le m} \alpha\tilde{s}_i.$$

This yields that the first inequality in the hypothesis of the lemma holds, and for $i < m$ we have $\tilde{q}_i < \tilde{s}_i$, verifying the second inequality. Then, there holds

$$\mathscr{B} \lesssim \Big(\sum_{F_1,\dots,F_{m-1}} \prod_{i=1}^{m-1} \lambda_{i,F_i}^{1/\alpha} \frac{\lfloor \vec{w} \rfloor_{\vec{q},\vec{p},\rho,\alpha,m,\mathcal{S}}}{2^r\rho(2^r)} \sum_{Q:\forall j, \pi_j(Q)=F_j} |Q| \prod_{i=1}^{m-1} \langle w_i \rangle_Q^{\tilde{s}_i - \tilde{q}_i} \Big)^{\alpha}. \tag{5.10}$$

Note that

$$\begin{aligned} &\lfloor \vec{w} \rfloor_{\vec{q},\vec{p},\rho,\alpha,m,\mathcal{S}} \\ &:= \sup_{Q\in\mathcal{S}} \prod_{i=1}^{m} \langle w_i \rangle_Q^{\frac{q_i}{\alpha}} \left(\frac{\int_Q \mathcal{M}^{(1/(\alpha p_i))_{i\neq m}}(\vec{w}\chi_Q)}{\int_Q \prod_{i=1}^{m-1} w_i^{1/(\alpha p_i)}} \right)^{\alpha} \rho\left(\left(\frac{\int_Q \mathcal{M}^{(1/(\alpha p_i))_{i\neq m}}(\vec{w}\chi_Q)}{\int_Q \prod_{i=1}^{m-1} w_i^{1/(\alpha p_i)}} \right)^{\alpha} \right) \\ &= \sup_{Q\in\mathcal{S}} \prod_{i=1}^{m} \langle w_i \rangle_Q^{\frac{q_i}{\alpha}} \left(\frac{\int_Q \mathcal{M}^{(1/(\alpha p_i))_{i\neq m}}(\vec{w}\chi_Q)}{\int_Q \prod_{i=1}^{m-1} w_i^{1/(\alpha p_i)}} \right) \rho\left(\left(\frac{\int_Q \mathcal{M}^{(1/(\alpha p_i))_{i\neq m}}(\vec{w}\chi_Q)}{\int_Q \prod_{i=1}^{m-1} w_i^{1/(\alpha p_i)}} \right)^{\alpha} \right)^{\frac{1}{\alpha}} \\ &\quad\times \left(\frac{\int_Q \mathcal{M}^{(1/(\alpha p_i))_{i\neq m}}(\vec{w}\chi_Q)}{\int_Q \prod_{i=1}^{m-1} w_i^{1/(\alpha p_i)}} \right)^{\alpha-1} \rho\left(\left(\frac{\int_Q \mathcal{M}^{(1/(\alpha p_i))_{i\neq m}}(\vec{w}\chi_Q)}{\int_Q \prod_{i=1}^{m-1} w_i^{1/(\alpha p_i)}} \right)^{\alpha} \right)^{1-\frac{1}{\alpha}} \\ &\lesssim \lfloor \vec{w} \rfloor_{\vec{q},\vec{p},\rho,\alpha,m}^{\frac{1}{\alpha}} (2^{r+1})^{1-\frac{1}{\alpha}} (\rho(2^{r+1}))^{1-\frac{1}{\alpha}}. \end{aligned}$$

The sparseness of $\mathcal{S}$ enables us to continue as follows

$$\begin{aligned} \sum_{Q:\forall j, \pi_j(Q)=F_j} |Q| \prod_{i=1}^{m-1} \langle w_i \rangle_Q^{\tilde{s}_i-\tilde{q}_i} &\lesssim \sum_{Q:\forall j, \pi_j(Q)=F_j} |E_Q| \Big(\prod_{i=1}^{m-1} \langle w_i \rangle_Q^{\frac{1}{p_i}} \Big)^{\frac{1}{\alpha}} \\ &\le \sum_{Q:\forall j, \pi_j(Q)=F_j} \int_{E_Q} \mathcal{M}^{(1/(\alpha p_i))_{i\neq m}}(\vec{w}\chi_Q) \le \int_{F_1\cap\dots\cap F_{m-1}} \mathcal{M}^{(1/(\alpha p_i))_{i\neq m}}(\vec{w}\chi_{F_1\cap\dots\cap F_{m-1}}). \end{aligned}$$

Thus, it follows from (5.9) and Hölder's inequality that

$$\begin{aligned} \mathscr{B} &\lesssim \frac{\lfloor \vec{w} \rfloor_{\vec{q},\vec{p},\rho,\alpha,m}}{\rho(2^r)} (2^{r+1})^{\alpha-1} 2^{-r\alpha} \left(\frac{\rho(2^{r+1})}{\rho(2^r)} \right)^{\alpha-1} \\ &\quad \Big(\sum_{F_1,\dots,F_{m-1}} \prod_{i=1}^{m-1} \lambda_{i,F_i}^{\frac{1}{\alpha}} 2^{\frac{r}{\alpha}} \int_{F_1\cap\dots\cap F_{m-1}} \prod_{i=1}^{m-1} w_i^{\frac{1}{\alpha p_i}} \Big)^{\alpha} \\ &\lesssim \frac{\lfloor \vec{w} \rfloor_{\vec{q},\vec{p},\rho,\alpha,m}}{\rho(2^r)} 2^{\alpha-1} \Big(\int \prod_{i=1}^{m-1} \sum_{F_i} 1_{F_i} \lambda_{i,F_i}^{\frac{1}{\alpha}} w_i^{\frac{1}{\alpha p_i}} \Big)^{\alpha} \end{aligned}$$

$$\lesssim \frac{\lfloor \vec{w} \rfloor_{\vec{q},\vec{p},\rho,\alpha,m}}{\rho(2^r)} \prod_{i=1}^{m-1} \left(\int (\sum_{F_i} 1_{F_i} \lambda_{i,F_i}^{\frac{1}{\alpha}})^{\alpha p_i} w_i \right)^{\frac{1}{p_i}}.$$

We end the proof noticing that

$$\left(\sum_{F_i} 1_{F_i} \lambda_{i,F_i}^{\frac{1}{\alpha}} \right)^{\alpha} \simeq M\lambda_i,$$

since at each point, the sum on the left-hand side is geometrically increasing and, consequently, it is comparable to the last term. □

Lemma 5.8 *Let $m \geq 2$ and $0 < p_i, s_i < \infty$, $1 \leq i < m$, and let $\alpha := \sum_{i=1}^{m-1} 1/p_i$. Suppose $q_i := s_i - 1/p_i > 0$ for $i < m$ and let $q_m := \alpha$. Then for every sparse family $\mathcal{S}$ and $\alpha \geq 1$,*

$$\left\| \sum_{Q \in \mathcal{S}} \prod_{i=1}^{m-1} \lambda_{i,Q} \langle w_i \rangle_Q^{s_i} \mathbf{1}_Q \right\|_{L^{1/\alpha}(w_m)} \lesssim \lfloor \vec{w} \rfloor_{\vec{q},\vec{p},\rho,\alpha,m} \prod_{i=1}^{m-1} \| M\lambda_i \|_{L^{p_i}(w_i)}, \tag{5.11}$$

provided that ρ is a monotonic increasing function on $(1, \infty)$, $\int_1^{\infty} \frac{dt}{\rho^{\frac{1}{\alpha}}(t)t} < \infty$ and $\rho(2t) \leq C\rho(t)$ for $t \geq 1$.

Proof First we split $\mathcal{S}$ as follows

$$\sum_{Q \in \mathcal{S}} \prod_{i=1}^{m-1} \lambda_{i,Q} \langle w_i \rangle_Q^{s_i} \mathbf{1}_Q = \sum_j \sum_{Q \in \mathcal{S}_j} \prod_{i=1}^{m-1} \lambda_{i,Q} \langle w_i \rangle_Q^{s_i} \mathbf{1}_Q,$$

where

$$Q \in \mathcal{S}_j \iff 2^j \leq \left(\frac{\int_Q \mathcal{M}^{(1/(\alpha p_i))_{i \neq m}}(\vec{w}\chi_Q)}{\int_Q \prod_{i=1}^{m-1} w_i^{1/(\alpha p_i)}} \right)^{\alpha} \leq 2^{j+1}.$$

Then one has

$$\int \left(\sum_{Q \in \mathcal{S}} \prod_{i=1}^{m-1} \lambda_{i,Q} \langle w_i \rangle_Q^{s_i} \mathbf{1}_Q \right)^{\frac{1}{\alpha}} w_m \leq \sum_j \int \left(\sum_{Q \in \mathcal{S}_j} \prod_{i=1}^{m-1} \lambda_{i,Q} \langle w_i \rangle_Q^{s_i} \mathbf{1}_Q \right)^{\frac{1}{\alpha}} w_m.$$

Note that each term in the right-hand side of the preceding equation can be estimated by

$$\int \Big(\sum_{F_1,\ldots,F_{m-1}} \prod_{i=1}^{m-1} \lambda_{i,F_i} \sum_{Q:\forall j, \pi_j(Q) = F_j} \prod_{i=1}^{m-1} \langle w_i \rangle_Q^{s_i} \mathbf{1}_Q \Big)^{1/\alpha} dw_m \Big).$$

By subadditivity of the function $x \mapsto x^{1/\alpha}$, this is bounded by

$$\sum_{F_1,\dots,F_{m-1}} \prod_{i=1}^{m-1} \lambda_{i,F_i}^{1/\alpha} \int \Big(\sum_{Q:\forall j, \pi_j(Q)=F_j} \prod_{i=1}^{m-1} \langle w_i \rangle_Q^{s_i} \mathbf{1}_Q \Big)^{1/\alpha} dw_m.$$

Therefore, Lemma 5.7 applied with $s_m = 1$ gives

$$\begin{aligned} \int \left(\sum_{Q\in\mathcal{S}} \prod_{i=1}^{m-1} \lambda_{i,Q} \langle w_i \rangle_Q^{s_i} \mathbf{1}_Q \right)^{\frac{1}{\alpha}} w_m &\lesssim \sum_j \lfloor \vec{w} \rfloor_{\vec{q},\vec{p},\rho,\alpha,m}^{\frac{1}{\alpha}} \frac{1}{\rho(2^r)^{\frac{1}{\alpha}}} \left(\prod_{i=1}^{m-1} \|M\lambda_i\|_{L^{p_i}(w_i)} \right)^{\frac{1}{\alpha}}, \\ &= \left(\sum_j \frac{1}{\rho(2^j)^{\frac{1}{\alpha}}} \right) \lfloor \vec{w} \rfloor_{\vec{q},\vec{p},\rho,\alpha,m}^{\frac{1}{\alpha}} \left(\prod_{i=1}^{m-1} \|M\lambda_i\|_{L^{p_i}(w_i)} \right)^{\frac{1}{\alpha}}. \end{aligned}$$

Consequently

$$\left(\int \left(\sum_{Q\in\mathcal{S}} \prod_{i=1}^{m-1} \lambda_{i,Q} \langle w_i \rangle_Q^{s_i} \mathbf{1}_Q \right)^{\frac{1}{\alpha}} w_m \right)^{\alpha} \lesssim \left(\sum_j \frac{1}{\rho(2^j)^{\frac{1}{\alpha}}} \right)^{\alpha} \lfloor \vec{w} \rfloor_{\vec{q},\vec{p},\rho,\alpha,m} \prod_{i=1}^{m-1} \|M\lambda_i\|_{L^{p_i}(w_i)}$$

and (5.11) holds as desired. □

Proof of Theorem 5.2 We rewrite

$$\|\mathcal{A}_{\mathcal{S}}^r(\overrightarrow{f\sigma})\|_{L^p(\nu)} = \left\| \sum_{Q\in\mathcal{S}} \left(\prod_{i=1}^m \langle f_i\sigma_i \rangle_Q \right)^r 1_Q \right\|_{L^{\frac{p}{r}}(\sigma_{m+1})}^{\frac{1}{r}}.$$

For $m+1$, $w_i = \sigma_i$, $w_{m+1} = \nu$, $\lambda_{i,Q} = \left(\langle f_i \rangle_Q^{\sigma_i} \right)^r$, $s_i = r$, and $\alpha = \frac{r}{p} = \sum_{i=1}^m \frac{r}{p_i}$, we have $q_i := r - r/p_i$ and by Lemma 5.8

$$\begin{aligned} \left\| \sum_{Q\in S} \left(\prod_{i=1}^m \langle f_i\sigma_i \rangle_Q \right)^r \mathbf{1}_Q \right\|_{L^{\frac{p}{r}}(\nu)} &\lesssim \lfloor \vec{\sigma}, \nu \rfloor_{\frac{r}{\vec{p}'},\vec{p},\rho,\frac{r}{p},m+1} \prod_{i=1}^m \|(M_{\sigma_i} f)^r\|_{L^{\frac{p_i}{r}}(\sigma_i)} \\ = \lfloor \vec{\sigma}, \nu \rfloor_{\frac{r}{\vec{p}'},\vec{p},\varepsilon,\frac{r}{p},m+1} \prod_{i=1}^m \|M_{\sigma_i} f\|_{L^{p_i}(\sigma_i)}^r &\lesssim \lfloor \vec{\sigma}, \nu \rfloor_{\frac{r}{\vec{p}'},\vec{p},\varepsilon,\frac{r}{p},m+1} \prod_{i=1}^m \|f\|_{L^{p_i}(\sigma_i)}^r. \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathcal{A}_{\mathcal{S}}^r(\overrightarrow{f\sigma})\|_{L^p(\nu)} &= \left\| \sum_{Q\in\mathcal{S}} \left(\prod_{i=1}^m \langle f_i\sigma_i \rangle_Q \right)^r \mathbf{1}_Q \right\|_{L^{\frac{p}{r}}(\nu)}^{\frac{1}{r}} \\ &\lesssim \lfloor \vec{\sigma}, \nu \rfloor_{\frac{r}{\vec{p}'},\vec{p},\varepsilon,\frac{r}{p},m+1}^{\frac{1}{r}} \prod_{i=1}^m \|f\|_{L^{p_i}(\sigma_i)} \end{aligned}$$

as we wanted to show. □

6 Mixed Weak Type Estimates

The goal of this section is devoted to presenting the proof of Theorem 1.5. To this end, we first establish a Coifman–Fefferman inequality with the precise A_∞ weight constant.

6.1 A Coifman–Fefferman Inequality

Theorem 6.1 *Let $\alpha \geq 1$. Then for every $0 < p < \infty$ and for every $w \in A_\infty$,*

$$\|S_\alpha(\vec{f})\|_{L^p(w)} \lesssim \alpha^{mn}(p+1)[w]_{A_\infty}^{\frac{1}{2}} \|\mathcal{M}(\vec{f})\|_{L^p(w)}. \tag{6.1}$$

• **Sparse approach for** $p \geq 2$. Considering (4.1), we are going to show that

$$\|\mathcal{A}_{\mathcal{S}}^2(\vec{f})\|_{L^p(w)} \lesssim [w]_{A_\infty}^{\frac{1}{2}} \|\mathcal{M}(\vec{f})\|_{L^p(w)}, \quad \forall p \geq 2. \tag{6.2}$$

Without loss of generality, we shall assume that $f_i \geq 0$, $i = 1, \ldots, m$. Note that

$$\|\mathcal{A}_{\mathcal{S}}^2(\vec{f})\|_{L^p(w)}^2 = \sup_{\substack{0 \leq g \in L^{(p/2)'}(w) \\ \|g\|_{L^{(p/2)'}(w)=1}}} \left| \sum_{Q \in \mathcal{S}} \prod_{i=1}^m \langle f_i \rangle_Q^2 ⨍_Q g\, dw\, w(Q) \right|. \tag{6.3}$$

Fix $0 \leq g \in L^{(p/2)'}(w)$ with $\|g\|_{L^{(p/2)'}(w)} = 1$. We are going to split the sparse family in terms of principal cubes. Set

$$\tau(P) := \prod_{i=1}^m \langle f_i \rangle_P^2 ⨍_P g\, dw,$$

and consider $\mathcal{F}_0$ the family of maximal cubes of $\mathcal{S}$. We define

$$\mathcal{F} := \bigcup_{i=0}^\infty \mathcal{F}_i \quad \text{and} \quad \mathcal{F}_i := \bigcup_{Q \in \mathcal{F}_{i-1}} \{P \subsetneq Q \text{ maximal} : \tau(P) > 2\tau(Q)\}.$$

For this family of cubes, we have that

$$\begin{aligned} &\sum_{Q \in \mathcal{S}} \prod_{i=1}^m \langle f_i \rangle_Q^2 ⨍_Q g\, dw\, w(Q) \\ &\leq \sum_{P \in \mathcal{F}} \prod_{i=1}^m \langle f_i \rangle_P^2 ⨍_P g\, dw \sum_{Q \in \mathcal{S}: \pi(Q) = P} w(Q) \\ &\lesssim [w]_{A_\infty} \sum_{P \in \mathcal{F}} \prod_{i=1}^m \langle f_i \rangle_P^2 ⨍_P g\, dw\, w(P) \end{aligned}$$

$$\lesssim [w]_{A_\infty} \int_{\mathbb{R}^n} \mathcal{M}(\vec{f})(x)^2 M_w g(x)\, w(x) dx$$
$$\lesssim [w]_{A_\infty} \|\mathcal{M}(\vec{f})^2\|_{L^{p/2}(w)} \|g\|_{L^{(p/2)'}(w)}. \tag{6.4}$$

Thus, (6.3) and (6.4) immediately lead (6.2). □

• $M_\delta^\sharp$ **approach.** We next deal with the general case $0 < p < \infty$. Recall that the sharp maximal function of f is defined by

$$M_\delta^\sharp(f)(x) := \sup_{x \in Q} \inf_{c \in \mathbb{R}} \left(\fint_Q |f^\delta - c| dx \right)^{\frac{1}{\delta}}.$$

It was proved in [41] that for every $0 < p < \infty$ and $\delta \in (0, 1)$,

$$\|f\|_{L^p(w)} \lesssim (p+1)[w]_{A_\infty} \|M_\delta^\sharp(f)\|_{L^p(w)}. \tag{6.5}$$

Let Φ be a fixed Schwartz function such that $\mathbf{1}_{B(0,1)}(x) \le \Phi(x) \le \mathbf{1}_{B(0,2)}(x)$. We define

$$\widetilde{S}_\alpha(\vec{f})(x) := \left(\iint_{\mathbb{R}^{n+1}_+} \Phi\Big(\frac{x-y}{\alpha t}\Big) |\psi_t(\vec{f})(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}. \tag{6.6}$$

It is easy to verify that

$$S_\alpha(\vec{f})(x) \le \widetilde{S}_\alpha(\vec{f})(x) \le S_{2\alpha}(\vec{f})(x). \tag{6.7}$$

We note here that

$$\|\widetilde{S}_\alpha(\vec{f})\|_{L^{1/m,\infty}(\mathbb{R}^n)} \lesssim \alpha^{mn} \prod_{j=1}^m \|f_j\|_{L^1(\mathbb{R}^n)}. \tag{6.8}$$

In fact, by [2, Lemma 3.1] and the endpoint estimate for S_1, we get

$$\|\widetilde{S}_\alpha(\vec{f})\|_{L^{1/m,\infty}(\mathbb{R}^n)} \le \|S_{2\alpha}(\vec{f})\|_{L^{1/m,\infty}(\mathbb{R}^n)}$$
$$\lesssim \alpha^{mn} \|S_1(\vec{f})\|_{L^{1/m,\infty}(\mathbb{R}^n)} \lesssim \alpha^{mn} \prod_{j=1}^m \|f_j\|_{L^1(\mathbb{R}^n)}.$$

Now, combining (6.7), (6.5) and Lemma 6.2 below, we conclude that

$$\|S_\alpha(\vec{f})\|_{L^p(w)} \le \|\widetilde{S}_\alpha(\vec{f})\|_{L^p(w)} \le \|\widetilde{S}_\alpha(\vec{f})^2\|_{L^{p/2}(w)}^{\frac{1}{2}}$$
$$\le (p+1)[w]_{A_\infty}^{\frac{1}{2}} \|M_\gamma^\sharp(\widetilde{S}_\alpha(\vec{f})^2)\|_{L^{p/2}(w)}^{\frac{1}{2}}$$
$$\lesssim \alpha^{mn}(p+1)[w]_{A_\infty}^{\frac{1}{2}} \|\mathcal{M}(\vec{f})^2\|_{L^{p/2}(w)}^{\frac{1}{2}}$$

$$= \alpha^{mn}(p+1)[w]_{A_\infty}^{\frac{1}{2}} \|\mathcal{M}(\vec{f})\|_{L^p(w)}$$

where we have used that for suitable choices of γ,

$$M_\gamma^\sharp(\widetilde{S}_\alpha(\vec{f})^2)(x) \lesssim \alpha^{2mn}\mathcal{M}(\vec{f})(x)^2, \quad x \in \mathbb{R}^n.$$

Hence to end the proof of Theorem 6.1, it remains to settle that pointwise estimate.

Lemma 6.2 *For every $\alpha \geq 1$ and $0 < \gamma < \frac{1}{2m}$, we have*

$$M_\gamma^\sharp(\widetilde{S}_\alpha(\vec{f})^2)(x) \lesssim \alpha^{2mn}\mathcal{M}(\vec{f})(x)^2, \quad x \in \mathbb{R}^n. \tag{6.9}$$

Proof Let $x \in Q$. It suffices to show that for some c_Q chosen later

$$\mathcal{J} := \left(\fint_Q |\widetilde{S}_\alpha(\vec{f})^2(x) - c_Q|^\gamma dx \right)^{\frac{1}{\gamma}} \lesssim \alpha^{2mn}\mathcal{M}(\vec{f})(x)^2. \tag{6.10}$$

For a cube $Q \subset \mathbb{R}^n$, we set $T(Q) = Q \times (0, \ell(Q))$. We then write

$$\widetilde{S}_\alpha(\vec{f})^2(x) = E(\vec{f})(x) + F(\vec{f})(x),$$

where

$$E(\vec{f})(x) := \iint_{T(2Q)} \Phi\Big(\frac{x-y}{\alpha t}\Big)|\psi_t(\vec{f})(y)|^2 \frac{dydt}{t^{n+1}},$$
$$F(\vec{f})(x) := \iint_{\mathbb{R}^{n+1}_+ \setminus T(2Q)} \Phi\Big(\frac{x-y}{\alpha t}\Big)|\psi_t(\vec{f})(y)|^2 \frac{dydt}{t^{n+1}}.$$

Let us choose $c_Q = F(\vec{f})(x_Q)$ where x_Q is the center of Q. Then we have that

$$\mathcal{J} \lesssim \left(\fint_Q |E(\vec{f})(x)|^\gamma dx \right)^{\frac{1}{\gamma}} + \left(\fint_Q |F(\vec{f})(x) - F(\vec{f})(x_Q)|^\gamma dx \right)^{\frac{1}{\gamma}} =: \mathcal{J}_1 + \mathcal{J}_2. \tag{6.11}$$

Let us first focus on $\mathcal{J}_1$. Set $\vec{f}^0 := (f_1^0, \ldots, f_m^0)$, $f_i^0 = f_i\chi_{Q^*}$, and $f_i^\infty = f_i\chi_{(Q^*)^c}$, $i = 1, \ldots, m$, where $Q^* = 8Q$. Then we have

$$E(\vec{f})(x) \lesssim E(\vec{f}^0)(x) + \sum_{\alpha \in \mathcal{I}_0} E(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(x), \tag{6.12}$$

where $\mathcal{I}_0 := \{\alpha = (\alpha_1, \ldots, \alpha_m) : \alpha_i \in \{0, \infty\}, \text{ and at least one } \alpha_i \neq 0\}$. Using Kolmogorov's inequality and (6.8), we have

$$\Big(\fint_Q |E(\vec{f}^0)(x)|^\gamma dx\Big)^{\frac{1}{\gamma}} \le \Big(\fint_Q |\widetilde{S}_\alpha(\vec{f}^0)|^{2\gamma} dx\Big)^{\frac{2}{2\gamma}}$$
$$\lesssim \|\widetilde{S}_\alpha(\vec{f}^0)\|^2_{L^{1/m,\infty}(Q,\frac{dx}{|Q|})} \lesssim \alpha^{2mn}\Big(\prod_{j=1}^m \fint_Q |f_j| dx\Big)^2. \quad (6.13)$$

On the other hand, for each $\alpha \in \mathcal{I}_0$,

$$\Big(\fint_Q |E(\vec{f}^\alpha)(x)|^\gamma dx\Big)^{\frac{1}{\gamma}} \lesssim \frac{1}{|Q|}\int_{\mathbb{R}^n}\iint_{T(2Q)} \Phi\Big(\frac{x-y}{\alpha t}\Big)|\psi_t(\vec{f}^\alpha)(y)|^2 \frac{dydt}{t^{n+1}} dx$$
$$\lesssim \frac{1}{|Q|}\iint_{T(2Q)} (\alpha t)^n |\psi_t(\vec{f}^\alpha)(y)|^2 \frac{dydt}{t^{n+1}}, \quad (6.14)$$

since $\int_{\mathbb{R}^n} \Phi\big(\frac{x-y}{\alpha t}\big) dx \le c_n(\alpha t)^n$. By size estimate, for $y \in 2Q$ and $\alpha \in \mathcal{I}_0$, one has

$$|\psi_t(\vec{f}^\alpha)(y)| \lesssim \Big(\frac{t}{\ell(Q)}\Big)^\delta \sum_{k=0}^\infty 2^{-k\delta}\Big(\prod_{j=1}^m \fint_{2^kQ} |f_j| dx\Big). \quad (6.15)$$

Then, (6.14) and (6.15) give that for every $\alpha \in \mathcal{I}_0$,

$$\Big(\fint_Q |E(\vec{f}^\alpha)(x)|^\gamma dx\Big)^{\frac{1}{\gamma}}$$
$$\lesssim \Big[\sum_{k=0}^\infty 2^{-k\delta}\Big(\prod_{j=1}^m \fint_{2^kQ}|f_j|\Big)\Big]^2 \frac{\alpha^n}{|Q|}\iint_{T(2Q)} \Big(\frac{t}{\ell(Q)}\Big)^{2\delta} dy\frac{dt}{t}$$
$$\lesssim \alpha^n \Big[\sum_{k=0}^\infty 2^{-k\delta}\Big(\prod_{j=1}^m \fint_{2^kQ}|f_j|\, dx\Big)\Big]^2$$
$$\lesssim \alpha^n \sum_{k=0}^\infty 2^{-k\delta}\Big(\prod_{j=1}^m \fint_{2^kQ}|f_j|\, dx\Big)^2, \quad (6.16)$$

where the Cauchy-Schwarz inequality was used in the last inequality. Gathering (6.12), (6.13) and (6.16), we obtain

$$\mathcal{J}_1 \lesssim \alpha^{2mn}\mathcal{M}(\vec{f})(x). \quad (6.17)$$

To complete the proof it remains to provide a bound for $\mathcal{J}_2$. From [2, eq. (4.6)], we have that for any $x \in Q$,

$$|F(\vec{f})(x) - F(\vec{f})(x_Q)| \lesssim \alpha^{2mn}\sum_{k=0}^\infty 2^{-k\delta}\Big(\prod_{j=1}^m \fint_{2^kQ}|f_j|\, dx\Big)^2. \quad (6.18)$$

Hence, (6.10) is a consequence of (6.11), (6.17) and (6.18). □

6.2 Proof of Theorem 1.5

In view of (3.6) and $\lambda > 2m$, it is enough to present the proof of (1.10). We use a hybrid of the arguments in [15] and [35]. Define

$$\mathcal{R}h(x) = \sum_{j=0}^{\infty} \frac{T_u^j h(x)}{2^j K_0^j},$$

where $K_0 > 0$ will be chosen later and $T_u f(x) := M(fu)(x)/u(x)$ if $u(x) \neq 0$, $T_u f(x) = 0$ otherwise. It immediately yields that

$$h \le \mathcal{R}h \quad \text{and} \quad T_u(\mathcal{R}h) \le 2K_0 \mathcal{R}h. \tag{6.19}$$

Moreover, we claim that for some $r > 1$,

$$\mathcal{R}h \cdot uv^{\frac{1}{mr'}} \in A_\infty \quad \text{and} \quad \|\mathcal{R}h\|_{L^{r',1}(uv^{\frac{1}{m}})} \le 2\|h\|_{L^{r',1}(uv^{\frac{1}{m}})}. \tag{6.20}$$

The proofs will be given at the end of this section.

Note that

$$\|f^q\|_{L^{p,\infty}(w)} = \|f\|^q_{L^{pq,\infty}(w)}, \quad 0 < p, q < \infty. \tag{6.21}$$

This implies that

$$\begin{aligned}
&\left\| \frac{S_\alpha(\vec{f})}{v} \right\|^{\frac{1}{mr}}_{L^{\frac{1}{m},\infty}(uv^{\frac{1}{m}})} \\
&= \left\| \left(\frac{S_\alpha(\vec{f})}{v} \right)^{\frac{1}{mr}} \right\|_{L^{r,\infty}(uv^{\frac{1}{m}})} = \sup_{\|h\|_{L^{r',1}(uv^{\frac{1}{m}})}=1} \left| \int_{\mathbb{R}^n} |S_\alpha(\vec{f})(x)|^{\frac{1}{mr}} h(x)u(x)v(x)^{\frac{1}{mr'}}\, dx \right| \\
&\le \sup_{\|h\|_{L^{r',1}(uv^{\frac{1}{m}})}=1} \int_{\mathbb{R}^n} |S_\alpha(\vec{f})(x)|^{\frac{1}{mr}} \mathcal{R}h(x)u(x)v(x)^{\frac{1}{mr'}}\, dx.
\end{aligned}$$

Invoking Theorem 6.1 and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} |S_\alpha(\vec{f})(x)|^{\frac{1}{mr}} \mathcal{R}h(x)u(x)v(x)^{\frac{1}{mr'}}\, dx \\
&\lesssim \int_{\mathbb{R}^n} \mathcal{M}(\vec{f})(x)^{\frac{1}{mr}} \mathcal{R}h(x)u(x)v(x)^{\frac{1}{mr'}}\, dx \\
&= \int_{\mathbb{R}^n} \left(\frac{\mathcal{M}(\vec{f})(x)}{v(x)} \right)^{\frac{1}{mr}} \mathcal{R}h(x)u(x)v(x)^{\frac{1}{m}}\, dx \\
&\le \left\| \left(\frac{\mathcal{M}(\vec{f})}{v} \right)^{\frac{1}{mr}} \right\|_{L^{r,\infty}(uv^{\frac{1}{m}})} \|\mathcal{R}h\|_{L^{r',1}(uv^{\frac{1}{m}})}
\end{aligned}$$

$$\leq \left\| \frac{\mathcal{M}(\vec{f})}{v} \right\|_{L^{\frac{1}{m},\infty}(uv^{\frac{1}{m}})}^{\frac{1}{mr}} \|h\|_{L^{r',1}(uv^{\frac{1}{m}})},$$

where we used (6.21) and (6.20) in the last inequality. Here we need to apply the weighted mixed weak type estimates for $\mathcal{M}$ proved in Theorems 1.4 and 1.5 in [35]. Consequently, collecting the above estimates, we get the desired result

$$\left\| \frac{S_\alpha(\vec{f})}{v} \right\|_{L^{\frac{1}{m},\infty}(uv^{\frac{1}{m}})} \lesssim \left\| \frac{\mathcal{M}(\vec{f})}{v} \right\|_{L^{\frac{1}{m},\infty}(uv^{\frac{1}{m}})} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^1(w_i)}.$$

It remains to show our foregoing claim (6.20). The proof follows the same scheme of that in [15]. For the sake of completeness we here give the details. Together with Lemma 2.2, the hypothesis (1) or (2) indicates that $u \in A_1$ and $v^{\frac{1}{m}} \in A_\infty$. The former implies that

$$\|T_u f\|_{L^\infty(uv^{\frac{1}{m}})} \leq [u]_{A_1} \|f\|_{L^\infty(uv^{\frac{1}{m}})}. \tag{6.22}$$

The latter yields that $v^{\frac{1}{m}} \in A_{q_0}$ for some $q_0 > 1$. It follows from A_p factorization theorem that there exist $v_1, v_2 \in A_1$ such that $v^{\frac{1}{m}} = v_1 v_2^{1-q_0}$.

Additionally, it follows from Lemma 2.3 in [15] that if $v_1, v_2 \in A_1$, then there exists $\epsilon_0 = \epsilon_0([v_1]_{A_1}, [v_2]_{A_1}) \in (0, 1)$ such that $v_1 u_1^\epsilon \in A_{p_1}$ and $v_2 u_2^\epsilon \in A_{p_2}$ for any $0 < \epsilon < \epsilon_0$, $u_1 \in A_{p_1}$ and $u_2 \in A_{p_2}$, $1 \leq p_1, p_2 < \infty$. Then $uv_2^{\frac{q_0-1}{p_0-1}} \in A_1$ if we set $p_0 > 1 + (q_0 - 1)/\epsilon_0$. Thus, we have

$$u^{1-p_0} v^{\frac{1}{m}} = v_1 \Big(uv_2^{\frac{q_0-1}{p_0-1}} \Big)^{1-p_0} \in A_{p_0}.$$

It immediately implies that

$$\|T_u f\|_{L^{p_0}(uv^{\frac{1}{m}})} = \|M(fu)\|_{L^{p_0}(u^{1-p_0} v^{\frac{1}{m}})} \leq c_1 \|f\|_{L^{p_0}(uv^{\frac{1}{m}})}. \tag{6.23}$$

By (6.22), (6.23) and Marcinkiewicz interpolation in [15, Proposition A.1], we have T_u is bounded on $L^{p,1}(uv^{\frac{1}{m}})$ for all $p \in (p_0, \infty)$ with the constant

$$K(p) = 2^{\frac{1}{p}} \left(c_1 \Big(\frac{1}{p_0} - \frac{1}{p} \Big)^{-1} + c_2 \right),$$

and $c_2 := [v]_{A_1}$. Note that $K(p)$ is decreasing with respect to p. Hence, we obtain

$$\|T_u f\|_{L^{p,1}(uv^{\frac{1}{m}})} \leq K_0 \|f\|_{L^{p,1}(uv^{\frac{1}{m}})}, \ \forall p \geq 2p_0, \tag{6.24}$$

where $K_0 := 4p_0(c_1 + c_2) > K(2p_0) \geq K(p)$.

The inequality (6.19) indicates that $\mathcal{R}h \cdot u \in A_1$ with $[\mathcal{R}h \cdot u]_{A_1} \leq 2K_0$. Let $0 < \epsilon < \min\{\epsilon_0, \frac{1}{2p_0}\}$, and $r = (\frac{1}{\epsilon})'$. Then $(\mathcal{R}h \cdot u)v_1^\epsilon \in A_1$, and the second inequality in (6.20) follows from (6.24). By A_p factorization theorem again, we obtain

$$\mathcal{R}h \cdot uv^{\frac{1}{mr'}} = [(\mathcal{R}h \cdot u)v_1^{\epsilon}] \cdot v_2^{1-[(q_0-1)\epsilon+1]} \in A_{(q_0-1)\epsilon+1} \subset A_\infty.$$

The proof is complete. □

7 Local Decay Estimates

To show Theorem 1.6, we need the following Carleson embedding theorem from [26, Theorem 4.5].

Lemma 7.1 *Suppose that the sequence $\{a_Q\}_{Q\in\mathcal{D}}$ of nonnegative numbers satisfies the Carleson packing condition*

$$\sum_{Q\in\mathcal{D}:Q\subset Q_0} a_Q \le Aw(Q_0), \quad \forall Q_0 \in \mathcal{D}.$$

Then for all $p \in (1,\infty)$ and $f \in L^p(w)$,

$$\left(\sum_{Q\in\mathcal{D}} a_Q \Big(\frac{1}{w(Q)}\int_Q f(x)w\,dx\Big)^p\right)^{\frac{1}{p}} \le A^{\frac{1}{p}} p' \|f\|_{L^p(w)}.$$

We also need a local version of Coifman–Fefferman inequality with the precise A_p norm.

Lemma 7.2 *For every $1 < p < \infty$ and $w \in A_p$, we have*

$$\|S_\alpha(\vec{f})\|_{L^2(Q,w)} \le c_{n,p}\alpha^{mn}[w]_{A_p}^{\frac{1}{2}} \|\mathcal{M}(\vec{f})\|_{L^2(Q,w)}, \tag{7.1}$$

$$\|g_\lambda^*(\vec{f})\|_{L^2(Q,w)} \le \frac{c_{n,p}}{1-2^{-n(\lambda-2m)/2}}[w]_{A_p}^{\frac{1}{2}} \|\mathcal{M}(\vec{f})\|_{L^2(Q,w)}, \tag{7.2}$$

for every cube Q and $f_j \in L_c^\infty$ with supp $f_j \subset Q$ $(j = 1,\dots,m)$.

Proof Let $w \in A_p$ with $1 < p < \infty$. Fix a cube $Q \subset \mathbb{R}^n$. Recall the definition of $\widetilde{S}_\alpha$ in (6.6). Pick $0 < \epsilon < \frac{1}{2m}$. By (2.1), Kolmogorov's inequality, (6.8) and $f_j \in L_c^\infty$ with supp $f_j \subset Q$, $j = 1,\dots,m$, we have

$$\begin{aligned} m_{\widetilde{S}_\alpha(\vec{f})^2}(Q) &\lesssim \|\widetilde{S}_\alpha(\vec{f})^2\|_{L^\epsilon(Q,\frac{dx}{|Q|})} \lesssim \|\widetilde{S}_\alpha(\vec{f})\|^2_{L^{1/m,\infty}(Q,\frac{dx}{|Q|})} \\ &\lesssim \alpha^{2mn}\Big(\prod_{i=1}^m \fint_Q |f_i|dx\Big)^2 \le \alpha^{2mn}\inf_{x\in Q}\mathcal{M}(\vec{f})(x)^2, \end{aligned}$$

which implies that

$$m_{\widetilde{S}_\alpha(\vec{f})^2}(Q)w(Q) \lesssim \alpha^{2mn} \int_Q \mathcal{M}(\vec{f})(x)^2 w(x)dx. \tag{7.3}$$

On the other hand, from [2, Proposition 4.1], one has for every cube Q',

$$\begin{aligned} \omega_\lambda(\widetilde{S}_\alpha(\vec{f})^2; Q') &\lesssim \alpha^{2mn} \sum_{j=0}^{\infty} 2^{-j\delta_0} \bigg(\prod_{i=1}^{m} ⨍_{2^j Q'} |f_i(y_i)| dy_i \bigg)^2 \\ &\lesssim \alpha^{2mn} \sum_{j=0}^{\infty} 2^{-j\delta_0} \inf_{Q'} \mathcal{M}(\vec{f})^2 \lesssim \alpha^{2mn} \inf_{Q'} \mathcal{M}(\vec{f})^2, \end{aligned} \tag{7.4}$$

where $0 < \delta_0 < \min\{\delta, \frac{1}{2}\}$. Thus, together with (7.3) and (7.4), the estimate (2.2) applied to $Q_0 = Q$ and $f = \widetilde{S}_\alpha(\vec{f})^2$ gives that

$$\begin{aligned} \|\widetilde{S}_\alpha(\vec{f})\|_{L^2(Q,w)}^2 &\lesssim m_{\widetilde{S}_\alpha(\vec{f})^2}(Q)w(Q) + \sum_{Q' \in \mathcal{S}(Q)} \omega_{2^{-n-2}}(\widetilde{S}_\alpha(\vec{f})^2; Q')w(Q') \\ &\lesssim \alpha^{2mn} \|\mathcal{M}(\vec{f})\|_{L^2(Q,w)}^2 + \alpha^{2mn} \sum_{Q' \in \mathcal{S}(Q)} \inf_{Q'} \mathcal{M}(\vec{f})^2 w(Q'). \end{aligned}$$

From this and (6.7), we see that to obtain (7.1), it suffices to prove

$$\sum_{Q' \in \mathcal{S}(Q)} \inf_{Q'} \mathcal{M}(\vec{f})^2 w(Q') \lesssim [w]_{A_p} \|\mathcal{M}(\vec{f})\|_{L^2(Q,w)}^2. \tag{7.5}$$

Recall that a new version of A_∞ was introduced by Hytönen and Pérez [26]:

$$[w]'_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\mathbf{1}_Q)(x)dx.$$

By [26, Proposition 2.2], there holds

$$c_n [w]'_{A_\infty} \le [w]_{A_\infty} \le [w]_{A_p}. \tag{7.6}$$

Observe that for every $Q'' \in \mathcal{D}$,

$$\begin{aligned} \sum_{Q' \in \mathcal{S}(Q): Q' \subset Q''} w(Q') &= \sum_{Q' \in \mathcal{S}(Q): Q' \subset Q''} \langle w \rangle_{Q'} |Q'| \lesssim \sum_{Q' \in \mathcal{S}(Q): Q' \subset Q''} \inf_{Q'} M(w\mathbf{1}_{Q''}) |E_{Q'}| \\ &\lesssim \int_{Q''} M(w\mathbf{1}_{Q''})(x)dx \le [w]'_{A_\infty} w(Q'') \lesssim [w]_{A_p} w(Q''), \end{aligned}$$

where we used the disjointness of $\{E_{Q'}\}_{Q' \in \mathcal{S}(Q)}$ and (7.6). This shows that the collection $\{w(Q')\}_{Q' \in \mathcal{S}(Q)}$ satisfies the Carleson packing condition with the constant

$c_n[w]_{A_p}$. As a consequence, this and Lemma 7.1 give that

$$\sum_{Q'\in\mathcal{S}(Q)} \inf_{Q'} \mathcal{M}(\vec{f})^2 w(Q') \le \sum_{Q'\in\mathcal{S}(Q)} \left(\frac{1}{w(Q')}\int_{Q'} \mathcal{M}(\vec{f})\,\mathbf{1}_Q w\,dx\right)^2 w(Q')$$
$$\lesssim [w]_{A_p} \|\mathcal{M}(\vec{f})\mathbf{1}_Q\|^2_{L^2(w)} = [w]_{A_p} \|\mathcal{M}(\vec{f})\mathbf{1}_Q\|^2_{L^2(Q,w)},$$

where the above implicit constants are independent of $[w]_{A_p}$ and Q. This shows (7.5) and completes the proof of (7.1).

Finally, the estimate (7.2) immediately follows from (7.1) and the fact that

$$g^*_\lambda(\vec{f})(x) \le S_1(\vec{f})(x) + \sum_{k=0}^\infty 2^{-\frac{k\lambda n}{2}} S_{2^{k+1}}(\vec{f})(x).$$

This completes the proof. □

Proof of Theorem 1.6 Let $p > 1$ and $r > 1$ be chosen later. Define the Rubio de Francia algorithm:

$$\mathcal{R}h = \sum_{k=0}^\infty \frac{M^k h}{2^k \|M\|^k_{L^{r'}\to L^{r'}}}.$$

Then it is obvious that

$$h \le \mathcal{R}h \quad \text{and} \quad \|\mathcal{R}h\|_{L^{r'}(\mathbb{R}^n)} \le 2\|h\|_{L^{r'}(\mathbb{R}^n)}. \tag{7.7}$$

Moreover, for any nonnegative $h \in L^{r'}(\mathbb{R}^n)$, we have that $\mathcal{R}h \in A_1$ with

$$[\mathcal{R}h]_{A_1} \le 2\|M\|_{L^{r'}\to L^{r'}} \le c_n\, r. \tag{7.8}$$

By Riesz representation theorem and the first inequality in (7.7), there exists some nonnegative function $h \in L^{r'}(Q)$ with $\|h\|_{L^{r'}(Q)} = 1$ such that

$$\begin{aligned}\mathscr{F}_Q^{\frac{1}{r}} &:= |\{x \in Q : S_\alpha(\vec{f})(x) > t\mathcal{M}(\vec{f})(x)\}|^{\frac{1}{r}} \\ &= |\{x \in Q : S_\alpha(\vec{f})(x)^2 > t^2\mathcal{M}(\vec{f})(x)^2\}|^{\frac{1}{r}} \\ &\le \frac{1}{t^2}\left\|\left(\frac{S_\alpha(\vec{f})}{\mathcal{M}(\vec{f})}\right)^2\right\|_{L^r(Q)} \le \frac{1}{t^2}\int_Q S_\alpha(\vec{f})^2\, h\, \mathcal{M}(\vec{f})^{-2} dx \\ &\le t^{-2}\|S_\alpha(\vec{f})\|^2_{L^2(Q,w)},\end{aligned} \tag{7.9}$$

where $w = w_1 w_2^{1-p}$, $w_1 = \mathcal{R}h$ and $w_2 = \mathcal{M}(\vec{f})^{2(p'-1)}$. Recall that the m-linear version of Coifmann–Rochberg theorem [40, Lemma 1] asserts that

$$[(\mathcal{M}(\vec{f}))^{\delta}]_{A_1} \le \frac{c_n}{1-m\delta}, \quad \forall \delta \in (0, \frac{1}{m}). \tag{7.10}$$

In view of (7.8) and (7.10), we see that $w_1, w_2 \in A_1$ provided $p > 2m+1$. Then the reverse A_p factorization theorem gives that $w = w_1 w_2^{1-p} \in A_p$ with

$$[w]_{A_p} \le [w_1]_{A_1}[w_2]_{A_1}^{p-1} \le c_n\, r. \tag{7.11}$$

Thus, gathering (7.1), (7.9) and (7.11), we obtain

$$\begin{aligned}
\mathscr{F}_Q^{\frac{1}{r}} &\le c_n t^{-2}\alpha^{2mn}[w]_{A_p}\|\mathcal{M}(\vec{f})\|_{L^2(Q,w)}^2 \\
&= c_n t^{-2}\alpha^{2mn}[w]_{A_p}\|\mathcal{R}h\|_{L^1(Q)} \\
&\le c_n t^{-2}\alpha^{2mn}[w]_{A_p}\|\mathcal{R}h\|_{L^{r'}(Q)}|Q|^{\frac{1}{r}} \\
&\le c_n t^{-2}\alpha^{2mn}[w]_{A_p}\|h\|_{L^{r'}(Q)}|Q|^{\frac{1}{r}} \\
&\le c_n r t^{-2}\alpha^{2mn}|Q|^{\frac{1}{r}}.
\end{aligned}$$

Consequently, if $t > \sqrt{c_n e}\,\alpha^{mn}$, choosing $r > 1$ so that $t^2/e = c_n\alpha^{2mn}r$, we have

$$\mathscr{F}_Q \le (c_n\alpha^{2mn} r t^{-2})^r |Q| = e^{-r}|Q| = e^{-\frac{t^2}{c_n e\alpha^{2mn}}}|Q|. \tag{7.12}$$

If $0 < t \le \sqrt{c_n e}\alpha^{mn}$, it is easy to see that

$$\mathscr{F}_Q \le |Q| \le e\cdot e^{-\frac{t^2}{c_n e\alpha^{2mn}}}|Q|. \tag{7.13}$$

Summing (7.12) and (7.13) up, we deduce that

$$\mathscr{F}_Q = |\{x \in Q : S_\alpha(\vec{f})(x) > t\mathcal{M}(\vec{f})(x)\}| \le c_1 e^{-c_2 t^2/\alpha^{2mn}}|Q|, \quad \forall t > 0.$$

This proves (1.12).

To obtain (1.13), we use the same strategy and (7.2) in place of (7.1). □

Next we present another proof of Theorem 1.6. In view of (4.1) and (4.2), following the approach in [42], it suffices to prove the following.

Lemma 7.3 *There exist $c_1 > 0$ and $c_2 > 0$ such that for every sparse family $\mathcal{S} \subset \mathcal{D}$ and for every cube Q_0,*

$$|\{x \in Q_0 : \mathcal{A}_{\mathcal{S}}^2(\vec{f}) > t\mathcal{M}(\vec{f})\}| \le c_1 e^{-c_2 t^2}|Q_0|.$$

where $\vec{f} = (f_1, \ldots, f_m)$ are supported on Q_0.

Proof Fix a sparse family $\mathcal{S} \subset \mathcal{D}$ and a cube Q_0. First we observe that

$$\mathcal{K} := |\{x \in Q_0 : \mathcal{A}^2_{\mathcal{S}}(\vec{f}) > t\mathcal{M}(\vec{f})\}| = \left|\left\{x \in Q_0 : \sum_{Q\in\mathcal{S}} \prod_{i=1}^{m} \langle |f_i|\rangle_Q^2 > t^2\mathcal{M}(\vec{f})^2\right\}\right|.$$

Now we consider the family of at most 3^n cubes $Q_j \in \mathcal{D}$ such that $|Q_j| \simeq |Q_0|$ and $|Q_j \cap Q_0| > 0$. We have that adding those cubes to $\mathcal{S}$ it remains a sparse family, we shall assume then that $Q_j \in \mathcal{S}$. For such Q_j, we define

$$T_j^1(\vec{f}) := \sum_{Q\in\mathcal{S}:Q\subset Q_j} \prod_{i=1}^{m} \langle |f_i|\rangle_Q^2 \mathbf{1}_Q \quad \text{and} \quad T_j^2(\vec{f}) := \sum_{Q\in\mathcal{S}:Q\supsetneq Q_j} \prod_{i=1}^{m} \langle |f_i|\rangle_Q^2 \mathbf{1}_Q.$$

Then, one has

$$\begin{aligned} \mathcal{K} &\le \sum_{j=1}^{3^n} \left|\left\{x \in Q_j : T_j^1(\vec{f}) + T_j^2(\vec{f}) > t^2\mathcal{M}(\vec{f})^2\right\}\right| \\ &\le \sum_{j=1}^{3^n}\sum_{i=1}^{2} \left|\left\{x \in Q_j \ : T_j^i(\vec{f}) > c_n t^2\mathcal{M}(\vec{f})^2\right\}\right| =: \sum_{j=1}^{3^n}(\mathcal{K}_j^1 + \mathcal{K}_j^2). \end{aligned}$$

We recall that in [41, Theorem 2.1], it was established that

$$\left|\left\{x \in Q \ : \sum_{Q'\in\mathcal{S},\, Q'\subseteq Q} \mathbf{1}_{Q'}(x) > t\right\}\right| \le ce^{-\alpha t}|Q|, \quad \forall Q. \tag{7.14}$$

For $\mathcal{K}_j^1$, taking into account (7.14), we obtain

$$\mathcal{K}_j^1 \le \left|\left\{x \in Q_j \ : \sum_{Q\in\mathcal{S}, Q\subset Q_j} \mathbf{1}_Q(x) > c_n t^2\right\}\right| \le ce^{-\alpha t^2}|Q_j| \simeq ce^{-\alpha t^2}|Q_0|.$$

For $\mathcal{K}_j^1$, since $\vec{f}$ is supported in Q_0, we deduce that

$$\begin{aligned} \mathcal{K}_j^2 &\le \left|\left\{x \in Q_j \ : T_j^2(\vec{f} \cdot \mathbf{1}_{Q_0}) > c_n t^2 \prod_{i=1}^{m} \langle |f_i|\rangle_{Q_0}^2\right\}\right| \\ &\le \left|\left\{x \in Q_j \ : \sum_{Q\in\mathcal{S}, Q\supsetneq Q_j} \left(\prod_{i=1}^{m} |Q_0|/|Q|\right)^2 \mathbf{1}_Q(x) > c_n t^2\right\}\right| \\ &\le \left|\left\{x \in Q_j \ : \sum_{j=1}^{\infty} 2^{-2mj} > c_n t^2\right\}\right|. \end{aligned}$$

Observe that if t is large enough, then

$$\left|\left\{x \in Q_j : \sum_{j=1}^{\infty} 2^{-2mj} > c_n t^2\right\}\right| = 0.$$

Consequently,

$$\mathcal{K}_j^2 \lesssim e^{-t^2} |Q_0|.$$

We are done. □

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