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Pointwise Convergence Along Restricted Directions for the Fractional Schrödinger Equation

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Abstract
We consider the pointwise convergence problem for the solution of Schrödinger-type equations along directions determined by a given compact subset of the real line. This problem contains Carleson's problem as the simplest case and was studied in general by Cho et al. We extend their result from the case of the classical Schrödinger equation to a class of equations which includes the fractional Schrödinger equations. To achieve this, we significantly simplify their proof by completely avoiding a time localization argument.

Keywords Fractional schrödinger equation · Pointwise convergence

Mathematics Subject Classification 35Q41

1 Introduction

Let $d \geq 1$, $a > 0$ and consider the fractional Schrödinger equation

$$\begin{cases} \partial_t u(x,t) = i(-\Delta_x)^{\frac{a}{2}} u(x,t) & (x,t) \in \mathbb{R}^d \times \mathbb{R} \\ u(x,0) = f(x) & x \in \mathbb{R}^d. \end{cases}$$

It is well-known that for a sufficiently nice initial data f, the solution can be written as

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$$u(x,t) = e^{it(-\Delta)^{\frac{a}{2}}} f(x) := \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^a)} \widehat{f}(\xi)\, d\xi,$$

where $\widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx$. When $a = 2$, this is the standard Schrödinger equation from quantum mechanics. The general case arose in recent years in physical models and turns out to be a fundamental equation in fractional quantum mechanics (fQM), and may be traced back to work of Laskin [20,21]. Motivated by this, the fractional Schrödinger equation and related nonlinear models have been the subject of numerous recent papers (see, for example, [5,7,13–16,18,23]). From a rather different viewpoint, certain nonlinear equations were the subject of study in recent work of Ionescu and Pusateri [17] and arise from models of water waves. In addition, the fractional Schrödinger equation is a model case in studies of more general dispersive equations; see, for example, [8,19].

Associated with the fractional Schrödinger equation, it is natural to try to determine the minimum level of regularity s which guarantees that the limit

$$\lim_{\substack{(y,t)\to(x,0)\\ (y,t)\in\Gamma_x}} e^{it(-\Delta)^{\frac{a}{2}}} f(y) = f(x) \qquad \text{a.e.} \tag{1}$$

holds whenever $f \in H^s(\mathbb{R}^d)$. Here, $H^s(\mathbb{R}^d)$ is the Sobolev space of order s whose norm is given by

$$\|f\|_{H^s(\mathbb{R}^d)} = \|(1-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^d)}$$

and $\Gamma_x \subset \mathbb{R}^d \times [-1, 1]$ is a convergence domain corresponding to each $x \in \mathbb{R}^d$.

The classical case, known widely as Carleson's problem, is concerned with the case of vertical lines $\Gamma_x = \{x\} \times \{0\}$. Here, when $d = 1$ and $a > 1$ it is known that (1) holds if and only if $s \geq \frac{1}{4}$; see the work of Carleson [3] and Dahlberg and Kenig [10] for the case $a = 2$, and also see the work of Sjölin [27] for general $a > 1$. The higher dimensional case $d \geq 2$ has been subject to a recent flurry of activity. When $a = 2$, Bourgain [2] showed that $s \geq \frac{1}{2} - \frac{1}{2(d+1)}$ is necessary for (1) for $d \geq 2$, and Du et al. [11] and Du and Zhang [12] have shown $s > \frac{1}{2} - \frac{1}{2(d+1)}$ is sufficient for (1) for $d = 2$ and $d \geq 3$, respectively (for important earlier contributions see, for example, papers by Vega [29], Lee [22] and Bourgain [1]). For, $a > 1$, Cho and Ko [4] proved analogous result that (1) holds if $s > \frac{1}{2} - \frac{1}{2(d+1)}$ and $d \geq 2$. In addition, we also note that Prestini [24] showed that for $d \geq 2$, $a > 1$ and f radial, (1) holds if and only if $s \geq \frac{1}{4}$. Results are also available for $0 < a \leq 1$ (see, for example, [9,25,30]) but these cases are of a rather different nature and from now on we focus entirely on the case $a > 1$.

Non-tangential convergence corresponds to the case

$$\Gamma_x = \{(x + t\theta, t) : t \in [-1, 1] \text{ and } \theta \in \mathbb{B}\},$$

where $\mathbb{B} \subset \mathbb{R}^d$ is a given euclidean ball which is centered at the origin, that is, Γ_x is a conical region with vertex at $(x, 0)$ and aperture determined by the radius of $\mathbb{B}$. In this case, it is known that (1) with $a > 1$ holds if and only if $s > \frac{d}{2}$. The sufficiency part of this claim follows easily by a well-known argument using Sobolev embedding and the delicate necessity part has been proved by Sjögren and Sjölin in [26] (strictly speaking, the case $a = 2$ was considered in [26] but their argument extends to $a > 1$ without difficulty).

When $d = 1$, the classical case and the non-tangential case were unified in a natural way by Cho et al.[6] who proved that (1) holds in the case

$$\Gamma_x = \{(x + t\theta, t) : t \in [-1, 1] \text{ and } \theta \in \Theta\}$$

when $a = 2$ and $s > \frac{1}{4} + \frac{\beta(\Theta)}{4}$. Here, $\Theta \subset \mathbb{R}$ is a given compact set and $\beta(\Theta)$ denotes the upper Minkowski dimension of Θ. We note that establishing the necessity of the condition $s > \frac{1}{4} + \frac{\beta(\Theta)}{4}$ is an interesting but still open problem. Our main goal in this paper is to improve the result in [6] by extending to a class of equations which includes the fractional Schrödinger equation for $a > 1$. We define the evolution operator S_t on appropriate input functions by

$$S_t f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x\xi + t\Phi(\xi))} \widehat{f}(\xi)\, d\xi.$$

Here, $\Phi : \mathbb{R} \to \mathbb{R}$ is a C^2 function which satisfies for some $C_1 > 0$,

$$|\xi||\Phi''(\xi)| \geq C_1 \tag{2}$$

for all $|\xi| \geq 1$. Moreover, for some $C_2 > 0$,

$$|\xi||\Phi''(\xi)| \geq C_2|\Phi'(\xi)| \tag{3}$$

for all $|\xi| \geq 1$. It is trivial to verify that $\Phi(\xi) = |\xi|^a$ satisfies these conditions when $a > 1$.

Our main result is the following.

Theorem 1 *Let $\Theta \subset \mathbb{R}$ be compact and suppose $\Phi \in C^2(\mathbb{R})$ satisfies (2) and (3). For any $q \in [1, 4]$ and $s > \frac{1}{4} + \frac{\beta(\Theta)}{4}$, there exists a constant $C_{q,s}$ such that*

$$\left\| \sup_{(t,\theta)\in[-1,1]\times\Theta} |S_t f(\cdot + t\theta)| \right\|_{L^q(-1,1)} \leq C_{q,s} \|f\|_{H^s(\mathbb{R})}$$

whenever $f \in H^s(\mathbb{R})$.

By standard arguments, we thus obtain the associated pointwise convergence.

Corollary 2 *Let $\Theta \subset \mathbb{R}$ be compact and suppose $\Phi \in C^2(\mathbb{R})$ satisfies (2) and (3). If $s > \frac{1}{4} + \frac{\beta(\Theta)}{4}$, then*

$$\lim_{\substack{(y,t)\to(x,0)\\ y-x\in t\Theta}} S_t f(y) = f(x) \qquad \text{a.e.} \tag{4}$$

whenever $f \in H^s(\mathbb{R})$.

Theorem 1 improves the result in [6] in two respects; the class of evolution operators has been widened from the case $\Phi(\xi) = |\xi|^2$ to those satisfying (2) and (3), and our maximal estimates are valid for $q \in [1, 4]$ (the estimate in [6] was proved in only the cases $q \in [1, 2]$). While the proof in [6] may be modified in a straightforward way to go beyond the classical case $\Phi(\xi) = |\xi|^2$ to a certain extent, it seems to us to be difficult to handle case $\Phi(\xi) = |\xi|^a$ with a close to 1. Indeed, the argument in [6] rests on a certain widely used time localization argument which becomes increasingly weak as a approaches 1. To overcome this significant obstacle, we remove the use of the time localization lemma; this simplification to the proof has allowed us to handle the case $\Phi(\xi) = |\xi|^a$ for any $a > 1$. Further explanation of this point will follow our proof of Theorem 1 in Sect. 3. Prior to that, we prepare for the proof of Theorem 1 in Sect. 2.

2 Preliminaries

Notation

Associated with the operator S_t given above by

$$S_t f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x\xi + t\Phi(\xi))} \widehat{f}(\xi)\, d\xi$$

and a fixed compact set $\Theta \subset \mathbb{R}$, we define the maximal operator M_Θ by

$$M_\Theta f(x) = \sup\{|S_t f(x + t\theta)| : -1 \le t \le 1,\ \theta \in \Theta\}.$$

Also, we recall that the upper Minkowski dimension of Θ is defined by

$$\beta(\Theta) = \inf\{r > 0 : \limsup_{\delta \to 0} N(\Theta, \delta)\delta^r = 0\},$$

where $N(\Theta, \delta)$ denotes the smallest number of δ-intervals which cover Θ.

We will use the following notation frequently:

- $I = [-1, 1]$.
- $q' = \frac{q}{q-1}$: Hölder conjugate of $q \in [1, \infty]$.
- $A \lesssim B$: $A \le CB$ for some constant $C > 0$.
- $A \gtrsim B$: $A \ge CB$ for some constant $C > 0$.
- $A \sim B$: $C^{-1}B \le A \le CB$ for some constant $C > 0$.
- $L^p_x L^q_t L^r_\theta$: The Lebesgue space with norm

$$\|F\|_{L^p_x L^q_t L^r_\theta} = \left(\int \left(\int \left(\int |F(x, t, \theta)|^r \, d\theta \right)^{\frac{q}{r}} dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}},$$

where the domains of integration will be clear from the context.

Useful Lemmas

The following lemmas will be crucial for the oscillatory integral estimates in the proof of Theorem 1. Applying these lemmas appropriately essentially allows us to avoid the time localization lemma, which is used in [6].

Lemma 3 (van der Corput's lemma) *Let $-\infty < a < b < \infty$, ϕ be a sufficiently smooth real-valued function and ψ be a bounded smooth complex-valued function. Suppose we have $|\phi^{(k)}(x)| \geq 1$ for all $x \in [a, b]$. If $k = 1$ and ϕ' is monotonic on (a, b), or simply $k \geq 2$, then there exists a constant C_k such that*

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x)\, dx \right| \leq C_k \lambda^{-\frac{1}{k}} \left(\int_a^b |\psi'(x)|\, dx + \|\psi\|_{L^\infty} \right)$$

for all $\lambda > 0$.

For a proof of van der Corput's lemma, we refer the reader to [28].

Lemma 4 *Let $1 \leq q \leq 4$. There exists a constant C_q such that*

$$\left| \iiiint g(x, t) h(x', t') |x - x'|^{-\frac{1}{2}}\, dx dt dx' dt' \right| \leq C_q \|g\|_{L_x^{q'} L_t^1} \|h\|_{L_x^{q'} L_t^1},$$

where the integrals are taken over $(x, t), (x', t') \in I \times I$.

Proof Denoting $G(x) = \|g(x, \cdot)\|_{L^1}$ and $H(x') = \|h(x', \cdot)\|_{L^1}$,

$$\left| \iiiint g(x, t) h(x', t') |x - x'|^{-\frac{1}{2}}\, dx dx' dt dt' \right| \leq \int_{-1}^1 \int_{-1}^1 G(x) H(x') |x - x'|^{-\frac{1}{2}}\, dx dx'.$$

By the Hardy et al. inequality,

$$\begin{aligned} \int_{-1}^1 \int_{-1}^1 G(x) H(x') |x - x'|^{-\frac{1}{2}}\, dx dx' &\lesssim \|G\|_{L^{\frac{4}{3}}(I)} \|H\|_{L^{\frac{4}{3}}(I)} \\ &\lesssim \|g\|_{L_x^{q'} L_t^1} \|h\|_{L_x^{q'} L_t^1}, \end{aligned}$$

where the last inequality is obtained by Hölder's inequality since $\frac{4}{3} \leq q'$ from our assumption. □

3 Proof of Theorem 1

Proof of Theorem 1 We fix $q \in [2, 4]$. The case $q \in [1, 2)$ follows immediately by Hölder's inequality.

The proof begins with a reduction to the case where f is frequency-localised to a large annulus and θ belongs to an interval of an appropriately small length. This reduction to the forthcoming Proposition 5 essentially follows the argument in [6]; our main novelty is the proof of Propositon 5.

Suppose $\psi_0 \in C_0^\infty(I)$ and $\psi \in C_0^\infty((-2,-\frac{1}{2}) \cup (\frac{1}{2},2))$ give rise to a standard dyadic partition of unity $\psi_0(\xi) + \sum_{k\geq 1}\psi_k \equiv 1$, where $\psi_k = \psi(\frac{\cdot}{2^{k-1}})$. For each $0 \leq k \in \mathbb{Z}$, the frequency localization operator P_k is defined by $\widehat{P_k f}(\xi) = \psi_k(\xi)\widehat{f}(\xi)$. Then,

$$\|M_\Theta f\|_{L^q(I)} \lesssim \|M_\Theta P_0 f\|_{L^q(I)} + \sum_{k\geq 1} \|M_\Theta P_k f\|_{L^q(I)}. \tag{5}$$

The first term is relatively easy to estimate. In fact,

$$\|M_\Theta P_0 f\|_{L^q(I)} \lesssim \int_{\mathbb{R}} \psi_0(\xi)|\widehat{f}(\xi)|\, d\xi \lesssim \|f\|_{L^2} \lesssim \|f\|_{H^s}$$

for $s \geq 0$, and thereby this term can be easily handled.

For the remaining terms, first note that for each $k \geq 1$, there exists a finite collection of intervals $\{\Omega_{k,j}\}_{j=1}^{N_k}$ which satisfies

$$\Theta \subset \bigcup_{j=1}^{N_k} \Omega_{k,j},$$

where $|\Omega_{k,j}| \leq 2^{-\frac{qk}{4}}$ for each j and $N_k = N(\Theta, 2^{-\frac{qk}{4}})$ is the smallest number of $2^{-\frac{qk}{4}}$-intervals which cover Θ. (The reason for the choice of scale $2^{-\frac{qk}{4}}$ will become clear as we proceed.) For $x \in I$,

$$M_\Theta P_k f(x)^q \leq \sum_{j=1}^{N_k} \sup_{\substack{t\in I \\ \theta\in\Omega_{k,j}}} |S_t P_k f(x+t\theta)|^q,$$

therefore

$$\sum_{k\geq 1} \|M_\Theta P_k f\|_{L^q(I)} \leq \sum_{k\geq 1} \left(\sum_{j=1}^{N_k} \|M_{\Omega_{k,j}} P_k f\|_{L^q(I)}^q \right)^{\frac{1}{q}}.$$

Now, we shall introduce the following crucial proposition.

Proposition 5 *Let $2 \leq q \leq 4$, $k \geq 1$ and Ω be an interval with $|\Omega| \leq 2^{-\frac{qk}{4}}$. Then, there exists a constant C_q such that*

$$\|M_\Omega P_k f\|_{L^q(I)} \leq C_q 2^{\frac{k}{4}} \|f\|_{L^2} \tag{6}$$

holds for all $f \in L^2(\mathbb{R})$.

Proof of Proposition 5 Set $\lambda = 2^k$ and

$$Tf(x,t,\theta) := \chi(x,t,\theta)\int_{\mathbb{R}} e^{i((x+t\theta)\xi + t\Phi(\xi))} f(\xi)\psi(\tfrac{\xi}{\lambda})\,d\xi,$$

where $\chi = \chi_{I\times I\times\Omega}$. Then (6) follows from

$$\|Tf\|_{L^q_x L^\infty_t L^\infty_\theta} \lesssim \lambda^{\frac{1}{4}}\|f\|_{L^2} \quad (\lambda \gtrsim 1) \tag{7}$$

since

$$\|M_\Omega P_k f\|_{L^q(I)} \sim \|T\widehat{f}\|_{L^q_x L^\infty_t L^\infty_\theta} \lesssim \lambda^{\frac{1}{4}}\|\widehat{f}\|_{L^2} \lesssim \lambda^{\frac{1}{4}}\|f\|_{L^2}$$

by Plancherel's theorem. Let us consider the dual form of (7), which is

$$\|T^*F\|_{L^2} \lesssim \lambda^{\frac{1}{4}}\|F\|_{L^{q'}_x L^1_t L^1_\theta} \tag{8}$$

where

$$T^*F(\xi) = \psi(\tfrac{\xi}{\lambda})\iiint \chi(x',t',\theta')e^{-i((x'+t'\theta')\xi + t'\Phi(\xi))}F(x',t',\theta')\,dx'dt'd\theta'.$$

Then,

$$\begin{aligned}
&\|T^*F\|_{L^2}^2\\
&= \lambda\int \psi^2(\xi)\iiint\iiint \chi(x,t,\theta)\chi(x',t',\theta')\\
&\quad\times e^{i(\lambda(x-x'+t\theta-t'\theta')\xi + (t-t')\Phi(\lambda\xi))}\bar{F}(x,t,\theta)F(x',t',\theta')\,dxdtd\theta dx'dt'd\theta'd\xi\\
&= \lambda\int_W\int_{W'} \chi(w)\chi(w')\bar{F}(w)F(w')K_\lambda(w,w')\,dwdw'\\
&= \sum_{\ell=1}^{3}\lambda\iint_{V_\ell} \chi(w)\chi(w')\bar{F}(w)F(w')K_\lambda(w,w')\,dwdw'\\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

Here, we denote $w = (x,t,\theta) \in W$ and $w' = (x',t',\theta') \in W$, where $W := I\times I\times\Omega$. Also,

$$K_\lambda(w,w') = \int_{\mathbb{R}} e^{i\phi(\lambda\xi,w,w')}\psi^2(\xi)\,d\xi,$$

$$\phi(\xi,w,w') = (x-x'+t\theta-t'\theta')\xi + (t-t')\Phi(\xi),$$

and

$$\begin{cases} V_1 &= \{(w, w') \in W \times W : |x - x'| < 4|t - t'|\}, \\ V_2 &= \{(w, w') \in W \times W : |x - x'| \geq 4|t - t'| \text{ and } |x - x'| \geq 4\lambda^{-\frac{q}{4}}\}, \\ V_3 &= \{(w, w') \in W \times W : |x - x'| \geq 4|t - t'| \text{ and } |x - x'| < 4\lambda^{-\frac{q}{4}}\}. \end{cases}$$

Thus, (8) follows from

$$A_\ell \lesssim \lambda^{\frac{1}{2}} \|F\|^2_{L_x^{q'} L_t^1 L_\theta^1} \quad \text{for each } \ell = 1, 2, 3.$$

The Term A_1

Let us start with an estimate of A_1. Since

$$|\phi''(\lambda\xi)| = \lambda^2 |t - t'||\Phi''(\lambda\xi)| \gtrsim \lambda |x - x'|$$

holds from (2), we are allowed to apply Lemma 3 to get

$$|K_\lambda(w, w')| \lesssim (\lambda|x - x'|)^{-\frac{1}{2}}.$$

By using Lemma 4, it follows that

$$\begin{aligned} A_1 &\leq \lambda^{\frac{1}{2}} \iint_{V_1} \chi(w')|F(w')|\chi(w)|\bar{F}(w)||x - x'|^{-\frac{1}{2}} \, dw dw' \\ &\lesssim \lambda^{\frac{1}{2}} \|F\|^2_{L_x^{q'} L_t^1 L_\theta^1}. \end{aligned}$$

The Term A_2

Next, we shall consider A_2. In this case, we firstly observe the following key relationship:

$$|x - x' + t\theta - t'\theta'| \sim |x - x'|. \tag{9}$$

Indeed,

$$\begin{aligned} |x - x' + t\theta - t'\theta'| &\geq |x - x'| - |t - t'| - |\theta - \theta'| \\ &\geq \frac{3}{4}|x - x'| - \lambda^{-\frac{q}{4}} \\ &\geq \frac{1}{2}|x - x'|. \end{aligned}$$

Similarly, the other way holds, too.

Now, let us observe that for all $(w, w') \in V_2$, we have

$$|K_\lambda(w, w')| \lesssim (\lambda|x - x'|)^{-\frac{1}{2}}. \tag{10}$$

Before proving (10), we note that

$$A_2 \lesssim \lambda^{\frac{1}{2}} \|F\|^2_{L_x^{q'} L_t^1 L_\theta^1}$$

immediately follows by using Lemma 4 as before.

To see (10), let us split K_λ into B_1 and B_2 as follows

$$\begin{aligned} K_\lambda(w, w') &= \int_{U_1} e^{i\phi(\lambda\xi, w, w')} \psi^2(\xi)\, d\xi + \int_{U_2} e^{i\phi(\lambda\xi, w, w')} \psi^2(\xi)\, d\xi \\ &=: B_1 + B_2, \end{aligned}$$

where

$$U_1 = \{\xi \in \operatorname{supp} \psi : |x - x' + t\theta - t'\theta'| \geq 2|t - t'||\Phi'(\lambda\xi)|\}$$

and

$$U_2 = \{\xi \in \operatorname{supp} \psi : |x - x' + t\theta - t'\theta'| < 2|t - t'||\Phi'(\lambda\xi)|\}.$$

For B_1, we have

$$\begin{aligned} |\phi'(\lambda\xi)| &\geq \lambda|x - x' + t\theta - t'\theta'| - \lambda|t - t'||\Phi'(\lambda\xi)| \\ &\geq \frac{\lambda}{2}|x - x' + t\theta - t'\theta'| \\ &\geq \frac{\lambda}{4}|x - x'| \\ &> \lambda^{1-\frac{q}{4}} \\ &\geq 1, \end{aligned}$$

where we have used the fact that $q \leq 4$. From (2) and the intermediate value theorem, $\Phi''(\xi)$ is single-signed on $(-\infty, -1]$ and $[1, \infty)$, which guarantees that $\Phi'(\xi)$ is monotone on these intervals. Hence, U_1 consists of at most two intervals. Invoking Lemma 3,

$$B_1 \lesssim (\lambda|x - x'|)^{-1} \lesssim (\lambda|x - x'|)^{-\frac{1}{2}}.$$

On the other hand, for B_2, it follows from (3) that

$$\begin{aligned} |\phi''(\lambda\xi)| &= \lambda^2|t - t'||\Phi''(\lambda\xi)| \\ &\gtrsim \lambda|t - t'||\Phi'(\lambda\xi)| \\ &\gtrsim \lambda|x - x' + t\theta - t\theta'| \\ &\gtrsim \lambda|x - x'|. \end{aligned}$$

Then, by using Lemma 3, we obtain

$$B_2 \lesssim (\lambda|x-x'|)^{-\frac{1}{2}}.$$

Therefore, (10) holds.

The Term A_3

It remains to show

$$A_3 \lesssim \lambda^{\frac{1}{2}} \|F\|^2_{L_x^{q'} L_t^1 L_\theta^1}.$$

Trivially,

$$|K_\lambda(w, w')| \lesssim 1$$

so by the dual form of Young's convolution inequality

$$\begin{aligned}&\int_{-1}^{1}\int_{-1}^{1} \|F(x,\cdot,\cdot)\|_{L_t^1 L_\theta^1} \|F(x',\cdot,\cdot)\|_{L_t^1 L_\theta^1} \chi_{[-4\lambda^{-\frac{q}{4}}, 4\lambda^{-\frac{q}{4}}]}(x-x')\, dx dx' \\ &\lesssim \|F\|^2_{L_x^{q'} L_t^1 L_\theta^1} \|\chi_{[-4\lambda^{-\frac{q}{4}}, 4\lambda^{-\frac{q}{4}}]}\|_{L^{\frac{q}{2}}} \\ &\sim \lambda^{-\frac{1}{2}} \|F\|^2_{L_x^{q'} L_t^1 L_\theta^1}.\end{aligned}$$

Here, we have used the fact that $q \geq 2$. Therefore, we conclude that

$$A_3 \lesssim \lambda^{\frac{1}{2}} \|F\|^2_{L_x^{q'} L_t^1 L_\theta^1}$$

as claimed. □

By the definition of the upper Minkowski dimension, for small $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ depending on ε such that

$$N(\Theta, 2^{-\frac{qk}{4}}) \leq C_\varepsilon 2^{\frac{qk}{4}(\beta(\Theta)+\varepsilon)}.$$

Thus, if we also let $\widehat{\tilde{P}_k f} = \tilde{\psi}_k \widehat{f}$, where $\tilde{\psi} \in C_0^\infty((-4, -\frac{1}{4}) \cup (\frac{1}{4}, 4))$ with $\tilde{\psi} \equiv 1$ on $(-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)$, then

$$\sum_{k\geq 1}\left(\sum_{j=1}^{N_k} \|M_{\Omega_{k,j}} P_k f\|^q_{L^q(I)}\right)^{\frac{1}{q}} = \sum_{k\geq 1}\left(\sum_{j=1}^{N_k} \|M_{\Omega_{k,j}} P_k \tilde{P}_k f\|^q_{L^q(I)}\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&\lesssim \sum_{k\geq 1}\left(\sum_{j=1}^{N_k} 2^{\frac{qk}{4}}\|\tilde{P}_k f\|_{L^2}^q\right)^{\frac{1}{q}}\\
&\lesssim \sum_{k\geq 1} 2^{k(\frac{1}{4}+\frac{\beta(\Theta)}{4}+\frac{\varepsilon}{4})}\|\tilde{P}_k f\|_{L^2}\\
&\sim \sum_{k\geq 1} 2^{-\frac{3}{4}k\varepsilon}\left(\int_{\operatorname{supp}\tilde{\psi}_k} 2^{2k(\frac{1}{4}+\frac{\beta(\Theta)}{4}+\varepsilon)}|\widehat{f}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}\\
&\lesssim \|f\|_{H^{\frac{1}{4}+\frac{\beta(\Theta)}{4}+\varepsilon}}.
\end{aligned}
$$

Therefore, for arbitrary $\varepsilon > 0$,

$$
\|M_\Theta f\|_{L^q(I)} \lesssim \|f\|_{H^{\frac{1}{4}+\frac{\beta(\Theta)}{4}+\varepsilon}}
$$

holds, which ends the proof. □

Remarks The crucial component in the above proof of Theorem 1 is Proposition 5. The corresponding result in [6] (Lemma 3.1), stated for $q = 2$ and $\Phi(\xi) = |\xi|^2$, is established through the following steps: TT^* argument, the time localization lemma, Schur's lemma and then an oscillatory integral argument. Following this approach in the case $\Phi(\xi) = |\xi|^a$, one may extend by simple modification to the range $a \geq \frac{3}{2}$. However, the time localization lemma reduces to the case of time intervals of length λ^{1-a}, and for a close to 1 this causes certain technical difficulties in the estimation of the oscillatory integrals which arise; in particular, the relationship (9) breaks down if we follow their argument as it stands. In order to overcome the significant technical difficulty, we removed the use of the time localization lemma and replaced this with appropriate decompositions of the domain $W \times W$.

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