



Norm Inflation for Nonlinear Schrödinger Equations in Fourier–Lebesgue and Modulation Spaces of Negative Regularity

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Abstract

We consider nonlinear Schrödinger equations in Fourier–Lebesgue and modulation spaces involving negative regularity. The equations are posed on the whole space, and involve a smooth power nonlinearity. We prove two types of norm inflation results. We first establish norm inflation results below the expected critical regularities. We then prove norm inflation with infinite loss of regularity under less general assumptions. To do so, we recast the theory of multiphase weakly nonlinear geometric optics for nonlinear Schrödinger equations in a general abstract functional setting.

Keywords Nonlinear Schrödinger equations · Ill-posedness · Fourier–Lebesgue spaces · Modulation spaces

Mathematics Subject Classification 35Q55 · 42B35 (primary) · 35A01 (secondary)

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1 Introduction

1.1 General Setting

We consider the nonlinear Schrödinger (NLS) equations of the form

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = \mu|\psi|^{2\sigma}\psi, \quad x \in \mathbb{R}^d; \quad \psi(0, x) = \psi_0(x), \quad (1.1)$$

where $\psi = \psi(t, x) \in \mathbb{C}$, $\sigma \in \mathbb{N}$, $\mu \in \{1, -1\}$. We prove some ill-posedness results in Fourier–Lebesgue and modulation spaces, involving negative regularity in space. We recall the notion of well-posedness in the sense of Hadamard.

Definition 1.1 Let $X, Y \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ be a Banach spaces. The Cauchy problem for (1.1) is well posed from X to Y if, for all bounded subsets $B \subset X$, there exist $T > 0$ and a Banach space $X_T \hookrightarrow C([0, T], Y)$ such that:

- (i) For all $\phi \in X$, (1.1) has a unique solution $\psi \in X_T$ with $\psi|_{t=0} = \phi$.
- (ii) The mapping $\phi \mapsto \psi$ is continuous from $(B, \|\cdot\|_X)$ to $C([0, T], Y)$.

The negation of the above definition is called a *lack of well-posedness* or *instability*. In connection with the study of ill-posedness of (1.1) and nonlinear wave equations Christ, Colliander, and Tao introduced in [14] the notion of *norm inflation* with respect to a given (Sobolev) norm, saying that there exist a sequence of smooth initial data $(\psi_n(0))_{n \geq 1}$ and a sequence of times $(t_n)_{n \geq 1}$, both converging to 0, so that the corresponding smooth solution ψ_n , evaluated at t_n , is unbounded (in the same space).

The solutions to (1.1) is invariant under the scaling transformation

$$\psi(t, x) \mapsto \lambda^{1/\sigma}\psi\left(\lambda^2 t, \lambda x\right), \quad \lambda > 0. \quad (1.2)$$

The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ is invariant exactly for $s = s_c$, where

$$s_c = \frac{d}{2} - \frac{1}{\sigma}.$$

Another important invariance of (1.1) is the Galilean invariance: if $\psi(t, x)$ solves (1.1), then so does

$$e^{iv \cdot x - i|v|^2 t/2} \psi(t, x - vt) \quad (1.3)$$

for any $v \in \mathbb{R}^d$. This transform does not alter the $L^2(\mathbb{R}^d)$ norm of the function. From these two invariances, well-posedness is not expected to hold in $H^s(\mathbb{R}^d)$ as soon as $s < \max(0, s_c)$. In this paper, we consider the case of negative regularity, $s < 0$, in Fourier–Lebesgue and modulation spaces, instead of Sobolev spaces.

Kenig, Ponce and Vega [26] established instability for the cubic NLS in $H^s(\mathbb{R})$ for $s < 0$. Christ, Colliander and Tao [14] generalized this result in $H^s(\mathbb{R}^d)$ for $s < 0$ and $d \geq 1$. In the periodic case $x \in \mathbb{T}^d$, instability in $H^s(\mathbb{T}^d)$ for $s < 0$ was established in [15] ($d = 1$) and [10] ($d \geq 1$). Stronger results for the cubic NLS on the circle were

proven by Molinet [29]. In [31, Theorem 1.1], Oh established norm-inflation for (1.1) in the cubic case $\sigma = 1$, in $H^s(\mathbb{R})$ for $s \leq -1/2$ and in $H^s(\mathbb{R}^d)$ for $s < 0$ if $d \geq 2$. He actually proved that the flow map fails to be continuous at any function in H^s , for s as above. Norm inflation in the case of mixed geometries, $x \in \mathbb{R}^d \times \mathbb{T}^n$, for sharp negative Sobolev regularity in (1.1), is due to Kishimoto [27], who also considers nonlinearities which are not gauge invariant.

The general picture to prove ill-posedness results is typically as following, as explained in e.g. [14]: at negative regularity, one relies on a transfer from high frequencies to low frequencies, while to prove ill-posedness at positive regularity, one uses a transfer from low frequencies to high frequencies. In particular, the proofs are different whether a negative or a positive regularity is considered.

Stronger phenomena than norm inflation have also been proved, showing that the flow map fails to be continuous at the origin from H^s to H^k even for (some) $k < s$, and so a loss of regularity is present. This was proven initially for $0 < s < s_c$ by Lebeau [28] in the case of the wave equation, then in [8] (cubic nonlinearity) and [1,36] for NLS. In the case of negative regularity, an infinite loss of regularity was established in [11] for (1.1) in $H^s(\mathbb{R}^d)$ ($d \geq 2$ and $s < -1/(2\sigma + 1)$), and in the periodic case $x \in \mathbb{T}^d$ in [12], in Fourier–Lebesgue spaces. Typically, the NLS flow map fails to be continuous at the origin from $H^s(\mathbb{R}^d)$ to $H^k(\mathbb{R}^d)$, for any $k \in \mathbb{R}$.

1.2 Fourier–Lebesgue Spaces

The Fourier–Lebesgue space $\mathcal{FL}_s^p(\mathbb{R}^d)$ is defined by

$$\mathcal{FL}_s^p(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathcal{FL}_s^p} := \|\hat{f}\langle \cdot \rangle^s\|_{L^p} < \infty \right\},$$

where the Fourier transform is defined as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

and where $1 \leq p \leq \infty$, $s \in \mathbb{R}$, and $\langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$ ($\xi \in \mathbb{R}^d$). For $p = 2$, $\mathcal{FL}_s^2 = H^s$ the usual Sobolev space. For $s = 0$, we write $\mathcal{FL}_0^p(\mathbb{R}^d) = \mathcal{FL}^p(\mathbb{R}^d)$.

The scaling (1.2) leaves the homogeneous $\mathcal{FL}_s^p(\mathbb{R}^d)$ -norm (replace the Japanese bracket $\langle \cdot \rangle^s$ with the length $|\cdot|^s$ in the definition of $\mathcal{FL}_s^p(\mathbb{R}^d)$) invariant for $s = s_c(p)$, where

$$s_c(p) := d \left(1 - \frac{1}{p} \right) - \frac{1}{\sigma}.$$

Of course when $p = 2$, we recover the previous value s_c . On the other hand, the Galilean transform (1.3) does not alter the $\mathcal{FL}^p(\mathbb{R}^d)$ norm of ψ , and so well-posedness is not expected to hold in \mathcal{FL}_s^p for $s < \max(0, s_c(p))$. Note however that the recent results from [22] show that this heuristical argument is not always correct: in the case $p = 2$, $d = 1 = \sigma$, well-posedness may hold for $\min(0, s_c(2)) = -\frac{1}{2} < s <$

$\max(0, s_c(2)) = 0$. Therefore, $s < \min(0, s_c(p))$ is a safer assumption to obtain ill-posedness results. In this paper, we consider cases where $s < 0$.

In [24, Theorem 1], Hyakuna-Tsutsumi established local well-posedness for the cubic NLS in $\mathcal{FL}^p(\mathbb{R}^d)$ for $p \in (4/3, 4) \setminus \{2\}$. Later this result is generalized in [23, Theorem 1.1] for $p \in [1, 2]$.

Our first results concern norm inflation of the type discussed above:

Theorem 1.2 *Assume that $1 \leq p \leq \infty$, $d, \sigma \in \mathbb{N}$ and $s < \min(0, s_c(p))$. For any $\delta > 0$, there exists $\psi_0 \in \mathcal{FL}_s^p(\mathbb{R}^d)$ and $T > 0$ satisfying*

$$\|\psi_0\|_{\mathcal{FL}_s^p} < \delta \quad \text{and} \quad 0 < T < \delta,$$

such that the corresponding solution ψ to (1.1) exists on $[0, T]$ and

$$\|\psi(T)\|_{\mathcal{FL}_s^p} > \delta^{-1}.$$

As discussed above, in the case $s_c(p) > 0$, norm inflation is expected in $\mathcal{FL}_s^p(\mathbb{R}^d)$ for $0 < s < s_c(p)$, but with different arguments. The proof of Theorem 1.2 is inspired by the two-scale analysis of Kishimoto [27]. We also prove norm inflation with an infinite loss of regularity: the initial regularity must be sufficiently small, and we leave out the cubic one-dimensional nonlinearity.

Theorem 1.3 *Let $\sigma \in \mathbb{N}$, $s < -\frac{1}{2\sigma+1}$ and assume $d\sigma \geq 2$. There exist a sequence of initial data $(\psi_n(0))_{n \geq 1}$ in $\mathcal{S}(\mathbb{R}^d)$ such that*

$$\|\psi_n(0)\|_{\mathcal{FL}_s^p} \xrightarrow{n \rightarrow \infty} 0, \quad \forall p \in [1, \infty],$$

and a sequence of times $t_n \rightarrow 0$ such that the corresponding solutions ψ_n to (1.1) satisfies

$$\|\psi_n(t_n)\|_{\mathcal{FL}_k^p} \xrightarrow{n \rightarrow \infty} \infty, \quad \forall k \in \mathbb{R}, \quad \forall p \in [1, \infty].$$

Remark 1.4 There is no general comparison between the assumptions on s in Theorems 1.2 and 1.3: for $p = 1$, $\min(0, s_c(1)) = -1/\sigma < -1/(2\sigma + 1)$, while if $s_c(p) \geq 0$, we obviously have $\min(0, s_c(p)) = 0 > -1/(2\sigma + 1)$.

1.3 Modulation Spaces

We now turn our attention to the theory of modulation spaces. The idea of modulation spaces is to consider the decaying properties of space variable and its Fourier transform simultaneously. Specifically, we consider the short-time Fourier transform (STFT) (sliding-window transform/wave packet transform) of f with respect to Schwartz class function $g \in \mathcal{S}(\mathbb{R}^d)$:

$$V_g f(x, \xi) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-i\xi \cdot t} dt, \quad (x, \xi) \in \mathbb{R}^{2d},$$

whenever the integral exists. Then the modulation spaces $M_s^{p,q}(\mathbb{R}^d)$ ($1 \leq p, q \leq \infty$, $s \in \mathbb{R}$) is defined as the collection of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M_s^{p,q}} = \left\| \|V_g f\|_{L_x^p} (1 + |\xi|^2)^{s/2} \right\|_{L_\xi^q} < \infty,$$

with natural modification if a Lebesgue index is infinite. For $s = 0$, we write $M_0^{p,q}(\mathbb{R}^d) = M^{p,q}(\mathbb{R}^d)$. When $p = q = 2$, modulation spaces coincide with usual Sobolev spaces $H^s(\mathbb{R}^d)$. For the last two decades, these spaces have made their own place in PDEs and there is a tremendous ongoing interest to use these spaces as a low regularity Cauchy data class for nonlinear dispersive equations; see e.g. [2,5,6,32,33,37,38]. Using the algebra property and boundedness of Schrödinger propagator on $M_s^{p,q}(\mathbb{R}^d)$, (1.1) is proved to be locally well-posed in $M_s^{p,1}(\mathbb{R}^d)$ for $1 \leq p \leq \infty$, $s \geq 0$, and in $M_s^{p,q}(\mathbb{R}^d)$ for $1 \leq p, q \leq \infty$ and $s > d(1 - 1/q)$, via fixed point argument; see [2,5,7]. Using uniform-decomposition techniques, Wang and Hudzik [37] established global well-posedness for (1.1) with small initial data in $M^{2,1}(\mathbb{R}^d)$. Guo [21] proved local well-posed for the cubic NLS in $M^{2,q}(\mathbb{R})$ ($2 \leq q \leq \infty$), and later Oh and Wang [32], established global existence for this result. In [13], Chaichenets et al. established global well-posedness for the cubic NLS in $M^{p,p'}(\mathbb{R})$ for p sufficiently close to 2. The well-posedness problems for some other PDEs in $M_s^{p,q}(\mathbb{R}^d)$ are widely studied by many authors, see for instance the excellent survey [33] and references therein. We complement the existing literature on well-posedness theory for (1.1) with Cauchy data in modulation spaces. First, observe that, in view of Proposition 2.6 below,

$$\|\psi(\lambda \cdot)\|_{M_s^{2,q}} \lesssim \begin{cases} \lambda^{-\frac{d}{2}} \max(1, \lambda^s) \|\psi\|_{M_s^{2,q}}, & \text{if } 1 \leq q \leq 2, \\ \lambda^{-d(1-\frac{1}{q})} \max(1, \lambda^s) \|\psi\|_{M_s^{2,q}}, & \text{if } 2 \leq q \leq \infty, \end{cases}$$

for all $\lambda \leq 1$ and $s \in \mathbb{R}$. Invoking the general belief that ill-posedness at positive regularity is due to the transfer from low frequencies ($0 < \lambda \ll 1$) to high frequencies, the scaling (1.2) suggests that ill-posedness occurs in $M_s^{2,q}(\mathbb{R}^d)$ if

$$s < \begin{cases} s_c = \frac{d}{2} - \frac{1}{\sigma} & \text{if } 1 \leq q \leq 2, \\ d \left(1 - \frac{1}{q}\right) & \text{if } 2 \leq q \leq \infty. \end{cases}$$

The following analogue of Theorem 1.2 then appears rather natural.

Theorem 1.5 *Let $d, \sigma \in \mathbb{N}$ and assume that*

- $s < \min\left(\frac{d}{2} - \frac{1}{\sigma}, 0\right)$ when $1 \leq q \leq 2$, and
- $s < \min\left(d\left(1 - \frac{1}{q}\right) - \frac{1}{\sigma}, 0\right)$ when $2 \leq q \leq \infty$.

For any $\delta > 0$, there exists $\psi_0 \in M_s^{2,q}(\mathbb{R}^d)$ and $T > 0$ satisfying

$$\|\psi_0\|_{M_s^{2,q}} < \delta \quad \text{and} \quad 0 < T < \delta$$

such that the corresponding solution ψ to (1.1) exists on $[0, T]$ and

$$\|\psi(T)\|_{M_s^{2,q}} > \delta^{-1}.$$

We also have some infinite loss of regularity of the flow map (1.1) at the level of modulation spaces with negative regularity. We no longer assume $p = 2$, and show a stronger result, provided that the negative regularity s is sufficiently small, and (again) that we discard the one-dimensional cubic case.

Theorem 1.6 *Let $\sigma \in \mathbb{N}$, $s < -\frac{1}{2\sigma+1}$ and assume $d\sigma \geq 2$. There exists a sequence of initial data $(\psi_n(0))_{n \geq 1}$ in $\mathcal{S}(\mathbb{R}^d)$ such that*

$$\|\psi_n(0)\|_{M_s^{p,q}} \xrightarrow{n \rightarrow \infty} 0, \quad \forall p, q \in [1, \infty],$$

and a sequence of times $t_n \rightarrow 0$ such that the corresponding solutions ψ_n to (1.1) satisfies

$$\|\psi_n(t_n)\|_{M_k^{p,q}} \xrightarrow{n \rightarrow \infty} \infty, \quad \forall k \in \mathbb{R}, \quad \forall p, q \in [1, \infty].$$

Remark 1.7 Contrary to the Fourier–Lebesgue case, the assumption regarding s is always weaker in Theorem 1.5 than in Theorem 1.6 (recall that the cubic one-dimensional case is ruled out in Theorem 1.6).

1.4 Comments and Outline of the Paper

As pointed out before, the numerology regarding the norm inflation phenomenon (Theorems 1.2 and 1.5) is probably sharp, up to the fact that the minimum should be replaced by a maximum in the assumption on s , and that at positive regularity, different arguments are required. On the other hand, we believe that the restriction $s < -\frac{1}{2\sigma+1}$ in Theorems 1.3 and 1.6 is due to our approach, and we expect that the result is true under the mere assumption $s < 0$ if $d\sigma \geq 2$, and for $s < -1/2$ if $d = \sigma = 1$.

The analogue of our results remains true if we replace Δ by the generalized dispersion of the form $\Delta_\eta = \sum_{j=1}^d \eta_j \partial_{x_j}^2$, $\eta_j = \pm 1$. The (1.1) associated Δ_η (with the non uniform signs of η_j) arises in the description of surface gravity waves on deep water, see e.g. [35].

In [34], Sugimoto–Wang–Zhang established some local well-posedness results for Davey–Stewartson equation in some weighted modulation spaces. We note that our method of proof can be applied to get norm-inflation results for Davey–Stewartson equation, and infinite loss of regularity in the spirit of [11], in some negative modulation and Fourier–Lebesgue spaces.

Theorems 1.3 and 1.6 cover any smooth power nonlinearity in multidimension, and power nonlinearities which are at least quintic in the one-dimensional case. Our method of proof seems too limited to prove loss of regularity in the case of the cubic nonlinearity on the line. It turns out that the method followed to treat the cubic nonlinearity on the

circle in [12] seems helpless in the case of the line. On the other hand, Theorems 1.2 and 1.5 include the cubic one-dimensional Schrödinger equation.

The rest of this paper is organized as follows, In Sect. 2, we recall various properties associated to modulation spaces. In Sect. 3, we prove Theorem 1.2, and we adapt the argument in Sect. 4 to prove Theorem 1.5. In Sect. 5, we show how the theory of weakly nonlinear geometric optics makes it possible to prove loss of regularity at negative regularity for (1.1). A general framework where multiphase weakly nonlinear geometric optics is justified is presented in Sect. 6, and it is applied in Sect. 7 to prove Theorems 1.3 and 1.6.

Notations The notation $A \lesssim B$ means $A \leq cB$ for a some constant $c > 0$, Let $(\Lambda^\epsilon)_{0 < \epsilon \leq 1}$ and $(\Upsilon^\epsilon)_{0 < \epsilon \leq 1}$ be two families of positive real numbers.

- We write $\Lambda^\epsilon \ll \Upsilon^\epsilon$ if $\limsup_{\epsilon \rightarrow 0} \Lambda^\epsilon / \Upsilon^\epsilon = 0$.
- We write $\Lambda^\epsilon \lesssim \Upsilon^\epsilon$ if $\limsup_{\epsilon \rightarrow 0} \Lambda^\epsilon / \Upsilon^\epsilon < \infty$.
- We write $\Lambda^\epsilon \approx \Upsilon^\epsilon$ if $\Lambda^\epsilon \lesssim \Upsilon^\epsilon$ and $\Upsilon^\epsilon \lesssim \Lambda^\epsilon$.

2 Preliminary: Modulation Spaces

Feichtinger [18] introduced a class of Banach spaces, the so-called modulation spaces, which allow a measurement of space variable and Fourier transform variable of a function, or distribution, on \mathbb{R}^d simultaneously, using the short-time Fourier transform (STFT). The STFT of a function f with respect to a window function $g \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$V_g f(x, y) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-iy \cdot t} dt, \quad (x, y) \in \mathbb{R}^{2d}, \tag{2.1}$$

whenever the integral exists. For $x, y \in \mathbb{R}^d$, the translation operator T_x , and the modulation operator M_y , are defined by $T_x f(t) = f(t-x)$ and $M_y f(t) = e^{iy \cdot t} f(t)$. In terms of these operators the STFT may be expressed as

$$V_g f(x, y) = \langle f, M_y T_x g \rangle = e^{-ix \cdot y} (f * M_y g^*)(x), \tag{2.2}$$

where $\langle f, g \rangle$ denotes the inner product for L^2 functions, or the action of the tempered distribution f on the Schwartz class function g , and $g^*(y) = \overline{g(-y)}$. Thus $V : (f, g) \mapsto V_g(f)$ extends to a bilinear form on $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$, and $V_g(f)$ defines a uniformly continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ whenever $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

Definition 2.1 (Modulation spaces) Let $1 \leq p, q \leq \infty, s \in \mathbb{R}$ and $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$. The weighted modulation space $M_s^{p,q}(\mathbb{R}^d)$ is defined to be the space of all tempered distributions f for which the following norm is finite:

$$\|f\|_{M_s^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, y)|^p dx \right)^{q/p} (1 + |y|^2)^{sq/2} dy \right)^{1/q},$$

for $1 \leq p, q < \infty$. If p or q is infinite, $\|f\|_{M_s^{p,q}}$ is defined by replacing the corresponding integral by the essential supremum.

Remark 2.2 The definition of the modulation space given above, is independent of the choice of the particular window function. See [20, Proposition 11.3.2(c)].

We recall an alternative definition of modulation spaces via the frequency-uniform localization techniques, providing another characterization which will be useful to prove Theorem 1.5. Let Q_n be the unit cube with the center at n , so $(Q_n)_{n \in \mathbb{Z}^d}$ constitutes a decomposition of \mathbb{R}^d , that is, $\mathbb{R}^d = \cup_{n \in \mathbb{Z}^d} Q_n$. Let $\rho \in \mathcal{S}(\mathbb{R}^d)$, $\rho : \mathbb{R}^d \rightarrow [0, 1]$ be a smooth function satisfying $\rho(\xi) = 1$ if $|\xi|_\infty \leq \frac{1}{2}$ and $\rho(\xi) = 0$ if $|\xi|_\infty \geq 1$. Let ρ_n be a translation of ρ , that is,

$$\rho_n(\xi) = \rho(\xi - n), \quad n \in \mathbb{Z}^d.$$

Denote

$$\sigma_n(\xi) = \frac{\rho_n(\xi)}{\sum_{\ell \in \mathbb{Z}^d} \rho_\ell(\xi)}, \quad n \in \mathbb{Z}^d.$$

Then $(\sigma_n(\xi))_{n \in \mathbb{Z}^d}$ satisfies the following properties:

$$\left\{ \begin{array}{l} |\sigma_n(\xi)| \geq c, \forall \xi \in Q_n, \\ \text{supp } \sigma_n \subset \{\xi : |\xi - n|_\infty \leq 1\}, \\ \sum_{n \in \mathbb{Z}^d} \sigma_n(\xi) \equiv 1, \forall \xi \in \mathbb{R}^d, \\ |D^\alpha \sigma_n(\xi)| \leq C_{|\alpha|}, \forall \xi \in \mathbb{R}^d, \alpha \in (\mathbb{N} \cup \{0\})^d. \end{array} \right. \tag{2.3}$$

The frequency-uniform decomposition operators can be exactly defined by

$$\square_n = \mathcal{F}^{-1} \sigma_n \mathcal{F}.$$

For $1 \leq p, q \leq \infty, s \in \mathbb{R}$, it is known [18] that

$$\|f\|_{M_s^{p,q}} \asymp \left(\sum_{n \in \mathbb{Z}^d} \|\square_n(f)\|_{L^p}^q (1 + |n|)^{sq} \right)^{1/q},$$

with natural modifications for $p, q = \infty$.

Lemma 2.3 ([20,33,38]) *Let $p, q, p_j, q_j \in [1, \infty], s, s_j \in \mathbb{R} (j = 1, 2)$. Then*

- (1) $M_{s_1}^{p_1, q_1}(\mathbb{R}^d) \hookrightarrow M_{s_2}^{p_2, q_2}(\mathbb{R}^d)$ whenever $p_1 \leq p_2$ and $q_1 \leq q_2$ and $s_2 \leq s_1$. In particular, $H^s(\mathbb{R}^d) \hookrightarrow M_s^{p, q}(\mathbb{R}^d)$ for $2 \leq p, q \leq \infty$ and $s \in \mathbb{R}$.
- (2) $M_{s_1}^{p_1, q_1}(\mathbb{R}^d) \hookrightarrow M_{s_2}^{p_2, q_2}(\mathbb{R}^d)$ for $q_1 > q_2, s_1 > s_2$ and $s_1 - s_2 > d/q_2 - d/q_1$.
- (3) $M^{p, q_1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p, q_2}(\mathbb{R}^d)$ holds for $q_1 \leq \min\{p, p'\}$ and $q_2 \geq \max\{p, p'\}$ with $\frac{1}{p} + \frac{1}{p'} = 1$.
- (4) $M^{\min\{p', 2\}, p}(\mathbb{R}^d) \hookrightarrow \mathcal{FL}^p(\mathbb{R}^d) \hookrightarrow M^{\max\{p', 2\}, p}(\mathbb{R}^d), \frac{1}{p} + \frac{1}{p'} = 1$.
- (5) $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p, q}(\mathbb{R}^d)$ if p and q are finite.

- (6) $M^{p,p}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,p'}(\mathbb{R}^d)$ for $1 \leq p \leq 2$ and $M^{p,p'}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,p}(\mathbb{R}^d)$ for $2 \leq p \leq \infty$.
- (7) The Fourier transform $\mathcal{F} : M_s^{p,p}(\mathbb{R}^d) \rightarrow M_s^{p,p}(\mathbb{R}^d)$ is an isomorphism.
- (8) The space $M_s^{p,q}(\mathbb{R}^d)$ is a Banach space.
- (9) The space $M_s^{p,q}(\mathbb{R}^d)$ is invariant under complex conjugation.

Theorem 2.4 (Algebra property) *Let $p, q, p_i, q_i \in [1, \infty]$ ($i = 0, 1, 2$). If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0}$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q_0}$, then*

$$M^{p_1,q_1}(\mathbb{R}^d) \cdot M^{p_2,q_2}(\mathbb{R}^d) \hookrightarrow M^{p_0,q_0}(\mathbb{R}^d); \tag{2.4}$$

with norm inequality $\|fg\|_{M^{p_0,q_0}} \lesssim \|f\|_{M^{p_1,q_1}} \|g\|_{M^{p_2,q_2}}$. In particular, the space $M^{p,q}(\mathbb{R}^d)$ is a pointwise $\mathcal{FL}^1(\mathbb{R}^d)$ -module, that is, we have

$$\|fg\|_{M^{p,q}} \lesssim \|f\|_{\mathcal{FL}^1} \|g\|_{M^{p,q}}.$$

Proof The product relation (2.4) between modulation spaces is well known and we refer the interested reader to [5] and since $\mathcal{FL}^1(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d)$, the desired inequality (2.4) follows. □

For $f \in \mathcal{S}(\mathbb{R}^d)$, the Schrödinger propagator $e^{i\frac{t}{2}\Delta}$ is given by

$$e^{i\frac{t}{2}\Delta} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-i\frac{t}{2}|\xi|^2} \hat{f}(\xi) d\xi.$$

The first point in the following statement was established in [4], and the second, in [37, Proposition 4.1].

Proposition 2.5 ([4,37])

- (1) Let $t \in \mathbb{R}$, $p, q \in [1, \infty]$. Then

$$\|e^{i\frac{t}{2}\Delta} f\|_{M^{p,q}} \leq C(t^2 + 1)^{d/4} \|f\|_{M^{p,q}}$$

where C is some constant depending on d .

- (2) Let $2 \leq p \leq \infty, 1 \leq q \leq \infty$. Then

$$\|e^{i\frac{t}{2}\Delta} f\|_{M^{p,q}} \leq (1 + |t|)^{-d\left(\frac{1}{p} - \frac{1}{2}\right)} \|f\|_{M^{p',q}}.$$

For $(1/p, 1/q) \in [0, 1] \times [0, 1]$, we define the subsets

$$\begin{aligned} I_1 &= \{(p, q); \max(1/p, 1/p') \leq 1/q\}, & I_1^* &= \{(p, q); \min(1/p, 1/p') \geq 1/q\}, \\ I_2 &= \{(p, q); \max(1/q, 1/2) \leq 1/p'\}, & I_2^* &= \{(p, q); \min(1/q, 1/2) \geq 1/p'\}, \\ I_3 &= \{(p, q); \max(1/q, 1/2) \leq 1/p\}, & I_3^* &= \{(p, q); \min(1/q, 1/2) \geq 1/p\}. \end{aligned}$$

We now define the indices:

$$\mu_1(p, q) = \begin{cases} -1/p & \text{if } (1/p, 1/q) \in I_1^*, \\ 1/q - 1 & \text{if } (1/p, 1/q) \in I_2^*, \\ -2/p + 1/q & \text{if } (1/p, 1/q) \in I_3^*, \end{cases}$$

and

$$\mu_2(p, q) = \begin{cases} -1/p & \text{if } (1/p, 1/q) \in I_1, \\ 1/q - 1 & \text{if } (1/p, 1/q) \in I_2, \\ -2/p + 1/q & \text{if } (1/p, 1/q) \in I_3. \end{cases}$$

The dilation operator f_λ is given by

$$f_\lambda(x) = f(\lambda x), \quad \lambda > 0.$$

Proposition 2.6 (See Theorem 3.2 in [17]) *Let $1 \leq p, q \leq \infty, s \in \mathbb{R}$. There exists a constant $C > 0$ such that for all $f \in M_s^{p,q}(\mathbb{R}^d), 0 < \lambda \leq 1$, we have*

$$C^{-1} \lambda^{d\mu_1(p,q)} \min\{1, \lambda^s\} \|f\|_{M_s^{p,q}} \leq \|f_\lambda\|_{M_s^{p,q}} \leq C \lambda^{d\mu_2(p,q)} \max\{1, \lambda^s\} \|f\|_{M_s^{p,q}}.$$

3 Norm Inflation in Fourier–Lebesgue Spaces

Define

$$\mu_\sigma(z_1, \dots, z_{2\sigma+1}) = \prod_{\ell=1}^{\sigma+1} z_\ell \prod_{m=\sigma+2}^{2\sigma+1} \bar{z}_m.$$

Definition 3.1 For $\psi_0 \in L^2(\mathbb{R}^d)$, define $U_1[\psi_0](t) = e^{i\frac{t}{2}\Delta}\psi_0$,

$$U_k[\psi_0](t) = -i \sum_{\substack{k_1, \dots, k_{2\sigma+1} \geq 1 \\ k_1 + \dots + k_{2\sigma+1} = k}} \int_0^t e^{i\frac{(t-\tau)}{2}\Delta} \mu_\sigma(U_{k_1}[\psi_0], \dots, U_{k_{2\sigma+1}}[\psi_0])(\tau) d\tau, \quad k \geq 2.$$

It is known that the solution ψ of (1.1) can be written as a power series expansion $\psi = \sum_{k=1}^\infty U_k[\psi_0]$, see [3], and [25,27] for later refinements of the method.

Definition 3.2 Let $A > 0$ be a dyadic number. Define the space M_A as the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|f\|_{M_A} = \sum_{\xi \in A\mathbb{Z}^d} \|\hat{f}\|_{L^2(\xi + Q_A)}, \quad Q_A = [-A/2, A/2)^d.$$

Lemma 3.3 ([3,27]) *Let $A > 0$ be a dyadic number.*

- (1) $M_A \sim_A M_1$, and for all $\epsilon > 0$, $H^{\frac{d}{2}+\epsilon} \hookrightarrow M_1 \hookrightarrow L^2$.
- (2) M_A is a Banach algebra under pointwise multiplication, and

$$\|fg\|_{M_A} \leq C(d)A^{d/2}\|f\|_{M_A}\|g\|_{M_A} \quad \forall f, g \in M_A.$$

- (3) Let $A \geq 1$ be a dyadic number and $\phi \in M_A$ with $\|\psi_0\|_{M_A} \leq M$. Then, there exists $C > 0$ independent of A and M such that

$$\|U_k[\psi_0](t)\|_{M_A} \leq t^{\frac{k-1}{2\sigma}}(CA^{d/2}M)^{k-1}M,$$

for any $t \geq 0$ and $k \geq 1$.

- (4) Let $(b_k)_{k=1}^\infty$ be a sequence of nonnegative real numbers such that

$$b_k \leq C \sum_{\substack{k_1, \dots, k_{2\sigma+1} \geq 1 \\ k_1 + \dots + k_{2\sigma+1} = k}} b_{k_1} \cdots b_{k_{2\sigma+1}} \quad \forall k \geq 2.$$

Then we have

$$b_k \leq b_1 C_0^{k-1}, \quad \forall k \geq 1, \text{ where } C_0 = \frac{\pi^2}{6} (C(2\sigma + 1)^2)^{1/(2\sigma)} b_1.$$

Corollary 3.4 (See Corollary 1 in [27]) *Let $A \geq 1$ be dyadic and $M > 0$. If $0 < T \ll (A^{d/2}M)^{-2\sigma}$, then for any $\psi_0 \in M_A$ with $\|\psi_0\|_{M_A} \leq M$:*

- (i) *A unique solution ψ to the integral equation associated with (1.1),*

$$\psi(t) = e^{i\frac{t}{2}\Delta}\psi_0 - i \int_0^t e^{i\frac{(t-\tau)}{2}\Delta}\mu_\sigma(\psi(\tau))d\tau$$

exists in $C([0, T], M_A)$.

- (ii) *The solution ψ given in (i) has the expression*

$$\psi = \sum_{k=1}^\infty U_k[\psi_0] = \sum_{\ell=0}^\infty U_{2\sigma\ell+1}[\psi_0] \tag{3.1}$$

which converges absolutely in $C([0, T], M_A)$.

Remark 3.5 By Definition 3.1, we obtain $U_k[\psi_0](t) = 0$ unless $k \equiv 1 \pmod{2\sigma}$. For instance, $U_k[\psi_0](t) \equiv 0$ for all $k \in 2\sigma\mathbb{N}$. To see this, fix $\sigma \in \mathbb{N}$. Then clearly $U_{2\sigma}[\psi_0] \equiv 0$ because there does not exist $k_j \geq 1$ such that $k_1 + \dots + k_{2\sigma+1} = 2\sigma$. Now since $U_{2\sigma}[\psi_0] \equiv 0$, it follows that $U_{4\sigma}[\psi_0] \equiv 0$ and so on. Thus, $U_k[\psi_0](t) \equiv 0$ for all $k \in 2\sigma\mathbb{N}$.

The general idea from [3] to prove instability is to show that one term in the sum (3.1) dominates the sum of the other terms, and rules out the continuity of the flow map. Usually, the first Picard iterate accounting for nonlinear effects, that is, $U_{2\sigma+1}[\psi_0]$ in

our case, does the job. The proof of Theorems 1.2 and 1.5 indeed relies on this idea, for a suitable ψ_0 as in [27].

Let N, A be dyadic numbers to be specified so that $N \gg 1$ and $0 < A \ll N$. We choose initial data of the following form

$$\widehat{\psi}_0 = RA^{-d/p}N^{-s}\chi_\Omega, \tag{3.2}$$

for a positive constant R and a set Ω satisfying

$$\Omega = \bigcup_{\eta \in \Sigma} (\eta + Q_A),$$

for some $\Sigma \subset \{\xi \in \mathbb{R}^d : |\xi| \sim N\}$ such that $\#\Sigma \leq 3$. Then we have

$$\|\psi_0\|_{\mathcal{F}L_s^p} \sim R, \quad \|\psi_0\|_{M_A} \sim RA^{d(\frac{1}{2}-\frac{1}{p})}N^{-s}.$$

In fact, we have

$$\begin{aligned} \|\psi_0\|_{\mathcal{F}L_s^p}^p &= R^p A^{-d} N^{-sp} \int_\Omega (1 + |\xi|^2)^{ps/2} d\xi \\ &= R^p A^{-d} N^{-sp} \sum_\eta \int_{\eta+Q_A} (1 + |\xi|^2)^{ps/2} d\xi. \end{aligned}$$

Since $A < N$ and $|\eta| \sim N$, we have $N^2 \lesssim (1 + |\xi|^2) \lesssim N^2$ for $\xi \in \eta + Q_A$ and so $N^{ps} \lesssim (1 + |\xi|^2)^{ps/2} \lesssim N^{ps}$. As $\#\Sigma \leq 3$ and $|\eta + Q_A| \sim A^d$, we infer that $\|\psi_0\|_{\mathcal{F}L_s^p}^p \sim R^p$.

Lemma 3.6 (See Lemma 3.6 in [27]) *There exists $C > 0$ such that for any ψ_0 satisfying (3.2) and $k \geq 1$, we have*

$$\left| \text{supp } \widehat{U_k[\psi_0]}(t) \right| \leq C^k A^d, \quad \forall t \geq 0.$$

The next result is the analogue of [27, Lemma 3.7].

Lemma 3.7 *Let ψ_0 given by (3.2), $s < 0$ and $1 \leq p \leq \infty$. Then there exists $C > 0$ depending only on d, σ and s such that following holds.*

$$\|U_1[\psi_0](T)\|_{\mathcal{F}L_s^p} \leq CR, \quad \forall T \geq 0, \tag{3.3}$$

$$\|U_k[\psi_0](T)\|_{\mathcal{F}L_s^p} \lesssim \rho_1^{k-1} C^k A^{-d/p} RN^{-s} \| \langle \cdot \rangle^s \|_{L^p(Q_A)}, \tag{3.4}$$

where $\rho_1 = RN^{-s} A^{d(1-\frac{1}{p})} T^{\frac{1}{2\sigma}}$.

Proof The Schrödinger group is a Fourier multiplier,

$$\|U_1[\psi_0](T)\|_{\mathcal{F}L_s^p} = \left\| \left(e^{i\frac{T}{2}|\cdot|^2} \widehat{\psi_0}(\cdot) \right)^s \right\|_{L^p} = \|\psi_0\|_{\mathcal{F}L_s^p} \leq CR,$$

hence (3.3). We note that

$$\begin{aligned} I &:= \|U_k[\psi_0](T)\|_{\mathcal{F}L_s^p} \\ &\leq \|\langle \cdot \rangle^s\|_{L^p(\text{supp } \widehat{U_k[\psi_0]}(t))} \sup_{\xi \in \mathbb{R}^d} \left| \widehat{U_k[\psi_0]}(t, \xi) \right| \\ &\leq \|\langle \cdot \rangle^s\|_{L^p(\text{supp } \widehat{U_k[\psi_0]}(t))} \sum_{k_1+\dots+k_{2\sigma+1}=k} \int_0^t \| |v_{k_1}(\tau)| * \dots * |v_{k_{2\sigma+1}}(\tau)| \|_{L^\infty} d\tau, \end{aligned}$$

where v_{k_ℓ} is either $\widehat{U_{k_\ell}[\psi_0]}$ or $\overline{\widehat{U_{k_\ell}[\psi_0]}}$. By Young and Cauchy-Schwarz inequalities,

$$\begin{aligned} \|v_{k_1} * \dots * v_{k_{2\sigma+1}}\|_{L^\infty} &\leq \|v_{k_1} * v_{k_2}\|_{L^\infty} \|v_{k_3} * \dots * v_{k_{2\sigma+1}}\|_{L^1} \\ &\leq \|v_{k_1}^\vee v_{k_2}^\vee\|_{L^1} \prod_{\ell=3}^{2\sigma+1} \|v_{k_\ell}\|_{L^1} \\ &\leq \|v_{k_1}\|_{L^2} \|v_{k_2}\|_{L^2} \prod_{\ell=3}^{2\sigma+1} \|v_{k_\ell}\|_{L^1} \\ &\leq \prod_{\ell=3}^{2\sigma+1} \left| \text{supp } \widehat{U_{k_\ell}[\psi_0]} \right|^{1/2} \prod_{\ell=1}^{2\sigma+1} \|U_{k_\ell}[\psi_0]\|_{L^2}. \end{aligned}$$

Thus, we have

$$I \leq \|\langle \cdot \rangle^s\|_{L^p(\text{supp } \widehat{U_k[\psi_0]}(t))} I_1,$$

where

$$I_1 := \sum_{k_1+\dots+k_{2\sigma+1}=k} \int_0^t \prod_{\ell=3}^{2\sigma+1} \left| \text{supp } \widehat{U_{k_\ell}[\psi_0]}(\tau) \right|^{1/2} \prod_{\ell=1}^{2\sigma+1} \|U_{k_\ell}[\psi_0](\tau)\|_{L^2} d\tau.$$

By Lemma 3.3 (3) (with $M = CRN^{-s} A^{\frac{d}{2}-\frac{d}{p}}$), we have, for all $k \geq 1$,

$$\|U_k[\psi_0](t)\|_{L^2} \leq \|U_k[\psi_0](t)\|_{M_A} \leq Ct^{\frac{k-1}{2\sigma}} \left(C^2 RA^{d/2} N^{-s} A^{\frac{d}{2}-\frac{d}{p}} \right)^{k-1} RN^{-s} A^{\frac{d}{2}-\frac{d}{p}}.$$

Note that, by Lemma 3.6,

$$\begin{aligned}
 I_1 &\lesssim \sum_{k_1+\dots+k_{2\sigma+1}=k} \int_0^t \prod_{\ell=3}^{2\sigma+1} A^{d/2} \prod_{\ell=1}^{2\sigma+1} \left[\tau^{\frac{k_\ell-1}{2\sigma}} \left(RA^{d(1-\frac{1}{p})} N^{-s} \right)^{k_\ell-1} RN^{-s} A^{\frac{d}{2}-\frac{d}{p}} \right] d\tau \\
 &\lesssim (RN^{-s})^k A^{\frac{d(2\sigma-1)}{2}} A^{d(1-\frac{1}{p})(k-2\sigma-1)} A^{(\frac{d}{2}-\frac{d}{p})(2\sigma+1)} \int_0^t \tau^{\frac{k-2\sigma-1}{2\sigma}} d\tau \\
 &\lesssim A^{d(1-\frac{1}{p})(k-1)} A^{-d/p} (RN^{-s})^k t^{\frac{k-1}{2\sigma}}.
 \end{aligned}$$

Since $s < 0$, for any bounded set $D \subset \mathbb{R}^d$, we have

$$|\{ \langle \xi \rangle^s > \lambda \} \cup D| \leq |\{ \langle \xi \rangle^s > \lambda \} \cup B_D|, \quad \forall \lambda > 0,$$

where $B_D \subset \mathbb{R}^d$ is the ball centered at origin with $|D| = |B_D|$. This implies that $\| \langle \xi \rangle^s \|_{L^p(D)} \leq \| \langle \xi \rangle^s \|_{L^p(B_D)}$. In view of this and performing simple change of variables ($\xi = C^{k/d} \xi'$), we obtain

$$\| \langle \cdot \rangle^s \|_{L^p(\text{supp } \widehat{U}_k[\psi_0](t))} \leq \| \langle \cdot \rangle^s \|_{L^p(\{ |\xi| \leq C^{k/d} A \})} \lesssim C^k \| \langle \cdot \rangle^s \|_{L^p(\{ |\xi| \leq A \})},$$

and the lemma follows. □

In the next lemma we establish a crucial lower bound on $U_{2\sigma+1}[\psi_0]$.

Lemma 3.8 *Let $1 \leq p \leq \infty$, $1 \leq A \ll N$ and $\Sigma = \{Ne_d, -Ne_d, 2Ne_d\}$ where $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$. If $0 < T \ll N^{-2}$, then we have*

$$\| U_{2\sigma+1}[\psi_0](T) \|_{\mathcal{F}L^p_\Sigma} \gtrsim RA^{-\frac{d}{p}} N^{-s} \rho_1^{2\sigma} \| \langle \cdot \rangle^s \|_{L^p(Q_A)},$$

where $\rho_1 = RN^{-s} A^{d-\frac{d}{p}} T^{\frac{1}{2\sigma}}$.

Proof Note that

$$U_{2\sigma+1}[\widehat{\psi_0}](T, \xi) = ce^{-i\frac{T}{2}|\xi|^2} \int_\Gamma \prod_{\ell=1}^{\sigma+1} \widehat{\psi_0}(\xi_\ell) \prod_{m=\sigma+2}^{2\sigma+1} \overline{\widehat{\psi_0}(\xi_m)} \int_0^T e^{i\frac{1}{2}\Phi} dt d\xi_1 \dots d\xi_{2\sigma},$$

where

$$\begin{aligned}
 \Gamma &= \left\{ (\xi_1, \dots, \xi_{2\sigma+1}) \in \mathbb{R}^{(2\sigma+1)d} : \sum_{\ell=1}^{\sigma+1} \xi_\ell - \sum_{m=\sigma+2}^{2\sigma+1} \xi_m = \xi \right\}, \\
 \Phi &= |\xi|^2 - \sum_{\ell=1}^{\sigma+1} |\xi_\ell|^2 + \sum_{m=\sigma+2}^{2\sigma+1} |\xi_m|^2.
 \end{aligned}$$

By the choice of initial data (3.2), we have

$$\begin{aligned} \int_{\Gamma} \prod_{\ell=1}^{\sigma+1} \widehat{\psi}_0(\xi_{\ell}) \prod_{m=\sigma+2}^{2\sigma+1} \overline{\widehat{\psi}_0(\xi_m)} &= \int_{\Gamma} \prod_{\ell=1}^{\sigma+1} RA^{-d/p} N^{-s} \chi_{\Omega}(\xi_{\ell}) \prod_{m=\sigma+2}^{2\sigma+1} RA^{-d/p} N^{-s} \chi_{\Omega}(\xi_m) \\ &= \left(RA^{-d/p} N^{-s} \right)^{2\sigma+1} \int_{\Gamma} \prod_{\ell=1}^{2\sigma+1} \chi_{\Omega}(\xi_{\ell}) d\xi_1 \dots d\xi_{2\sigma} \\ &= \left(RA^{-d/p} N^{-s} \right)^{2\sigma+1} \sum_C \int_{\Gamma} \prod_{\ell=1}^{2\sigma+1} \chi_{\eta_{\ell} + Q_A}(\xi_{\ell}) d\xi_1 \dots d\xi_{2\sigma}, \end{aligned}$$

where the sum is taken over the non-empty set

$$C = \left\{ (\eta_1, \dots, \eta_{2\sigma+1}) \in \{\pm Ne_d, 2Ne_d\}^{2\sigma+1} : \sum_{\ell=1}^{\sigma+1} \eta_{\ell} - \sum_{m=\sigma+2}^{2\sigma+1} \eta_m = 0 \right\}.$$

For $\xi \in Q_A$, we have $|\xi_i|^2 \leq |\xi|^2 \leq A^2 \ll N^2$ and so $|\Phi| \lesssim N^2$. Then $|\frac{t}{2}\Phi(\xi)| \ll 1$ for $0 < T \ll N^{-2}$. In view of this, together with the fact that the cosine function decreasing on $[0, \pi/4]$, we obtain

$$\left| \int_0^T e^{i\frac{t}{2}\Phi(\xi)} dt \right| \geq \operatorname{Re} \int_0^T e^{i\frac{t}{2}\Phi(\xi)} dt \geq \frac{1}{2}T.$$

Taking the above inequalities into account, we infer

$$\left| U_{2\sigma+1}[\widehat{\psi}_0](T, \xi) \right| \gtrsim \left(RA^{-d/p} N^{-s} \right)^{2\sigma+1} (A^d)^{2\sigma} T \chi_{(2\sigma+1)^{-1}Q_A}(\xi). \tag{3.5}$$

Hence, we have

$$\begin{aligned} \|U_{2\sigma+1}[\psi_0](T)\|_{\mathcal{F}L_s^p} &\gtrsim \left(RA^{-d/p} N^{-s} \right)^{2\sigma+1} (A^d)^{2\sigma} T \|\langle \cdot \rangle^s\|_{L^p((2\sigma+1)^{-1}Q_A)} \\ &\gtrsim RA^{-\frac{d}{p}} N^{-s} \rho_1^{2\sigma} \|\langle \cdot \rangle^s\|_{L^p(Q_A)}, \end{aligned}$$

where $\rho_1 = RN^{-s} A^{d-\frac{d}{p}} T^{\frac{1}{2\sigma}}$. □

For the convenience of reader, we compute the L^p -norm of weight $\langle \cdot \rangle^s$ on the cube Q_A .

Lemma 3.9 *Let $A \gg 1, d \geq 1, s < 0$ and $1 \leq p < \infty$. We define*

$$f_s^p(A) = \begin{cases} 1 & \text{if } s < -\frac{d}{p}, \\ (\log A)^{1/p} & \text{if } s = -\frac{d}{p}, \\ A^{d/p+s} & \text{if } s > -\frac{d}{p}. \end{cases}$$

Then we have $f_s^p(A) \lesssim \|(\cdot)^s\|_{L^p(Q_A)} \lesssim f_s^p(A)$ and $f_s^\infty(A) = \|(\cdot)^s\|_{L^\infty(Q_A)} \sim 1$. In particular, $f_s^p(A) \gtrsim A^{\frac{d}{p}+s}$ for any $s < 0$.

Proof We first compute the $\|\cdot\|_{L^p}$ -norm on ball of radius R_1 in \mathbb{R}^d , say $B_{R_1}(0)$. Since $(\cdot)^s$ is radial, we have

$$I(R_1) := \int_{B_{R_1}(0)} \frac{1}{(1 + |\xi|^2)^{-sp/2}} d\xi = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{R_1} \frac{r^{d-1}}{(1 + r^2)^{-sp/2}} dr.$$

Notice that $(1 + r^2)^{-sp/2} \geq \max\{1, r^{-sp}\}$, and assuming that $R_1 \gg 1$, we obtain:

$$\begin{aligned} I(R_1) &\lesssim \int_0^1 \frac{r^{d-1}}{\max\{1, r^{-sp}\}} dt + \int_1^{R_1} \frac{r^{d-1}}{\max\{1, r^{-sp}\}} dr \\ &= \int_0^1 r^{d-1} dr + \int_1^{R_1} \frac{1}{r^{-sp-d+1}} dr. \end{aligned}$$

Using conditions on s , we have $I(R_1) \lesssim (f_s^p(R_1))^p$. Notice that $Q_A \subset B_{\sqrt{d}A/2}(0)$, we have $\|(\cdot)^s\|_{L^p(Q_A)} \leq \left(I(\sqrt{d}A/2)\right)^{1/p} \lesssim f_s^p(A)$. On the other hand, we notice that $1 + r^2 \leq 2$ if $0 < r < 1$ and $1 + r^2 \leq 2r^2$ if $1 < r < R_2$ for some appropriate R_2 . Using this together with the above ideas, we obtain $f_s^p(A) \lesssim \|(\cdot)^s\|_{L^p(Q_A)}$. This completes the proof. \square

Proof of Theorem 1.2 By Corollary 3.4, we have the existence of a unique solution to (1.1) in M_A up to time T whenever $\rho_1 = RN^{-s}A^{d(1-\frac{1}{p})}T^{1/(2\sigma)} \ll 1$. In view of Lemma 3.7 and since $\rho_1 < 1$, $\sum_{\ell=2}^\infty \|U_{2\sigma\ell+1}[\psi_0](T)\|_{\mathcal{F}L_s^p}$ can be dominated by the sum of the geometric series. Specifically, we have

$$\begin{aligned} \left\| \sum_{\ell=2}^\infty U_{2\sigma\ell+1}[\psi_0](T) \right\|_{\mathcal{F}L_s^p} &\lesssim A^{-d/2} RN^{-s} f_s^p(A) \sum_{\ell=2}^\infty \rho_1^{2\sigma\ell} \\ &\lesssim A^{-d/2} RN^{-s} f_s^p(A) \rho_1^{4\sigma}. \end{aligned} \tag{3.6}$$

By Corollary 3.4 and the triangle inequality, we obtain

$$\begin{aligned} \|\psi(T)\|_{\mathcal{F}L_s^p} &= \left\| \sum_{\ell=0}^\infty U_{2\sigma\ell+1}[\psi_0] \right\|_{\mathcal{F}L_s^p} \\ &\geq \|U_{2\sigma+1}[\psi_0](T)\|_{\mathcal{F}L_s^p} - \|U_1[\psi_0](T)\|_{\mathcal{F}L_s^p} - \left\| \sum_{\ell=2}^\infty U_{2\sigma\ell+1}[\psi_0](T) \right\|_{\mathcal{F}L_s^p}. \end{aligned}$$

In order to ensure

$$\|\psi(T)\|_{\mathcal{F}L_s^p} \gtrsim \|U_{2\sigma+1}[\psi_0](T)\|_{\mathcal{F}L_s^p},$$

we rely on the conditions

$$\|U_{2\sigma+1}[\psi_0](T)\|_{\mathcal{F}L_s^p} \gg \|U_1[\psi_0](T)\|_{\mathcal{F}L_s^p}, \tag{3.7}$$

$$\|U_{2\sigma+1}[\psi_0](T)\|_{\mathcal{F}L_s^p} \gg \left\| \sum_{\ell=2}^{\infty} U_{2\sigma\ell+1}[\psi_0](T) \right\|_{\mathcal{F}L_s^p}. \tag{3.8}$$

To use Lemma 3.8, we require

(i) $T \ll N^{-2}$.

In view of Lemma 3.7, to prove (3.7) it is sufficient to prove

(ii) $R\rho_1^{2\sigma} A^{-\frac{d}{p}} N^{-s} f_s^p(A) \gg R$, with $\rho_1 = RN^{-s} A^{d-\frac{d}{p}} T^{\frac{1}{2\sigma}}$.

Finally, in view of Lemmas 3.7, 3.8 and 3.9, and (3.6), to prove (3.8) it is sufficient to prove:

(iii) $\rho_1 \ll 1$,

(iv) $R\rho_1^{2\sigma} A^{-\frac{d}{p}} N^{-s} f_s^p(A) \gg R\rho_1^{4\sigma} A^{-\frac{d}{p}} N^{-s} f_s^p(A)$.

We now choose A , R and T so that conditions (i)- (iv) are satisfied. To this end, we set

$$R = (\log N)^{-1}, \quad A \sim (\log N)^{-\frac{2\sigma+2}{|s|}} N, \quad T = (A^{d(\frac{1}{p}-1)} N^s)^{2\sigma}.$$

Then we have

$$\rho_1 = RN^{-s} A^{d-\frac{d}{p}} T^{\frac{1}{2\sigma}} = (\log N)^{-1} \ll 1.$$

Hence, condition (iii) is satisfied and so condition (iv). Note that

$$T = (\log N)^{-\frac{2\sigma+2}{|s|}d(\frac{1}{p}-1)2\sigma} N^{d(\frac{1}{p}-1)2\sigma+2\sigma s}.$$

Since $s < d\left(1 - \frac{1}{p}\right) - \frac{1}{\sigma}$ and $\log N = \mathcal{O}(N^\epsilon)$ for any $\epsilon > 0$, we have

$$T \ll N^{-2},$$

and hence (i) is satisfied. By Lemma 3.9, we have $f_s^p(A) \gtrsim A^{\frac{d}{p}+s}$ for any $s < 0$ and $A \geq 1$ and so

$$R\rho_1^{2\sigma} A^{-\frac{d}{p}} N^{-s} f_s^p(A) \gtrsim \log N \gg (\log N)^{-1} = R$$

and hence (ii) is satisfied. Thus, we have $\|\psi(T)\|_{\mathcal{F}L_s^p} \gtrsim \|U_{2\sigma+1}[\psi_0](T)\|_{\mathcal{F}L_s^p} \gtrsim \log N$. Since $\|\psi_0\|_{\mathcal{F}L_s^p} \sim R = (\log N)^{-1}$ and $T \ll N^{-2}$, we get norm inflation by letting $N \rightarrow \infty$. □

4 Norm Inflation in Modulation Spaces

The proof of Theorem 1.5 follows the same general lines as the proof of Theorem 1.2 from the previous section. Let N, A be dyadic numbers to be specified so that $N \gg 1$ and $0 < A \ll N$. We choose initial data of the following form

$$\widehat{\psi}_0 = \begin{cases} RA^{-d/2}N^{-s}\chi_\Omega, & \text{if } 1 \leq q \leq 2, \\ RA^{-d/q}N^{-s}\chi_\Omega, & \text{if } 2 \leq q \leq \infty, \end{cases} \tag{4.1}$$

where

$$\Omega = \bigcup_{\eta \in \Sigma} (\eta + Q_A), \quad Q_A = [-A/2, A/2],$$

for some $\Sigma \subset \{\xi \in \mathbb{R}^d : |\xi| \sim N\}$ such that $\#\Sigma \leq 3$.

4.1 A Priori Estimates: $1 \leq q \leq 2$

Then we have, for any $s \in \mathbb{R}$,

$$\|\psi_0\|_{H^s} \sim R, \quad \|\psi_0\|_{M_A} \sim RN^{-s}.$$

Lemma 4.1 *Let $q \in [1, 2]$, ψ_0 given by (4.1), $s < 0$. Then there exists $C > 0$ depending only on d, σ and s such that following holds.*

$$\|U_1[\psi_0](T)\|_{M_s^{2,q}} \leq CR, \quad \forall T \geq 0, \tag{4.2}$$

$$\|U_k[\psi_0](T)\|_{M_s^{2,q}} \lesssim \rho^{k-1} C^k A^{-d/2} RN^{-s} \|(1 + |n|)^s\|_{\ell^q(0 \leq |n| \leq A)}, \tag{4.3}$$

where $\rho = RN^{-s} A^{d/2} T^{\frac{1}{2\sigma}}$.

Proof By Lemma 2.3 and Proposition 2.5, we have

$$\|U_1[\psi_0](T)\|_{M_s^{2,q}} \lesssim \|\psi_0(T)\|_{M_s^{2,q}} \lesssim \|\psi_0(T)\|_{M_s^{2,1}} \lesssim R,$$

hence (4.2). By Plancherel theorem and (2.3), for $s < 0$, we have

$$\begin{aligned} \|U_k[\psi_0](T)\|_{M_s^{2,q}} &= \left\| (1 + |n|)^s \|\sigma_n \widehat{U_k[\psi_0]}(T)\|_{L^2} \right\|_{\ell^q} \\ &\leq \sup_{\xi \in \mathbb{R}^d} \left| \widehat{U_k[\psi_0]}(t, \xi) \right| \left\| (1 + |n|)^s \|\sigma_n\|_{L^2(Q_n \cap \text{supp } \widehat{U_k[\psi_0]}(t))} \right\|_{\ell^q} \\ &\leq \sup_{\xi \in \mathbb{R}^d} \left| \widehat{U_k[\psi_0]}(t, \xi) \right| \|(1 + |n|)^s\|_{\ell^q(0 \leq |n| \leq CA)}. \end{aligned}$$

This yields the desired inequality in (4.3). □

Lemma 4.2 *Let $s < 0$, $q \in [1, 2]$, $2 \leq A \ll N$ and $\Sigma = \{Ne_d, -Ne_d, 2Ne_d\}$ where $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$. If $0 < T \ll N^{-2}$, then we have*

$$\|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}} \gtrsim RA^{-\frac{d}{2}}N^{-s}\rho^{2\sigma} \|(1 + |n|)^s\|_{\ell^q(0 \leq |n| \leq A)},$$

where $\rho = RN^{-s}A^{d/2}T^{\frac{1}{2\sigma}}$.

Proof By (2.3), we note that

$$\begin{aligned} \|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}}^q &= \sum_{n \in \mathbb{Z}^d} \|\square_n(U_{2\sigma+1}[\psi_0](T))\|_{L^2}^q (1 + |n|)^{sq} \\ &= \sum_{n \in \mathbb{Z}^d} \|\sigma_n \widehat{U_{2\sigma+1}[\psi_0]}(T)\|_{L^2}^q (1 + |n|)^{sq} \\ &\gtrsim \sum_{n \in \mathbb{Z}^d} \frac{1}{(1 + |n|)^{-sq}} \left(\int_{Q_n} |\widehat{U_{2\sigma+1}[\psi_0]}(\xi, T)|^2 d\xi \right)^{q/2}, \end{aligned}$$

where Q_n is a unit cube centered at $n \in \mathbb{Z}^d$. Arguing as before in the proof of Lemma 3.8 (specifically, by (3.5)), for $\xi \in Q_A = [-A/2, A/2]^d$, we have

$$\left| \widehat{U_{2\sigma+1}[\psi_0]}(T, \xi) \right| \gtrsim \left(RA^{-d/2}N^{-s} \right)^{2\sigma+1} (A^d)^{2\sigma} T \chi_{(2\sigma+1)^{-1}Q_A}(\xi).$$

It follows that

$$\begin{aligned} \|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}} &\gtrsim \left(RA^{-d/2}N^{-s} \right)^{2\sigma+1} (A^d)^{2\sigma} T \left(\sum_{|n|=\lfloor -A/2 \rfloor}^{\lfloor A/2 \rfloor} \frac{1}{(1 + |n|)^{-sq}} \right)^{1/q} \\ &\gtrsim RA^{-d/2}N^{-s}\rho^{2\sigma} \left(\sum_{|n|=\lfloor -A/2 \rfloor}^{\lfloor A/2 \rfloor} \frac{1}{(1 + |n|)^{-sq}} \right)^{1/q}, \end{aligned}$$

where the floor function is $\lfloor x \rfloor = \max \{m \in \mathbb{Z} \mid m \leq x\}$ and $\rho = RN^{-s}A^{d/2}T^{\frac{1}{2\sigma}}$. \square

4.2 A Priori Estimates: $2 \leq q \leq \infty$

Then we have, for any $s \in \mathbb{R}$,

$$\|\psi_0\|_{\mathcal{F}L_s^q} \sim R, \quad \|\psi_0\|_{M_A} \sim RA^{d(\frac{1}{2}-\frac{1}{q})}N^{-s}.$$

Lemma 4.3 *Let $s < 0$ and $2 \leq q \leq \infty$. Then there exists $C > 0$ depending only on d, σ and s such that following holds.*

$$\|U_1[\psi_0](T)\|_{M_s^{2,q}} \leq CR, \quad \forall T \geq 0, \tag{4.4}$$

$$\|U_k[\psi_0](T)\|_{M_s^{2,q}} \lesssim \rho_2^{k-1} C^k A^{-d/q} RN^{-s} \|(1 + |n|)^s\|_{\ell^q(0 \leq |n| \leq A)}, \tag{4.5}$$

where $\rho_2 = RN^{-s} A^{d(1-\frac{1}{q})} T^{\frac{1}{2\sigma}}$.

Proof By Lemma 2.3, we have

$$\|U_1[\psi_0](T)\|_{M_s^{2,q}} \lesssim \|U_1[\psi_0](T)\|_{\mathcal{FL}^q} \leq CR.$$

The proof of (4.5) is similar to Lemmas 4.1, (4.3) and 3.7, (3.4), we omit the details. □

The next lemma is the analogue of Lemmas 3.8 and 4.2, so we leave out its proof.

Lemma 4.4 *Let $s < 0$, $2 \leq q \leq \infty$, $1 \leq A \ll N$ and $\sum = \{Ne_d, -Ne_d, 2Ne_d\}$ where $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$. If $0 < T \ll N^{-2}$, then we have*

$$\|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}} \gtrsim RA^{-\frac{d}{q}} N^{-s} \rho_2^{2\sigma} \|(1 + |n|)^s\|_{\ell^q(0 \leq |n| \leq A)},$$

where $\rho_2 = RN^{-s} A^{d-\frac{d}{q}} T^{\frac{1}{2\sigma}}$.

4.3 Proof of Theorem 1.5

Lemma 4.5 *Let $s < 0$. In the limit $A \rightarrow \infty$, we have, for $1 \leq q < \infty$,*

$$\|(1 + |n|)^s\|_{\ell^q(0 \leq |n| \leq A)} \underset{A \rightarrow \infty}{\sim} g_s^q(A) := \begin{cases} 1 & \text{if } -sq > d, \\ (\log A)^{1/q} & \text{if } -sq = d, \\ A^{1/q+s} & \text{if } -sq < d, \end{cases}$$

and $\|(1 + |n|)^s\|_{\ell^\infty(0 \leq |n| \leq A)} \underset{A \rightarrow \infty}{\sim} 1$.

Proof Since $(1 + |\xi|)^{sq}$ is a decreasing function in $|\xi|$, in view of the integral test and Lemma 3.9, we have, for $1 \leq q < \infty$,

$$\begin{aligned} \|(1 + |n|)^s\|_{\ell^q(0 \leq |n| \leq A)} &= \left(\sum_{0 \leq |n| \leq A} \frac{1}{(1 + |n|)^{-sq}} \right)^{1/q} \underset{A \rightarrow \infty}{\sim} \left(\int_{|\xi| \leq A} \frac{d\xi}{(1 + |\xi|)^{-sq}} \right)^{1/q} \\ &\underset{A \rightarrow \infty}{\sim} \left(\int_0^A \frac{r^{d-1} dr}{(1 + r)^{-sq}} \right)^{1/q}, \end{aligned}$$

hence the result for q finite. The case $q = \infty$ is straightforward. □

To prove Theorem 1.5, we distinguish two cases.

First case: $1 \leq q \leq 2$. By Corollary 3.4, we have the existence of solution to (1.1) in M_A up to time T whenever $\rho = RN^{-s} A^{d/2} T^{1/(2\sigma)} \ll 1$. In view of Lemma 3.7

and since $\rho < 1$, $\sum_{\ell=2}^{\infty} \|U_{2\sigma\ell+1}[\psi_0](T)\|_{M_s^{2,q}}$ can be dominated by the sum of a geometric series. Specifically, we have

$$\begin{aligned} \left\| \sum_{\ell=2}^{\infty} U_{2\sigma\ell+1}[\psi_0](T) \right\|_{M_s^{2,q}} &\lesssim A^{-d/2} RN^{-s} g_s^q(A) \sum_{\ell=2}^{\infty} \rho^{2\sigma\ell} \\ &\lesssim A^{-d/2} RN^{-s} g_s^q(A) \rho^{4\sigma}. \end{aligned} \tag{4.6}$$

By Corollary 3.4 and the triangle inequality, we obtain

$$\begin{aligned} \|\psi(T)\|_{M_s^{2,q}} &= \left\| \sum_{\ell=0}^{\infty} U_{2\sigma\ell+1}[\psi_0] \right\|_{M_s^{2,q}} \\ &\geq \|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}} - \|U_1[\psi_0](T)\|_{M_s^{2,q}} - \left\| \sum_{\ell=2}^{\infty} U_{2\sigma\ell+1}[\psi_0](T) \right\|_{M_s^{2,q}}. \end{aligned}$$

In order to ensure

$$\|\psi(T)\|_{M_s^{2,q}} \gtrsim \|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}},$$

we rely on the conditions

$$\|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}} \gg \|U_1[\psi_0](T)\|_{M_s^{2,q}}, \tag{4.7}$$

$$\|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}} \gg \left\| \sum_{\ell=2}^{\infty} U_{2\sigma\ell+1}[\psi_0](T) \right\|_{M_s^{2,q}}. \tag{4.8}$$

In view of Lemmas 4.1 and 4.2, (4.7) amount to the condition

$$R\rho^{2\sigma} A^{-\frac{d}{2}} N^{-s} g_s^q(A) \gg R \tag{4.9}$$

In view of Lemmas 4.1 and 4.2, and (4.6), (4.8) amount to the condition

$$T \ll N^{-2}, \quad \rho \ll 1, \quad R\rho^{2\sigma} A^{-\frac{d}{2}} N^{-s} g_s^q(A) \gg R\rho^{4\sigma} A^{-d/2} N^{-s} g_s^q(A). \tag{4.10}$$

We now choose A, R and T so that conditions (4.9) and (4.10) are satisfied. To this end, we set

$$R = (\log N)^{-1}, \quad A \sim (\log N)^{-\frac{2\sigma+2}{|s|}} N, \quad T = (A^{-d/2} N^s)^{2\sigma}.$$

Then we have

$$\rho = RN^{-s} A^{d/2} T^{\frac{1}{2\sigma}} = (\log N)^{-1} \ll 1.$$

Note that

$$T = (\log N)^{-\frac{2\sigma+2}{|s|}d(\frac{1}{2}-1)2\sigma} N^{-d\sigma+2\sigma s}.$$

Since $s < \frac{d}{2} - \frac{1}{\sigma}$ and $\log N = \mathcal{O}(N^\epsilon)$ for any $\epsilon > 0$, we have

$$T \ll N^{-2}.$$

We have $g_s^q(A) \gtrsim A^{\frac{d}{q}+s}$ for any $s < 0$ and $A \geq 1$. Thus, for $1 \leq q \leq 2$, we have

$$\begin{aligned} R\rho^{2\sigma} A^{-\frac{d}{2}} N^{-s} g_s^q(A) &\gtrsim (\log N)^{-(2\sigma+1)} (\log N)^{2\sigma+2} A^{d(\frac{1}{q}-\frac{1}{2})} \gtrsim \log N \\ &\gg (\log N)^{-1} = R, \end{aligned}$$

and (4.9) is satisfied. Thus, we have $\|\psi(T)\|_{M_s^{2,q}} \gtrsim \|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}} \gtrsim \log N$. Since $\|\psi_0\|_{M_s^{2,q}} \lesssim R = (\log N)^{-1}$ and $T \ll N^{-2}$, we get norm inflation by letting $N \rightarrow \infty$. This completes the proof for $q \in [1, 2]$.

Second case: $2 \leq q \leq \infty$. By Corollary 3.4, we have the existence of solution to (1.1) in M_A up to time T whenever $\rho_2 = RN^{-s} A^{d(1-\frac{1}{q})} T^{1/(2\sigma)} \ll 1$. By Lemmas 4.3 and 4.4, the conditions

$$T \ll N^{-2}, \quad \rho_2 \ll 1 \quad \text{and} \quad R\rho_2^{2\sigma} A^{-\frac{d}{q}} N^{-s} g_s^q(A) \gg R \tag{4.11}$$

ensures that

$$\|\psi(T)\|_{M_s^{2,q}} \gtrsim \|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}} \sim R\rho_1^{2\sigma} A^{-\frac{d}{q}} N^{-s} g_s^q(A).$$

We now choose A, R and T so that conditions (4.11) satisfied. To this end, we set

$$R = (\log N)^{-1}, \quad A \sim (\log N)^{-\frac{2\sigma+2}{|s|}d(\frac{1}{q}-1)2\sigma} N, \quad T = (A^{d(\frac{1}{q}-1)} N^s)^{2\sigma}.$$

Then we have

$$\rho_2 = RN^{-s} A^{d-\frac{d}{q}} T^{\frac{1}{2\sigma}} = (\log N)^{-1} \ll 1.$$

Note that

$$T = (\log N)^{-\frac{2\sigma+2}{|s|}d(\frac{1}{q}-1)2\sigma} N^{d(\frac{1}{q}-1)2\sigma+2\sigma s}.$$

Since $s < d(1 - \frac{1}{q}) - \frac{1}{\sigma}$ and $\log N = \mathcal{O}(N^\epsilon)$ for any $\epsilon > 0$, we have

$$T \ll N^{-2}.$$

Note that $g_s^q(A) \gtrsim A^{\frac{d}{q}+s}$ for any $s < 0$ and $A \geq 1$ and so

$$R\rho_2^{2\sigma} A^{-\frac{d}{q}} N^{-s} g_s^q(A) \gtrsim \log N \gg (\log N)^{-1} = R.$$

Thus, we have $\|\psi(T)\|_{M_s^{2,q}} \gtrsim \|U_{2\sigma+1}[\psi_0](T)\|_{M_s^{2,q}} \gtrsim \log N$. Since $\|\psi_0\|_{M_s^{2,q}} \lesssim R = (\log N)^{-1}$ and $T \ll N^{-2}$, we get norm inflation by letting $N \rightarrow \infty$. This completes the proof of Theorem 1.5.

5 Norm Inflation as a By-product of Geometric Optics

The proof of Theorems 1.3 and 1.6 follows the same strategy as in [11]: through a suitable rescaling, we turn the ill-posedness result into an asymptotic result, which can be expressed in the framework of weakly nonlinear geometric optics. More precisely, we change the unknown function ψ to u , via

$$u^\varepsilon(t, x) = \varepsilon^{\frac{2-J}{2\sigma}} \psi(\varepsilon t, x), \tag{5.1}$$

where the parameter ε will tend to zero. For ψ solution to (1.1), u^ε solves

$$i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = \mu \varepsilon^J |u^\varepsilon|^{2s} u^\varepsilon. \tag{5.2}$$

The case $J = 1$ corresponds to *weakly nonlinear geometric optics* (WNLGO), as defined in [9]. As noticed in [11] in the framework of Sobolev spaces, a phenomenon of infinite loss of regularity can be proved via this WNLGO setting, under the assumption $s < -1/(2\sigma)$ in the analogue of Theorems 1.3 and 1.6. Like in that paper, in order to weaken the assumption on s to $s < -1/(2\sigma + 1)$, we will have to consider some value $J > 1$, and perform some ‘‘asymptotic sin’’, in the sense that we change the hierarchy in an asymptotic expansion involving the limit $\varepsilon \rightarrow 0$.

The heuristic idea is the same as in [15]: when negative regularity is involved, the zero Fourier mode plays a stronger role than (large) non-zero modes, which come with a small factor. With the scaling (5.1) in mind, our goal is to show that we may consider initial data (for u) of the form

$$u(0, x) = \sum_{j \neq 0} e^{ij \cdot x/\varepsilon} \alpha_j(x),$$

that is containing only rapidly oscillatory terms, and such that the evolution under (5.2) creates a non-trivial non-oscillatory term.

To be more specific, recall the strategy of multiphase nonlinear geometric optics (see [10] for more details): we plug an *ansatz* of the form

$$u(t, x) = \sum_j e^{i\phi_j(t,x)/\varepsilon} a_j(t, x)$$

into (5.2), and order the powers of ε . The most singular term is of order ε^0 , it is the eikonal equation:

$$\partial_t \phi_j + \frac{1}{2} |\nabla \phi_j|^2 = 0.$$

In the case of an initial phase $\phi_j(0, x) = j \cdot x$, no caustic appears, and the global solution is given by

$$\phi_j(t, x) = j \cdot x - \frac{|j|^2}{2} t. \tag{5.3}$$

In the sequel, we consider such phases, for $j \in \mathbb{Z}^d$. The next term in the hierarchy is of order ε^1 , but as evoked above, we “cheat”, and incorporate some nonlinear effects even if $J > 1$ (and $J < 2$),

$$\partial_t a_j + j \cdot \nabla_x a_j = -i\mu\varepsilon^{J-1} \sum_{\phi_{k_1} - \phi_{k_2} + \dots + \phi_{k_{2\sigma+1}} = \phi_j} a_{k_1} \bar{a}_{k_2} \dots a_{k_{2\sigma+1}}, \tag{5.4}$$

where we have used $\nabla \phi_j(t, x) = j$. Again in view of the specific form of the phase (5.3), the condition on the sum involves a *resonant condition*, $(k_1, k_2, \dots, k_{2\sigma+1}) \in \mathcal{R}_j$, where

$$\mathcal{R}_j = \left\{ (k_\ell)_{1 \leq \ell \leq 2\sigma+1}, \sum_{\ell=1}^{2\sigma+1} (-1)^{\ell+1} k_\ell = j, \sum_{\ell=1}^{2\sigma+1} (-1)^{\ell+1} |k_\ell|^2 = |j|^2 \right\}.$$

In the cubic case $\sigma = 1$, those sets are described exactly:

Lemma 5.1 (See [10,16]) *Suppose $\sigma = 1$.*

- *If $d = 1$, then $\mathcal{R}_j = \{(j, \ell, \ell), (\ell, \ell, j) ; \ell \in \mathbb{Z}\}$.*
- *If $d \geq 2$, then $(k_1, k_2, k_3) \in \mathcal{R}_j$ precisely when the endpoints of the vectors k_1, k_2, k_3, j for four corners of a non-degenerate rectangle with k_2 and j opposing each other, or when this quadruplet corresponds to one of the following two degenerate cases: $(k_1 = j, k_2 = k_3)$ or $(k_1 = k_2, k_3 = j)$.*

The above lemma explains why our approach distinguishes the one-dimensional case and the multi-dimensional case, and in particular why the cubic one-dimensional case is left out in Theorems 1.3 and 1.6.

5.1 Multi-dimensional Case

The leading idea in [11] is to start from three non-trivial modes only, in the case $d \geq 2$, and create at least one new mode (possibly more if $\sigma \geq 2$), corresponding to $j = 0$.

Lemma 5.2 *Let $d \geq 2$ and $\sigma \in \mathbb{N}^*$. Define $k_1, k_2, k_3 \in \mathbb{Z}^d$ as*

$$k_1 = (1, 0, \dots, 0), \quad k_2 = (1, 1, 0, \dots, 0), \quad k_3 = (0, 1, 0, \dots, 0).$$

For initial data of the form

$$u(0, x) = \alpha(x) \sum_{j=1}^3 e^{ik_j \cdot x/\varepsilon}, \quad \alpha \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\},$$

we have $a_{0|t=0} = 0$ and $\partial_t a_{0|t=0} = \varepsilon^{J-1} c_0 \alpha(x)$, with $c_0 = \sharp \mathcal{R}_0 \geq 1$.

This lemma is straightforward, in view of (5.4), and since $(k_1, k_2, k_3) \in \mathcal{R}_0$ if $\sigma = 1$, $(k_1, k_2, k_3, k_1, k_1, \dots, k_1) \in \mathcal{R}_0$ if $\sigma \geq 2$.

5.2 One-Dimensional Case

In the one-dimensional case, we have a similar result, provided that the nonlinearity is at least quintic, in view of [12, Lemma 4.2] (and Example 4.3 there):

Lemma 5.3 *Let $d = 1$ and $\sigma \geq 2$. Define $k_1, k_2, k_3, k_4, k_5 \in \mathbb{Z}$ as*

$$(k_1, k_2, k_3, k_4, k_5) = (2, -1, -2, 4, 3).$$

For initial data of the form

$$u(0, x) = \alpha(x) \sum_{j=1}^5 e^{ik_j x/\varepsilon}, \quad \alpha \in \mathcal{S}(\mathbb{R}) \setminus \{0\},$$

we have $a_{0|t=0} = 0$ and $\partial_t a_{0|t=0} = \varepsilon^{J-1} c_0 \alpha(x)$, with $c_0 = \sharp \mathcal{R}_0 \geq 1$.

5.3 How to Conclude

Supposing that we can prove that the geometric expansion recalled above provides an approximation u_{app} for the solution u to (5.2), suitable in the sense that the error is measured in a sufficiently strong norm, the idea is that both $u|_{t=0}$ and $u - u_{\text{app}}$ are small in spaces involving negative regularity in x , while Lemma 5.2 or 5.3 implies that u_{app} is large in many spaces.

6 (Very) Weakly Nonlinear Geometric Optics

6.1 A Convenient Functional Framework

Throughout this section, we denote by X a Banach algebra in the space variable, that is

$$\exists C > 0, \quad \|fg\|_X \leq C \|f\|_X \|g\|_X, \quad \forall f, g \in X. \tag{6.1}$$

We suppose that X is translation invariant and, denoting by $\tau_k f(x) = f(x - k)$,

$$\|\tau_k f\|_X = \|f\|_X, \quad \forall k \in \mathbb{R}^d, \quad \forall f \in X. \tag{6.2}$$

We assume in addition that the multiplication by plane wave oscillations leaves the X -norm invariant,

$$\|f e_k\|_X = \|f\|_X, \quad \forall k \in \mathbb{R}^d, \text{ where } e_k(x) = e^{ik \cdot x}. \tag{6.3}$$

Note that this assumption rules out Sobolev spaces $H^s(\mathbb{R}^d)$, unless $s = 0$. Finally, we assume that the Schrödinger group acts on X , at least locally in time:

Assumption 6.1 There exists T_0 such that $e^{i\frac{t}{2}\Delta}$ maps X to X for $t \in [0, T_0]$, and

$$\exists C > 0, \quad \|e^{i\frac{t}{2}\Delta}\|_{\mathcal{L}(X,X)} \leq C, \quad \forall t \in [0, T_0].$$

In [10,19], the case $X = \mathcal{FL}^1(M)$ was considered, with $M = \mathbb{T}^d$ (a choice resumed in [12]) or \mathbb{R}^d . In [11,30], the choice $X = \mathcal{FL}^1 \cap L^2(\mathbb{R}^d)$ was motivated by the presence of more singular nonlocal nonlinearities. We shall consider later two sorts of X spaces: $\mathcal{FL}^1 \cap \mathcal{FL}^p$ or $\mathcal{FL}^1 \cap M^{1,1}$.

We denote by

$$Y = \left\{ (a_j)_{j \in \mathbb{Z}^d}, \sum_{j \in \mathbb{Z}^d} \|a_j\|_X < \infty \right\} = \ell^1(X),$$

and

$$Y_2 = \left\{ (a_j)_{j \in \mathbb{Z}^d} \in Y, \sum_{j \in \mathbb{Z}^d} \left(\langle j \rangle^2 \|a_j\|_X + \langle j \rangle \|\nabla a_j\|_X + \|\Delta a_j\|_X \right) < \infty \right\}.$$

We suppose that u solves (5.2) with initial data

$$u(0, x) = \sum_{j \in \mathbb{Z}^d} \alpha_j(x) e^{ij \cdot x / \varepsilon}.$$

6.2 Construction of the Approximate Solution

The approximate solution is given by

$$u_{\text{app}}(t, x) = \sum_{j \in \mathbb{Z}^d} a_j(t, x) e^{i\phi_j(t,x)/\varepsilon},$$

where ϕ_j is given by (5.3) and the a_j 's solve (5.4), with initial data α_j .

Lemma 6.2 Let $d \geq 1$, $\sigma \in \mathbb{N}^*$ and $J \geq 1$.

- If $(\alpha_j)_{j \in \mathbb{Z}^d} \in Y$, then there exists $T > 0$ independent of $\varepsilon \in [0, 1]$ and a unique solution $(a_j)_{j \in \mathbb{Z}^d} \in C([0, T]; Y)$ to the system (5.4), such that $a_j|_{t=0} = \alpha_j$ for all $j \in \mathbb{Z}^d$.

- If in addition $(\alpha_j)_{j \in \mathbb{Z}^d} \in Y_2$, then $(a_j)_{j \in \mathbb{Z}^d} \in C([0, T]; Y_2)$.

Proof In view of Duhamel’s formula for (5.4),

$$a_j(t, x) = a_j(0, x - jt) - i\lambda \sum_{(k_1, k_2, \dots, k_{2\sigma+1}) \in \mathcal{R}_j} \int_0^t (a_{k_1} \bar{a}_{k_2} \dots a_{k_{2\sigma+1}})(s, x - j(t - s)) ds,$$

the first point of the lemma is straightforward, as an easy consequence of (6.2) and (6.1), and a fixed point argument. The second point requires a little bit more care: it was proven in [10, Proposition 5.12] in the case $X = \mathcal{FL}^1$ (Wiener algebra), and the proof relies only on the properties of X required at the beginning of this section. □

From now on, we assume $(\alpha_j)_{j \in \mathbb{Z}^d} \in Y_2$.

6.3 Error Estimate

First, in view of the assumptions made in this section, a standard fixed point argument yields, in view of (6.1) and Assumption 6.1:

Lemma 6.3 *Let $d \geq 1$, $\sigma \in \mathbb{N}^*$ and $J \geq 1$. If $u_0 \in X$, then there exists $T^\varepsilon > 0$ and a unique solution $u \in C([0, T^\varepsilon]; X)$ to (5.2) such that $u|_{t=0} = u_0$.*

To construct the approximate solution u_{app} , we have discarded two families of terms:

- Non-resonant terms, involving the source term

$$r_1 := \mu \varepsilon^J \sum_j \sum_{(k_1, k_2, \dots, k_{2\sigma+1}) \notin \mathcal{R}_j} a_{k_1} \bar{a}_{k_2} \dots a_{k_{2\sigma+1}} e^{i(\phi_{k_1} - \phi_{k_2} + \dots + \phi_{2\sigma+1})/\varepsilon}.$$

- Higher order terms, involving

$$r_2 := \frac{\varepsilon^2}{2} \sum_j \Delta a_j e^{i\phi_j/\varepsilon}.$$

Indeed, u_{app} solves

$$i\varepsilon \partial_t u_{\text{app}} + \frac{\varepsilon^2}{2} \Delta u_{\text{app}} = \mu \varepsilon^J |u_{\text{app}}|^{2\sigma} u_{\text{app}} + r_1 + r_2. \tag{6.4}$$

Duhamel’s formula for $u - u_{\text{app}} =: w$ reads

$$\begin{aligned} w(t) &= -i\mu \varepsilon^{J-1} \int_0^t e^{i\varepsilon \frac{t-\tau}{2} \Delta} (|u|^{2\sigma} u - |u_{\text{app}}|^{2\sigma} u_{\text{app}})(\tau) d\tau \\ &\quad - i\mu \varepsilon^{J-1} \sum_{j=1,2} \int_0^t e^{i\varepsilon \frac{t-\tau}{2} \Delta} r_j(\tau) d\tau. \end{aligned}$$

In view of our assumptions on X , we readily have, thanks to Minkowski inequality,

$$\begin{aligned} \|w(t)\|_X \leq & C \int_0^t \left(\|u_{\text{app}}(\tau)\|_X^{2\sigma} + \|w(\tau)\|_X^{2\sigma} \right) \|w(\tau)\|_X d\tau \\ & + C\varepsilon^{-1} \sum_{j=1,2} \left\| \int_0^t e^{i\varepsilon \frac{t-\tau}{2} \Delta} r_j(\tau) d\tau \right\|_X, \end{aligned}$$

for some C independent of $\varepsilon \in [0, 1]$ and $t \in [0, T_0]$. In view of the second point of Lemma 6.2, we readily have

$$\left\| \int_0^t e^{i\varepsilon \frac{t-\tau}{2} \Delta} r_2(\tau) d\tau \right\|_X \lesssim \varepsilon^2.$$

By construction, r_1 is the sum of terms of the form $g(t, x)e^{ik \cdot x/\varepsilon - \omega t/(2\varepsilon)}$, with $k \in \mathbb{Z}^d$, $\omega \in \mathbb{Z}$, and the non-resonance property reads exactly $|k|^2 \neq \omega$.

Lemma 6.4 *Let $k \in \mathbb{R}^d$, $\omega \in \mathbb{R}$, with $|k|^2 \neq \omega$. Define*

$$D^\varepsilon(t, x) = \int_0^t e^{i\varepsilon \frac{t-\tau}{2} \Delta} \left(g(\tau, x)e^{ik \cdot x/\varepsilon - i\omega\tau/(2\varepsilon)} \right) d\tau.$$

Then we have

$$\begin{aligned} D^\varepsilon(t, x) = & \frac{-2i\varepsilon}{|k|^2 - \omega} e^{i\varepsilon \frac{t-\tau}{2} \Delta} \left(g(\tau, x)e^{ik \cdot x/\varepsilon - i\omega\tau/(2\varepsilon)} \right) \Big|_0^t \\ & + \frac{2i\varepsilon}{|k|^2 - \omega} \int_0^t e^{i\varepsilon \frac{t-\tau}{2} \Delta} \left(e^{ik \cdot x/\varepsilon - i\omega\tau/(2\varepsilon)} \left(\frac{i}{2} (\varepsilon \Delta g + 2k \cdot \nabla g) + \partial_t g \right) (\tau, x) \right) d\tau. \end{aligned}$$

In particular, for $t \in [0, T_0]$,

$$\begin{aligned} \|D^\varepsilon(t)\|_X \lesssim & \frac{\varepsilon}{\left| |k|^2 - \omega \right|} \left(\|g\|_{L^\infty([0,t];X)} + \|\Delta g\|_{L^\infty([0,t];X)} + |k| \|\nabla g\|_{L^\infty([0,t];X)} \right. \\ & \left. + \|\partial_t g\|_{L^\infty([0,t];X)} \right). \end{aligned}$$

Proof The last estimate follows directly from the identity of the lemma, (6.3) and Assumption 6.1, so we only address the identity, which is essentially established in [10, Lemma 5.7] (up to the typos there). Setting $\eta = \xi - k/\varepsilon$, the (spatial) Fourier transform of D is given by

$$\begin{aligned} \widehat{D}^\varepsilon(t, \xi) &= e^{-i\varepsilon t|\eta+k/\varepsilon|^2/2} \int_0^t e^{i\varepsilon\tau|\eta+k/\varepsilon|^2/2} \widehat{b}(\tau, \eta) e^{-i\omega\tau/(2\varepsilon)} d\tau \\ &= e^{-i\varepsilon t|\eta+k/\varepsilon|^2/2} \int_0^t e^{i\tau\theta/2} \widehat{b}(\tau, \eta) d\tau \\ &= e^{-i\varepsilon t|\eta+k/\varepsilon|^2/2} \int_0^t e^{i\tau\theta_2/2} e^{i\tau\theta_1/2} \widehat{b}(\tau, \eta) d\tau, \end{aligned}$$

where we have denoted

$$\theta = \varepsilon \left| \eta + \frac{k}{\varepsilon} \right|^2 - \frac{\omega}{\varepsilon} = \underbrace{\varepsilon|\eta|^2 + 2k \cdot \eta}_{\theta_1} + \underbrace{\frac{|k|^2 - \omega}{\varepsilon}}_{\theta_2}.$$

Integrate by parts, by first integrating $e^{i\tau\theta_2/2}$:

$$e^{i\varepsilon \frac{t}{2} |\xi|^2} \widehat{D}^\varepsilon(t, \xi) = -\frac{2i}{\theta_2} e^{i\tau\theta_2/2} \widehat{b}(\tau, \eta) \Big|_0^t + \frac{2i}{\theta_2} \int_0^t e^{i\tau\theta_2/2} \left(i \frac{\theta_1}{2} \widehat{b}(\tau, \eta) + \widehat{\partial_t b}(\tau, \eta) \right) d\tau.$$

The identity follows by inverting the Fourier transform. □

We infer:

Proposition 6.5 *Let $d \geq 1$, $\sigma \in \mathbb{N}^*$, $J \geq 1$, and $(\alpha_j)_{j \in \mathbb{Z}^d} \in Y_2$. Then for T as in Lemma 6.2,*

$$\|u - u_{\text{app}}\|_{L^\infty([0, T]; X)} \lesssim \varepsilon.$$

Proof First, Lemma 6.2 and (5.4) imply that we also have $(\partial_t a_j)_{j \in \mathbb{Z}^d} \in C([0, T]; Y)$. Then, in view of these properties and Lemma 6.4, we have

$$\left\| \int_0^t e^{i\varepsilon \frac{t-\tau}{2} \Delta} r_1(\tau) d\tau \right\|_X \lesssim \varepsilon^{J+1},$$

where we have used the fact that in the application of Lemma 6.4, $\left| |k|^2 - \omega \right| \geq 1$, since now $k \in \mathbb{Z}^d$ and $\omega \in \mathbb{Z}$. We infer

$$\|w(t)\|_X \leq C \int_0^t \left(\|u_{\text{app}}(\tau)\|_X^{2\sigma} + \|w(\tau)\|_X^{2\sigma} \right) \|w(\tau)\|_X d\tau + C\varepsilon^J + C\varepsilon,$$

where C is independent of $\varepsilon \in [0, 1]$ and $t \in [0, T]$. Lemmas 6.2 and 6.3 yield $w \in C([0, \min(T, T^\varepsilon)]; X)$. Since $w|_{t=0} = 0$, the above inequality and a standard continuity argument imply that $u \in C([0, T]; X)$ provided that $\varepsilon > 0$ is sufficiently small, along with the announced error estimate. □

7 Norm Inflation with Infinite Loss of Regularity

7.1 Proof of Theorem 1.3

For $1 < J < 2$ to be fixed later, let u^ε defined by (5.1), and consider the initial data given by Lemma 5.2 (if $d \geq 2$) or Lemma 5.3 (if $d = 1$ and $\sigma \geq 2$). We apply the analysis from Sect. 6 with $X = \mathcal{FL}^1 \cap \mathcal{FL}^\infty$. This is obviously a Banach algebra, (6.1) holds, thanks to Young inequality, the X -norm is invariant by translation, and by multiplication by plane wave oscillations as in (6.3). Assumption 6.1 is satisfied with $C = 1$ for any $T_0 > 0$, since the Schrödinger group is a Fourier multiplier of modulus one. We can therefore invoke the conclusions of Lemma 5.2 (if $d \geq 2$), Lemma 5.3 (if $d = 1$ and $\sigma \geq 2$), and Proposition 6.5 (in all cases). In order to translate these properties involving u solving (5.2) in terms of ψ solving (1.1), we use the following lemma:

Lemma 7.1 *Let $d \geq 1$. For $f \in \mathcal{S}'(\mathbb{R}^d)$ and $j \in \mathbb{R}^d$, denote*

$$I^\varepsilon(f, j)(x) = f(x)e^{ij \cdot x/\varepsilon}.$$

- For all $s \in \mathbb{R}$, $p \in [1, \infty]$, and $f \in \mathcal{FL}_s^p(\mathbb{R}^d)$, $\|I^\varepsilon(f, 0)\|_{\mathcal{FL}_s^p} = \|f\|_{\mathcal{FL}_s^p}$.
- Let $j \in \mathbb{R}^d \setminus \{0\}$. For all $s \leq 0$, there exists $C = C(j)$ independent of $p \in [1, \infty]$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|I^\varepsilon(f, j)\|_{\mathcal{FL}_s^p} \leq C\varepsilon^{|s|} \|f\|_{\mathcal{FL}_{|s|}^p}.$$

Proof We have obviously

$$I^\varepsilon(\widehat{f, j})(\xi) = \widehat{f}\left(\xi - \frac{j}{\varepsilon}\right).$$

The first point is thus trivial. For the second one, if p is finite,

$$\|I^\varepsilon(f, j)\|_{\mathcal{FL}_s^p}^p = \int \langle \xi \rangle^{ps} \left| \widehat{f}\left(\xi - \frac{j}{\varepsilon}\right) \right|^p d\xi.$$

Note that, for $s \leq 0$,

$$\begin{aligned} \|I^\varepsilon(f, j)\|_{\mathcal{FL}_s^p}^p &= \int \langle \xi \rangle^{ps} \left\langle \xi - \frac{j}{\varepsilon} \right\rangle^{ps} \left\langle \xi - \frac{j}{\varepsilon} \right\rangle^{p|s|} \left| \widehat{f}\left(\xi - \frac{j}{\varepsilon}\right) \right|^p d\xi \\ &\leq \sup_{\xi \in \mathbb{R}^d} \left(\langle \xi \rangle^{-1} \left\langle \xi - \frac{j}{\varepsilon} \right\rangle^{-1} \right)^{p|s|} \|f\|_{\mathcal{FL}_{|s|}^p}^p \end{aligned}$$

For $j \neq 0$,

$$\inf_{\xi \in \mathbb{R}^d} \langle \xi \rangle \left\langle \xi - \frac{j}{\varepsilon} \right\rangle \gtrsim \frac{1}{\varepsilon},$$

hence the second point of the lemma in the case p finite. The case $p = \infty$ follows from the same estimate, controlling the supremum of a product by the product of the suprema. \square

With $u|_{t=0}$ as in Lemmas 5.2 or 5.3, the above lemma yields, in view of (5.1), and for $s < 0$,

$$\|\psi(0)\|_{\mathcal{FL}_s^p} \lesssim \varepsilon^{\frac{J-2}{2\sigma}+|s|}.$$

This sequence of initial data is small (in \mathcal{FL}_s^p for all p) provided that

$$|s| > \frac{2-J}{2\sigma}. \tag{7.1}$$

Lemmas 5.2 and 5.3 show that there exists $\tau > 0$ independent of ε such that

$$\|a_0(\tau)\|_{\mathcal{FL}_k^p} \gtrsim \varepsilon^{J-1}, \quad \forall p \in [1, \infty], \quad \forall k \in \mathbb{R}.$$

By construction,

$$\hat{u}_{\text{app}}(t, \xi) = \sum_{j \in \mathbb{Z}^d} e^{-it \frac{|j|^2}{2\varepsilon}} \hat{a}_j \left(t, \xi - \frac{j}{\varepsilon} \right),$$

so we infer, at least for ε sufficiently small ($(a_j(\tau))_{j \in \mathbb{Z}^d} \in Y$ from Lemma 6.2),

$$\|u_{\text{app}}(\tau)\|_{\mathcal{FL}_k^p} \gtrsim \varepsilon^{J-1}, \quad \forall p \in [1, \infty], \quad \forall k \in \mathbb{R}.$$

On the other hand, since $X = \mathcal{FL}^1 \cap \mathcal{FL}^\infty \subset \mathcal{FL}^p$, Proposition 6.5 yields

$$\|u(\tau) - u_{\text{app}}(\tau)\|_{\mathcal{FL}_k^p} \lesssim \|u(\tau) - u_{\text{app}}(\tau)\|_X \lesssim \varepsilon, \quad \forall k \leq 0,$$

hence, if $J < 2$,

$$\|u(\tau)\|_{\mathcal{FL}_k^p} \gtrsim \varepsilon^{J-1}, \quad \forall p \in [1, \infty], \quad \forall k \leq 0.$$

Therefore,

$$\|\psi(\varepsilon\tau)\|_{\mathcal{FL}_k^p} \gtrsim \varepsilon^{\frac{J-2}{2\sigma}} \times \varepsilon^{J-1}, \quad \forall p \in [1, \infty], \quad \forall k \in \mathbb{R}.$$

The right hand side is unbounded as $\varepsilon \rightarrow 0$ provided that

$$\frac{J-2}{2\sigma} + J - 1 < 0, \quad \text{that is,} \quad J < \frac{2\sigma + 2}{2\sigma + 1}.$$

Then given $s < -1/(2\sigma + 1)$, we can always find a $J \in]1, 2[$ satisfying (7.1) and the above constraint. Theorem 1.3 follows in the case $k \leq 0$, by taking for instance $\varepsilon_n = 1/n$ and $t_n = \varepsilon_n \tau$. In the case $k > 0$, we just recall the obvious estimate

$$\|\psi(\varepsilon\tau)\|_{\mathcal{FL}_k^p} \geq \|\psi(\varepsilon\tau)\|_{\mathcal{FL}^p},$$

and the proof of Theorem 1.3 is complete.

7.2 Proof of Theorem 1.6

In the case of modulation spaces, the proof goes along the same lines as above, up to adapting the space X and Lemma 7.1.

We choose $X = \mathcal{FL}^1 \cap M^{1,1}$. Theorem 2.4 shows that the Banach algebra property (6.1) is satisfied. \mathcal{FL}^1 is translation invariant, and for $M^{1,1}$,

$$\begin{aligned} V_g(\tau_k f)(x, y) &= \int_{\mathbb{R}^d} f(t-k) \overline{g(t-x)} e^{-iy \cdot t} dt = e^{-iy \cdot k} \int_{\mathbb{R}^d} f(t) \overline{g(t+k-x)} e^{-iy \cdot t} dt \\ &= e^{-iy \cdot k} V_g(f)(x-k, y), \end{aligned}$$

and thus

$$\|\tau_k f\|_{M^{1,1}} = \|V_g(\tau_k f)\|_{L^1_{x,y}} = \|f\|_{M^{1,1}}.$$

We have used already the fact that (6.3) is satisfied on \mathcal{FL}^1 . On $M^{1,1}$, this is the case too, since for $k \in \mathbb{R}^d$,

$$V_g(fe_k)(x, y) = \int_{\mathbb{R}^d} f(t) e^{ik \cdot t} \overline{g(t-x)} e^{-iy \cdot t} dt = V_g(f)(x, y-k),$$

and so

$$\|fe_k\|_{M^{1,1}} = \|V_g(fe_k)\|_{L^1_{x,y}} = \|V_g(f)\|_{L^1_{x,y}} = \|f\|_{M^{1,1}}.$$

Finally, Assumption 6.1 is satisfied thanks to Proposition 2.5, and we can again invoke Lemmas 5.2, 5.3, and Proposition 6.5.

Like before, Lemmas 5.2 and 5.3 show that there exists $\tau > 0$ independent of ε such that

$$\|a_0(\tau)\|_{M_k^{p,q}} \gtrsim \varepsilon^{J-1}, \quad \forall p, q \in [1, \infty], \forall k \in \mathbb{R}.$$

By the same asymptotic decoupling phenomenon as in the case of \mathcal{FL}^p spaces, we infer

$$\|u_{\text{app}}(\tau)\|_{M_k^{p,q}} \gtrsim \varepsilon^{J-1}, \quad \forall p, q \in [1, \infty], \forall k \in \mathbb{R}.$$

In view of Lemma 2.3, $X \hookrightarrow M_k^{p,q}$ for all $p, q \geq 1$ and all $k \leq 0$, and so

$$\|u(\tau) - u_{\text{app}}(\tau)\|_{M_k^{p,q}} \lesssim \|u(\tau) - u_{\text{app}}(\tau)\|_X \lesssim \varepsilon, \quad \forall p, q \in [1, \infty], \forall k \leq 0.$$

The analogue of Lemma 7.1 is the following:

Lemma 7.2 *Let $d \geq 1$. For $f \in \mathcal{S}'(\mathbb{R}^d)$ and $j \in \mathbb{R}^d$, denote*

$$I^\varepsilon(f, j)(x) = f(x) e^{ij \cdot x / \varepsilon}.$$

- For all $s \in \mathbb{R}$, $p, q \in [1, \infty]$, and $f \in M_s^{p,q}(\mathbb{R}^d)$, $\|I^\varepsilon(f, 0)\|_{M_s^{p,q}} = \|f\|_{M_s^{p,q}}$.

- Let $j \in \mathbb{R}^d \setminus \{0\}$. For all $s \leq 0$, there exists $C = C(j)$ independent of $p, q \in [1, \infty]$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|I^\varepsilon(f, j)\|_{M_s^{p,q}} \leq C\varepsilon^{|s|} \|f\|_{M_s^{p,q}}.$$

Proof The first point is proven like (6.3) above. For the second point, write

$$\begin{aligned} \|I^\varepsilon(f, j)\|_{M_s^{p,q}} &= \left\| \left\| V_g f \left(x, y - \frac{j}{\varepsilon} \right) \right\|_{L_x^p} \langle y \rangle^s \right\|_{L_y^q} = \left\| \|V_g f(x, y)\|_{L_x^p} \left\langle y + \frac{j}{\varepsilon} \right\rangle^s \right\|_{L_y^q} \\ &\leq \left\langle \frac{j}{\varepsilon} \right\rangle^s \left\| \|V_g f(x, y)\|_{L_x^p} \langle y \rangle^{|s|} \right\|_{L_y^q}, \end{aligned}$$

where we have used Peetre inequality $\langle a + b \rangle^s \leq \langle a \rangle^s \langle b \rangle^{|s|}$. The lemma follows. \square

At this stage, we can repeat the same arguments as in the previous subsection, and Theorem 1.6 follows.

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