Journal of Fourier Analysis and Applications (2021) 27:43
https://doi.org/10.1007/s00041-021-09842-1

Boundedness of Sparse and Rough Operators on Weighted Lorentz Spaces

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Received: 4 April 2020 / Revised: 13 March 2021 / Accepted: 15 March 2021 /
Published online: 30 April 2021

Abstract
We present new estimates in the setting of weighted Lorentz spaces for important operators in Harmonic Analysis such as sparse operators, Bochner–Riesz at the critical index, Hörmander multipliers and rough singular integrals among others.

Keywords Weights · Restricted weak type Rubio de Francia extrapolation · Weighted Lorentz spaces · Hardy–Littlewood maximal operator

Mathematics Subject Classification 42B99 · 46E30

1 Introduction

The main purpose of this paper is to prove boundedness of important operators in harmonic analysis in weighted Lorentz spaces $\Lambda^p(w)$, $0 < p < \infty$, defined by

$$\Lambda^p(w) = \left\{ f \in \mathcal{M}(\mathbb{R}^n) : ||f||_{\Lambda^p(w)} = \left(\int_0^\infty f^*(t)^p w(t)\, dt \right)^{\frac{1}{p}} < \infty \right\}.$$

Communicated by Mieczyslaw Mastylo.

The authors were supported by Grants MTM2016-75196-P.

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In particular, we shall present new estimates for operators such as the Bochner–Riesz operator at the critical index $B_{\frac{n-1}{2}}$, the rough operators T_Ω or the sparse operators $A_{\mathcal{S}}$ among many others.

There are many results in the literature about boundedness of operators in $\Lambda^p(w)$ or even in rearrangement invariant (r.i.) spaces but they do not usually include the above ones. Let us just mention the classical paper [42], including the case of the Hilbert transform and Riesz transforms, and a very recent one [26] which contains a rather complete list of papers on this topic, among which we should mention [1,10,24].

Now, the three examples mentioned above have one important property in common: they all satisfy that, for some (and hence for all) $1 < p_0 < \infty$, and for every $v \in A_{p_0}$,

$$T : L^{p_0}(v) \longrightarrow L^{p_0}(v) \tag{1.1}$$

is bounded, where A_{p_0} is the class of Muckenhoupt weights defined in Sect. 2 (see [38]). Let us just mention here that these weights A_{p_0} characterize the boundedness on $L^{p_0}(v)$ of the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy, \qquad f \in L^1_{\text{loc}}(\mathbb{R}^n),\ x \in \mathbb{R}^n,$$

where the supremum is taken overall cubes $Q \subseteq \mathbb{R}^n$ containing x. An operator T satisfying (1.1) will be called a Rubio de Francia operator [41]. Such operators satisfy the following result:

Theorem 1.1 [25] *If T is a Rubio de Francia operator and $\mathbb{X}$ is a Banach function space such that $M : \mathbb{X} \to \mathbb{X}$ and $M : \mathbb{X}' \to \mathbb{X}'$, with $\mathbb{X}'$ being the associated space of $\mathbb{X}$, then*

$$T : \mathbb{X} \longrightarrow \mathbb{X}$$

is bounded.

This result is very useful to prove the boundedness of operators for which condition (1.1) has been widely studied while this is not the case in other contexts such as, for example, of weighted Lorentz spaces or more generally r.i. spaces. Let us explain, as an example, the case of the sparse operators $A_{\mathcal{S}}$ introduced by Lerner in [33]. These operators have become very useful since they dominate many operators such as Calderón–Zygmund operators. Independently of the fact that the decreasing rearrangement of an sparse operator $A_{\mathcal{S}}$ has not been estimated, the above theorem implies, for example, that if we consider the weighted Lorentz spaces, then if

$$M : \Lambda^p(w) \longrightarrow \Lambda^p(w), \qquad M : (\Lambda^p(w))' \longrightarrow (\Lambda^p(w))' \tag{1.2}$$

are bounded operators, we have that

$$A_{\mathcal{S}} : \Lambda^p(w) \longrightarrow \Lambda^p(w)$$

is bounded, where

$$(\Lambda^p(w))' = \Big\{ f \in \mathcal{M}(\mathbb{R}^n) \,:\, \|f\|_{(\Lambda^p(w))'} = \sup_{g\in\Lambda^p(w)} \frac{1}{\|g\|_{\Lambda^p(w)}} \int_{\mathbb{R}^n} |f(x)g(x)|\,dx < \infty \Big\}$$

is the associated space of $\Lambda^p(w)$. Now, on many occasions the difficulty to apply Theorem 1.1 to a concrete space $\mathbb{X}$ is precisely to characterize w so that (1.2) holds. In the case of $\mathbb{X} = \Lambda^p(w)$ this has already been done, along the years, in several papers:

i) The boundedness of M in $\Lambda^p(w)$ was first characterized for $p > 1$, by Ariño and Muckenhoupt [4] in 1990, who showed

$$M : \Lambda^p(w) \longrightarrow \Lambda^p(w) \iff w \in B_p, \tag{1.3}$$

where

$$||w||_{B_p} = \sup_{t>0} \frac{\int_0^\infty w(r)\min\left(1, \frac{t^p}{r^p}\right)dr}{\int_0^t w(r)\,dr}.$$

This class of weights has been widely studied (see for instance [11,14,39,40,44]) and now it is known that the same result holds for every $p > 0$. Moreover, from (1.3), it can be easily seen that if $p > 1$ and $w \in B_p$, then $\Lambda^p(w)$ is a Banach function space. Besides, the reciprocal is also known to be true (see [42]). In addition, when $w \in B_1$, $\Lambda^1(w)$ is a Banach function space as well (see [16]).

ii) To characterize the boundedness M on $(\Lambda^p(w))'$ we can use the following. Given a r.i. Banach function space $\mathbb{X}$ on $\mathbb{R}^n$, the Lorentz–Shimogaki theorem (see [7, Chap.3 p. 154]) asserts that

$$M : \mathbb{X} \longrightarrow \mathbb{X}$$

is bounded if and only if the upper Boyd index $\alpha_{\mathbb{X}} < 1$. Therefore, since $\alpha_{\mathbb{X}'} = 1 - \beta_{\mathbb{X}}$ ([7]),

$$M : \mathbb{X}' \longrightarrow \mathbb{X}'$$

is bounded if and only if the lower Boyd index $\beta_{\mathbb{X}} > 0$ and it is known [2] that when $W(t) = \int_0^t w(r)\,dr$ and $\mathbb{X} = \Lambda^p(w)$, this holds if and only if $w \in B_\infty^*$; that is,

$$\sup_{t>0} \frac{1}{W(t)} \int_0^t \frac{W(r)}{r}\,dr < \infty.$$

From all the above results, we can conclude that if $A_{\mathcal{S}}$ is an sparse operator and $p \geq 1$,

$$w \in B_p \cap B_\infty^* \quad \Longrightarrow \quad A_{\mathcal{S}} : \Lambda^p(w) \longrightarrow \Lambda^p(w).$$

In fact, the same result is true for every Rubio de Francia operator and it is sharp since it is known [42] that

$$w \in B_p \cap B_\infty^* \quad \Longleftrightarrow \quad H : \Lambda^p(w) \longrightarrow \Lambda^p(w),$$

with H the Hilbert transform, which is a Rubio de Francia operator. Now, up to now, we have considered $p \geq 1$ since Theorem 1.1 concerns the case of Banach function spaces. But, what can we say if $0 < p < 1$? Moreover, in the case $p = 1$, it is known that

$$A_{\mathcal{S}} : L^1 \longrightarrow L^{1,\infty}$$

is bounded, and this case is not covered by Theorem 1.1. To study these two cases are the purposes of this paper, since they will have as a consequence new estimates for all the above mentioned operators (see also [19] for related results). Now, it is known that (1.1) does not imply, in general, the weak boundedness of T from L^1 into $L^{1,\infty}$, but using the results in [20] we now know that we can arrive to the endpoint $p = 1$ if we assume a slightly stronger condition on T which is satisfied by the sparse operators among many others. This condition reads:

$$T : L^{p_0,1}(v) \longrightarrow L^{p_0,\infty}(v) \tag{1.4}$$

need to be bounded for every $v \in A_{p_0}^{\mathcal{R}}$, a slightly bigger class than A_{p_0} defined by

$$\|v\|_{A_{p_0}^{\mathcal{R}}} = \sup_Q \frac{1}{|Q|} \|\chi_Q\|_{L^{p_0}(v)} \|\chi_Q v^{-1}\|_{L^{p_0',\infty}(v)} < \infty.$$

This class was introduced in [23,31] where it was proved that

$$M : L^{p_0,1}(v) \longrightarrow L^{p_0,\infty}(v) \quad \Longleftrightarrow \quad v \in A_{p_0}^{\mathcal{R}}.$$

Operators satisfying (1.4) will be called restricted weak type Rubio de Francia operators or RWT-Rubio de Francia operators, in short.

So a question we want to answer is the following: If T is a RWT-Rubio de Francia operator, what conditions do we need on w in order to have that some restricted weak type boundedness on weighted Lorentz spaces

$$T : \Lambda^{p,1}(w) \longrightarrow \Lambda^{p,\infty}(w)$$

holds? At this point we have to recall that if $w = 1$, then $w \in B_\infty^*$ and although $w \notin B_1$ it belongs to a slightly bigger class $B_1^{\mathcal{R}}$ defined in Sect. 2. Also, we have the following extrapolation result [20]: Let T be a sublinear operator satisfying that, for some $p_0 > 1$ and every $v \in A_{p_0}^{\mathcal{R}}$,

$$T : L^{p_0,1}(v) \rightarrow L^{p_0,\infty}(v)$$

is bounded, with constant less than or equal to $\varphi_{p_0}(||v||_{\hat{A}_{p_0}})$, with φ_{p_0} an increasing function on $(0, \infty)$. Then,

$$T : L^{1,\frac{1}{p_0}} \to L^{1,\infty}$$

is bounded. Hence, if T is a RWT-Rubio de Francia operator, it is natural to expect that

$$w \in B_1^{\mathcal{R}} \cap B_\infty^* \quad \Longrightarrow \quad T : \Lambda^{1,q}(w) \longrightarrow \Lambda^{1,\infty}(w), \qquad \forall\, 0 < q < 1.$$

In fact, our main theorems show this result holds true.

Theorem 1.2 *Let T be an operator satisfying that, for every $v \in A_1$,*

$$T : L^1(v) \to L^{1,\infty}(v)$$

is bounded, with constant less than or equal to $\varphi(||v||_{A_1})$, with φ an increasing function on $(0, \infty)$. Then, for every $0 < p < \infty$ and every $w \in B_p^{\mathcal{R}} \cap B_\infty^*$,

$$T : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w)$$

is bounded with constant less than or equal to $C_1||w||_{B_p^{\mathcal{R}}}\varphi\left(C_2||w||_{B_\infty^*}\right)$ for some positive constants C_1, C_2 independent of w.

Theorem 1.3 *Let T be an operator satisfying that, for some $1 < p_0 < \infty$ and every $v \in A_{p_0}^{\mathcal{R}}$,*

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v)$$

is bounded, with constant less than or equal to $\varphi(||v||_{A_{p_0}^{\mathcal{R}}})$, with φ an increasing function on $(0, \infty)$. Then, for every $0 < p < \infty$, every $w \in B_p^{\mathcal{R}} \cap B_\infty^*$ and every $0 < q < 1$,

$$T : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w)$$

is bounded with constant less than or equal to

$$\Phi_{p_0,p,q}(||w||_{B_p^{\mathcal{R}}}, ||w||_{B_\infty^*}) \le \frac{C_1}{1-q}||w||_{B_p^{\mathcal{R}}}\varphi\left(C_2||w||_{B_\infty^*}^{\max\left(\frac{1}{p_0},q\right)}\right),$$

for some positives constants C_1, C_2 independent of w.

The paper is organized as follows: in Sect. 2 we present some technical lemmas and previous results which shall be used later and also the statements of our main results. Their proof will be given in Sect. 3, and Sect. 4 will be devoted to apply our results to the boundedness not only of sparse operators but also of Bochner–Riesz operator at the critical index, Hörmander multipliers, and many others.

As usual, we write $A \lesssim B$ if there exists an universal constant $C > 0$ which may depends on the exponents but is independent of A and B, such that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

2 Definitions, Previous Results and Some Technical Lemmas

Let us recall the definition of the spaces which are going to be important for us. Given $0 < p, q < \infty$, $L^{p,q}$ is the Lorentz space of measurable functions such that

$$||f||_{L^{p,q}} = \left(p \int_0^\infty y^q \lambda_f(y)^{\frac{q}{p}} \frac{dy}{y} \right)^{\frac{1}{q}} = \left(\int_0^\infty f^*(t)^q t^{\frac{q}{p}-1} dt \right)^{\frac{1}{q}} < \infty,$$

and $L^{p,\infty}$ is the Lorentz space of measurable functions such that

$$||f||_{L^{p,\infty}} = \sup_{y>0} y\lambda_f(y)^{\frac{1}{p}} = \sup_{t>0} f^*(t) t^{\frac{1}{p}} < \infty,$$

where f^* is the decreasing rearrangement of f defined by

$$f^*(t) := \inf\{y > 0 : \lambda_f(y) \le t\}, \qquad \lambda_f(t) := |\{|f| > t\}|, \qquad t > 0.$$

For further information about these notions and related topics see [7]. These spaces are, in fact, a particular case of the so called weighted Lorentz spaces $\Lambda^{p,q}(w)$ defined for $0 < p, q < \infty$ by

$$||f||_{\Lambda^{p,q}(w)} = \left(p \int_0^\infty y^q W(\lambda_f(y))^{\frac{q}{p}} \frac{dy}{y} \right)^{\frac{1}{q}} = \left(\int_0^\infty f^*(t)^q W(t)^{\frac{q}{p}-1} w(t) dt \right)^{\frac{1}{q}},$$

and for $q = \infty$,

$$||f||_{\Lambda^{p,\infty}(w)} = \sup_{y>0} yW(\lambda_f(y))^{\frac{1}{p}} = \sup_{t>0} f^*(t) W(t)^{\frac{1}{p}},$$

where w is a positive locally integrable function defined on $(0, \infty)$ and $W(t) = \int_0^t w(r)\, dr$. We should emphasize here that for $0 < q < \infty$,

$$\Lambda^{p,q}(w) = \Lambda^q(\widetilde{w}), \qquad \widetilde{W}(t) \approx W(t)^{\frac{q}{p}}, \tag{2.1}$$

and, similar, $\Lambda^{p,\infty}(w) = \Lambda^{1,\infty}(\widetilde{w})$, with $\widetilde{W}(t) \approx W(t)^{\frac{1}{p}}$. Besides, these spaces satisfy the embeddings

$$\Lambda^{p,q_0}(w) \hookrightarrow \Lambda^{p,q_1}(w) \tag{2.2}$$

continuously for $0 < q_0 \leq q_1 \leq \infty$. For more details on these spaces, we refer to [18].

Let us consider the Hardy–Littlewood maximal operator M, defined for locally integrable functions on $\mathbb{R}^n$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy, \qquad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ containing x. It is known [38] that for every $1 < p < \infty$,

$$M : L^p(v) \longrightarrow L^p(v) \quad \Longleftrightarrow \quad v \in A_p,$$

where $v \in A_p$ if v is a positive and locally integrable function such that

$$||v||_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q v(x)dx \right) \left(\frac{1}{|Q|} \int_Q v(x)^{1-p'}dx \right)^{p-1} < \infty.$$

Moreover, if $1 \leq p < \infty$,

$$M : L^p(v) \longrightarrow L^{p,\infty}(v) \quad \Longleftrightarrow \quad v \in A_p,$$

where $v \in A_1$ if

$$Mv(x) \leq Cv(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

and the infimum of all such constants C in the above inequality is denoted by $||v||_{A_1}$. Also, in the context of restricted weak type inequalities the following result was proved in [23,31]:

$$M : L^{p,1}(v) \longrightarrow L^{p,\infty}(v) \quad \Longleftrightarrow \quad v \in A_p^{\mathcal{R}},$$

where a weight $v \in A_p^{\mathcal{R}}$ if

$$\|v\|_{A_p^{\mathcal{R}}} = \sup_Q \frac{1}{|Q|} \|\chi_Q\|_{L^p(v)} \|\chi_Q v^{-1}\|_{L^{p',\infty}(v)} < \infty.$$

Moreover,

$$\|M\|_{L^{p,1}(v) \to L^{p,\infty}(v)} \lesssim \|v\|_{A_p^{\mathcal{R}}}.$$

Now, given $p \geq 1$, it was proved in [15,20] that $\widehat{A}_p \subseteq A_p^{\mathcal{R}}$, where

$$\widehat{A}_p := \{0 < v \in L^1_{loc} : \exists h \in L^1_{loc}, u \in A_1; \; v = (Mh)^{1-p}u\},$$

with

$$\|v\|_{A_p^{\mathcal{R}}} \lesssim \|v\|_{\widehat{A}_p} := \inf ||u||_{A_1}^{\frac{1}{p}},$$

where the infimum is taken over all $u \in A_1$ for which there exists $h \in L^1_{loc}$ such that $v = (Mh)^{1-p}u$. Observe that $A_1^{\mathcal{R}} = \widehat{A}_1 = A_1$.

Also, the following extrapolation result is going to be important in our main theorems.

Theorem 2.1 [20] *Let T be an operator satisfying that, for some $1 < p_0 < \infty$ and every $v \in A_{p_0}^{\mathcal{R}}$,*

$$T : L^{p_0,1}(v) \to L^{p_0,\infty}(v)$$

is bounded, with constant less than or equal to $\varphi_{p_0}(||v||_{A_{p_0}^{\mathcal{R}}})$, with φ_{p_0} an increasing function on $(0, \infty)$. Then, for every $1 < p \le p_0$ and every $v \in \widehat{A}_p$

$$T : L^{p,1}(v) \to L^{p,\infty}(v)$$

is bounded with constant

$$\varphi_{p_0,p}(t) \lesssim t^{1-\frac{p}{p_0}} \varphi_{p_0}\left(Ct^{\frac{p}{p_0}}\right).$$

Concerning boundedness of the Hardy–Littlewood maximal operator on weighted Lorentz spaces we have, as mentioned in the introduction, that for every $p > 0$,

$$M : \Lambda^p(w) \longrightarrow \Lambda^p(w) \iff w \in B_p.$$

Now, in order to characterize the boundedness of

$$M : \Lambda^{p,1}(w) \longrightarrow \Lambda^{p,\infty}(w),$$

by (2.1), $\Lambda^{p,1}(w) = \Lambda^1(\widetilde{w})$ where $\widetilde{w}$ is such that its primitive $\widetilde{W}(t) \approx W(t)^{\frac{1}{p}}$ and hence, since $(Mf)^*(t) \approx f^{**}(t) := \frac{1}{t}\int_0^t f^*(r)\,dr$, this boundedness is equivalent to the embedding

$$\Lambda^1(\tilde{w}) \subset \Gamma^{p,\infty}(w) := \left\{ f \in \mathcal{M}(\mathbb{R}^n) \,:\, \|f^{**}\|_{\Lambda^{p,\infty}(w)} < \infty \right\},$$

which is known [17] to be characterized by the condition $w \in B_p^{\mathcal{R}}$ defined by

$$||w||_{B_p^{\mathcal{R}}} = \sup_{0<s\le t<\infty} \frac{sW(t)^{\frac{1}{p}}}{tW(s)^{\frac{1}{p}}} < \infty. \tag{2.3}$$

In fact, $||M||_{\Lambda^{p,1}(w)\longrightarrow\Lambda^{p,\infty}(w)} \leq ||w||_{B_p^{\mathcal{R}}}$. Also,

$$\|g\|_{(\Lambda^{p,1}(w))'} = \sup_{f\in\Lambda^1(\tilde{w})} \frac{\int_{\mathbb{R}^n} |f(x)g(x)|\,dx}{\int_0^\infty f^*(t)\widetilde{w}(t)dt} = \sup_{t>0} \frac{\int_0^t g^*(r)\,dr}{\int_0^t \widetilde{w}(r)dr} \approx \sup_{t>0} \frac{\int_0^t g^*(r)\,dr}{W(t)^{\frac{1}{p}}},$$

where in the second equality we have used [18, Corollary 1.2.12]. Hence

$$||Mf||_{(\Lambda^{p,1}(w))'} \approx \sup_{t>0} \frac{\int_0^t \frac{1}{r}\int_0^r f^*(s)dsdt}{W(t)^{\frac{1}{p}}} \lesssim ||f||_{(\Lambda^{p,1}(w))'} \left(\sup_{t>0} \frac{\int_0^t W(r)^{\frac{1}{p}}\frac{dr}{r}}{W(t)^{\frac{1}{p}}} \right),$$

and consequently

$$||M||_{(\Lambda^{p,1}(w))'} := ||M||_{(\Lambda^{p,1}(w))'\to(\Lambda^{p,1}(w))'} \lesssim \sup_{t>0} \frac{\int_0^t W(r)^{\frac{1}{p}}\frac{dr}{r}}{W(t)^{\frac{1}{p}}}. \tag{2.4}$$

Now, the last expression is finite if and only if $w \in B_\infty^*$ (see [3]). So let us recall several facts on this class of weights and give an expression for the quantity $||w||_{B_\infty^*}$ which is going to be important for our purposes.

Definition 2.2 A weight $w \in B_\infty^*$ if and only if

$$||w||_{B_\infty^*} = \sup_{t>0} \frac{1}{W(t)} \int_0^t \frac{W(r)}{r}\,dr < \infty.$$

Now, many equivalent definitions have appeared in the literature and, as a consequence, we have the following result [3]:

Lemma 2.3 *For* $0 < p < \infty$,

$$||M||_{(\Lambda^{p,1}(w))'} \lesssim ||w||_{B_\infty^*}.$$

Proof In virtue of estimate (2.4), it would be enough to see that

$$\sup_{t>0} \frac{\int_0^t W(r)^{\frac{1}{p}}\frac{dr}{r}}{W(t)^{\frac{1}{p}}} \lesssim ||w||_{B_\infty^*}.$$

If $0 < p \leq 1$, then the result is immediate, since for every $t > 0$,

$$\int_0^t W(r)^{\frac{1}{p}}\frac{dr}{r} \leq W(t)^{\frac{1}{p}-1} \int_0^t \frac{W(r)}{r}\,dr \leq W(t)^{\frac{1}{p}} ||w||_{B_\infty^*}.$$

So let us consider the case $p > 1$. First, integrating by parts,

$$\int_0^t \frac{W(r)^{\frac{1}{p}}}{r}\,dr = \lim_{s\to 0}\log\left(\frac{t}{s}\right)W(s)^{\frac{1}{p}} + \frac{1}{p}\int_0^t \log\left(\frac{t}{r}\right)W(r)^{\frac{1}{p}-1}w(r)\,dr$$
$$\leq \left(\int_0^t \left(\log\frac{t}{r}\right)^p w(r)\,dr\right)^{\frac{1}{p}} + \frac{1}{p}\int_0^t \log\left(\frac{t}{r}\right)W(r)^{\frac{1}{p}-1}w(r)\,dr, \tag{2.5}$$

an observe that by means of the Hölder's inequality,

$$\int_0^t \log\left(\frac{t}{r}\right)W(r)^{\frac{1}{p}-1}w(r)\,dr$$
$$\leq \left(\int_0^t \left(\log\frac{t}{r}\right)^{2p} w(r)\,dr\right)^{\frac{1}{2p}}\left(\int_0^t W(r)^{\frac{1}{2p-1}-1}w(r)\,dr\right)^{\frac{2p-1}{2p}} \tag{2.6}$$
$$\leq (2p-1)^{\frac{2p-1}{2p}}\left(\int_0^t \left(\log\frac{t}{r}\right)^{2p} w(r)\,dr\right)^{\frac{1}{2p}} W(t)^{\frac{1}{2p}}.$$

We claim that for every $q \geq 2$,

$$\int_0^t \left(\log\frac{t}{r}\right)^q w(r)\,dr \leq q(q-1)\cdots(q-[q]+1)\|w\|_{B_\infty^*}^q W(t), \tag{2.7}$$

from which putting together (2.5) and (2.6), we will conclude that

$$\int_0^t \frac{W(r)^{\frac{1}{p}}}{r}\,dr \lesssim \left(\int_0^t \left(\log\frac{t}{r}\right)^{2p} w(r)\,dr\right)^{\frac{1}{2p}} W(t)^{\frac{1}{2p}} \lesssim \|w\|_{B_\infty^*} W(t)^{\frac{1}{p}}.$$

So let us see (2.7). Observe that since $q \geq 2$, using Fubini twice we get

$$\int_0^t \left(\log\frac{t}{r}\right)^q w(r)\,dr = q\int_0^t \int_r^t \left(\log\frac{t}{s}\right)^{q-1}\frac{ds}{s}w(r)\,dr$$
$$= q\int_0^t \left(\log\frac{t}{s}\right)^{q-1} W(s)\,\frac{ds}{s}$$
$$= q(q-1)\int_0^t \left(\log\frac{t}{r}\right)^{q-2}\int_0^r W(s)\,\frac{ds}{s}\,\frac{dr}{r}$$
$$\leq q(q-1)||w||_{B_\infty^*}\int_0^t \left(\log\frac{t}{r}\right)^{q-2} W(r)\,\frac{dr}{r}.$$

Hence, iterating in that way we obtain

$$\int_0^t \left(\log \frac{t}{r}\right)^q w(r)\,dr$$
$$\leq q(q-1)\cdots(q-[q]+1)\|w\|_{B_\infty^*}^{[q]-1} \int_0^t \left(\log \frac{t}{r}\right)^{q-[q]} W(r)\frac{dr}{r},$$

with $[q]$ being the integer part of q. Finally,

$$\begin{aligned}\int_0^t \left(\log \frac{t}{r}\right)^{q-[q]} \frac{W(r)}{r}\,dr &\leq \left(\int_0^t \log\frac{t}{r}\,\frac{W(r)}{r}\,dr\right)^{q-[q]} \left(\int_0^t \frac{W(r)}{r}\,dr\right)^{1+[q]-q} \\ &\leq \left(\int_0^t \int_0^u \frac{W(s)}{s}ds\frac{du}{u}\right)^{q-[q]} \left(||w||_{B_\infty^*} W(t)\right)^{1+[q]-q} \\ &\leq \left(\|w\|_{B_\infty^*}^2 W(t)\right)^{q-[q]} ||w||_{B_\infty^*}^{1+[q]-q} W(t)^{1+[q]-q} \\ &= ||w||_{B_\infty^*}^{1+q-[q]} W(t),\end{aligned}$$

and the result follows. □

3 Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2 Since $w \in B_\infty^*$, we have that M is bounded in $(\Lambda^{p,1}(w))'$, and hence we can define, for every nonnegative function $h \in \left(\Lambda^{p,1}(w)\right)'$,

$$Rh(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{\left(2||M||_{(\Lambda^{p,1}(w))'}\right)^k}$$

to be the function resulting from the Rubio de Francia algorithm [41]. Then, $h(x) \leq Rh(x)$ a.e., $Rh \in A_1$ with

$$||Rh||_{A_1} \lesssim ||M||_{(\Lambda^{p,1}(w))'} \lesssim ||w||_{B_\infty^*}$$

by Lemma 2.3, and

$$\|Rh\|_{(\Lambda^{p,1}(w))'} \leq 2||h||_{(\Lambda^{p,1}(w))'}.$$

Let $y > 0$ and set $F := F_{Tf,y} = \{x \in \mathbb{R}^n : |Tf(x)| > y\}$. Then, $v = R(\chi_F) \in A_1$ and,

$$\begin{aligned}\lambda_{Tf}(y) &\leq \int_{\{|Tf(x)|>y\}} R\left(\chi_F\right)(x)\,dx \leq \frac{\varphi\left(||v||_{A_1}\right)}{y} \int_{\mathbb{R}^n} |f(x)|R\left(\chi_F\right)(x)\,dx \\ &= \frac{\varphi\left(||v||_{A_1}\right)}{y} \int_0^\infty \left[\int_{\{|f(x)|>z\}} R(\chi_F)(x)\,dx\right] dz.\end{aligned}$$

Now, in the inner integral we apply the definition of associate space to obtain that

$$\int_{\{|f(x)|>z\}} R(\chi_F)(x)\,dx \le ||\chi_{\{|f|>z\}}||_{\Lambda^{p,1}(w)}||R(\chi_F)||_{(\Lambda^{p,1}(w))'},$$

and hence,

$$\lambda_{Tf}(y) \lesssim \frac{\varphi\left(||v||_{A_1}\right)}{y}||R(\chi_F)||_{(\Lambda^{p,1}(w))'}||f||_{\Lambda^{p,1}(w)}$$
$$\lesssim \frac{\varphi\left(||v||_{A_1}\right)}{y}||\chi_F||_{(\Lambda^{p,1}(w))'}||f||_{\Lambda^{p,1}(w)}.$$

Now, since $w \in B_p^{\mathcal{R}}$, we have by (2.3) that

$$||\chi_F||_{(\Lambda^{p,1}(w))'} \lesssim ||w||_{B_p^{\mathcal{R}}} \frac{\lambda_{Tf}(y)}{W(\lambda_{Tf}(y))^{\frac{1}{p}}}, \tag{3.1}$$

and hence,

$$\sup_{y>0} yW(\lambda_{Tf}(y))^{\frac{1}{p}} \lesssim ||w||_{B_p^{\mathcal{R}}}\varphi\big(C||w||_{B_\infty^*}\big)||f||_{\Lambda^{p,1}(w)}$$

as we wanted to see. □

Remark 3.1 (1) We observe that, since

$$\Lambda^{p,1}(w) = \Lambda^1(\widetilde{w}), \quad \text{and} \quad \Lambda^{p,\infty}(w) = \Lambda^{1,\infty}(\widetilde{w}),$$

with $\widetilde{w}$ such that $\widetilde{W}(t) \approx W(t)^{\frac{1}{p}}$ (see (2.1)), and we also have that

$$w \in B_p^{\mathcal{R}} \cap B_\infty^* \iff \widetilde{w} \in B_1^{\mathcal{R}} \cap B_\infty^*,$$

the condition

$$w \in B_p^{\mathcal{R}} \cap B_\infty^* \implies T : \Lambda^{p,1}(w) \longrightarrow \Lambda^{p,\infty}(w)$$

is equivalent to

$$w \in B_1^{\mathcal{R}} \cap B_\infty^* \implies T : \Lambda^1(w) \longrightarrow \Lambda^{1,\infty}(w).$$

(2) Using interpolation theory of weighted Lorentz spaces (see for instance [18, Theorem 2.6.5]) we can deduce that if T is a quasi-linear operator satisfying the hypothesis of Theorem 1.2, then

$$w \in B_1 \cap B_\infty^* \implies T : \Lambda^1(w) \longrightarrow \Lambda^1(w),$$

and similarly, for every $0 < p < \infty$,

$$w \in B_p \cap B_\infty^* \implies T : \Lambda^p(w) \longrightarrow \Lambda^p(w).$$

Proof of Theorem 1.3 Let $y > 0$ and set $F := F_{Tf,y} = \{x \in \mathbb{R}^n : |Tf(x)| > y\}$. Then, $v = (Mf)^{1-p_0} R(\chi_F) \in \widehat{A}_{p_0} \subset A_{p_0}^{\mathcal{R}}$ and, for every $\gamma > 0$,

$$\begin{aligned}
\lambda_{Tf}(y) &\leq \lambda_{Mf}(\gamma y) + \int_{\{|Tf(x)|>y,\, Mf(x)\leq \gamma y\}} R(\chi_F)(x)\,dx \\
&\leq \lambda_{Mf}(\gamma y) + \gamma^{p_0-1}\frac{y^{p_0}}{y}\int_F Mf(x)^{1-p_0} R(\chi_F)(x)\,dx \\
&\lesssim \lambda_{Mf}(\gamma y) + \frac{\gamma^{p_0-1}\varphi\left(||v||_{\widehat{A}_{p_0}}\right)^{p_0}}{y} ||f||^{p_0}_{L^{p_0,1}(Mf^{1-p_0}R(\chi_F))} \\
&= \lambda_{Mf}(\gamma y) + \frac{\gamma^{p_0-1}\varphi\left(||v||_{\widehat{A}_{p_0}}\right)^{p_0}}{y} \\
&\quad \times \left(p_0 \int_0^\infty \left[\int_{\{|f(x)|>z\}} Mf(x)^{1-p_0} R(\chi_F)(x)\,dx\right]^{\frac{1}{p_0}} dz\right)^{p_0} \\
&\lesssim \lambda_{Mf}(\gamma y) + \frac{\gamma^{p_0-1}\varphi\left(||v||_{\widehat{A}_{p_0}}\right)^{p_0}}{y} \\
&\quad \left(\int_0^\infty z^{\frac{1}{p_0}-1}\left[\int_{\{|f(x)|>z\}} R(\chi_F)(x)\,dx\right]^{\frac{1}{p_0}} dz\right)^{p_0},
\end{aligned}$$

where in the last estimate we have used that for every $x \in \{|f(x)| > z\}$,

$$Mf(x)^{1-p_0} \leq |f(x)|^{1-p_0} \leq z^{1-p_0}.$$

Now, in the inner integral we apply the definition of associate space to obtain that

$$\int_{\{|f(x)|>z\}} R(\chi_F)(x)\,dx \leq ||\chi_{\{|f|>z\}}||_{\Lambda^{p,1}(w)} ||R(\chi_F)||_{(\Lambda^{p,1}(w))'},$$

and hence, using (3.1) and the boundedness of the operator R,

$$\begin{aligned}
\lambda_{Tf}(y) &\lesssim \lambda_{Mf}(\gamma y) + p^{1-p_0}\frac{\gamma^{p_0-1}\varphi\left(||v||_{\widehat{A}_{p_0}}\right)^{p_0}}{y} ||f||_{\Lambda^{p,\frac{1}{p_0}}(w)} ||R(\chi_F)||_{(\Lambda^{p,1}(w))'} \\
&\lesssim \lambda_{Mf}(\gamma y) + p^{1-p_0}\frac{\gamma^{p_0-1}\varphi\left(||v||_{\widehat{A}_{p_0}}\right)^{p_0}}{y} ||f||_{\Lambda^{p,\frac{1}{p_0}}(w)} ||w||_{B_p^{\mathcal{R}}} \frac{\lambda_{Tf}(y)}{W(\lambda_{Tf}(y))^{\frac{1}{p}}},
\end{aligned}$$

so that using that $w \in B_p^{\mathcal{R}}$ and the continuous embedding $\Lambda^{p,\frac{1}{p_0}}(w) \subset \Lambda^{p,1}(w)$ (see (2.2)),

$$\sup_{y>0} yW(\lambda_{Mf}(\gamma y))^{\frac{1}{p}} = \frac{1}{\gamma}||Mf||_{\Lambda^{p,\infty}(w)} \lesssim \frac{||w||_{B_p^{\mathcal{R}}}}{\gamma}||f||_{\Lambda^{p,1}(w)} \lesssim \frac{||w||_{B_p^{\mathcal{R}}}}{\gamma}||f||_{\Lambda^{p,\frac{1}{p_0}}(w)}.$$

Therefore, we conclude

$$||Tf||_{\Lambda^{p,\infty}(w)} \lesssim \max\left(\frac{1}{\gamma}, \gamma^{p_0-1}\varphi\left(||v||_{\widehat{A}_{p_0}}\right)^{p_0}\right)||w||_{B_p^{\mathcal{R}}}||f||_{\Lambda^{p,\frac{1}{p_0}}(w)},$$

and taking the infimum in $\gamma > 0$,

$$||Tf||_{\Lambda^{p,\infty}(w)} \lesssim ||w||_{B_p^{\mathcal{R}}}\varphi\left(||v||_{\widehat{A}_{p_0}}\right)||f||_{\Lambda^{p,\frac{1}{p_0}}(w)}.$$

Now,

$$||v||_{\widehat{A}_{p_0}} \lesssim ||R(\chi_F)||_{A_1}^{1/p_0} \lesssim ||w||_{B_\infty^*}^{1/p_0}.$$

Therefore,

$$||Tf||_{\Lambda^{p,\infty}(w)} \lesssim ||w||_{B_p^{\mathcal{R}}}\varphi\left(C||w||_{B_\infty^*}^{1/p_0}\right)||f||_{\Lambda^{p,\frac{1}{p_0}}(w)}.$$

Observe that if $0 < q \leq \frac{1}{p_0}$, by the continuous embeddings of the weighted Lorentz spaces (see (2.2)), then

$$\Lambda^{p,q}(w) \hookrightarrow \Lambda^{p,\frac{1}{p_0}}(w)$$

and the result follows. Otherwise, take $\frac{1}{p_0} < q < 1$. Hence, by Theorem 2.1,

$$T : L^{\frac{1}{q},1}(v) \to L^{\frac{1}{q},\infty}(v)$$

is bounded for every $v \in \widehat{A}_{\frac{1}{q}}$ with constant less than or equal to $\varphi_{p_0,\frac{1}{q}}\left(||v||_{\widehat{A}_{\frac{1}{q}}}\right)$. Therefore, applying the first part of the proof we have that

$$T : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w)$$

is bounded with constant less than or equal to $\Phi_{p_0,p,q}\left(||w||_{B_p^{\mathcal{R}}}, ||w||_{B_\infty^*}\right)$. □

4 Boundedness of Operators on Weighted Lorentz Spaces

The main result of this section is the following:

Theorem 4.1 *Let T be any of the following operators:*

(i) *Sparse operators $A_{\mathcal{S}}$.*
(ii) *Bochner–Riesz at the critical index $B_{\frac{n-1}{2}}$.*
(iii) *Radial Fourier multipliers satisfying (4.2).*
(iv) *Hörmander multipliers with $m \in HC(s, k)$.*
(v) *Rough singular integrals.*

Then, for every $p > 0$ and every $w \in B_p^{\mathcal{R}} \cap B_\infty^*$

$$T : \Lambda^{p,1}(w) \longrightarrow \Lambda^{p,\infty}(w) \tag{4.1}$$

is bounded.

Let us give the definition of all the above mentioned operators, some references related to them and, when possible, an estimate for the norm of the operator T in (4.1).

4.1 Sparse Operators

These operators have become very popular due to their role in the so called A_2 conjecture consisting in proving that if T is a Calderón–Zygmund operator then

$$||Tf||_{L^2(v)} \lesssim ||v||_{A_2} ||f||_{L^2(v)}.$$

This result was first obtained by Hytönen [29] and then simplified by Lerner [33,34], who proved that the norm of a Calderón–Zygmund operator in a Banach function space $\mathbb{X}$ is dominated by the supremum of the norm in $\mathbb{X}$ of all the possible sparse operators and then proved that every sparse operator is bounded on $L^2(v)$ for every weight $v \in A_2$ with sharp constant. Let us give the precise definition. A general dyadic grid $\mathcal{D}$ is a collection of cubes in $\mathbb{R}^n$ satisfying the following properties:

(i) For any cube $Q \in \mathcal{D}$, its side length is 2^k for some $k \in \mathbb{Z}$.
(ii) Every two cubes in $\mathcal{D}$ are either disjoint or one is wholly contained in the other.
(iii) Given $x \in \mathbb{R}^n$, for every $k \in \mathbb{Z}$ there is only one cube in $\mathcal{D}$ of side length 2^k containing it.

Let $0 < \eta < 1$, a collection of cubes $\mathcal{S} \subset \mathcal{D}$ is called η-sparse if one can choose pairwise disjoint measurable sets $E_Q \subset Q$ with $|E_Q| \geq \eta |Q|$, where $Q \in \mathcal{S}$.

Definition 4.2 Given a sparse family of cubes $\mathcal{S} \subset \mathcal{D}$, the sparse operator is defined by

$$A_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y)\, dy \right) \chi_Q(x).$$

Proposition 4.3 [30] *For every $v \in A_1$, it holds that*

$$||A_{\mathcal{S}} f||_{L^{1,\infty}(v)} \lesssim ||v||_{A_1}(1+\log ||v||_{A_1})||f||_{L^1(v)}.$$

In [35], this same bound was proved for a Calderón–Zygmund operator and in [36] the authors have proved that it is sharp. Hence, using the domination property of the sparse operators, we conclude that the bound in the above proposition is also sharp. We thank A.K. Lerner for this information.

Therefore, as a consequence of Theorem 1.2, we get the following result.

Corollary 4.4 *Let $0 < p < \infty$ and $w \in B_p^{\mathcal{R}} \cap B_\infty^*$. Then,*

$$A_{\mathcal{S}} : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w)$$

is bounded with constant controlled by

$$C_1 ||w||_{B_p^{\mathcal{R}}} ||w||_{B_\infty^*} \left(1+\log\left(C_2 ||w||_{B_\infty^*}\right)\right),$$

with C_1, C_2 constants independent of w.

4.2 The Bochner–Riesz Operator

Let

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n,$$

be the Fourier transform of an integrable function $f \in L^1(\mathbb{R}^n)$ and let $a_+ = \max\{a, 0\}$ denote the positive part of $a \in \mathbb{R}$. Given $\lambda > 0$, the Bochner–Riesz operator B_λ on $\mathbb{R}^n$ is defined by

$$\widehat{B_\lambda f}(\xi) = \left(1-|\xi|^2\right)_+^\lambda \hat{f}(\xi).$$

They were first introduced by Bochner in [8] and, since then, they have been widely studied. The case $\lambda = 0$ corresponds to the so-called ball multiplier, which is known to be unbounded on $L^p(\mathbb{R}^n)$ if $n \geq 2$ and $p \neq 2$ ([27]). It is known that when $\lambda > \frac{n-1}{2}$, $B_\lambda f$ is controlled by the Hardy–Littlewood maximal function Mf, and hence B_λ satisfies the same weighted estimates as M. We will focus on the value $\lambda = \frac{n-1}{2}$, which is called the critical index. In this case, Christ [22] showed that $B_{\frac{n-1}{2}}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, and although we do not have the control of $B_{\frac{n-1}{2}}$ by M, Shi and Sun proved in [43] that $B_{\frac{n-1}{2}}$ is bounded on $L^p(v)$ for every $v \in A_p$ when $1 < p < \infty$. The corresponding weak-type inequality for $p = 1$ was obtained by Vargas in [45], where she proved that $B_{\frac{n-1}{2}}$ is bounded from $L^1(v)$ to $L^{1,\infty}(v)$ for every $v \in A_1$.

Proposition 4.5 [37] *For every* $n > 1$,

$$B_{\frac{n-1}{2}} : L^1(v) \to L^{1,\infty}(v)$$

is bounded with constant less than or equal to $||v||_{A_1}^2 \log_2(C||v||_{A_1} + 1)$.

Therefore, as a consequence of Theorem 1.2, we get the following result.

Corollary 4.6 *For every* $n > 1$, $0 < p < \infty$ *and* $w \in B_p^{\mathcal{R}} \cap B_\infty^*$,

$$B_{\frac{n-1}{2}} : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w)$$

is bounded with constant controlled by

$$C_1||w||_{B_p^{\mathcal{R}}}||w||_{B_\infty^*}^2(1 + \log(C_2||w||_{B_\infty^*})),$$

with C_1, C_2 *constants independent of* w.

4.3 Radial Multipliers

Given a multiplier m, we say that T_m is a radial Fourier multiplier if $\widehat{T_m f}(\xi) = m(|\xi|)\hat{f}(\xi)$.

Definition 4.7 Given $0 \le \delta < 1$ and $r > 0$, we define the truncated fractional integral of order $1 - \delta$ of a locally integrable function f on $\mathbb{R}$ by

$$I_r^{1-\delta} f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\delta)} \int_{-r}^{r} (s-t)_+^{-\delta} f(s)\, ds, & t < r, \\ 0, & t \ge r, \end{cases} \qquad t \in \mathbb{R},$$

with Γ being the Gamma function. Moreover, if $\alpha = [\alpha] + \delta > 0$, with $[\alpha]$ being its integer part and δ its fractional part, we define the fractional derivative of f of order α by

$$D^\alpha f(t) = -\left(\frac{d}{dt}\right)^{[\alpha]} \lim_{r\to\infty} \frac{d}{dt} I_r^{1-\delta} f(t),$$

whenever the right-hand side exists.

Let AC_{loc} be the space of functions which are absolutely continuous on every compact subset of $(0, \infty)$.

Proposition 4.8 ([13]) *Fix* $n \ge 2$ *and* $\alpha = \frac{n+1}{2}$. *Let* m *be a bounded, continuous function on* $(0, \infty)$ *which vanishes at infinity and satisfies that*

$$D^{\alpha - j} m \in AC_{loc} \qquad \forall j = 1, \dots, [\alpha].$$

Then, if $D^\alpha m$ *exists and*

$$\Phi(t) = t^{\alpha-1} D^\alpha m(t) \in L^1(0,\infty), \tag{4.2}$$

the operator T_m *defined by*

$$\widehat{T_m f}(\xi) = m(|\xi|^2)\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n,$$

satisfies that

$$T_m : L^1(v) \to L^{1,\infty}(v)$$

is bounded for every $v \in A_1$ *with constant controlled by* $C||\Phi||_{L^1(0,\infty)}||u||_{A_1}^5$.

Therefore, as a consequence of Theorem 1.2, we get the following result.

Corollary 4.9 *Let* $0 < p < \infty$ *and* $w \in B_p^{\mathcal{R}} \cap B_\infty^*$. *Then if* m *satisfies the hypothesis of Proposition 4.8 we have that*

$$T_m : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w)$$

is bounded with constant controlled by

$$C||\Phi||_{L^1(0,\infty)}||w||_{B_p^{\mathcal{R}}}||w||_{B_\infty^*}^5,$$

with C *independent of* w.

4.4 Fourier Multipliers of Hörmander Type

Let us use the standard notation $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and if $x \in \mathbb{R}^n$,

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Then, the Hörmander condition for a multiplier m is the following.

Definition 4.10 Let $k \in \mathbb{N}$ such that $k > n/2$ and let $m : \mathbb{R}^n \to \mathbb{R}$ be a bounded function of $\mathcal{C}^k$ class on $\mathbb{R}^n \setminus \{0\}$. Given $1 < s \le 2$, we say that m satisfies the Hörmander condition respect s and k, and denote it by $m \in HC(s,k)$, if

$$\sup_{r>0} \left(r^{2|\alpha|-n} \int_{r<|x|<2r} \left| \left(\frac{\partial}{\partial x}\right)^\alpha m(x) \right|^s dx \right)^{1/s} < \infty, \quad |\alpha| \le k.$$

The Fourier multipliers operators of Hörmander type are those defined by

$$\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$

where $m \in HC(s, k)$ for $k > n/2$ and $1 < s \leq 2$. The classical Hörmander theorem (see for example [28, Theorem 5.2.7]) says that when $s = 2$, in the unweighted case, T_m is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and satisfies the weak-type inequality

$$T_m : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$$

whenever $m \in HC(2, k)$. The generalization of the condition to $1 < s < 2$ was introduced by Calderón and Torchinsky in [12], where the authors see that in order to the classical result to be true for $m \in HC(s, k)$ is needed that $k > n/s$.

Proposition 4.11 [32, Theorem 1] *Let $1 < s \leq 2$ and $m \in HC(s, n)$. Then, for every $v \in A_1$,*

$$T_m : L^1(v) \to L^{1,\infty}(v)$$

is bounded with constant less than or equal to $\varphi(||v||_{A_1})$ with φ being an increasing function in $(0, \infty)$.

Therefore, as a consequence of Theorem 1.2, we get the following result.

Corollary 4.12 *Let $0 < p < \infty$ and $w \in B_p^{\mathcal{R}} \cap B_\infty^*$. Then*

$$T_m : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w)$$

is bounded with constant controlled by

$$C_1||w||_{B_p^{\mathcal{R}}}\, \varphi(C_2||w||_{B_\infty^*}),$$

with C_1 and C_2 independent of w.

At this point we have to say that although the property of φ being increasing is known, the sharp expression for such φ is unknown.

4.5 Rough Singular Integrals

Definition 4.13 Let $\Sigma^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ and given $\Omega \in L^\infty(\Sigma^{n-1})$ such that

$$\int_{\Sigma^{n-1}} \Omega(x)\, dx = 0.$$

The rough singular integral is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x - y)\, dy,$$

where $y' = \dfrac{y}{|y|}$. In [37], the authors obtained the following result:

Proposition 4.14 *For every $v \in A_1$,*

$$T_\Omega : L^1(v) \to L^{1,\infty}(v)$$

is bounded with constant controlled by $||v||_{A_1}^2 \log_2(||v||_{A_1} + 1)$.

Therefore, as a consequence of Theorem 1.2, we get the following result.

Corollary 4.15 *Let $0 < p < \infty$ and $\Omega \in L^\infty(\Sigma^{n-1})$ such that $\int_{\Sigma^{n-1}} \Omega = 0$. If $w \in B_p^{\mathcal{R}} \cap B_\infty^*$, then*

$$T_\Omega : \Lambda^{p,1}(w) \to \Lambda^{p,\infty}(w)$$

is bounded with constant controlled by

$$C_1 ||w||_{B_p^{\mathcal{R}}} ||w||_{B_\infty^*}^2 \left(1 + \log\left(C_2 ||w||_{B_\infty^*}\right)\right),$$

with C_1, C_2 constants independent of w.

4.6 Assani Operator

There is a very interesting operator which satisfies the hypothesis of Theorem 1.3 but it is not of weak type (1, 1) and hence we cannot apply Theorem 1.2. This operator is related with the Return time theorem of Bourgain [9], and we refer to [5] for a very interesting review on the topic and also to [6], where the following related operator was introduced:

$$Af(x) = \left\| \frac{f(\cdot)\chi_{(0,x)}(\cdot)}{x - \cdot} \right\|_{L^{1,\infty}(0,1)}.$$

Since one can easily check that $A\chi_E \leq M\chi_E$, we have that A satisfies (1.4) and consequently, we can deduce that, for every $1 < p < \infty$ and $v \in A_p^{\mathcal{R}}$,

$$||Af||_{L^{p,\infty}(v)} \lesssim \frac{1}{p-1} ||v||_{A_p^{\mathcal{R}}} ||f||_{L^{p,1}(v)}.$$

Therefore, as a consequence of Theorem 1.3, we get the following result.

Corollary 4.16 *Let $0 < p < \infty$, $0 < q < 1$ and $w \in B_p^{\mathcal{R}} \cap B_\infty^*$, then*

$$A : \Lambda^{p,q}(w) \to \Lambda^{p,\infty}(w)$$

is bounded with constant less than or equal to

$$\frac{C}{(1-q)^2} ||w||_{B_p^{\mathcal{R}}} ||w||_{B_\infty^*}^q,$$

with C independent of w.

At this point, we have to say that, in the case $w = 1$, the boundedness of A in $L^{1,q}$ was obtained in [21] and, as a consequence, it was proved that the space $L^{1,q}$ satisfies the Return Time Property for the Tail, while this is not the case for L^1 (see [6]).

On the other hand, it is an interesting open question in the area whether the space $L \log\log\log L$ satisfies this property. Since

$$L \log\log\log L = \Lambda^1 \left(1 + \log\left(1 + \log\left(1 + \log\frac{1}{t}\right)\right)\right),$$

it will be very interesting to study, for which weights w, the Assani operator A is bounded on $\Lambda^1(w)$.

Acknowledgements The authors would like to thank the anonymous referees who provided useful, detailed and insightful comments on an earlier version of the manuscript.

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