Harmonic Analysis Invariants for Infinite Graphs Via Operators and Algorithms

Sergey Bezuglyi[1] · Palle E. T. Jorgensen[1]

Abstract

We present recent advances in harmonic analysis on infinite graphs. Our approach combines combinatorial tools with new results from the theory of unbounded Hermitian operators in Hilbert space, geometry, boundary constructions, and spectral invariants. We focus on particular classes of infinite graphs, including such weighted graphs which arise in electrical network models, as well as new diagrammatic graph representations. We further stress some direct parallels between our present analysis on infinite graphs, on the one hand, and, on the other, specific areas of potential theory, probability, harmonic functions, and boundary theory. The limit constructions, finite to infinite, and local to global, can be used in various applications.

1 Introduction

Our presentation here is based in part on new joint research dealing with a geometric and computational harmonic analysis on infinite graphs and related network-models. This entails multiple recent research collaborations. Our emphasis is on both new and recent results from collaborations between the present co-authors, as well as joint

✉ Palle E. T. Jorgensen
palle-jorgensen@uiowa.edu

Sergey Bezuglyi
sergii-bezuglyi@uiowa.edu

1   Department of Mathematics, University of Iowa, Iowa City, IA, USA

Birkhäuser

results by the second named author and Erin Pearse; see especially [7,8,10,13,30,31, 33–35,39]. In the present paper, we aim to address readers and researchers from diverse neighboring areas, as well as from applied areas. We have further added a discussion of outlook and of new perspectives.

In rough outline, we consider here the following general framework for combinatorial graphs G. A graph G will be specified by two countable infinite sets, vertices V, edges E, with edges "connecting" neighboring pairs of vertices. While there is also a rich literature for finite graphs, our setting will be that of countable infinite graphs. The motivation derives in part from new problems dealing with "large networks". We have been especially motivated by applications to electrical network models. But the framework is much more general than that, encompassing for example social networks. In addition to a choice of a pair of vertices and edges, (V, E), we shall also introduce a specific positive and symmetric function, say c, defined on the set E of edges. In electrical network systems, the function c will represent conductance. For our graph-systems, we shall use the terminology (V, E, c).

Analysis on infinite graphs has diverse applications in both pure and applied mathematics, see e.g. [5]. This includes numerical analysis, signal/image processing, as well as Monte Carlo simulations in physics and in financial mathematics (pricing of derivative securities). Some important ideas of classical analysis on $\mathbb{R}^n$ and compact and non-compact manifolds which already have numerous applications to signal and image processing were extended to finite and infinite graphs in [24,49,50]. Indeed, our present framework includes cases of graph networks arising as discrete samples within ambient "continuous" networks, i.e., the case sets of vertex points arising from optimal sampling. Other notions of optimal sampling also arise in probabilistic frameworks. The reader can find other applications in [23,48,55].

Our analysis will be presented in this framework, and we shall further discuss several parallels between analysis on combinatorial graphs, on the one hand, and more classical notions from harmonic analysis, on the other: In our analysis, the classical Laplacian has a counterpart for graphs, the graph-Laplacian, Δ. It will play an important role in the results we present below. Every choice of conductance function c induces a choice of graph-Laplacian. Each choice of graph-Laplacian also entails a new and important study of harmonic functions on graphs.

Our motivations include: boundary-conditions and boundary value problems, Markov random walk models, and compactifications, for the classical Laplacian in PDE theory, e.g., Greens–Gauss–Stokes, we show has a counterpart for graphs. There are two sides to this, one is to make precise the notions of "boundary" for combinatorial graphs; and the second entails a precise study of, and identification of, graph-Laplacians as unbounded semibounded operators in suitable choices of Hilbert spaces.

The list of examples of the use of infinite graphs in numerical analysis of *partial differential equations* (PDEs) problems includes what goes by the name multigrid methods, finite element methods, and finite difference methods. We refer to popular tools in *numerical analysis* involving groups of *algorithms* for solving classical PDEs with the use of graph-hierarchies of discretizations. Each of these techniques, applied to a specific partial differential operator, leads in turn to an associated discretized operator (for example a graph Laplacian), with the discretization referring to a choice of an

infinite weighted graph. These techniques are also called *multiresolution* methods, and they are very useful in problems exhibiting *multiple scales* of behavior. Since this is a large and diverse area, we will not explore details here. Interested readers will find extensive detailed treatments in, for example, [41].

In more detail, starting with a specific graph network (V, E, c), this may be anyone in a variety of network models, there is then a naturally corresponding *Laplace operator* Δ. We outline how this operator reflects both geometric and analytic features of the network under consideration.

Of course, the corresponding spectral theory and harmonic analysis of the *Laplace operator* Δ will depend on choice of Hilbert space associated with the graph network (V, E, c). In fact, there are three useful choices of Hilbert spaces, each one serving its own purpose: We consider (i) the standard ℓ^2 Hilbert spaces, $\ell^2(V)$ and $\ell^2(V, c)$, (ii) the finite energy Hilbert space $\mathcal{H}_E$, and (iii) a certain path-space, the dissipation Hilbert space $\mathcal{H}_D$. (We refer readers to Sect. 8.2 below as well as to the paper [33] for details.) As outlined above, both the operator Δ, and the three Hilbert spaces, depend on, and reflect, the particular graph network at hand $G = (V, E, c)$. So our analysis will therefore also depend on the specification of three parts of G: (a) the set of vertices V, (b) the set of edges E for G, and (c) a choice of *conductance function c* (as it is defined as a function on E).

Our present proposed graph-harmonic analysis will therefore also depend on such associated spectral theory for the operator Δ. We shall outline how Δ may be realized as an unbounded symmetric operator in each of the three Hilbert spaces. As a result, there will be choices, reflecting the particular graph network at hand. This in turn entails several technical issues from the theory of unbounded operators with dense domain in Hilbert space, and entailing choices of domains and generalized boundary conditions. In a way, the theory for the first Hilbert space $\ell^2(V)$ is easier as it turns out that Δ will automatically be essentially self-adjoint on its natural domain in $\ell^2(V)$. By contrast, as an operator in the energy Hilbert space $\mathcal{H}_E$, Δ will generally not be essentially self-adjoint. Nonetheless, we will show that it still has a natural family of self-adjoint extensions; each one with its own spectral theory, and each of which yields a graph-harmonic analysis.

It is the realization of Δ in the *energy Hilbert space* $\mathcal{H}_E$ which best reflects both metric (resistance metric) and analytic data: Harmonic analysis of boundary conditions, random walk properties, and useful correspondences between (i) geometric features of G on the one hand, and (ii) spectra data on the other.

A particular and versatile choice of graph networks is considered in Sect. 6, the *Bratteli diagrams*, denoted by $G(B)$ where $B = B(V, E)$ stands for a Bratteli diagram. As we outline, the choice of this setting for network graph-analysis is dictated by a variety of applications, e.g., to reversible random walk models, to general classes of dynamical systems in symbolic dynamics, to combinatorics, and to commutative and non-commutative harmonic analysis; see the papers cited below in Sect. 6. A further advantage of the specialization to Bratteli diagrams is that it allows for algorithmic computations, for example, it yields (semi) explicit formulas for computation of harmonic functions on this class of graph systems, $G(B)$.

1.1 Organization of the Paper

In Sect. 2, we define the main notions and tools we shall need for our graph analysis; this includes the theory of weighted (electrical, resistance) networks, resistance metrics, random walk analysis, boundaries, Green kernel, Dirichlet spaces, and graph Laplacians. Involved in our analysis of graph Laplacians will be three Hilbert spaces $\ell^2(V)$, where V is the set of vertices, an energy Hilbert space $\mathcal{H}_E$, and a dissipation Hilbert space $\mathcal{H}_D$.

Starting with a given infinite graph, and a prescribed conductance function, in Sect. 3, we introduce a more detailed potential theoretic analysis; it is based on an identification of systems of dipoles and monopoles. a graph-theoretic Gauss–Green Identity.

In Sect. 4, starting with a graph, and an associated graph-Laplacian, we show how the three Hilbert spaces ($\ell^2(V)$, the energy Hilbert space, and the dissipation Hilbert space) enter into our spectral theoretic determination of the graph Laplacian.

In Sect. 5 we use a probabilistic approach for a detailed study of the Green's function, dipoles, and monopoles for a transient weighted network.

In Sects. 6–8, we discuss a special class of infinite graphs, Bratteli diagrams. We include the precise definitions and references on Bratteli diagrams in Sect. 6. In rough outline, the key feature which characterizes the Bratteli diagrams, dictates a particular configuration of the corresponding sets of vertices V and edges E as follows. In a Bratteli diagram B, the set of vertices V (and the set of edges E) admits an arrangement into a disjoint partition consisting of levels V_n (respectively, E_n), and indexed by discrete time. The further property characterizing Bratteli diagrams B is that the edges only link vertices from neighboring levels.

The results from the previous sections are then applied, in Sects. 6, 7, and 8 to the case of graphs of particular forms. The focus of Sects. 6 and 7 is an algorithmic approach to finding particular harmonic functions, for this class of countably graded graphs.

The ideas and methods, which are discussed in the first part of the paper, are applied to several combinatorial graphs. In Sects. 7 and 8, we considered trees, the Pascal graph, and some special realizations of Bratteli diagrams.

Section 8 is focused on the existence of harmonic functions of finite energy for some classes of weighted networks.

2 Basic Concepts on Networks and Related Hilbert Spaces

In this section, we define the main notions of the theory of weighted (electrical, resistance) networks. We would like to mention that the material of this section is not a comprehensive introduction to the theory of weighted networks. Our goal is to prepare the reader for the main results in the present paper, as well as in related papers. We do not mention here several key concepts of this theory such as resistance metrics, boundaries, and physical interpretations of the considered objects.

The reader can find more information on this subject, for example, in [3,4,16,20, 26,30,33,42,43,51], and many other papers and books.

2.1 Electrical Networks, Laplace and Markov Operators

Let G be a countably infinite graph. We assume in this paper that all graphs are connected, undirected, and locally finite. Moreover, G has single edges between connected vertices. Denote by V the set of all vertices of G, and by E the edge set of G. The set E has no loops. If two vertices x and y from V are connected neighbors, we write $y \sim x$. It allows us to identify such pairs of vertices with edges $e = (xy) \in E$. Local finiteness of G means that the set $\{y \in V : y \sim x\}$ of all neighbors of x is finite for every vertex x. For any two vertices $x, y \in V$, there exists a finite path $\gamma = (x_0, x_1, ..., x_n)$ such that $x_0 = x$, $x_n = y$ and $(x_i x_{i+1}) \in E$ for all i. On the other hand, we can define the set X_G of all infinite paths.

Definition 2.1 An *electrical network* (or *weighted network*)[1] (G, c) is, by definition, a graph G equipped with a *symmetric function* $c : V \times V \to [0, \infty)$, i.e., $c_{xy} = c_{yx}$ for any $(x, y) \in V \times V$. Moreover, it is required that $c_{xy} > 0$ if and only if $(xy) \in E$. This means that c is actually defined on the edge set E of G. The function c will be called a *conductance function*. The reciprocal value $r_{xy} = 1/c_{xy}$ is called the *resistance* of the edge $e = (xy)$. For any $x \in V$, we define the *total conductance* at x as

$$c(x) := \sum_{y \sim x} c_{xy}.$$

The function c is defined for every $x \in V$ since this sum is always finite. If necessary, one can extend the definition of c to the set $V \times V$ setting $c_{xy} = 0$ if x, y are not neighbors in G.

Definition 2.2 Let (G, c) be an electric network. (1) The *Laplacian* (or *graph-Laplace operator*) Δ on (G, c) is the linear operator defined on the space of functions[2] $f : V \to \mathbb{R}$ by the formula

$$(\Delta f)(x) := \sum_{y \sim x} c_{xy}(f(x) - f(y)). \tag{2.1}$$

A function $f : V \to \mathbb{R}$ is called *harmonic* on (G, c) if $\Delta f(x) = 0$ for every $x \in V$. If (2.1) holds at each vertex x of a subset $W \subset V$, then we say that f is harmonic on W.

(2) A *Markov* operator P is defined on the space of functions $f : V \to \mathbb{R}$ by the relation

$$(Pf)(x) = \sum_{y \sim x} p(x, y) f(y), \tag{2.2}$$

---

[1] The terms electrical network, weighted network, electrical resistance network are used as synonyms in many papers and books on this subject.

[2] One can consider complex-valued functions in this (and other) definition; obvious changes can be easily made.

where $\{p(x, y) : x, y \in V\}$ is a transition probability kernel, i.e., $\sum_{y \sim x} p(x, y) = 1$ for all $x \in V$. A function f is called *harmonic* if $P(f) = f$. If $P = (p(x, y))$ where $p(x, y) = \dfrac{c_{xy}}{c(x)}$, then a function f is harmonic if and only if $\Delta(f) = 0$.

(3) The space of all harmonic functions will be denoted by $\mathcal{H}arm$. Relations (2.1) and (2.2) should be viewed as pointwise ones. Usually the operators Δ and P and harmonic functions are considered in some Hilbert spaces, see below.

(4) Let W be a subset of $V = V(G)$. The solutions $\varphi : V \to \mathbb{R}$ of the problem

$$\Delta\varphi(x) = 0 \ \text{ on } \ V \setminus W \tag{2.3}$$

are called harmonic on exterior domain functions.

It is important to find solutions of the equation $\Delta\varphi = f$ for a given function f of simplest form. If $W = \{x_0\}$, then the solution $\overline{w} = \overline{w}_{x_0}$ to the problem $\Delta w(x) = \delta_{x_0}$ is called a *monopole* at the point x_0. If $W = \{x_1, x_2\}, x_1 \neq x_2$, then the solution $\overline{v} = \overline{v}_{x_1, x_2}$ to $\Delta v(x) = \delta_{x_1} - \delta_{x_2}$ is called a *dipole*. In Sect. 3 we give an equivalent definition of monopoles and dipoles, see Proposition 3.6. They are also extensions to the present graph context of Green's functions in PDE theory.

Naturally, an analysis of infinite weighted graphs entails subtle choices on infinite dimensional function spaces. And, in infinite dimensions, existence of solutions is delicate. In fact, this subtlety is at the core of our understanding of a systematic harmonic analysis for infinite weighted graphs. Indeed, this is a central theme, both in the present discussion, in earlier papers by Jorgensen and Pearse [30,31,33–35], as well as in earlier papers by the present co-authors, see e.g., [8,10]. While inclusion of full details here (of these matters) would take us too far afield, we wish to stress the following four key features, important for our discussion inside the paper: (i) Following parallels to PDE theory, readers will notice that the role played here by monopoles and dipoles is similar to that of Green's functions for elliptic boundary value problems; so "inverting" elliptic PDEs. (ii) Moreover (for infinite graphs), a systematic study of monopoles/dipoles reflects a precise graph theoretic analysis of aspects of "boundaries". Here, boundaries understood in the sense of Markov processes. Hence, to make precise our notions of boundaries, we show that our present graph Laplacians are directly related to Markov operators, referring to associated reversible Markov processes and their boundaries. With these facts in mind, we make the choice of the Hilbert space $\mathcal{H}_E$ (energy Hilbert space), see also [10,13,35], and we show that $\mathcal{H}_E$ will always contain the dipole-vectors; in fact, the span of the system of dipoles is dense in $\mathcal{H}_E$. By contrast, except for trivial special cases, the span of the Dirac point masses is not dense in $\mathcal{H}_E$. The latter fact highlights the role played by the harmonic functions contained in $\mathcal{H}_E$. (iii) For some problems, $\mathcal{H}_E$ will contain monopoles, but not in general. (iv) There are alternative Hilbert spaces relevant for an analysis and spectral theory of graph Laplacians, for example $l^2(V)$, but they typically do not accommodate the solutions discussed above.

In the following remark we collect several properties of the objects defined above.

Remark 2.3 (1) The space $\mathcal{H}arm$ of harmonic function always contains constant functions. We will call them trivial functions. If only trivial harmonic functions exist, then $\mathcal{H}arm$ is called trivial.

(2) Given a weighted network (G, c), one can define a transition probability kernel by setting $p(x, y) := \dfrac{c_{xy}}{c(x)}$. In this case, the kernel $(p(x, y) : x, y \in V)$ is *reversible*, i.e., the relation $c(x)p(x, y) = c(y)p(y, x)$ holds because of the symmetry of the function c_{xy}.

(3) It follows from (2.1) and (2.2) that

$$\Delta(f) = c(f - P(f))$$

where c is considered as a multiplication operator.

(4) The domain of operators Δ and P will be discussed later when we consider these operators acting in specific Hilbert spaces.

Harmonic functions give an important tool for the study of networks (G, c). In this connection, we discuss the *maximum principle* and *Dirchlet problem* here.

Let G_1 be a connected subgraph of G with the vertex set $V_1 \subset V$. Define the *outer boundary* of V_1:

$$outbd(V_1) := \{x \in V \setminus V_1 : \exists y \in V_1 \text{ such that } x \sim y\}, \tag{2.4}$$

and the *inner boundary* of V_1

$$inbd(V_1) := \{x \in V_1 : \exists y \in V \setminus V_1 \text{ such that } x \sim y\}. \tag{2.5}$$

Following these definitions, one can introduce the interior $Int(V_1) := V_1 \setminus inbd(V_1)$ and closure $cl(V_1) := V_1 \cup outbd(V_1)$.

Suppose that $h : V \to \mathbb{R}$ is a function that is harmonic on V_1, and the supremum of h is achieved at a point from V_1. Then the maximum principle states that h is a constant on $V_1 \cup outbd(V_1)$.

Remark 2.4 For infinite graphs (V, E, c) there is a rich variety of corresponding analyses, e.g., harmonic analysis, and path-space analysis. For each one in a list of settings of analysis on infinite graphs, there is a corresponding variety of important notions of boundaries. For details, see the cited references. Each one is covered by an extensive literature, and each one serves different purposes. The simplest of these boundary notions refers to boundary of a finite subset, say W of V (there one must distinguish between an inner and an outer variant). More important cases are Cantor space boundaries at "infinity", or infinite-dimensional path-space. Both notions are important in the analysis of the special case of infinite graphs which may be realized as Bratteli diagrams (see below Sects. 6–8). A detailed discussion of the extensive literature in all the diverse settings is far beyond the scope of the present paper, but interested readers are referred to the sources cited above. For the specific analyses covered here we shall be specific, and we will cite the literature, e.g., [2,4,5,8,17–20,28,30–32,34,35,45,56].

The second fact is related to the *Dirichlet problem*. Let (G, c) and V_1 be as above. The Dirichlet problem consists of solving the following boundary problem:

$$\begin{cases} (\Delta u)(x) = g(x) & \text{for all } x \in V_1, \\ u(x) = f(x) & \text{for all } x \in bd(V_1), \end{cases} \tag{2.6}$$

where $u : V \to \mathbb{R}$ is an unknown function, and the functions $g : V_1 \to \mathbb{R}$ and $f : bd(V_1) \to \mathbb{R}$ are given.

Lemma 2.5 *If V_1 is finite, the Dirichlet problem* (2.6) *has a unique solution for all functions g and f. In particular, in the case of finite V_1, there exists a unique harmonic function h on V_1 such that $h = f$ on $bd(V_1)$.*

The reader can find more information about the Dirichlet problem and maximum principle in [42,43].

2.2 Three Hilbert Spaces Associated with Electrical Networks

Given an electrical network $(G, c) = (V, E, c)$ with a fixed conductance function c, the following three Hilbert spaces will be considered in the paper: $\ell^2(V)$, $\ell^2(V, c)$, and the finite energy space $\mathcal{H}_E$. Recall that we work with real-valued functions, so that we set

$$\ell^2(V) := \text{all functions } u \text{ on } V \text{ such that } \|u\|_{\ell^2}^2 = \sum_{x \in V} (u(x))^2 < \infty,$$

and the inner product $\langle u_1, u_2 \rangle_{\ell^2(V)}$ is $\sum_{x \in V} u_1(x) u_2(x)$. Similarly,

$$\ell^2(V, c) := \text{all functions } u \text{ on } V \text{ such that } \|u\|_{\ell^2(V,c)}^2 = \sum_{x \in V} c(x)(u(x))^2 < \infty,$$

and

$$\langle u_1, u_2 \rangle_{\ell^2(V,c)} = \sum_{x \in V} c(x) u_1(x) u_2(x).$$

One of our main objects is the *finite energy* Hilbert space $\mathcal{H}_E$. The definition of $\mathcal{H}_E$ (and its properties) requires more details.

Definition 2.6 For an electrical network (G, c), consider a quadratic form $\mathcal{E}$ defined on functions $u, v : V \to \mathbb{R}$ by

$$\mathcal{E}(u, v) := \frac{1}{2} \sum_{x, y \in V} c_{xy}(u(x) - u(y))(v(x) - v(y)).$$

Set $\mathcal{E}(u) = \mathcal{E}(u, u)$. Then $\mathcal{E}$ is called an *energy form*. The domain of $\mathcal{E}$ is the set of all functions u such that $\mathcal{E}(u) < \infty$. Furthermore, $\mathcal{E}(u) = 0$ if and only if u is

a constant function. This observation leads to the definition of the pre-Hilbert vector space $\mathcal{H}_E := Dom(\mathcal{E})/\mathbb{R}\mathbb{1}$ equipped with the inner product

$$\langle u_1, u_2 \rangle_{\mathcal{H}_E} = \mathcal{E}(u_1, u_2), \quad ||u||_{\mathcal{H}_E} = \mathcal{E}(u)^{1/2}.$$

The completion of $\mathcal{H}_E$ with respect this norm is called the *finite energy Hilbert space*. We use the same notation $\mathcal{H}_E$ for it.

Lemma 2.7 *Denote by δ_x, $x \in V$, the Kronecker delta-function on V: $\delta_x(y) = \delta_{xy}$. Then $\delta_x \in \mathcal{H}_E$, and*

$$\mathcal{E}(\delta_x, \delta_y) = -c_{xy}, \quad \mathcal{E}(\delta_x) = c(x),$$

and $\mathcal{E}(\delta_x, \delta_y) = 0$ if $(xy) \notin E$.

In Remark 4.6, we give the countably infinite matrix M, see (4.2), which defines the operator Δ acting in $\ell^2(V)$. The entries of M are $\mathcal{E}(\delta_x, \delta_y) : x, y \in V$.

In the following remark we formulate several facts to explain why the finite energy Hilbert space plays an important role in the study of graph-Laplacian.

Remark 2.8 In this remark, we gathered several properties of the finite energy space.

(1) In many models of electrical networks harmonic functions do not belong to ℓ^2 space. On the other hand there are interesting networks with wide collection of harmonic functions of finite energy.

(2) The Hilbert space $\mathcal{H}_E$ does not have a canonical orthonormal basis. It follows from Lemma 2.7 that functions δ_x are in $\mathcal{H}_E$ for $x \in V$, but they are not orthogonal and their span is not dense in $\mathcal{H}_E$, in general. Indeed, as we will see in Proposition 3.6, the finite energy space $\mathcal{H}_E$ admits an orthogonal decomposition $Fin \oplus \mathcal{H}arm_0$ where Fin is the span of $\{\delta_x : x \in V\}$ and $\mathcal{H}arm_0$ is the set of harmonic functions of finite energy. For details, readers are referred to the cited literature.

(3) The multiplication operator $(M_f u)(x) = f(x)u(x)$ is not Hermitian if $f \neq 0$.

(4) The reader should distinguish pointwise identities and identities in the Hilbert space $\mathcal{H}_E$. Because the vectors of $\mathcal{H}_E$ are equivalence classes of functions, there are pointwise identities that do not hold in $\mathcal{H}_E$.

For a network (G, c), the Hilbert space $\mathcal{H}_E$ is defined on classes of equivalence of functions on the vertex set V. It turns out that this space can be embedded into another Hilbert space defined on the set of functions on the edge set E of G. It is called the *dissipation Hilbert space* and denoted by $\mathcal{H}_D$.

The *dissipation space* $\mathcal{H}_D$, whose vectors can be viewed as current functions. We note that vectors from the energy space $\mathcal{H}_E$ represent voltage difference. To define $\mathcal{H}_D$, we denote by $r_{xy} = c_{xy}^{-1}$ the resistance of the edge $e = (xy)$ and set

$$\mathcal{H}_D := \{f : E \to \mathbb{R} : f(x, y) = -f(y, x) \text{ and } ||f||_{\mathcal{H}_D} < \infty\}$$

where

$$||f||_{\mathcal{H}_D}^2 = \frac{1}{2} \sum_{x,y} r_{xy} f(x, y)^2 = \frac{1}{2} \sum_{e \in E} r_e f(e)^2.$$

Equivalently, $\mathcal{H}_D$ can be defined by using an orientation on G. Let $\vec{e} = (\vec{xy})$ denote an edge with orientation. Then

$$||f||_{\mathcal{H}_D}^2 = \sum_{\vec{e} \in E} r_{\vec{e}} f(\vec{e})^2.$$

Moreover, the vectors $\{\sqrt{c_e} \delta_{\vec{e}}\}$ form an orthonormal basis in $\mathcal{H}_D$.

To embed the finite energy space into the dissipation space, we use the drop operator $\partial : \mathcal{H}_E \to \mathcal{H}_D$ where

$$(\partial u)(x, y) = c_{xy}(u(x) - u(y)).$$

Lemma 2.9 *The operator ∂ is an isometry, $||u||_{\mathcal{H}_E} = ||\partial u||_{\mathcal{H}_D}$.*

Let C be a *cycle*, i.e., C is a finite closed path in (G, c).

Lemma 2.10 *For every $f \in \mathcal{H}_E$,*

$$\langle \partial f, \chi_C \rangle_{\mathcal{H}_D} = 0,$$

where χ_C is the characteristic function of a cycle C.

The proof of this lemma follows from the following computation: let $C = \{x_0, \ldots, x_n, x_0\}$ where $(x_i x_{i+1}) \in E$, then

$$\langle \partial f, \chi_C \rangle_{\mathcal{H}_D} = \sum_e c(e) r(e) f(e) \chi_C(e) = \sum_i (f(x_i) - f(x_{i+1})) = 0.$$

It follows from Lemma 2.10 that the orthogonal compliment $Cyc := \mathcal{H}_D \ominus \partial(\mathcal{H}_E)$ is generated by the characteristic functions of cycles in (G, c).

More details about the dissipation space $\mathcal{H}_D$ are given in Sect. 4. The reader can find more information about the dissipation space in [44,46].

2.3 Path Space

For a network (G, c), we defined the Markov operator P in Definition 2.2. We can use it to define a probability measure on the path space of the graph G.

Let $\Omega \subset V^\infty$ be the set of all infinite sequences $\omega = (x_i)_{i \geq 0}$ where $(x_i x_{i+1}) \in E$ for all i. We call Ω the *path space*. Define random variables $X_n : \Omega \to V$ by setting $X_n(\omega) = x_n$. Let $\Omega_x := \{\omega \in \Omega : X_0 = x\}$ be the subset of all paths beginning at x; then Ω is the disjoint union of $\Omega_x, x \in V$.

The transition probability kernel $P = (p(x, y) : x, y \in V)$, where $p(x, y) = \dfrac{c_{xy}}{c(x)}$ defines the family of *Markov measures* $\{\mathbb{P}_x : x \in V\}$ such that $\mathbb{P}_x$ is supported by the corresponding set Ω_x. Indeed, $\mathbb{P}_x$ is determined on cylinder sets of Ω_x by the formula

$$\mathbb{P}_x(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n \mid X_0 = x) = p(x, x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n),$$

The sequence of random variables (X_n) defines a *Markov chain* on $(\Omega_x, \mathbb{P}_x)$:

$$\mathbb{P}_x(X_{n+1} = y \mid X_n = z) = p(z, y), \quad y, z \in V.$$

Since G is a connected graph, the Markov chain defined by (X_n) is *irreducible*, that is, for any $x, y \in V$ there exists $n \in \mathbb{N}$ such that $p^{(n)}(x, y) > 0$, where $p^{(n)}(x, y)$ is the xy-entry of P^n.

Let $\lambda = (\lambda_x : x \in V)$ be a positive probability vector. Define the measure $\mathbb{P} = \sum_{x \in V} \lambda_x \mathbb{P}_x$ on Ω.

Lemma 2.11 *The measure $\mathbb{P}$ is a Markov measure if and only if the probability distribution λ satisfies the relation $\lambda P = \lambda$, or $\sum_{y \sim x} \lambda_y p(y, x) = \lambda_x$. Then*

$$\mathbb{P}(X_0 = x, X_1 = x_1, \ldots, X_n = x_n) = \lambda_x p(x, x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n).$$

In particular, $p^{(n)}(x, y) = \mathbb{P}_x(X_n = y \mid X_0 = x)$.

We remark that the equation $\lambda P = \lambda$ may not have solutions in the set of positive probability vectors λ.

A Markov kernel $P = (p(x, y))_{x, y \in V}$ determines a *random walk* on the weighted graph (G, c).

Definition 2.12 It is said that the random walk on $G = (V, E)$ defined by the transition matrix P is *recurrent* if, for any vertex $x \in V$, it returns to x infinitely often with probability one. Otherwise, it is called *transient*. Equivalently, the random walk is recurrent if and only if, for all $x, y \in V$,

$$\mathbb{P}_x(X_n = y \text{ for infinitely many } n) = 1, \tag{2.7}$$

and it is transient if and only if, for every finite set $F \subset V$ and for all $x \in V$,

$$\mathbb{P}_x(X_n \in F \text{ for infinitely many } n) = 0. \tag{2.8}$$

We say that an electrical network (G, c) is *recurrent/transient* if the random walk (X_n) defined on the vertices of G by the transition probability matrix P is recurrent/transient.

The reader will find more information and results about transient and recurrent random walks in [19,27,29,40].

Remark 2.13 (1) Let $G = (V, E, c)$ be an infinite transient weighted network, and the sequence of transition probabilities $p^{(n)}(x, y)$ is defined as in Lemma 2.11. Define the *Green kernel*:

$$\mathcal{G}(x, y) = \sum_{n=0}^{\infty} p^{(n)}(x, y).$$

Then, for any fixed $y \in V$, the function $\mathcal{G}(x, y)$ satisfies $\Delta \mathcal{G}(x, y) = \delta_y$ pointwise in V, and has finite energy.

(2) As shown in [57], a network supports monopoles if and only if the Green kernel $\mathcal{G}(x, y)$ exists (see Sect. 4 for the definition), which is equivalent to transience of the random walk. It was proved in [45] that the network is transient if and only if there exists a finite current flow to infinity.

(3) For a connected network G, the property of transience/recurrence of a random walk is independent of a point where is is started. Hence, a monopole exists at some vertex z if and only if there exists a monopole at any other vertex x.

Various notions of boundary for random walk models are discussed, e.g. in the papers [14,17,18].

3 Harmonic Functions of Finite Energy, Monopoles, Dipoles, and Laplacian

As sketched above, our present setting is that of infinite and connected graphs $G = (V, E, c)$, i.e., we specify an infinite set V of vertices (countable discrete), as well as a fixed and associated system of edges E. In the case of electrical networks (our main focus), we will also specify a positive function c on E which may represent a prescribed conductance in the network. Starting with a fixed conduction function c, we show that there are then two operators, one a graph-Laplacian, Δ and the other a transition operator P. Both induce key structures as graph theoretic harmonic analysis tools. Our present focus is that of harmonic functions, boundaries of graphs, and associated transforms. However, the precise formulation of these analysis tools require that we first specify appropriate Hilbert spaces. There will be three, $\ell^2(V)$, our energy Hilbert space $\mathcal{H}_E$, and our dissipation Hilbert space, $\mathcal{H}_D$. Each one serves its own purpose, for example $\ell^2(V)$ helps us make precise infinite matrix representations. We note that $\ell^2(V)$ is contained in $\mathcal{H}_E$, but the inclusion turns out to be an unbounded operator, in general. The range is non-closed in $\mathcal{H}_E$, and not dense. However, with our assumptions, we find that non-constant harmonic functions cannot be in $\ell^2(V)$. Hence, we shall study the of harmonic functions as a subspace of $\mathcal{H}_E$.

The third Hilbert space $\mathcal{H}_D$, will be a Hilbert space of functions on the edge set E. We shall need this, and the transition operator P, in order to make precise the notions of "boundary" which are part of our graph harmonic analysis.

The existing literature in the area is relatively recent. Here we shall follow mainly the papers [1,7,8,10–13,30,31,33–37,39]. For an analysis of reversible random processes, see [40,43,45]. For realization of boundaries as fractals, see e.g., [42]. General background papers and books on graph analysis include [29,51,54,56].

3.1 Harmonic Functions in the Energy Space

The following result gives formulas for computation of energy for harmonic functions on graph networks.

Proposition 3.1 (i) *Let* $f \in Harm \cap \mathcal{H}_E$ *on* (G, c). *Then*

$$\|f\|_{\mathcal{H}_E}^2 = \frac{1}{2} \sum_{x \in V} c(x)((Pf^2)(x) - f^2(x)), \qquad (3.1)$$

and

$$\|f\|_{\mathcal{H}_E}^2 = -\frac{1}{2} \sum_{x \in V} (\Delta f^2)(x). \qquad (3.2)$$

(ii) *If a given function* f *on* V *is harmonic off a finite set* $F \subset V$, *then it has finite energy if and only if the sums in* (3.1) *and* (3.2) *are finite.*

Hence, a harmonic function f on (G, c) has finite energy if and only if the function $x \mapsto P(f^2)(x) - f^2(x)$ belongs to $\ell^1(V, c)$.

Let Ω, Ω_x, $\mathbb{P}_x$, and X_n be as in Sect. 2.3. It was proved in [2] that, for a transient electrical network (G, c) and a given function $f \in \mathcal{H}_E$, the sequence $(f \circ X_n)$ converges a.e. and in L^2 on the space $(\Omega_x, \mathbb{P}_x)$ for any x. We set

$$\widetilde{F}(\omega) = \lim_{n \to \infty} (f \circ X_n(\omega)). \qquad (3.3)$$

Then the function $\widetilde{F}(\omega)$ is defined a.e.

Let $\sigma : \Omega \to \Omega$; $\sigma(\omega_0, \omega_1, \ldots) = (\omega_1, \omega_2, \ldots)$ be the shift. Then σ is a *finite-to-one endomorphism* of Ω such that

$$\sigma(\Omega_x) = \bigcup_{y \sim x} \Omega_y$$

(recall that G is locally finite). For every $x \in V$ and $y \sim x$, there exists an inverse branch $\tau_{xy} : \Omega_y \to \Omega_x$ of σ:

$$\tau_{xy}(y, y_1, y_2, \ldots) = (x, y, y_1, y_2, \ldots).$$

Remark that τ_{xy} is a one-to-one map such that $\tau_{xy}(\Omega_y) \subset \Omega_x$, $\sigma \circ \tau_{xy} = \mathrm{id}|_{\Omega_y}$, and

$$\Omega_x = \bigcup_{y \sim x} \tau_{xy}(\Omega_y).$$

One can show that

$$d\mathbb{P}_x(\omega) = \sum_{y \sim x} p(x, y) \, d(\mathbb{P}_y \circ \tau_{xy}^{-1})(\omega) = \sum_{y \sim x} p(x, y) \, d\mathbb{P}_y(\sigma(\omega)). \qquad (3.4)$$

Applying (3.4), we obtain the following characterization of harmonic functions.

Theorem 3.2 (i) *Suppose that a function $\widetilde{f}$ on Ω satisfies the condition $\widetilde{f} \in L^1(\Omega_x, \mathbb{P}_x)$ for every $x \in V$. Then*

$$f(x) := \int_{\Omega_x} \widetilde{f}(\omega) d\mathbb{P}_x(\omega)$$

is harmonic on (G, c) if and only if

$$\int_{\Omega_x} \widetilde{f}(\omega) \, d\mathbb{P}_x(\omega) = \int_{\Omega_x} \widetilde{f}(\sigma(\omega)) \, d\mathbb{P}_x(\omega).$$

(ii) *Suppose that a function f, defined on the vertex set of (G, c), belongs to $\mathcal{H}_E$, and let $\widetilde{F}$ be defined by* (3.3). *Then the function $f = \int_{\Omega_x} \widetilde{F} d\mathbb{P}_x$ is harmonic if and only if $\widetilde{F} = \widetilde{F} \circ \sigma$.*

3.2 More Results on Dipoles and Monopoles

Let (G, c) be a weighted connected graph as defined in Sect. 2. Let x, y be arbitrary distinct vertices of an electrical network (G, c). Define the linear functional $L = L_{xy} : \mathcal{H}_E \to \mathbb{R}$ by setting $L(u) = u(x) - u(y)$. It can be shown using connectedness of G that $|L(u)| \leq k\|u\|_{\mathcal{H}_E}$ where k is a constant depending on x and y. By the Riesz theorem, there exists a unique element $v_{xy} \in \mathcal{H}_E$ such that

$$\langle v_{xy}, u \rangle_{\mathcal{H}_E} = u(x) - u(y) \tag{3.5}$$

(we say that differences are *reproduced* by $v_{xy} \in \mathcal{H}_E$). This element v_{xy} is called a *dipole*.[3] If o is a fixed vertex from V, we will use the notation v_x instead of v_{xo}. Since for any u, $\langle v_{xy}, u \rangle_{\mathcal{H}_E} = \langle v_x, u \rangle_{\mathcal{H}_E} - \langle v_y, u \rangle_{\mathcal{H}_E}$, we see that $v_{xy} = v_x - v_y$, and it suffices to study function v_x, $x \in V$, only. We note that for any network (G, c) a dipole v_x is always in $\mathcal{H}_E$, and moreover, the set $\{v_x : x \in V\}$ is dense in $\mathcal{H}_E$:

$$\overline{\text{span}\{v_x : x \in V\}} = \overline{\text{span}\{v_{xy} : x, y \in V\}} = \mathcal{H}_E,$$

where the closure is taken in $\mathcal{H}_E$-norm. Indeed, if a function $f \in \mathcal{H}_E$ is orthogonal to $\text{span}\{v_{xy} : x, y \in V\}$, then

$$f(x) - f(y) = \langle v_{xy}, f \rangle_{\mathcal{H}_E} = 0,$$

i.e., $f(x)$ is pointwise constant, and therefore $f = 0$ in $\mathcal{H}_E$.

The uniqueness of the dipole v_{xy} in $\mathcal{H}_E$ allows one to define the *resistance distance* in V (see, e.g. [31]):

---

[3] It follows from Proposition 3.6 that the definition of a dipole given in Definition 2.2 agrees with that mentioned in (3.5)

Lemma 3.3 *For any $x, y \in V$, let*

$$\mathrm{dist}(x, y) = ||v_{xy}||^2_{\mathcal{H}_E}.$$

Then $\mathrm{dist}(x, y)$ *is a metric on V which is called the resistance distance.*

Definition 3.4 A function $w = w_x \in \mathcal{H}_E$ is called a *monopole* at $x \in V$ if it satisfies the equation

$$\langle w_x, u \rangle_{\mathcal{H}_E} = u(x) \tag{3.6}$$

for any $u \in \mathcal{H}_E$.

In contrast to case of dipoles, there are networks (G, c) that do not have monopoles in $\mathcal{H}_E$. In general, the following classical result holds.

In Proposition 3.6(4), we show that this definition of a monopole is equivalent to that given above in Definition 2.2.

Lemma 3.5 *An electrical network (G, c) is transient if and only if there exists a monopole in $\mathcal{H}_E$.*

In this connection we refer to the classical paper [47] on networks where it is proved that transience is equivalent to the existence of a flow to infinity of finite energy. We also refer to [54, Theorem 2.12], where this and other relevant results are discussed.

The roles and properties of dipoles and monopoles can be seen from the following statement involving these functions and the Laplacian.

Proposition 3.6 (1) *Let (G, c) be a weighted graph; choose and fix a vertex $o \in V$. Let $v_x \in \mathcal{H}_E$ be a dipole corresponding to a vertex $x \in V$. Then*

$$\Delta v_x = \delta_x - \delta_o. \tag{3.7}$$

More generally, the dipole v_{xy} satisfies the equation $\Delta v_{xy} = \delta_x - \delta_y$. The set $\mathrm{span}\{v_x\}$ is dense in $\mathcal{H}_E$.

(2) *Let u be a function from $\mathcal{H}_E$. Then, for any $x \in V$,*

$$(\Delta u)(x) = \langle u, \delta_x \rangle_{\mathcal{H}_E}. \tag{3.8}$$

(3) *For any $x \in V$, the function δ_x is in $\mathcal{H}_E$, and*

$$c(x)v_x - \sum_{y \sim x} c_{xy} v_y = \delta_x.$$

(4) *If w_x is a monopole corresponding to $x \in V$, then $\Delta w_x = \delta_x$. Moreover, $v_{xy} = w_x - w_y$, $x, y \in V$; thus if a monopole w_{x_0} exists as an element of $\mathcal{H}_E$ for some x_0, then w_x exists in $\mathcal{H}_E$ for every vertex x.*

(5) *Functions $\delta_x(\cdot)$, $x \in V$, belong to $\mathrm{span}\{v_x : x \in V\}$.*

(6) *Let $\mathcal{F}in$ denote the closure of* $\mathrm{span}\{\delta_x\}$ *with respect to the norm* $\|\cdot\|_{\mathcal{H}_E}$ *and* $\mathcal{H}arm_0 = \mathcal{H}arm \cap \mathcal{H}_E$. *Then*

$$\mathcal{H}_E = \mathcal{F}in \oplus \mathcal{H}arm_0, \tag{3.9}$$

where orthoganality is considered with respect to $\mathcal{H}_E$*-inner product. Relation* (3.9) *is called the Royden decomposition.*

Proof The proof of these and more results can be found in [31,33]. We show here (1) and (5) only.

To see that (3.7) holds, we take any function f from $\mathcal{H}_E$ and compute

$$\begin{aligned}\langle f, \Delta v_{xy} - (\delta_x - \delta_y)\rangle_{\mathcal{H}_E} &= \langle f, \Delta v_{xy}\rangle_{\mathcal{H}_E} - \langle f, \delta_x - \delta_y\rangle_{\mathcal{H}_E}\\ &= \Delta f(x) - \Delta f(y) - (\Delta f(x) - \Delta f(y)) = 0\end{aligned}$$

in view of (3.8).

For (5), one can check that

$$\delta_x = c(x)v_x - \sum_{y \sim x} c_{xy} v_y.$$

$\square$

Remark 3.7 (1) We observe that, in the space of functions u on V, the solution set of the equation $(\Delta u)(z) = (\delta_x - \delta_y)(z)$ is, in general, infinite because the function $u + h$ satisfies the same equation for any $h \in \mathcal{H}arm$. The meaning of Proposition 3.6 (1) is the fact that the dipole v_{xy} from (3.5) is a unique solution of this equation if it is considered as an element of the space $\mathcal{H}_E$.

(2) It is worth noting that we will use the same terms, monopoles and dipoles, for functions w_x and v_x on V that satisfy the relations $\Delta w_x = \delta_x$ and $\Delta v_x = \delta_x - \delta_o$, respectively.

(3) Since functions from the energy space $\mathcal{H}_E$ are defined up to a constant, we can assume, without loss of generality, that all functions lying in $\mathcal{H}_E$ take value zero at a fixed vertex $o \in V$.

(4) The family of functions $\{v_x : x \in V\}$ defines a reproducing kernel for $\mathcal{H}_E$ in virtue of the equality $\langle v_x, f\rangle_{\mathcal{H}_E} = f(x) - f(o)$.

(5) The dissipation space $\mathcal{H}_D$ admits the orthogonal decomposition

$$\mathcal{H}_D = \partial(\mathcal{F}in) \oplus \partial(\mathcal{H}arm) \oplus Cyc.$$

Corollary 3.8 *If P is the orthogonal projection from $\mathcal{H}_D$ onto $\partial(\mathcal{H}_E)$, then the adjoint of the drop operator $\partial^* : \mathcal{H}_D \to \mathcal{H}_E$ is*

$$(\partial^* f)(x) - (\partial^* f)(y) = \frac{1}{c_{xy}} P f(x, y).$$

Corollary 3.9 *Let $x_0 \in V$ be a fixed vertex. Then w_{x_0} is a monopole if and only if it is a finite energy harmonic function on $V \setminus \{x_0\}$.*

It is not hard to see that the notions of monopoles and dipoles can be extended to more general classes of functions.

Proposition 3.10 *Let $F = \{x_0, \ldots, x_N\}$ be a finite subset of V with $N + 1$ distinct vertices. Let α_i be positive numbers such that $\sum_{i=1}^N \alpha_i = 1$. Then there exists a unique solution $v = v_{F,\alpha} \in \mathcal{H}_E$ such that*

$$\langle v, f \rangle_{\mathcal{H}_E} = f(x_0) - \sum_{i=1}^N \alpha_i f(x_i) \tag{3.10}$$

hold for all $f \in \mathcal{H}_E$. Moreover, the solution v to (3.10) satisfies

$$\Delta v = \delta_{x_0} - \sum_{i=1}^N \alpha_i \delta_{x_i}.$$

Proof The argument is based on the Riesz' theorem applied to the Hilbert space $\mathcal{H}_E$, and is analogous the proof of existence of dipoles in $\mathcal{H}_E$. Assuming v satisfies (3.10), we verify that

$$w := \Delta v - \left(\delta_{x_0} - \sum_{i=1}^N \alpha_i \delta_{x_i} \right)$$

satisfies $\langle w, v_{oy} \rangle_{\mathcal{H}_E}$ for all $y \in V \setminus \{o\}$, where $\{v_{oy}\}$ is the system of dipoles. $\qquad \square$

3.3 Gauss–Green Identity

This subsection deals with boundary value problem on infinite graphs. We will follow the paper [33]. We begin this section with a fact about Laplacians acting on finite networks. If the network (G, c) is defined on a *finite graph*, then

$$\langle u, v \rangle_{\mathcal{H}_E} = \sum_{x \in V} u(x)(\Delta v)(x), \tag{3.11}$$

and all harmonic functions of finite energy are constant ($\mathcal{H}_E$ is trivial). Relation (3.11) fails if G is infinite. There is a version of (3.11) which includes a boundary term, see [33].

Let H be a subgraph of G. To simplify notation, the outer boundary of H is called boundary and denoted by $bd(H)$. For vertices in the boundary of a subgraph, we have the notion of the normal derivative of a function:

$$\frac{\partial v}{\partial n}(x) = \sum_{y \in H} c_{xy}(v(x) - v(y)), \quad x \in bd(H).$$

Suppose (G_k) is a sequence of finite connected subgraphs of G such that $G_k \subset G_{k+1}$ and $G = \bigcup_k G_k$. We call (G_k) an *exhaustion*. For the vertices V_k of G_k, the notation

$$\sum_{x \in V} := \lim_{k \to \infty} \sum_{x \in V_k}$$

is used whenever the limit in independent of the choice of exhaustion (G_k).

A *boundary sum* is computed in terms of an exhaustion by

$$\sum_{x \in bd(G)} := \lim_{k \to \infty} \sum_{x \in bd(G_k)},$$

whenever the limit in independent of the choice of exhaustion (G_k).

The following result is a discrete analogue of Gauss-Green formula. We use here the fact that dipoles and monopoles form a dense subset in $\mathcal{H}_E$, see Sect. 3.2.

Theorem 3.11 [33] *Let (G_k) be an exhaustion. If $u \in \mathcal{H}_E$ and v is in the dense subspace of $\mathcal{H}_E$ spanned by dipoles and monopoles, then*

$$\langle u, v \rangle_{\mathcal{H}_E} = \sum_{x \in V} u(x)(\Delta v)(x) + \sum_{x \in bd(G)} u(x) \frac{\partial v}{\partial n}(x). \qquad (3.12)$$

If a representative u is chosen such that $u(o) \neq 0$, then

$$\langle u, v \rangle_{\mathcal{H}_E} = \sum_{x \in V}(u(x) - u(o))(\Delta v)(x) + \sum_{x \in bd(G)}(u(x) - u(o))\frac{\partial v}{\partial n}(x).$$

The following corollary describes a boundary representation of harmonic functions. We use the notation $h_x = P_{Harm} v_x$ where P_{Harm} is the orthogonal projection from $\mathcal{H}_E$ onto the space of harmonic functions.

Corollary 3.12 (Boundary representation of harmonic functions, [31]) *For any function $f \in Harm$,*

$$f(x) = \sum_{bd(V)} u \frac{\partial h_x}{\partial n} + f(0).$$

It is said that the boundary term is *nonvanishing* if (3.12) holds with nonzero boundary term.

Theorem 3.13 *The network (G, c) is transient if and only if the boundary term is nonvanishing.*

The detailed justification of these results and properties can be found in the papers [30,31,33,39]. Interested readers are referred to these papers. A complete discussion here would take us too far afield.

4 Spectral Properties of Laplacians in ℓ^2 and $\mathcal{H}_E$

4.1 Unbounded Operators

For the reader's convenience, we recall main notions and facts about unbounded linear operators acting in a Hilbert space.

An unbounded densely defined operator $L : H \to H$ on a Hilbert space H is called *symmetric* if, for all $f, g \in \mathrm{dom}(L)$,

$$\langle Lf, g \rangle_H = \langle f, Lg \rangle_H, \tag{4.1}$$

that is $L \subset L^*$.

A symmetric operator L is called *semibounded* if [4] $\langle Lf, f \rangle_H \geq 0$, $f \in \mathrm{dom}(L)$.

A symmetric operator L is always *closable*. A symmetric operator L is *essentially self-adjoint* if the closure of L is self-adjoint. In other words, L has a unique self-adjoint extension.

The following fact is a useful criterion: A symmetric semibounded operator L is self-adjoint if and only if the only solution of the equation $L^* g = -g$ is trivial, $g = 0$.

Suppose H_1 and H_2 are two Hilbert spaces and A, B are operators with dense domains such that

$$A : \mathrm{dom}(A) \subset H_1 \longrightarrow H_2, \quad B : \mathrm{dom}(B) \subset H_2 \longrightarrow H_1.$$

Then (A, B) is called a *symmetric pair* if

$$\langle Af, g \rangle_{H_2} = \langle f, Bg \rangle_{H_1}, \quad f \in \mathrm{dom}(A), g \in \mathrm{dom}(B).$$

It is known that if (A, B) is a symmetric pair, then A and B are closable operators and

(i) $A^* \overline{A}$ is densely defined and self-adjoint with $\mathrm{dom}(A^* \overline{A}) \subset H_1$
(ii) $B^* \overline{B}$ is densely defined and self-adjoint with $\mathrm{dom}(B^* \overline{B}) \subset H_2$.

4.2 The Laplacians in $\ell^2(V)$ and $\mathcal{H}_E$

Let (G, c) be an electrical network and Δ the corresponding Laplacian, see (2.1). We collect in this section the results describing the spectral properties of the Laplacian which is considered as a linear operator acting in Hilbert spaces ℓ^2 and $\mathcal{H}_E$.

Starting with a fixed weighted infinite graph $G = (V, E, c)$, a natural question is if, or when, the associated graph Laplacian Δ might be a bounded operator (i.e., continuous with respect to the Hilbert norms on $l^2(V)$ or $\mathcal{H}_E$). While there is an answer (it is expressed in the language of quadratic forms), the case of bounded Δ is much too restrictive. More precisely, the infinite weighted graphs for which the corresponding Δ is a bounded operator miss most interesting applications of a systematic harmonic analysis of such infinite weighted graphs. For details, see e.g., [37–39].

---

[4] More generally, L is semibounded if for some number c one has $\langle Lf, f \rangle_H \geq c \langle f, f \rangle_H$.

Nonetheless, it is possible to show that, when the Laplacian Δ is realized as a densely defined symmetric operator in $l^2(V)$, and in $\mathcal{H}_E$, then the following three facts hold: (i) Δ will be lower semibounded in both of its incarnations. (ii) Δ has an upper bound relative to $l^2(V)$ if and only if it does with respect to $\mathcal{H}_E$; and (iii) if there is an upper bound, then it will be the same for the two Hilbert spaces, $l^2(V)$ and $\mathcal{H}_E$.

Remark 4.1 Consider the three Hilbert spaces: $\ell^2(V)$, $\mathcal{H}_E$, and $\mathcal{H}_D$. The set of functions spanned by $\delta_x, x \in V$, is dense in $\ell^2(V)$. On the other hand, every function δ_x defines an element of the energy space $\mathcal{H}_E$. Hence, we have a natural the embedding $T : \ell^2(V) \to \mathcal{H}_E : T(\delta_x) = \delta_x$ is defined. The operator T is, in general, unbounded.

Moreover, the drop operator ∂ determines an isometric embedding of $\mathcal{H}_E$ to $\mathcal{H}_D$, see Lemma 2.9:

$$\ell^2(V) \xrightarrow{T} \mathcal{H}_E \xrightarrow{\partial} \mathcal{H}_D.$$

Lemma 4.2 *Let the operator T be as in Remark 4.1, and define $S : \operatorname{span}\{v_x : x \in V\} \to \ell^2(V)$ by $S(v_x) = \delta_x - \delta_o$. Then (S, T) is a symmetric pair of operators.*

It suffices to check that

$$\langle T\delta_x, v_y \rangle_{\mathcal{H}_E} = \langle \delta_x, Sv_y \rangle_{\ell^2(V)}.$$

We will use the notation Δ_2 for the Laplacian defined in ℓ^2. The space $\ell^2(V)$ is equipped with the inner product

$$\langle u, v \rangle_{\ell^2(V)} := \sum_{x \in V} u(x)v(x).$$

The notation $\Delta_{\mathcal{H}_E}$ is used for the Laplacian acting in $\mathcal{H}_E$.

Definition 4.3 We define the operator Δ_2 in $\ell^2(V)$ as the graph closure of the operator Δ which is considered as on $\operatorname{span}\{\delta_x : x \in V\}$, the subspace of finite linear combinations.

The closed operator $\Delta_{\mathcal{H}_E}$ is obtained by taking the graph closure of Δ which is considered on the dense subset $\operatorname{span}(v_x : x \in V)$ where v_x is a dipole corresponding to x, see Proposition 3.6.

Theorem 4.4 *Let S and T be as in Lemma 4.2. Then $T^*\overline{T}$ is a self-adjoint extension of Δ_2 and $S^*\overline{S}$ is a self-adjoint extension of $\Delta_{\mathcal{H}_E}$.*

More details about the properties of the Laplacians Δ_2 and $\Delta_{\mathcal{H}_E}$ can be found in the following statement (see Sect. 4.1 for the definition of an Hermitian operator).

Theorem 4.5 (1) *The Laplacian $\Delta_{\mathcal{H}_E}$ is a well-defined non-negative closed and Hermitian operator acting in $\mathcal{H}_E$. The operator $\Delta_{\mathcal{H}_E}$ is, in general, unbounded and not self-adjoint.*

(2) *The operator $\Delta_2 : \ell^2(V) \to \ell^2(V)$ is essentially self-adjoint, generally unbounded with dense domain.*

(3) *The Markov operator P is self-adjoint and bounded in $\ell^2(V, c)$. Moreover, the spectrum of P is a subset of $[-1, 1]$, and $-I \leq P \leq I$ on $\ell^2(V, c)$ where I is the identity operator.*

Proof (sketch) To see that the Laplacians[5] are symmetric operators, we will check that (4.1) holds. For $v_x, v_y, x, y \in V \setminus \{o\}$, we have

$$\langle \Delta v_x, v_y \rangle_{\mathcal{H}_E} = \langle \delta_x - \delta_o, v_y \rangle_{\mathcal{H}_E} = (\Delta v_y)(x) - (\Delta v_y)(o) = \delta_{xy} + 1 = \langle v_x, \Delta v_y \rangle_{\mathcal{H}_E}.$$

The property of semi-boundness for the operator $\Delta_{\mathcal{H}_E}$ follows from the equality

$$\left\langle \sum_x a_x v_{xy}, \ \Delta_{\mathcal{H}_E}\left(\sum_x a_x v_x\right) \right\rangle_{\mathcal{H}_E} = \sum_x a_x^2 + \left(\sum_x a_x\right)^2.$$

The fact that P is a self-adjoint operator follows from the relations $c(y)p(y, x) = c(x)p(x, y)$, i.e., P is reversible.

Next, it follows, from the inequality

$$2\sum_{x \in V}\left(c(x)u(x)^2 - \langle u, Pu \rangle_{\ell^2(V,c)}\right) = \sum_{x, y \in V} c_{xy}(u(x) - u(y))^2 \geq 0,$$

that

$$\langle u, Pu \rangle_{\ell^2(V,c)} \leq \|u\|_{\ell^2(V,c)}^2.$$

Similarly, one can prove that $\langle u, Pu \rangle_{l^2(c)} \geq -\|u\|_{\ell^2(V,c)}^2.$ $\qquad\square$

Remark 4.6 (1) The operator Δ_2 admits a simple representation by an infinite matrix M whose entries are indexed by the set $V \times V$. Using $\{\delta_x : x \in V\}$ as a basis, we find that

$$M = \begin{pmatrix} \ddots & & \ddots & & 0 \\ & c(x) & \cdots & -c_{xy} & \\ \ddots & & \ddots & & \ddots \\ & -c_{xy} & \cdots & c(y) & \\ 0 & & \ddots & & \ddots \end{pmatrix}, \tag{4.2}$$

so that M is a countable banded matrix.

(2) It was shown in [36] that $\Delta_{\mathcal{H}_E}$ is a bounded operator in $\mathcal{H}_E$ if and only if $\ell^2(V)$ is boundedly contained in $\mathcal{H}_E$. Recall that, in general, the Laplacian $\Delta_{\mathcal{H}_E}$ is an unbounded operator.

---

[5] We deal with $\Delta_{\mathcal{H}_E}$ here; the operator Δ_2 is considered similarly.

Example 4.7 We mention here several examples with various properties of Laplacians and harmonic functions.

(1) Let the set of vertices of G be $\mathbb{N}_0$ and edges connect only neighboring vertices. For $c_{n,n+1} = n + 1$, one can compute $||\delta_n||^2_{\mathcal{H}_E} = 2n + 1$. This example shows that $\ell^2 \not\subseteq \mathcal{H}_E$.

(2) Let (G, c) be a network where $V = \mathbb{Z}$ and E is the set of edges connecting nearest neighbors. Suppose $c_{i,i+1} = 1$ for all i. Then

$$\Delta f(n) = 2f(n) - f(n-1) - f(n+1)$$

and

$$||u||^2_{\mathcal{H}_E} = \sum_{n \in \mathbb{Z}} (u(n) - u(n+1))^2.$$

For $n \geq 0$, the dipole v_n is defined by

$$v_n(i) = \begin{cases} i, & 0 \leq i \leq n \\ n, & i \geq n \\ 0 & i \leq 0 \end{cases} \tag{4.3}$$

and $v_n(i) = -v_{-n}(-i)$ for $n < 0$. Clearly, $v_n \in \mathcal{H}_E$ but $v_n \notin \ell^2(\mathbb{Z})$, hence $\ell^2(\mathbb{Z})$ is a proper subset of $\mathcal{H}_E$.

(3) The space *Harm* of harmonic functions on $(\mathbb{Z}, c)$ is nontrivial if and only if

$$\sum_{n,n+1} c^{-1}_{n,n+1} < \infty.$$

Moreover, in this case, the space *Harm* is 1-dimensional. To see this, set $u(0) = 0$, define $u(1) = c^{-1}_{0,1}$, and take

$$u(n) - u(n-1) = \frac{1}{c_{n-1,n}}.$$

Then u is a harmonic function of finite energy.

The boundary term for this function is $\sum_{bd(\mathbb{Z})} h \dfrac{\partial h}{\partial n} = 1$.

(4) We modify now the above example (2) and consider the network $(\mathbb{Z}, c)$ where $c_{i,i+1} = \lambda^{\max(|i|,|i+1|)}$, $i \in \mathbb{Z}$. For this example the following results can be deduced:

(i) The energy kernel (dipoles) $(v_n : n \in \mathbb{Z})$ is defined by

$$v_n(i) = \begin{cases} 0, & k \leq 0, \\ \dfrac{1 - r^{k+1}}{1 - r}, & 1 \geq k \geq n, n > 0, \\ \dfrac{1 - r^{n+1}}{1 - r} & k \geq n, \end{cases}$$

and similarly for $n \leq 0$. A similar result can be obtained for dipoles on a stationary Bratteli diagrams, see Sects. 6 and 7 for definitions.

(ii) The function $w_0(n) = \dfrac{r}{2(1-r)} r^{|n|}$ defines a monopole, and $h(n) = \mathrm{sgn}(n)(1 - w_0(n))$ is an element of $Harm$.

(iii) The Laplacian Δ in the energy space $\mathcal{H}_E$ is not essentially self-adjoint. One can explicitly construct a defect function g such that $\Delta g = -g$.

(5) Take now $V = \mathbb{N}_0$ and E as above. For a fixed $\lambda > 1$, set $c_{i,i+1} = \lambda^i$. Then

$$\Delta f(n) = (\lambda^n + \lambda^{n+1}) f(n) - \lambda^n f(n-1) - \lambda^{n+1} f(n+1)$$

and

$$\|u\|_{\mathcal{H}_E}^2 = \sum_{n \in \mathbb{Z}} \lambda^{n+1} (u(n) - u(n+1))^2.$$

One can see that, in this case, the spaces ℓ^2 and $\mathcal{H}_E$ are different.

5 Green's Function, Dipoles, and Monopoles for Transient Networks

We recall that the Hilbert space $\mathcal{H}_E$ always contains dipoles (see Sect. 5, Proposition 3.6). Here we will show how a dipole can be found in the space $\mathcal{H}_E$ by an explicit formula assuming that the electrical network (G, c) is transient. The results of this section are proved mostly in [8].

Given a weighted network $(G, c) = (V, E, c)$ we define the matrix $P = (p(x, y) : x, y \in V)$ of the transition probabilities where $p(x, y) = \dfrac{c_{xy}}{c(x)}$. Then P defines a random walk (X_n) on V such that $X_n(\omega) = x_n$ where the sequence $\omega = (x_0, \dots, x_n, \dots) \in \Omega$ (the path space Ω is defined in 2.3). For a fixed vertex $a \in V$, we consider the probability space $(\Omega_a, \mathbb{P}_a)$ where Ω_a consists of infinite paths that start at a, and $\mathbb{P}_a$ is the corresponding Markov measure on Ω_a, see, e.g. [54,56] for details.

Let F be a subset of V (we will be primarily interested in the case when $F = \{x_1, \dots, x_N\}$ is finite). For a probability space $(\Omega_a, \mathbb{P}_a)$, define the *stopping time*

$$\tau(F)(\omega) = \min\{n \geq 0 : X_n(\omega) \in F\}$$

with $\omega \in \Omega_a$. The *hitting time* is defined by

$$T(F) = \min\{n \geq 1 : X_n(\omega) \in F\}.$$

If $F = \{x\}$ is a singleton, then we write $\tau(x)$ and $T(x)$ for the stopping and hitting times, respectively.

Let $f^{(n)}(x, y) = \mathbb{P}_x[\tau(y) = n]$, $u^{(n)}(x, x) = \mathbb{P}_x[T(x) = n]$, and $p^{(n)}(x, y) = \mathbb{P}_x[X_n = y]$. Then the following quantities are crucial for the study of Markov chains:

$$\mathcal{G}(x, y) = \sum_{n \in \mathbb{N}_0} p^{(n)}(x, y), \quad F(x, y) = \sum_{n \in \mathbb{N}_0} f^{(n)}(x, y), \quad U(x, x) = \sum_{n \in \mathbb{N}} u^{(n)}(x, x)$$

The following properties of these functions are well known (see e.g. [43,54]).

Lemma 5.1 *Let $(G, c) = (V, E, c)$ be an electrical network. Then, for any pair of vertices $x, y \in V$,*

$$\mathcal{G}(x, x) = \frac{1}{1 - U(x, x)},$$
$$\mathcal{G}(x, y) = F(x, y)\mathcal{G}(y, y),$$
$$U(x, x) = \sum_{y \sim x} p(x, y)F(y, x),$$
$$F(x, y) = \sum_{z \sim x} p(x, z)F(z, y) = \Pr[x \to y], \quad (x \neq y).$$

Remark 5.2 It follows from the reversibility of the Markov chain (X_n) defined by transition probabilities $P = (p(x, y))$ that

$$c(x)F(x, y) = c(y)F(y, x) \quad \text{and} \quad c(x)\mathcal{G}(x, y) = c(y)\mathcal{G}(y, x).$$

Lemma 5.3 *(1) Let F be a subset of V and let $x \in F$ be any fixed vertex. We define*

$$h_x(a) := \mathbb{E}_a(\chi_{\{x\}} \circ X_{\tau(F)}).$$

Then

$$h_x(a) = F(a, x), \quad \forall a \in V.$$

(2) Given a subset $F = \{x_1, \ldots, x_N\}$ of vertices from V, we set

$$h_i(a) := \mathbb{E}_a(\chi_{\{x_i\}} \circ X_{\tau(F)}) = \int_{\Omega_a} \chi_{\{x_i\}}(X_{\tau(F)}(\omega))d\mathbb{P}_a(\omega), \quad i = 1, \ldots, N,$$

where a is an arbitrary vertex in V. Then

$$(\Delta h_i)(a) = 0, \quad a \in V \setminus F, \quad \text{and} \quad h_i(x_j) = \delta_{ij}.$$

Corollary 5.4 *Let $F = \{x_1, \ldots, x_N\}$ be a finite subset of V, and h_{x_i} is defined as in Lemma 5.3. Suppose that $\varphi : F \to \mathbb{R}$ is a given function on F. Define*

$$\Phi(a) := \sum_{i=1}^{N} \varphi(x_i)h_{x_i}(a).$$

Then Φ is a solution of the Dirichlet problem

$$\begin{cases} (\Delta\Phi)(a) = 0, & a \in V \setminus F, \\ \Phi(a) = \varphi(a), & a \in F. \end{cases}$$

Lemma 5.5 *Let h_x be defined as in Lemma 5.3. Then*

$$\|h_x\|_{\mathcal{H}_E}^2 < \frac{1}{2}c(x) \sum_{a \in V} \Pr[x \to a](1 - \Pr[a \to x]).$$

Proof We use the equalities $c_{ab} = c(a)p(a,b)$ and $\sum_b p(a,b)F(b,x) = F(a,x)$ (Lemma 5.1, and the inequality $F(b,x)^2 < F(b,x)$ in order to estimate the energy of h_x:

$$\|h_x\|_{\mathcal{H}_E}^2 = \frac{1}{2} \sum_{a,b} c_{ab}(h_x(a) - h_x(b))^2$$

$$= \frac{1}{2} \sum_{a,b} c_{ab}(\mathbb{E}_a(\chi_{\{x\}} \circ X_{\tau(x)}) - \mathbb{E}_b(\chi_{\{x\}} \circ X_{\tau(x)}))^2$$

$$= \frac{1}{2} \sum_{a,b} c_{ab}(F(a,x) - F(b,x))^2$$

$$= \frac{1}{2} \sum_a \left[c(a)F(a,x)^2 - 2F(a,x) \sum_{b \sim a} c_{ab}F(b,x) + \sum_{b \sim a} c_{ab}F(b,x)^2 \right]$$

$$= \frac{1}{2} \sum_a \left[-c(a)F(a,x)^2 + \sum_{b \sim a} c(a)p(a,b)F(b,x)^2 \right]$$

$$< \frac{1}{2} \sum_a \left[-c(a)F(a,x)^2 + c(a)F(a,x) \right]$$

$$= \frac{1}{2}c(x) \sum_{a \in V} \Pr[x \to a](1 - \Pr[a \to x]).$$

The last equality follows from Remark 5.2. □

For a transient network (G, c), one can see that the function h_x belongs to $\mathcal{H}_E$, see Theorem 5.7 below.

Our goal is to find formulas for dipoles v_{x_1,x_2} and monopoles w_x looking for solutions of equations $\Delta v_{x_1,x_2} = \delta_{x_1} - \delta_{x_2}$ and $\Delta w_x = \delta_x$ in the space $\mathcal{H}_E$.

Suppose that $F = \{x_1, x_2\}$, and $x_1 \neq x_2$. Let h_1 and h_2 be the function defined in Lemma 5.3. Consider the matrix

$$D := \begin{pmatrix} (\Delta h_1)(x_1) & (\Delta h_2)(x_1) \\ (\Delta h_1)(x_2) & (\Delta h_2)(x_2) \end{pmatrix} \tag{5.1}$$

and compute its entries using Lemma 5.3. To do this, we apply the relation $\Delta h = c(I - P)h$, which is valid for any function h on V.

Theorem 5.6 *The matrix M defined in* (5.1) *is represented as follows:*

$$D = \begin{pmatrix} c(x_1) & 0 \\ 0 & c(x_2) \end{pmatrix} \begin{pmatrix} 1 - U(x_1, x_1) & -F(x_1, x_2) \\ -F(x_2, x_1) & 1 - U(x_2, x_2) \end{pmatrix}. \tag{5.2}$$

Moreover,

$$\det M = \frac{c(x_1)c(x_2)(1 - \mathcal{G}(x_1, x_2)\mathcal{G}(x_2, x_1))}{\mathcal{G}(x_1, x_1)\mathcal{G}(x_2, x_2)} \tag{5.3}$$

and

$$\det M = 0 \iff \mathcal{G}(x_1, x_2) = \sqrt{\frac{c(x_2)}{c(x_1)}}.$$

The main results of this section are included in the following theorem.

Theorem 5.7 *Let (G, c) be a transient network, and x a fixed vertex in V. Let h_x be the function defined in Lemma 5.3, i.e.,*

$$h_x(a) := \mathbb{E}_a(\chi_{\{x\}} \circ X_{\tau(x)}), \quad a \in V.$$

Then the function $a \mapsto w_x(a) := \dfrac{\mathcal{G}(a, x)}{c(x)}$ is a monopole at x and satisfies the relation

$$w_x(a) = \frac{1}{c(x)(1 - U(x, x))} h_x(a), \quad a \in V. \tag{5.4}$$

In other words, $w_x \in \mathcal{H}_E$ and it satisfies the equation $\Delta w_x = \delta_x$.

Remark 5.8 (1) It follows from Theorem 5.7 that the monopole w_x has finite energy and $\|w_x\|_{\mathcal{H}_E} = \dfrac{\mathcal{G}(x, x)}{c(x)} \|h_x\|_{\mathcal{H}_E}$.

(2) Moreover, we deduce from relation (5.4) the following result: an electrical network (G, c) is transient if and only if the function $a \mapsto \mathcal{G}(a, x)$ has finite energy for every fixed $x \in V$.

Corollary 5.9 *Let x_1, x_2 be any distinct vertices in V. Set*

$$v_{x_1, x_2}(a) := (w_{x_1} - w_{x_2})(a) = \frac{\mathcal{G}(a, x_1)}{c(x_1)} - \frac{\mathcal{G}(a, x_2)}{c(x_2)}.$$

Then v_{x_1, x_2} is a dipole in $\mathcal{H}_E$.

We finish this section with another result regarding dipoles.

Theorem 5.10 *Let (G, c) be a transient electrical network, and let x_1, x_2 be any two distinct vertices in V such that the Green's function G (see Lemma 5.1) satisfies the relation*

$$\mathcal{G}(x_1, x_2) \neq \sqrt{\frac{c(x_2)}{c(x_1)}}.$$

Let D be the matrix defined by (5.1). Then the function

$$\overline{v}_{x_1, x_2}(a) = \alpha h_1(a) + \beta h_2(a), \quad a \in V,$$

is a dipole defined on V, where the coefficients α and β are determined as the solution to the equation

$$D \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{5.5}$$

Equivalently, we can write

$$\overline{v}_{x_1, x_2}(a) = [h_1(a), h_2(a)]D^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad a \in V.$$

Remark 5.11 It follows from Corollary 5.9 and Theorem 5.10 that the functions v_{x_1, x_2} and $\overline{v}_{x_1, x_2}$ satisfy the same equation. Therefore their difference $f = v_{x_1, x_2} - \overline{v}_{x_1, x_2}$ is a harmonic function which again is a linear combination of h_1 and h_2.

6 Combinatorial Diagrams and Electrical Networks

6.1 Fundamentals on Bratteli Diagrams

In the introduction, we described the notion of a Bratteli diagram considering it as an infinite graded graph. We give here the detailed definition and discuss the two principal cases: (i) finite number of vertices at each level, and (ii) countably many vertices at each level. In case (i), we deal with "classical" Bratteli diagrams, and case (ii) is considered as generalized Bratteli diagrams. The following definition is given for the second case, but it can be easily adopted to case (i). The notion of Bratteli diagrams is widely used in operator algebras, dimension groups, the theory of dynamical systems, and other adjoint areas. The reader can find more information in [7,9–12,22,28,32,52].

The results given below in Sect. 6 are taken mostly from [8,10].

Definition 6.1 Let V_0 be a countable vertex set (which may be identified with either $\mathbb{N}$ or $\mathbb{Z}$ when it is convenient). Set $V_i = V_0$ for all $i \geq 1$. A countable graded graph $B = (V, E)$ is called a *generalized Bratteli diagram* if it satisfies the following properties.

(i) The set of vertices V of B is a disjoint union of subsets V_i: $V = \bigsqcup_{i=0}^{\infty} V_i$.

(ii) The set of edges E of B is represented as $\bigsqcup_{i=0}^{\infty} E_i$ where E_i is the subset of edges connecting vertices of the levels V_i and V_{i+1}.

(iii) For every $w \in V_i, v \in V_{i+1}$, the set of edges $E(w, v)$ between the vertices w and v is finite (or empty); set $|E(w, v)| = f_{vw}^{(i)}$. It defines a sequence of infinite (countable-by-countable) *incidence matrices* $(F_n; n \in \mathbb{N}_0)$ whose entries are non-negative integers:

$$F_i = \left(f_{vw}^{(i)} : v \in V_{i+1}, w \in V_i \right), \quad f_{vw}^{(i)} \in \mathbb{N}_0.$$

(iv) The matrices F_i have *finitely many non-zero entries in each row*.

(v) The maps $r, s : E \to V$ are defined on the diagram B: for every $e \in E$, there are w, v such that $e \in E(w, v)$; then $s(e) = w$ and $r(e) = v$. They are called the *range* (r) and *source* (s) maps.

(vi) For every $w \in V_i$, $i \geq 0$, there exists an edge $e \in E_i$ such that $s(e) = w$, and for $i > 0$ there exists an edge $e' \in E_{i-1}$ such that $r(e') = w$. In other words, every incidence matrix F_i has no zero row and zero column.

Remark 6.2 (1) If V_0 is a single vertex, and each V_n is a finite set, then we obtain the standard definition of a Bratteli diagram originated in [9]. Later it was used in the theory of C^*-algebras and dynamical systems for solving some classification problems and the construction of models of transformations in ergodic theory, Cantor, and Borel dynamics (the references are given above and in Introduction 1).

(2) Property (iv) of Definition 6.1 allows to multiply matrices F_n. We emphasize that no restriction on the entries of columns of the incidence matrices F_n are assumed in the case when generalized Bratteli diagram is considered in the framework of dynamical systems. But interpreting $B = (V, E)$ as a network, we will usually assume that every vertex has finitely many neighbors.

(3) It follows from Definition 6.1 that every generalized Bratteli diagram is uniquely determined by a sequence of matrices (F_n) such that every matrix satisfies (iii) and (iv). When such a sequence is given, one can easily restore a Bratteli diagram with the prescribed incidence matrices. For this, one uses the rule that the entry $f_{vw}^{(n)}$ indicates the number of edges between the vertex $w \in V_n$ and vertex $v \in V_{n+1}$. It defines the set $E(w, v)$; then one takes

$$E_n = \bigcup_{w \in V_n, v \in V_{n+1}} E(w, v)$$

(4) A generalized Bratteli diagram is called *stationary* if $F_n = F$ for all $n \in \mathbb{N}_0$.

On Fig. 1, we give an example of a generalized Bratteli diagram. This example is a small (finite) part of a diagram since every Bratteli diagram has infinitely many levels and every level is a countably infinite set. We note that horizontal lines on Fig. 1 are not edges, they are used to show the levels of a diagram.
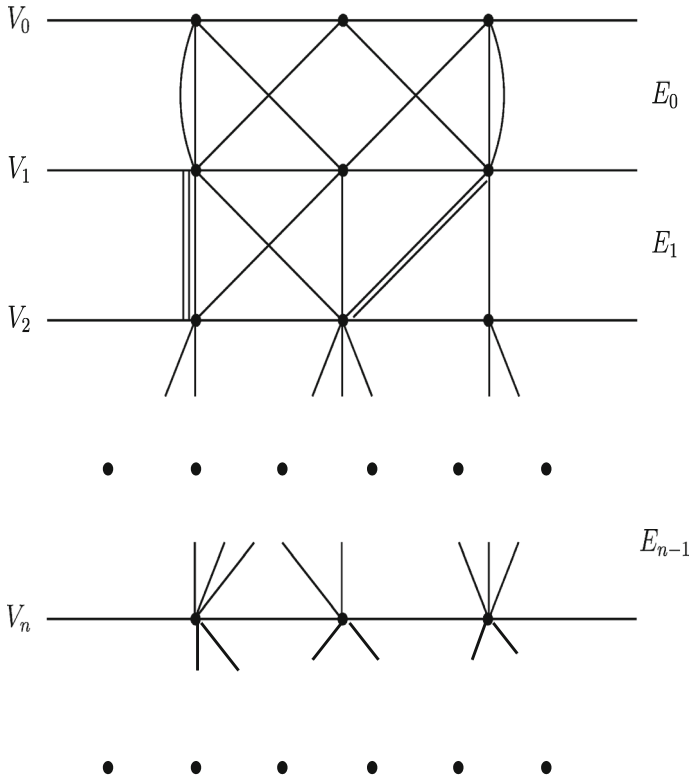
Fig. 1 Example of a Bratteli diagram: levels, verices, and edges (see Definition 6.1)

Definition 6.3 A finite or infinite *path* in a Bratteli diagram $B = (V, E)$ is a sequence of edges $(e_i : i \geq 0)$ such that $r(e_i) = s(e_{i+1})$. Denote by X_B the set of all infinite paths. Every finite path $\bar{e} = (e_0, \ldots, e_n)$ determines a cylinder subset $[\bar{e}]$ of X_B:

$$[\bar{e}] := \{x = (x_i) \in X_B : x_0 = e_0, \ldots, x_n = e_n\}.$$

The collection of all cylinder subsets forms a base of neighborhoods for a topology on X_B. In this topology, X_B is a Polish zero-dimensional space and every cylinder set is clopen. The metric on X_B can be defined by the formula:

$$\text{dist}(x, y) = \frac{1}{2^N}, \quad N = \min\{i \in \mathbb{N}_0 : x_i \neq y_i\}.$$

Remark 6.4 (1) It follows from Definition 6.3 that X_B is a 0-dimensional Polish space. We consider diagrams B for which X_B has no isolated points. In case (i) of classical Bratteli diagrams, X_B is compact.

(2) For a generalized Bratteli diagram, i.e., case (ii) is considered, every point $x = (x_i) \in X_B$ is represented as follows:

$$\{x\} = \bigcap_{n \geq 0} [\bar{e}]_n$$

where $[\bar{e}]_n = [x_0, \ldots, x_n]$. But, in general, it is not true that any nested sequence of clopen sets determines a point in X_B because the intersection may be empty.

Definition 6.5 It is said that a generalized Bratteli diagram B is *irreducible* if, for any two vertices v and w, there exists a level V_m such that $v \in V_0$ and $w \in V_m$ are connected by a finite path. This is equivalent to the property that, for any fixed v, w, there exists $m \in \mathbb{N}$ such that the product of matrices $F_{m-1} \ldots F_0$ has non-zero (w, v)-entry.

Remark 6.6 The structure of a generalized Bratteli diagram allows us to visualize the definitions of boundaries defined in (2.4) and (2.5). If $W = V_0 \cup \ldots \cup V_n$ is a subgraph of $B = (V, E, c)$, then

$$outbd(W) = V_{n+1}, \quad inbd(W) = V_n,$$
$$Int(W) = V_0 \cup \ldots \cup V_{n-1}, \quad cl(W) = V_0 \cup \ldots \cup V_{n+1}.$$

The path-space X_B is a 0-dimensional space which is the boundary of finite paths in the graph B.

In this paper, we focus on the case of 0-1 *Bratteli diagrams*, i.e., the entries of matrices F_n are either 0 or 1. There exists a simple procedure that allows to transform any Bratteli diagram to this case [28].

In what follows we discuss the question about conditions under which a countable locally finite graph G can be represented as a Bratteli diagram with finite levels.

For a finite path $\gamma(x, y)$ between $x, y \in V$, define its length $\ell(\gamma)$ as the number of edges from E that form γ. Define

$$\text{dist}(x, y) = \min\{\ell(\gamma) : \gamma \in E(x, y)\},$$

where $E(x, y)$ is the set of all finite paths γ from x to y.

Proposition 6.7 [8] (1) *A connected locally finite graph $G(V, E)$ has the structure of a Bratteli diagram if and only if:*

(i) $\deg(x) \geq 2$ *for all but at most one $o \in V$;*

(ii) *there exists a vertex $x_0 \in V$ such that, for any $n \geq 1$, there are no edges between any vertices from the set $V_n := \{y \in V : \text{dist}(x_0, y) = n\}$;*

(iii) *for any vertex $x \in V_n$ there exists an edge $e_{(xy)}$ connecting x with some vertex $y \in V_{n+1}, n \in \mathbb{N}$.*

(2) *In general, the vertex x_0 is not unique: there are graphs $G(V, E)$ that satisfy (i)–(iii) for different vertices x_0 and y_0 from V.*

To illustrate Proposition 6.7, we note that $\mathbb{Z}^d$, $d \geq 2$, can be represented as a Bratteli diagram. Another example of this kind is the Cayley graph $\mathcal{C}(S)$ of finitely generated group F with a finite generating set $S \subset F$ such that $S = S^{-1}$ and $SS \cap S = \emptyset$.

6.2 Weighted Bratteli Diagrams

Let $B = (V, E)$ be a graph which is represented by a Bratteli diagram with *finite levels* V_n for all n which is constructed by the incidence matrices F_n. The case of countably infinite sets V_n will be considered later. Recall that we denote by A_n the matrix transpose to F_n. We assume that $B = (V, E)$ is a 0-1 Bratteli diagram, i.e., every entry of A_n is either 0 or 1.

The conductance function c is defined on E and takes positive value at every edge e. Since every edge e is uniquely determined by a pair of vertices (x, y), we write also $c_e = c_{xy} = c_{yx}$. Based on the structure of the vertex set $V = \bigsqcup_{n \geq 0} V_n$ and edge set $E = \bigsqcup_{n \geq 0} E_n$ of the Bratteli diagram B, we define a sequence of matrices $(C_n)_{n \geq 0}$, $C_n = (c_{xy}^{(n)})$, which is naturally related to the matrices (A_n) and the conductance function c:

$$c_{xy}^{(n)} := \begin{cases} c_{xy}, & x = s(e), y = r(e), e \in E_n \\ 0, & \text{otherwise} \end{cases}$$

Then $c_{xy}^{(n)} > 0$ if and only if $a_{xy}^{(n)} = 1$ and the size of C_n is $|V_n| \times |V_{n+1}|$, $n \geq 0$. In particular, C_0 is a row matrix with entries $(c_{ox}^{(0)} : x \in V_1)$ where o is the root of the diagram. It is helpful to remember that for every n, the matrix C_n determines a linear transformation $C_n : \mathbb{R}^{|V_{n+1}|} \to \mathbb{R}^{|V_n|}$.

We note that the order of indexes in $c_{xy}^{(n)}$ is important: although the values of the conductance function c depend on edges only, the entry $c_{yx}^{(n)}$ belongs to the matrix C_n^T transpose of C_n.

It is said that the sequence of matrices (C_n) is *associated to the weighted Bratteli diagram* (B, c).

Together with the sequence of associated matrices (C_n), we will consider two other sequences of matrices. They are denoted by $(\overleftarrow{P}_n)$ and $(\overrightarrow{P}_{n-1})$, and their entries are defined by the formulas
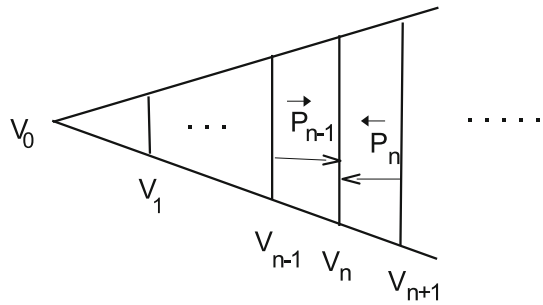
$$\overleftarrow{p}_{xz}^{(n)} = \frac{c_{xz}^{(n)}}{c_n(x)}, \quad x \in V_n, z \in V_{n+1}, \tag{6.1}$$

$$\overrightarrow{p}_{xy}^{(n-1)} = \frac{c_{yx}^{(n-1)}}{c_n(x)}, \quad x \in V_n, y \in V_{n-1}. \tag{6.2}$$

This means, in particular, that $\overleftarrow{P}_0$ is a row matrix, and, for all n, $\overrightarrow{P}_n = \overleftarrow{P}_{n+1}^T$ where T stands for the transpose matrix.

Remark 6.8 (1) We represent a Bratteli diagram as an infinite graph that is expanding in the "horizontal" direction from left to right, see Fig. 2.

Then the arrows used in the notation of the matrices show how the transformations defined by the matrices act: $\overleftarrow{P}_n$ sends $\mathbb{R}^{|V_{n+1}|}$ to $\mathbb{R}^{|V_n|}$ and $\overrightarrow{P}_{n-1}$ sends $\mathbb{R}^{|V_{n-1}|}$ to $\mathbb{R}^{|V_n|}$,

(2) The matrix P of transition probabilities has a simple form. It can be schematically represented as follows

$$
P = \begin{pmatrix}
0 & \overleftarrow{P}_0 & 0 & 0 & \cdots & \cdots \\
\overrightarrow{P}_0 & 0 & \overleftarrow{P}_1 & \cdots & \cdots \\
0 & \overrightarrow{P}_1 & 0 & \overleftarrow{P}_2 & \cdots & \cdots \\
0 & 0 & \overrightarrow{P}_2 & 0 & \overleftarrow{P}_3 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}. \tag{6.3}
$$

Here every entry $P_{ij}, i, j = 0, 1, 2, \ldots$, corresponds to a block matrix whose rows are enumerated by vertices from V_i and columns are enumerated by vertices from V_j.

6.3 Existence of Harmonic Functions on a Bratteli Diagram

In this subsection, we give some algebraic criteria under which a function f, defined on the set of vertices, is harmonic, i.e., $Pf = f$ where P is as in (6.3). All results in this section are taken from [8].

Let D_n be the diagonal matrix with $d_{xx}^{(n)} = c(x), x \in V_n$. Every function f on the vertices of $B = (V, E)$ is represented as a sequence of vectors $(f_n : n \in \mathbb{N}_0)$.

Proposition 6.9 (1) *Let $(B, c) = (V, E, c)$ be a weighted 0-1 Bratteli diagram with conductance function c. Let (C_n) be the sequence of matrices associated to (B, c). Then a function $f : V \to \mathbb{R}$ is harmonic if and only the sequence of vectors (f_n) where $f_n = f|_{V_n}$ satisfies*

$$
C_n f_{n+1} = D_n f_n - C_{n-1}^T f_{n-1}, \quad n \in \mathbb{N}. \tag{6.4}
$$

(2) *Let the matrices $(\overrightarrow{P}_n)$ and $(\overleftarrow{P}_n)$ are defined as in (6.1) and (6.2). Then a sequence of vectors (f_n) $(f_n \in \mathbb{R}^{|V_n|})$ represents a harmonic function $f = (f_n)$:*

$V \to \mathbb{R}$ *if and only if for any* $n \geq 1$

$$f_n - \overrightarrow{P}_{n-1} f_{n-1} = \overleftarrow{P}_n f_{n+1}. \tag{6.5}$$

Harmonic functions exist on a Bratteli diagram if the following criterion holds. Let $\mathcal{N}_1$ be the solution set of $C_0 f_1 = 0$ and $\mathcal{N}_2 = \{f_2 \in \mathbb{R}^{|V_2|} : \overleftarrow{P}_1 f_2 \in \mathcal{N}_1\}$. We define $\mathcal{N}_{n+1} = \{f_{n+1} \in \mathbb{R}^{|V_{n+1}|} : \overleftarrow{P}_n f_{n+1} \in \mathcal{G}_n\}$ where $\mathcal{G}_n = \{f_n - \overrightarrow{P}_{n-1} f_{n-1} \in \mathbb{R}^{|V_n|} : f_n \in \mathcal{N}_n, \ f_{n-1} \in \mathcal{N}_{n-1}\}$. Then we have the following result.

Proposition 6.10 *The space* $\mathcal{H}arm$ *of harmonic functions on a weighted Bratteli diagram* (B, c) *is nontrivial if and only if for every* n

$$\mathrm{Col}(\overleftarrow{P}_n) \cap \mathcal{G}_n \neq \{0\} \tag{6.6}$$

where $\mathrm{Col}(\overleftarrow{P}_n)$ *is the column space of* $\overleftarrow{P}_n$. *In particular, if* $\mathrm{Rank}(\overleftarrow{P}_n) = |V_n|$ *for all* $n \geq 1$ *then* (6.6) *is automatically satisfied.*

Remark 6.11 (1) We can refine the definition of a stationary Bratteli diagrams given above in this section: a weighted Bratteli diagram (B, c) is called *stationary* if $A_n = A$ and $C_n = C$ for all $n \geq 1$. It can be seen that there are stationary weighted Bratteli diagrams such that the space $Harm$ is finitely dimensional.

(2) On the other hand, if a Bratteli diagram has the property that $|V_n| < |V_{n+1}|$ for all n, then the space $\mathcal{H}arm$ is infinite-dimensional.

Now we consider some properties of harmonic functions defined on a Bratteli diagram (B, c). Given a function $f : V \to \mathbb{R}$, define the *current* $I(x)$ through $x \in V$ as

$$I(x) := \sum_{y \sim x} c_{xy}(f(x) - f(y)).$$

The following statement represents a form of the Kirchhoff law and can serve as a characterization of harmonic functions defined on vertices of a Bratteli diagram.

Lemma 6.12 *A function* $f : V \to \mathbb{R}$ *is harmonic on a weighted Bratteli diagram* (B, c) *if and only if for every* $x \in V_n, n \geq 1$,

$$I_{in}(x) := \sum_{y \in V_{n-1}} c_{xy}(f(x) - f(y)) = \sum_{z \in V_{n+1}} c_{xz}(f(z) - f(x)) =: I_{out}(x).$$

Hence, the incoming current is equal to outgoing current for every vertex if and only if the function f *is harmonic.*

Based on this result, we can define, for $x \in V_n$,

$$I_n(x) := I_{in}(x), \quad \text{and} \quad I_n = \sum_{x \in V_n} I_n(x).$$

Lemma 6.13 *Let f be a harmonic function on a weighted Bratteli diagram (B, c). Then, for any $n \geq 1$, we have $I_n = I_1 = \sum_{x \in V_1} c_{ox}(f(x) - f(o))$ and*

$$\sum_{x \in V_n} (I_n(x))^2 \geq \frac{I_1^2}{|V_n|}. \tag{6.7}$$

We formulate the following statement for harmonic functions only although it can be given in more general terms of subharmonic functions (that is $(\Delta f)(x) \leq 0$ for every x) and superharmonic functions (that is $(\Delta f)(x) \geq 0$ for every x) functions which are not discussed here.

Proposition 6.14 [8] *Let (B, c) be a weighted Bratteli diagram and $G_n = \{o\} \cup V_1 \cup \cdots \cup V_n$. Then for any nontrivial harmonic function $f : V \to \mathbb{R}$*

$$\max\{f(x) : x \in G_n\} = \max\{f(x) : x \in \partial G_n = V_n\} =: M_n(f).$$
$$\min\{f(x) : x \in G_n\} = \min\{f(x) : x \in \partial G_n = V_n\} =: m_n(f).$$

Moreover, for any $x, y \in G_n$,

$$f(x) - f(y) \leq M_n(f) - m_n(f), \quad n \in \mathbb{N}. \tag{6.8}$$

The sequence $\{M_n(f)\}$ is strictly increasing, and the sequence $\{m_n(f)\}$ is strictly decreasing.

It can be noticed that in conditions of Proposition 6.14 one can always assert that the sequence $\{M_n(f)\}$ is formed by positive numbers and the sequence $\{m_n(f)\}$ has only negative terms provided $f(o) = 0$.

Corollary 6.15 *Let $(B(V, E), c)$ be a weighted Bratteli diagram. A harmonic function $f : V \to \mathbb{R}$ belongs to $\ell^\infty(V)$ if and only if the sequences $\{M_n(f)\}, \{m_n(f)\}$ have finite limits.*

6.4 Explicit Integral Representation of Harmonic Functions

In this subsection we will assume that the networks considered on Bratteli diagrams (B, c) are *transient*; see Definition 2.12. We use the paper [2] and find an integral representation of harmonic functions in terms of a Poisson kernel, and investigate the convergence of harmonic functions on the path space of a Bratteli diagram.

The transition probabilities matrix $P = (p_{xy} : x, y \in V)$ defines a random walk on the set of all vertices V. Let $\Omega \subset V^\infty$ be the set of all paths $\omega = (x_0, x_1, \ldots, x_n, \ldots)$ where $(x_{i-1}x_i) \in E$. By Ω_x we denote the subset of Ω formed by those paths that starts with x. Then $\mathbb{P}_x$ denotes the Markov measure on Ω_x generated by P (see Sect. 2.3 for details).

Let $X_i : \Omega_x \to V$ be the random variable on $(\Omega_x, \mathbb{P}_x)$ such that $X_i(\omega) = x_i$. For a given vertex $x \in V$ and some level $V_n \subset V$ such that $x \notin V_n$, we determine the

function of stopping time (more information on this notion can be found, for instance, in [21,53]):

$$\tau(V_n)(\omega) = \min\{i \in \mathbb{N} : X_i(\omega) \in V_n\}, \quad \omega \in \Omega_x.$$

For $x \in V_n$, we set $\tau(V_n)(\omega) = 0$. The value $\tau(V_n)(\omega)$ shows when the orbit ω reaches V_n at the first time.

Lemma 6.16 *Let (B, c) be a transient network, and $W_{n-1} = \bigcup_{i=0}^{n-1} V_i$. Then for every $n \in \mathbb{N}$ and any $x \in W_{n-1}$, there exists $m > n$ such that for $\mathbb{P}_x$-a.e. $\omega \in \Omega_x$*

$$\tau(V_{i+1})(\omega) = \tau(V_i)(\omega) + 1, \quad i \geq m. \tag{6.9}$$

Now we fix a vector $f_n \in \mathbb{R}^{|V_n|}$ and define the function $h_n : X \to \mathbb{R}$ by setting

$$h_n(x) := \mathbb{E}_x(f_n \circ X_{\tau(V_n)}) = \int_{\Omega_x} f_n(X_{\tau(V_n)}(\omega))d\mathbb{P}_x(\omega), \quad n \in \mathbb{N}. \tag{6.10}$$

Lemma 6.17 *For a given function $f = (f_n)$, and, for every n, the function $h_n(x)$ is harmonic on $V \setminus V_n$ and $h_n(x) = f_n(x), x \in V_n$. Furthermore, $h_n(x)$ is uniquely defined on W_{n-1}.*

Proof (Sketch) We see that $h_n(x) = f_n(x)$ when $x \in V_n$ because in the relation $h_n(x) = \mathbb{E}_x(f_n \circ X_{\tau(V_n)}(\omega))$ the right side does not depend on ω and $\tau(V_n) = 0$. Then we compute

$$h_n(x) = \sum_{y \sim x} p(x, y)\mathbb{E}_x(f_n \circ X_{\tau(V_n)} \mid X_1 = y)$$

$$= \sum_{y \sim x} p(x, y)\mathbb{E}_y(f_n \circ X_{\tau(V_n)}) \text{ (using the Markov property)}$$

$$= \sum_{y \sim x} p(x, y)h_n(y)$$

$$= (Ph_n)(x)$$

The fact that $h_n(x)$ is uniquely determined on W_{n-1} follows from the uniqueness of the solution of the Dirichlet problem

$$(\Delta u)(x) = 0, \; x \in W_{n-1}, \text{ and } u(x) = f_n(x), \; x \in V_n$$

where $V_n = \partial W_{n-1}$. $\square$

Theorem 6.18 [8] *Let $f = (f_n) \geq 0$ be a function on V such that $\overleftarrow{P}_n f_{n+1} = f_n$. Then the sequence $(h_n(x))$ defined in (6.10) converges pointwise to a harmonic function $H(x)$. Moreover, for every $x \in V$, there exists $n(x)$ such that $h_i(x) = H(x), i \geq n(x)$. Equivalently, the sequence $(f_n \circ X_{\tau(V_n)})$ converges in $L^1(\Omega_x, \mathbb{P}_x)$.*

Remark 6.19 (1) We notice that the condition $\overleftarrow{P}_n f_{n+1} = f_n$ need not to be true for all n. It suffices to have this property for all sufficiently large n; the function f_i can be chosen arbitrary for a finite set of i's.

(2) In [2], the following statement was proved: If a reversible Markov chain X_n is transient and f is a (harmonic) function of finite energy, then $(f \circ X_n)$ converges almost everywhere. Our result above is of the same nature, but we do not require that the harmonic function has finite energy.

7 Harmonic Functions on Trees, Pascal Graph, Stationary Bratteli Diagrams

The goal of this section: we will show that the algorithm of finding harmonic functions and monopoles/dipoles described in Sect. 6. We use the notations introduced in Sect. 6. The results of this section are obtained in [8].

7.1 Harmonic Functions on Trees

Proposition 7.1 *Let T be a tree with conductance function c. The space $Harm$ of harmonic functions on the electrical network (T, c) is infinite-dimensional. Any harmonic function can be found according to Proposition 6.10.*

Example 7.2 (Symmetric harmonic functions on the binary tree) Let x_0 be the root of the binary tree T, and let V_n denote the set of vertices on the distance n from the root. Next, we assume that the conductance function $c = c(e), e \in E$, has the property: $c(e) = \lambda^n$ for all $e \in E_n, n \geq 0$. Hence, the associated matrices C_n are of the size $2^n \times 2^{n+1}$, and the i-th row of C_n consists of all zeros but $c_{i,2i-1}^{(n)} = c_{i,2i}^{(n)} = \lambda^n$. Denote by $x_n(1), ..., x_n(2^n)$ the vertices of V_n enumerated from the top to the bottom, see Fig. 3.

Proposition 7.3 *Let (T, c) be the weighted binary tree defined above.*

(1) *For each positive λ there exists a unique harmonic function $f = f_\lambda$ satisfying the following conditions:*

   (i) *$f(x_0) = 0$;*
   (ii) *$f(x_1(1)) = -f(x_1(2)) = \lambda$ and*

$$f(x_n(1)) = -f(x_n(2^n)) = \frac{1 + \cdots + \lambda^{n-1}}{\lambda^{n-2}}, \quad n \geq 2;$$

   (iii) *function f is constant on each of subtrees T_i and T_i' whose all infinite paths start at the roots $x_i(1)$ and $x_i(2^i)$, respectively, and go through the vertices $x_{i+1}(2)$ and $x_{i+1}(2^{i+1} - 1)$, $i \geq 1$ (see Fig. 3).*

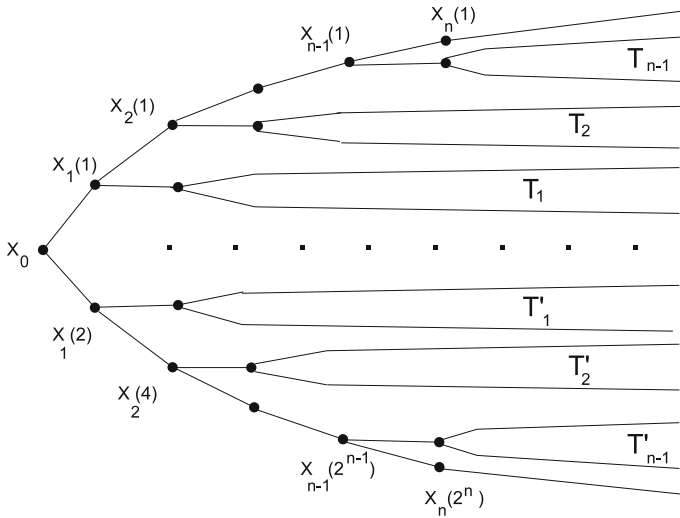(2) *The function f_λ has finite energy if and only if $\lambda > 1$.*

Fig. 3 Symmetric harmonic function on the binary tree

7.2 Harmonic Functions on the Pascal Graph

By definition, the Pascal graph is the 0-1 Bratteli diagram with the sequence of matrices $(A_n)_{n\geq 0}$ (A_n is transpose to the incidence matrix F_n) of the size $(n+1) \times (n+2)$ where

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

Every vertex $v \in V_n$ of the Pascal graph can be enumerated by two numbers (coordinates) (n, i), where $0 \leq i \leq n$ is the position of v in V_n (it is assumed that the set of vertices $\{(x, 0) : x \in \mathbb{N}_0\}$ is the upper bound line of the graph).
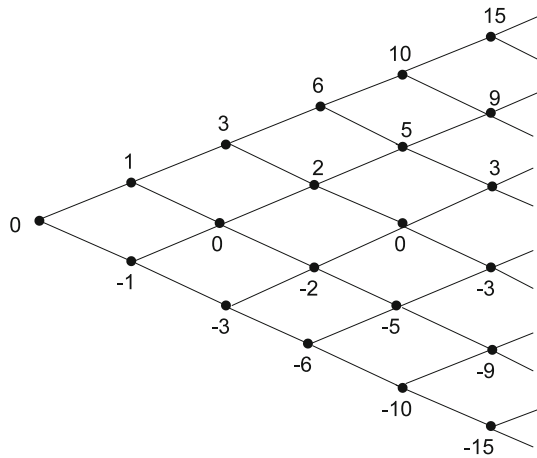
It can be proved that the algorithm of finding harmonic functions from Sect. 6 is applicable for the Pascal graph. In other words, the equation

$$\overleftarrow{P}_n f_{n+1} = f_n - \overrightarrow{P}_{n-1} f_{n-1} \tag{7.1}$$

always has a solution for f_{n+1} assuming that f_n and f_{n-1} have been determined in the previous steps. Moreover, the solution set of this equation is one-dimensional for every n.

Equation (7.1) becomes more transparent if we additionally require that the conductance function c is defined by the rule $c(e) = \lambda^n$, for any $e \in E_n$, and the harmonic function f vanishes at $(0, 0)$. Then one can easily find the explicit form of $\overleftarrow{P}_n$ for

Fig. 4 Harmonic function on the Pascal graph

any $n \geq 1$:

$$
\overleftarrow{P}_n =
\begin{pmatrix}
\frac{\lambda}{1+\lambda} & \frac{\lambda}{1+\lambda} & 0 & 0 & \cdots & 0 \\
0 & \frac{\lambda}{2+\lambda} & \frac{\lambda}{2+\lambda} & 0 & \cdots & 0 \\
0 & 0 & \frac{\lambda}{2+\lambda} & \frac{\lambda}{2+\lambda} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \frac{\lambda}{1+\lambda} & \frac{\lambda}{1+\lambda}
\end{pmatrix}
$$

We say that a harmonic function h on the Pascal graph (B, c) is *symmetric* if it satisfies the condition $h(n, i) = -h(n, n - i)$ for any n and $0 \leq i \leq n$.

Lemma 7.4 *Let (B, c) be a weighted Pascal graph. Then $Harm$ is a non-empty infinite dimensional space containing the subspace of symmetric harmonic functions. Moreover, this subspace is also infinite dimensional.*

For the case when $c = 1$, we can find an explicit formula of symmetric harmonic function.

Proposition 7.5 *Define $h(0, 0) = 0$ and set, for every vertex $v = (n, i)$,*

$$
h(n, i) := \frac{n(n + 1)}{2} - i(n + 1), \tag{7.2}
$$

where $0 \leq i \leq n$ and $n \geq 1$. Then $h : V \to \mathbb{R}$ is an integer-valued harmonic function on $(B, 1)$ satisfying the symmetry condition $h(n, i) = -h(n, n - i)$ (see Fig. 4).

7.3 Harmonic Functions on Stationary Bratteli Diagrams

Let A be the incidence matrix of a stationary Bratteli diagram B, and suppose A has $d \times d$ size. Assume that the conductance function c has the property: $c_e = c_{xy} = \lambda^n$

for any $e \in E_n$ with $s(e) = x \in V_n, r(e) = y \in V_{n+1}$. Then the associated matrix $C_n = \lambda^n A$ (C_n is defined in Sect. 6).

We start with rewriting relations (6.4) and (6.5) for the stationary case and the chosen c

$$A^T (f_{n-1} - f_n(x)\mathbf{1}_d) + \lambda A(f_{n+1} - f_n(x)\mathbf{1}_d) = 0 \tag{7.3}$$

where $x \in V_n, n \in \mathbb{N}$ and $\mathbf{1}_d = (1, \ldots, 1)^T \in \mathbb{R}^d$.

Proposition 7.6 ([8]) *Let the weighted stationary Bratteli diagram (B, c) be as defined above. Suppose that $A = A^T$, and A is invertible. Then any harmonic function $f = (f_n)$ on (B, c) can be found by the formula:*

$$f_{n+1}(x) = f_1(x) \sum_{i=0}^{n} \lambda^{-i}, \quad x \in V. \tag{7.4}$$

Proof We first observe that, as $V_n = V$ for $n \geq 1$, we can interpret relation (7.3) as a sequence of relations between vectors f_n that hold on the same space $\mathbb{R}^d$, where x is any vertex from V. Moreover, using the properties of A, we obtain

$$(f_{n-1} - f_n(x)\mathbf{1}_d) + \lambda(f_{n+1} - f_n(x)\mathbf{1}_d) = 0.$$

From this equation between vectors, we deduce that it holds for any coordinate $y \in V$, in particular, for $y = x$. Hence, for every $n \geq 1$,

$$\lambda(f_{n+1}(x) - f_n(x)) = f_n(x) - f_{n-1}(x),$$
$$f_{n+1}(x) - f_n(x) = \frac{1}{\lambda^n}(f_1(x) - f_0(x)) = \frac{1}{\lambda^n} f_1(x).$$

Summation of these relations gives

$$f_{n+1}(x) = f_1(x) \sum_{i=0}^{n} \lambda^{-i}.$$

$\square$

It follows from Proposition 7.6 that we can explicitly describe elements of the space *Harm* for this class of weighted stationary Bratteli diagrams.

Corollary 7.7 *Let (B, c) be as in Proposition 7.6.*

(1) *The dimension of the space Harm is $d - 1$ where $d = |V|$.*
(2) *If $\lambda > 1$, then every harmonic function on (B, c) is bounded.*

8 Harmonic Functions of Finite and Infinite Energy on Combinatorial Graphs

8.1 Energy Space for a Bratteli Diagram

Recall that, given a function $u : V \to \mathbb{R}$ defined on the vertex set V of an electrical network (G, c), its energy $\|u\|_{\mathcal{H}_E}$ is computed by

$$\|u\|_{\mathcal{H}_E}^2 = \frac{1}{2} \sum_{x,y} c_{xy} (u(x) - u(y))^2.$$

In case of a harmonic function $h \in Harm$, one can also use the formulas from Lemma 3.1 to find the energy of h.

Let (B, c) be a weighted Bratteli diagram. Denote

$$\beta_n = \max\{c(x) : x \in V_n\}. \tag{8.1}$$

Theorem 8.1 [8] *Let f be a harmonic function on a weighted Bratteli diagram (B, c). Then*

$$\sum_{n=0}^{\infty} \frac{I_1^2}{\beta_n |V_n|} \le \|f\|_{\mathcal{H}_E}^2, \tag{8.2}$$

where $I_1 = \sum_{x \in V_1} c_{ox} (f(x) - f(o))$ was defined in Lemma 6.13.

It immediately follows from the proved inequality (see (8.2)) that the following result holds.

Corollary 8.2 *Suppose that a weighted Bratteli diagram (B, c) satisfies the condition*

$$\sum_{n=0}^{\infty} (\beta_n |V_n|)^{-1} = \infty \tag{8.3}$$

where $V = \bigcup_n V_n$ and $\beta_n = \max\{c(x) : x \in V_n\}$. Then any nontrivial harmonic function has infinite energy, i.e., $Harm \cap \mathcal{H}_E = \{const\}$. In other words, such a (B, c) does not support non-constant harmonic functions of finite energy.

It is not difficult to realize the condition of Corollary 8.2 for a weighted Bratteli diagram (B, c). For instance, suppose that $c_{xy} = 1$ for any edge $e = (xy)$ If the sequence $(|V_n|)$ satisfies $\sum_{n=0}^{\infty} |V_n|^{-1} = \infty$, then any harmonic function has infinite energy.

For example, if $c = 1$ for the Pascal graph considered in Sect. 7, then Proposition 7.5 gives us the explicitly defined harmonic function h. It follows from Corollary 8.2 that $\|h\|_{\mathcal{H}_E} = \infty$. Moreover, there is no harmonic functions of finite energy on the Pascal graph with the conductance function $c = 1$.

The following results can be obtained from Proposition 3.1.

Lemma 8.3 *For the Pascal graph,*

$$\|f\|_{\mathcal{H}_E}^2 = \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} c_{xy}(f^2(y) - f^2(x))$$

for any harmonic function.

In Proposition 7.6, we described arbitrary harmonic function on a class of stationary Bratteli diagrams with $c_e \in \{\lambda^n : n \in \mathbb{N}_0\}$.

Proposition 8.4 *Suppose that a stationary weighted Bratteli diagram (B, c) satisfies conditions of Proposition 7.6 with $c_e = \lambda^n$, $e \in E_n$, and $\lambda > 1$. Let $f = (f_n)$ be a harmonic function defined by (7.4). Then $\|f\|_{\mathcal{H}_E} < \infty$ if and only if the vector $f_1(x)$ is constant.*

Remark 8.5 In [26], it was proved that if for an electrical network (G, c) the total conductance is finite, i.e., $\sum_{e \in E} c(e) < \infty$, then there is no nontrivial harmonic function of finite energy. If the network G is represented by a weighted Bratteli diagram (B, c), we have

$$\sum_{e \in E} c(e) = \frac{1}{2} \sum_{x} \sum_{y} c_{xy} = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{x \in V_n} c(x).$$

Then we deduce that if, in particular, $\sum_{n=0}^{\infty} \beta_n |V_n| < \infty$, then there is no non-constant harmonic function of finite energy on (B, c), see also [26].

Thus, we obtain the following qualitative observation: there two classes of Bratteli diagrams when all harmonic functions have infinite energy: (i) the sequence $(\beta_n |V_n|)$ is either decreasing sufficiently fast, or (ii) it is not growing too fast.

8.2 Graph Laplacians and Associated Harmonic Functions on Generalized Bratteli Diagrams

Let $B = (V, E)$ be a 0-1 generalized Bratteli diagram with infinite levels V_n. Let $q^{(0)} = (q_v^{(0)} : v \in V_0)$ be a strictly positive vector. Consider a sequence of non-negative matrices $R_n = (r_{w,v}^{(n)})$, $w \in V_n$, $v \in V_{n+1}$ such that (i) $r_{w,v}^{(n)} > 0$ if and only if $(wv) \in E_n$, and (ii) $\sum_{v \in V_{n+1}} r_{w,v}^{(n)} = 1$ for every $w \in V_n$.

Define inductively the vectors $q^{(n)}$ indexed by vertices of the level V_n by the relation

$$q_v^{(n+1)} = \sum_{w, (wv) \in E_n} q_w^{(n)} r_{w,v}^{(n)}.$$

If $q^{(0)}$ is a probability vector, then all $q^{(n)}$ are probability.

The sequences (R_n) and $(q^{(n)})$ determine a dual sequence of matrices (S_n) where $S_n = (s_{v,w}^{(n)} : w \in V_n, v \in V_{n+1})$ and the entries are

$$s_{v,w}^{(n)} = \frac{q_w^{(n)}}{q_v^{(n+1)}} r_{w,v}^{(n)}.$$

The matrices R_n and S_n generate operators which are defined by the formulas: if f is a bounded function on V_{n+1}, then for any $w \in V_n$,

$$(T_{R_n} f)(w) = \sum_{v,(wv) \in E_n} r_{w,v}^{(n)} f(v).$$

Similarly,

$$(T_{S_n} g)(v) = \sum_{w,(wv) \in E_n} s_{v,w}^{(n)} g(w).$$

Let $\mathcal{H}_n$ be the linear space of all functions $f = (f(v) : v \in V_n)$ such that

$$\|f\|_{\mathcal{H}_n}^2 := \sum_{v \in V_n} q_v^{(n)} f(v)^2 < \infty. \tag{8.4}$$

Then $\mathcal{H}_n$, equipped with this norm, is a Hilbert space with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}_n} = \sum_{v \in V_n} \varphi(v) \psi(v) q_v^{(n)}.$$

Proposition 8.6 [10]

(1) *The operators* $T_{R_n} : \mathcal{H}_{n+1} \to \mathcal{H}_n$ *and* $T_{S_n} : \mathcal{H}_n \to \mathcal{H}_{n+1}$ *are positive and contractive for all* $n \in \mathbb{N}_0$.
(2) $(T_{R_n})^* = T_{S_n}$ *and* $(T_{S_n})^* = T_{R_n}$.

We use now the graph $B = (V, E)$ and turn it in an electrical network $G(B)$ by defining the conductance function $c := c_{w,u}^{(n)}$.

Definition 8.7 For $w \in V_{n-1}, v \in V_n, u \in V_{n+1}$, we set

$$c_{vu}^{(n)} = \frac{1}{2} q_v^{(n)} r_{v,u}^{(n)}, \quad c_{vw}^{(n-1)} = \frac{1}{2} q_v^{(n)} s_{v,w}^{(n-1)}. \tag{8.5}$$

Lemma 8.8 (1) $c_{vu}^{(n)} = c_{uv}^{(n)}$, *i.e., the conductance function c is correctly defined on edges from E.*
(2)

$$c_n(v) = q_v^{(n)}, \quad v \in V_n, \ n \in \mathbb{N}_0.$$

Lemma 8.8 shows that $c(v) = (c_n(v))$ is finite for every $v \in V$. We will omit the index n in $c_n(v)$ if it is clear that v is taken from V_n.

Definition 8.9 For $G = G(B)$ and the conductance function c as above, we define a *reversible Markov kernel* $M = \{m(v, u) : v, u \in V\}$ by setting

$$
m(v, u) = \begin{cases} \dfrac{c_{vu}^{(n)}}{c_n(v)} = \frac{1}{2} r_{v,u}^{(n)}, & v \in V_n, u \in V_{n+1}, \\[3mm] \dfrac{c_{uv}^{(n-1)}}{c_n(v)} = \frac{1}{2} s_{v,u}^{(n-1)}, & v \in V_n, u \in V_{n-1}. \end{cases}
$$

The Markov kernel M is *reversible*, i.e.,

$$
c(v)m(v, u) = c(u)m(u, v), \quad \forall v, u, \ v \sim u.
$$

Definition 8.10 Let $f = (f_n)$ be a function defined on vertices of the graph $G(B)$ (or the Bratteli diagram). Suppose that a Markov kernel M is defined as in 8.9. The operator

$$
(Mf)(v) = \sum_{u \sim v} m(v, u) f(u), \quad v \in V,
$$

is called a *Markov operator* acting on the weighted network (G, c).

Define a *Laplacian operator* Δ:

$$
(\Delta f)(v) = \sum_{u \sim v} c_{vu} (f(v) - f(u)) = c(v)[f(v) - (Mf)(v)], \quad v \in V.
$$

A function f is called *harmonic*, if $M(f) = f$ (equivalently, $\Delta f = 0$).

Theorem 8.11 [10] *Let $f = (f_n)$ be a function on the vertex set of $G(B)$. Then f is harmonic if and only if $2f_n = R_n f_{n+1} + S_{n-1} f_{n-1}$.*

In particular,

$$
q^{(n)} M = \frac{1}{2} (q^{(n+1)} + q^{(n-1)}), \quad n \in \mathbb{N}. \tag{8.6}
$$

The next result characterizes functions of finite energy for the network $(G(B), c)$.

Proposition 8.12 [10] *Suppose that a function $f = (f_n)$ is such that every f_n belongs to $\mathcal{H}_n, n \geq 0$, where the Hilbert space $\mathcal{H}_n$ is defined in (8.4). Then $f \in \mathcal{H}_E$ if and only if*

$$
\sum_{n \geq 0} \left(||f_n||^2_{\mathcal{H}_n} - 2\langle f_n, T_{R_n}(f_{n+1})\rangle_{\mathcal{H}_n} + ||f_{n+1}||^2_{\mathcal{H}_{n+1}} \right) < \infty
$$

where the operator $T_{R_n} : \mathcal{H}_{n+1} \to \mathcal{H}_n$ is defined in Proposition 8.6.

Remark 8.13 We mention here two results which characterize weighted networks with nontrivial (or trivial) space of harmonic functions in $\mathcal{H}_E$.

It is a well known fact that if an electrical network (G, c) is recurrent, then $\mathcal{H}_E$ is trivial (this result can be found, for instance, in [43]).

(1) In [15] the author characterizes the locally finite connected networks (C, c) that admit non-constant harmonic functions of finite energy. More precisely, it is proved (G, c) has such harmonic functions if and only if there exist transient vertex-disjoint subnetworks $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that the graph obtained from G by contracting each of the sets V_1 and V_2 to a vertex admits a function φ of finite energy such that $\varphi(A) \neq \varphi(B)$.

(2) The reader can find interesting results in [25] and [6] about the graphs that do not support non-constant functions of finite energy and harmonic functions on them.

References

1. Alpay, D., Jorgensen, P., Volok, D.: Relative reproducing kernel Hilbert spaces. Proc. Am. Math. Soc. **142**(11), 3889–3895 (2014)
2. Anandam, V.: Harmonic functions and potentials on finite or infinite networks. Lecture Notes of the Unione Matematica Italiana, vol. 12. Heidelberg: Springer, UMI, Bologna (2011)
3. Anandam, V.: Subordinate harmonic structures in an infinite network. In: Complex analysis and potential theory, volume 55 of CRM Proc. Lecture Notes, pp. 301–314. American Mathematcial Society, Providence, RI (2012)
4. Ancona, A., Lyons, R., Peres, Y.: Crossing estimates and convergence of Dirichlet functions along random walk and diffusion paths. Ann. Probab. **27**(2), 970–989 (1999)
5. Arcozzi, N., Rochberg, R., Sawyer, E. T., Wick, B. D.: The Dirichlet space and related function spaces. Mathematical Surveys and Monographs, vol. 239. American Mathematical Society, Providence, RI (2019)
6. Benjamini, I., Duminil-Copin, H., Kozma, G., Yadin, A.: Minimal growth harmonic functions on lamplighter groups. New York J. Math. **23**, 833–858 (2017)
7. Beznea, L., Vlădoiu, S.: Markov processes on the Lipschitz boundary for the Neumann and Robin problems. J. Math. Anal. Appl. **455**(1), 292–311 (2017)
8. Bezuglyi, S., Jorgensen, P. E. T.: Graph Laplace and Markov operators on a measure space. In: Linear systems, signal processing and hypercomplex analysis, volume 275 of Oper. Theory Adv. Appl., pp. 67–138. Birkhäuser/Springer, Cham (2019)
9. Bezuglyi, S., Jorgensen, P.E.T.: Monopoles, dipoles, and harmonic functions on Bratteli diagrams. Acta Appl. Math. **159**, 169–224 (2019)
10. Bezuglyi, S., Jorgensen, P.E.T.: Harmonic analysis on graphs via Bratteli diagrams and path-space measures. arXiv:2007.15566 (2020)
11. Bezuglyi, S., Karpel, O.: Bratteli diagrams: structure, measures, dynamics. In: Dynamics and numbers, volume 669 of Contemp. Math., pp. 1–36. American Mathematical Society, Providence, RI, (2016)
12. Bezuglyi, S., Karpel, O.: Invariant measures for Cantor dynamical systems. In Dynamics: topology and numbers, volume 744 of Contemp. Math., pp. 259–295. American Mathematcial Society, [Providence], RI, (2020)
13. Bezuglyi, S., Jorgensen, Palle E. T.: Representations of Cuntz-Krieger relations, dynamics on Bratteli diagrams, and path-space measures. In: Trends in harmonic analysis and its applications, volume 650 of *Contemp. Math.*, pp. 57–88. American Mathematcial Society, Providence, RI (2015)

14. Bratteli, O.: Inductive limits of finite dimensional C^*-algebras. Trans. Am. Math. Soc. **171**, 195–234 (1972)

15. Carmesin, J.: A characterization of the locally finite networks admitting non-constant harmonic functions of finite energy. Potential Anal. **37**(3), 229–245 (2012)

16. Cho, I.: Algebras, Graphs and Their Applications. CRC Press, Boca Raton, FL (2014)

17. Chung, K.L.: On the boundary theory for Markov chains II. Acta Math. **115**, 111–163 (1966)

18. Chung, K.L.: Boundary behavior of Markov chains and its contributions to general processes. In: Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 499–505 (1971)

19. Downham, D.Y., Fotopoulos, S.B.: The transient behaviour of the simple random walk in the plane. J. Appl. Probab. **25**(1), 58–69 (1988)

20. Du, J.: On non-zero-sum stochastic game problems with stopping times. ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–University of Southern California (2012)

21. Durand, F.: Combinatorics on Bratteli diagrams and dynamical systems. In: Combinatorics, automata and number theory, volume 135 of Encyclopedia Math. Appl., pp. 324–372. Cambridge Univ. Press, Cambridge (2010)

22. Ervin Dutkay, D., Jorgensen, P.E.T.: Spectral theory for discrete Laplacians. Complex Anal. Oper. Theory **4**(1), 1–38 (2010)

23. Friedrich, T., Göbel, A., Quinzan, F., Wagner, M.: Heavy-tailed mutation operators in single-objective combinatorial optimization. In: Parallel problem solving from nature—PPSN XV. Part I, volume 11101 of Lecture Notes in Comput. Sci., pp. 134–145. Springer, Cham (2018)

24. Führ, H., Pesenson, I.Z.: Poincaré and Plancherel-Polya inequalities in harmonic analysis on weighted combinatorial graphs. SIAM J. Discrete Math. **27**(4), 2007–2028 (2013)

25. Georgakopoulos, A.: Lamplighter graphs do not admit harmonic functions of finite energy. Proc. Am. Math. Soc. **138**(9), 3057–3061 (2010)

26. Georgakopoulos, A.: Uniqueness of electrical currents in a network of finite total resistance. J. Lond. Math. Soc. (2) **82**(1), 256–272 (2010)

27. Gerl, P.: Natural spanning trees of $\mathbf{z}^d$ are recurrent. Discrete Math. **61**(2–3), 333–336 (1986)

28. Giordano, T., Putnam, I.F., Skau, C.F.: Topological orbit equivalence and C^*-crossed products. J. Reine Angew. Math. **469**, 51–111 (1995)

29. Herman, R.H., Putnam, I.F., Skau, C.F.: Ordered Bratteli diagrams, dimension groups and topological dynamics. Internat. J. Math. **3**(6), 827–864 (1992)

30. Jorgensen, P.E.T.: Unbounded graph-Laplacians in energy space, and their extensions. J. Appl. Math. Comput. **39**(1–2), 155–187 (2012)

31. Jorgensen, P.E.T., Pearse, E.P.J.: A Hilbert space approach to effective resistance metric. Complex Anal. Oper. Theory **4**(4), 975–1013 (2010)

32. Jorgensen, P.E. T., Pearse, E.P. J.: Resistance boundaries of infinite networks. In: Random walks, boundaries and spectra, volume 64 of Progr. Probab., pp. 111–142. Birkhäuser/Springer Basel AG, Basel (2011)

33. Jorgensen, P.E.T., Pearse, E.P.J.: A discrete Gauss–Green identity for unbounded Laplace operators, and the transience of random walks. Israel J. Math. **196**(1), 113–160 (2013)

34. Jorgensen, P.E.T., Pearse, E.P.J.: Spectral comparisons between networks with different conductance functions. J. Operator Theory **72**(1), 71–86 (2014)

35. Jorgensen, P.E.T., Pearse, E.P.J.: Symmetric pairs and self-adjoint extensions of operators, with applications to energy networks. Complex Anal. Oper. Theory **10**(7), 1535–1550 (2016)

36. Jorgensen, P.E.T., Pearse, E.P.J.: Symmetric pairs of unbounded operators in Hilbert space, and their applications in mathematical physics. Math. Phys. Anal. Geom. 20(2), 14 (2017)

37. Jorgensen, P.E.T., Pearse, E.P.J.: Continuum versus discrete networks, graph Laplacians, and reproducing kernel Hilbert spaces. J. Math. Anal. Appl. **469**(2), 765–807 (2019)

38. Jorgensen, P., Pearse, E., Tian, F.: Unbounded operators in Hilbert space, duality rules, characteristic projections, and their applications. Anal. Math. Phys. **8**(3), 351–382 (2018)

39. Keane, M.: Reinforced random walk. In *Entropy, search, complexity*, vol. 16 of *Bolyai Soc. Math. Stud.*. Springer, Berlin pp. 151–158 (2007)

40. Kigami, J.: Analysis on fractals. Cambridge Tracts in Mathematics, vol. 143. Cambridge University Press, Cambridge (2001)

41. Kok, J., Sudev, N.K., Chithra, K.P., Mary, A.: Jaco-type graphs and black energy dissipation. Adv. Pure Appl. Math. **8**(2), 141–152 (2017)

42. Kumar, V., Gavrilova, M.L., Tan, C.J.K., L'Ecuyer, P. (eds.): Computational science and its applications–ICCSA: Part I. Lecture Notes in Computer Science, vol. 2667. Springer, Berlin p. 2003 (2003)

43. Lyons, T.: A simple criterion for transience of a reversible Markov chain. Ann. Probab. **11**(2), 393–402 (1983)

44. Lyons, R., Peres, Y.: Probability on trees and networks. Cambridge Series in Statistical and Probabilistic Mathematics, vol. 42. Cambridge University Press, New York (2016)

45. Miller, G.L., Peng, R.: Approximate maximum flow on separable undirected graphs. In: Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms. SIAM, Philadelphia, PA, pp. 1151–1170 (2012)

46. O'Connor, M., Zhang, G.., Bastiaan K.., Abhayapala, W., Thushara, D.: Function splitting and quadratic approximation of the primal-dual method of multipliers for distributed optimization over graphs. IEEE Trans. Signal Inform. Process. Netw. 4(4), 656–666 (2018)

47. Pesenson, I.: Sampling in Paley–Wiener spaces on combinatorial graphs. Trans. Am. Math. Soc. **360**(10), 5603–5627 (2008)

48. Pesenson, I.: Variational splines and Paley–Wiener spaces on combinatorial graphs. Constr. Approx. **29**(1), 1–21 (2009)

49. Petit, C.: Harmonic functions on hyperbolic graphs. Proc. Am. Math. Soc. **140**(1), 235–248 (2012)

50. Putnam, I.F.: Cantor minimal systems. University Lecture Series, vol. 70. American Mathematical Society, Providence, RI (2018)

51. Sokol, A.: An elementary proof that the first hitting time of an open set by a jump process is a stopping time. In: Séminaire de Probabilités XLV, volume 2078 of Lecture Notes in Math. Springer, Cham, pp. 301–304 (2013)

52. St, C., Nash-Williams, J.A.: Random walk and electric currents in networks. Proc. Cambridge Philos. Soc. 55, 181–194 (1959)

53. Veldt, N., Gleich, D.F., Wirth, A., Saunderson, J.: Metric-constrained optimization for graph clustering algorithms. SIAM J. Math. Data Sci. **1**(2), 333–355 (2019)

54. Woess, W.: Random walks on infinite graphs and groups. Cambridge Tracts in Mathematics, vol. 138. Cambridge University Press, Cambridge (2000)

55. Woess, W.: Denumerable Markov chains. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2009. Generating functions, boundary theory, random walks on trees

56. Yamasaki, M.: Discrete potentials on an infinite network. Mem. Fac. Sci. Shimane Univ. **13**, 31–44 (1979)

57. Grigor'yan, A.: Introduction to analysis on graphs. University Lecture Series, vol. 71. American Mathematical Society, Providence, RI (2018)