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Concentration Estimates for Band-Limited Spherical Harmonics Expansions via the Large Sieve Principle

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Abstract
We study a concentration problem on the unit sphere $\mathbb{S}^2$ for band-limited spherical harmonics expansions using large sieve methods. We derive upper bounds for concentration in terms of the maximum Nyquist density. Our proof uses estimates of the spherical harmonics coefficients of certain zonal filters. We also demonstrate an analogue of the classical large sieve inequality for spherical harmonics expansions.

Keywords Large sieve inequalities · Concentration estimates · Spherical harmonics · Legendre polynomials · Signal recovery

Mathematics Subject Classification 33C55 · 33C45 · 46E15 · 46E20 · 42C10 · 11N36

1 Introduction

1.1 Main Contributions

Let $\mathbb{S}^2$ be the unit sphere in space, $\Omega \subset \mathbb{S}^2$ a measurable set, and let $\mathcal{S}$ be a Banach subspace of $L^p(\mathbb{S}^2)$, where $1 < p < \infty$. The concentration problem for the sphere is concerned with estimating the quantity

$$\lambda_{\mathcal{S}}^{(p)}(\Omega) := \sup_{f \in \mathcal{S} \setminus \{0\}} \frac{\int_\Omega |f|^p d\sigma}{\int_{\mathbb{S}^2} |f|^p d\sigma}. \tag{1}$$

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Following ideas of [10], we define the *maximum Nyquist density* on $\mathbb{S}^2$ as

$$\rho(\Omega, L) = \sup_{y\in\mathbb{S}^2} \frac{|\Omega \cap \mathcal{C}_{t_{L,L}}(y)|}{|\mathcal{C}_{t_{L,L}}(y)|}, \tag{2}$$

where $t_{L,L}$ denotes the largest zero of the Legendre polynomial P_L, $L = 1, 2, \ldots$, and $\mathcal{C}_{t_{L,L}}(y)$ denotes the spherical cap with the apex $y \in \mathbb{S}^2$ and the polar angle $\arccos(t_{L,L})$. A similar concept of density is considered in [27].

Let $\mathcal{S}_L$ denote the space of spherical harmonics expansions with the maximum degree L. In this paper, we derive upper bounds for the concentration constants $\lambda^{(p)}_{\mathcal{S}_L}(\Omega)$, $1 < p < \infty$, in terms of the maximum Nyquist density $\rho(\Omega, L)$. Our approach is to adapt the large sieve principle, that was first used by Donoho and Logan [10] to study the concentration problem for band-limited functions on the real line.

Our main result, which is given in Theorem 3.3, states that for $L = 1, 2, \ldots$

$$\lambda^{(2)}_{\mathcal{S}_L}(\Omega) \leqslant B_L \cdot \rho(\Omega, L), \tag{3}$$

where

$$B_L := (1 - t_{L,L}) \left(\int_{t_{L,L}}^1 P_L(t)^2 dt \right)^{-1}. \tag{4}$$

In Lemma 3.4, we show that

$$\lim_{L\to\infty} B_L = J_1(j_{0,1})^{-2} \approx 3.71038068570948, \tag{5}$$

where J_1 is the Bessel function of the first kind, and $j_{0,1}$ denotes the smallest positive zero of the Bessel function J_0. We then derive L^p-estimates by interpolation and duality. Specifically, we demonstrate that for $1 < p < \infty$

$$\lambda^{(p)}_{\mathcal{S}_L}(\Omega) \leqslant \big(B_L \cdot \rho(\Omega, L)\big)^{\min(p-1,\,1)}. \tag{6}$$

Donoho and Logan showed that their constants are optimal within their approach using the Beurling-Selberg function [31] and related extremal functions. Similarly, we show that for $p = 2$, the constant B_L in (3) is also optimal and solves an extremal problem that can be seen as a spherical analogue of the Beurling-Selberg problem, and also as a Fourier dual of the problem considered in [22, Theorem 4].

From Theorem 3.3, we derive an analogue of the classical large sieve inequality [24, (2)] for spherical harmonics expansions. Specifically, if

$$S(x) = \sum_{l=0}^{L} \sum_{m=-l}^{l} a_l^m Y_l^m(x),$$

and $x_1, \dots, x_R \in \mathbb{S}^2$ are θ-separated on the sphere with $\theta \in (0, \pi]$, i.e. $\langle x_k, x_l \rangle \leqslant \cos\theta$, $k \neq l$, then

$$\sum_{k=1}^{R} |S(x_k)|^2 \leqslant D(\theta, L) \cdot \sum_{l=0}^{L} \sum_{m=-l}^{l} |a_l^m|^2. \tag{7}$$

The constant $D(\theta, L)$ is given explicitly in Theorem 4.1. Our proof relies on estimating the maximum number of θ-separated points lying in a spherical cap, which can be viewed as a packing problem with spherical caps [7].

1.2 Previous Work

The concentration problem dealing with the quantity

$$\mu(\Omega, T) := \sup_{f \in \mathcal{S}_\Omega \setminus \{0\}} \frac{\int_{-T/2}^{T/2} |f|^2 dt}{\int_{\mathbb{R}} |f|^2 dt}, \tag{8}$$

where $\mathcal{S}_\Omega = \left\{ f \in L^2(\mathbb{R}) : \widehat{f}(\xi) = 0, \text{ for } |\xi| > \Omega \right\}$, was first studied in a series of papers by Landau, Slepian and Pollak, now commonly known as the Bell-Lab papers [21,29].

The largest eigenvalue of the product of the lowpassing operator and the timelimiting operator corresponds to the solution of (8). The eigenfunctions of the product - called Slepian functions - have appeared in various contexts, for example in spectral estimation with the multitaper method [2,5,30], in time-frequency/time-scale concentration problems [8,9], and in the study of spatial concentration of spherical harmonics expansions [6,28]. The Bell-Lab approach has had several generalizations, for example [1,16–18,23].

There is one common thread throughout the aforementioned papers. They all exploit specific geometry of concentration domains in order to solve the concentration problem. For a general concentration domain, it is hard to explicitly calculate the eigenvalues following the Bell-Lab theory. Moreover, in many applications, it is not necessary to know the exact solution to the concentration problem, and it is enough to have a good estimate. Take for example the task of reconstructing functions from incomplete observations. If a signal is not well-concentrated in a missing region Ω, then it can be reconstructed by the method of alternating projections, and the convergence rate is governed by $\lambda_S^{(2)}(\Omega) < 1$, see [11, Section 4].

The large sieve principle can be viewed as a class of inequalities satisfied by trigonometric polynomials T with complex coefficients

$$T(t) = \sum_{n=1}^{N} a_n e^{2\pi i n t}.$$

Trigonometric polynomials are defined on the interval $[0, 1]$ modulo 1, which is endowed with the distance $\operatorname{dist}(t, s) := \min_{n \in \mathbb{Z}} |t - s - n|$. If $\delta > 0$ and $t_1, \dots, t_R \in$

[0, 1] satisfy

$$\operatorname{dist}(t_i, t_j) \geqslant \delta, \qquad 1 \leqslant i < j \leqslant R,$$

then [24, Theorem 3]

$$\sum_{k=1}^{R} |T(t_k)|^2 \leqslant \left(N - 1 + \delta^{-1}\right) \sum_{n=1}^{N} |a_n|^2. \tag{9}$$

This is a basic form of the large sieve inequality, and the constant $N - 1 + \delta^{-1}$ is sharp. Montgomery [24] used (9) to study the distribution of prime numbers on large intervals. A multidimensional version of this estimate can be found in [19, Theorem 5].

Donoho and Logan first recognized that (9) can be used to *'control the size of trigonometric polynomials on "sparse" sets'* [10], which lead them to derive novel concentration estimates for band-limited functions. This rationale has recently inspired a study of the time-frequency concentration problem of the short-time Fourier transform with Hermite windows [3,4], and is also a central idea of this paper.

2 Preliminaries

Throughout this paper, we use the convention that x and y denote points on the unit sphere $\mathbb{S}^2$ in space, and t denotes numbers in the interval $[-1, 1]$.

2.1 Legendre Polynomials and the Mehler–Heine Formula

Legendre polynomials can be defined via the following three term recurrence [15, 8.914 (1)]

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t), \qquad n = 1, 2, \ldots, \tag{10}$$

with $P_0(t) = 1$, and $P_1(t) = t$. The derivative P_n' satisfies [15, 8.915 (2)]

$$P_n' = (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + \ldots. \tag{11}$$

For $t \in [-1, 1]$, we have [15, 8.917 (5)]

$$|P_n(t)| \leqslant 1, \tag{12}$$

which, combined with (11), gives

$$|P_n'(t)| \leqslant (2n-1) + (2n-5) + (2n-9) + \ldots = \frac{1}{2}n(n+1). \tag{13}$$

It is known that all zeros of P_n lie in the interval $(-1, 1)$ [26, 18.2(vi)]. For $n \geqslant 1$, we denote by $t_{n,n}$ the largest zero of P_n. It follows from [26, 18.2(vi)] that $t_{n,n} < t_{n+1,n+1}$. The following lemma demonstrates certain monotonicity properties of Legendre polynomials.

Lemma 2.1 *If $n \geqslant 1$ and $t \in (t_{n,n}, 1)$, then*

$$0 < P_n(t), \tag{14}$$

$$P_n(t) < P_{n-1}(t). \tag{15}$$

Consequently,

$$0 < P_n(t) < \ldots < P_0(t). \tag{16}$$

Proof Since $P_n(1) = 1$, (14) follows from the fact that $t_{n,n}$ is the largest zero of P_n. We now show (15) by induction with respect to n. For $n = 1$, we have $t_{1,1} = 0$, $P_0(t) = 1$, $P_1(t) = t$, so (15) is true. Let us assume that (15) holds for a fixed $n \geqslant 1$ and every $t \in (t_{n,n}, 1)$. Combining (10) and (15), we obtain

$$\begin{aligned}(n+1)P_{n+1}(t) &= (2n+1)tP_n(t) - nP_{n-1}(t)\\ &< (2n+1)tP_n(t) - nP_n(t)\\ &< (2n+1)P_n(t) - nP_n(t)\\ &= (n+1)P_n(t).\end{aligned}$$

The second inequality above follows from (14). Since $t_{n,n} < t_{n+1,n+1}$, we infer that

$$P_{n+1}(t) < P_n(t)$$

for every $t \in (t_{n+1,n+1}, 1)$. This completes the inductive proof of (15). Finally, (16) follows from (14) and (15). ◻

For $\theta_{n,1} := \arccos(t_{n,n})$, we have the following asymptotics [26, 18.16.5]

$$\theta_{n,1} = \frac{j_{0,1}}{n} + \mathcal{O}(n^{-2}),$$

where $j_{0,1} \approx 2.404825557695772$ denotes the smallest positive zero of the Bessel function of the first kind J_0. Taking the cosine of both sides yields

$$t_{n,n} = 1 - \frac{j_{0,1}^2}{2n^2} + \mathcal{O}\left(n^{-3}\right). \tag{17}$$

The Mehler–Heine formula [26, 18.11.5] describes the asymptotic behavior of P_n at arguments approaching 1

$$\lim_{n\to\infty} P_n\left(1 - \frac{z^2}{2n^2}\right) = J_0(z). \tag{18}$$

2.2 Spherical Harmonics and Spherical Caps

Expanding functions in terms of the spherical harmonics is a natural extension of Fourier series from the unit circle to the three dimensional sphere. The *complex spherical harmonics* Y_l^m are given in spherical coordinates by [26, 14.30.1]

$$Y_l^m(\theta,\varphi) := \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\varphi}, \qquad \theta\in[0,\pi),\ \varphi\in[0,2\pi),$$

where $0 \leqslant |m| \leqslant l$, $l = 0, 1, \ldots$, and P_l^m denotes the associated Legendre function of degree l and order m [26, 14.7.10]

$$P_l^m(t) = \frac{(-1)^{m+l}}{2^l l!}\left(1-t^2\right)^{m/2}\frac{d^{m+l}}{dt^{m+l}}\left(1-t^2\right)^l. \tag{19}$$

In particular, P_l^0 coincides with the Legendre polynomial P_l [26, 18.5.5]

$$P_l(t) = P_l^0(t) = \frac{(-1)^l}{2^l l!}\cdot\frac{d^l}{dt^l}\left(1-t^2\right)^l.$$

From (19), we infer that $P_l^m(1) = 0$ if $m \neq 0$. Consequently,

$$P_l^m(1) = \delta_{m,0}\cdot P_l^0(1) = \delta_{m,0}\cdot P_l(1) = \delta_{m,0},$$

where $\delta_{m,0}$ denotes the Kronecker delta function.

The family $\{Y_l^m\}_{0\leqslant|m|\leqslant l}$ forms an orthonormal basis of $L^2(\mathbb{S}^2)$, where $\mathbb{S}^2$ is equipped with the rotation invariant surface measure $d\sigma$. The basis coefficients of a function $f \in L^2(\mathbb{S}^2)$ are given by

$$\widehat{f}(l,m) = \int_{\mathbb{S}^2} f(x)\overline{Y_l^m(x)}d\sigma(x) = \int_0^\pi\int_0^{2\pi} f(\theta,\varphi)\overline{Y_l^m(\theta,\varphi)}\sin\theta\, d\varphi d\theta. \tag{20}$$

In particular,

$$Y_l^0(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta), \qquad \theta\in[0,\pi),\ \varphi\in[0,2\pi), \tag{21}$$

and

$$\widehat{f}(l,0) = \sqrt{\frac{2l+1}{4\pi}}\int_0^\pi\int_0^{2\pi} f(\theta,\varphi)\, P_l(\cos\theta)\sin\theta\, d\varphi d\theta. \tag{22}$$

Let $\mathcal{S}_L$ be the space of band-limited functions with the maximum degree L, i.e. $f \in \mathcal{S}_L$, if and only if $\widehat{f}(l,m) = 0$ whenever $l > L$ and $|m| \leqslant l$.

We denote the north pole (0, 0, 1) of the sphere $\mathbb{S}^2$ by η. For $\delta \in [-1, 1]$, we define the *spherical cap* with the apex $x \in \mathbb{S}^2$ and the polar angle $\arccos \delta$ as follows

$$\mathcal{C}_\delta(x) := \{y \in \mathbb{S}^2 : \langle x, y\rangle \geqslant \delta\}.$$

Thus the polar angle is the angle between the ray from the origin to the apex and the ray from the origin to any point on the boundary of the cap. The surface area of the spherical cap $\mathcal{C}_\delta(x)$ does not depend on the location of the apex x, and is given by the formula

$$\begin{aligned} |\mathcal{C}_\delta(x)| = |\mathcal{C}_\delta(\eta)| &= \int_0^\pi \int_0^{2\pi} \chi_{[\delta,1]}(\cos\theta) \sin\theta \, d\varphi d\theta \\ &= 2\pi \int_0^{\arccos\delta} \sin\theta \, d\theta = 2\pi(1-\delta). \end{aligned} \tag{23}$$

2.3 Convolution on $\mathbb{S}^2$

In this paper, we use a concept of convolution with a zonal function on $\mathbb{S}^2$ that is studied in [13,20,25]. One advantage of this approach is that it admits a convolution theorem.

Let g be a *zonal filter*, i.e. a function on $\mathbb{S}^2 \subset \mathbb{R}^3$ that only depends on the z-coordinate. A zonal filter can be viewed as a function defined on the interval $[-1, 1]$. Thus, with a slight abuse of notation, we write $g(x) = g(\langle x, \eta\rangle)$, where η denotes the north pole of $\mathbb{S}^2$.

We define convolution with the zonal function g as follows

$$(f * g)(x) := \int_{\mathbb{S}^2} f(y) g(\langle x, y\rangle) d\sigma(y), \qquad x \in \mathbb{S}^2. \tag{24}$$

Two numbers $1 \leqslant p, q \leqslant \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ are called *conjugate exponents*. From Hölder's inequality, we infer that if p and q are conjugate exponents, then

$$|(f * g)(x)| \leqslant \|f\|_{L^q(\mathbb{S}^2)} \cdot \|g\|_{L^p(\mathbb{S}^2)}, \qquad x \in \mathbb{S}^2.$$

Since

$$\|g\|_{L^p(\mathbb{S}^2)} = \|g(\langle\cdot, \eta\rangle)\|_{L^p(\mathbb{S}^2)} = (2\pi)^{\frac{1}{p}} \|g\|_{L^p([-1,1])},$$

zonal functions in $L^p(\mathbb{S}^2)$ may be regarded as functions in $L^p([-1, 1])$, $1 \leqslant p \leqslant \infty$.

Regarding the Legendre polynomial P_k as a zonal function on $\mathbb{S}^2$, we have

$$\widehat{P_k}(l, 0) = \sqrt{\frac{2l+1}{4\pi}} \int_0^{2\pi} d\varphi \int_0^\pi P_k(\cos\theta)\, P_l(\cos\theta) \sin\theta \, d\theta = \sqrt{\frac{4\pi}{2l+1}} \delta_{k,l}. \tag{25}$$

The following lemma shows that a convolution theorem holds.

Lemma 2.2 *If p and q are conjugate exponents, $f \in L^q(\mathbb{S}^2)$ and $g \in L^p(\mathbb{S}^2)$, then*

$$(f * g)\widehat{\ }(l, m) = \sqrt{\frac{4\pi}{2l+1}}\, \widehat{f}(l, m)\, \widehat{g}(l, 0) \tag{26}$$

for $|m| \leqslant l$ and $l = 0, 1, \ldots$.

Proof We may assume that $g(x) = P_k(\langle x, \eta\rangle)$, where η is the north pole and $k \geqslant 0$. The general case follows from this by a standard approximation argument. According to an addition theorem for spherical harmonics [26, 14.30.9], we have

$$P_k(\langle x, y\rangle) = \frac{4\pi}{2k+1} \sum_{n=-k}^{k} Y_k^n(x)\overline{Y_k^n(y)}.$$

Combining this with (20) and (24), we obtain

$$(f * P_k)\widehat{\ }(l, m) = \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} f(y)P_k(\langle x, y\rangle)d\sigma(y)\, \overline{Y_l^m(x)}d\sigma(x) \tag{27}$$

$$= \frac{4\pi}{2k+1} \sum_{n=-k}^{k} \int_{\mathbb{S}^2} f(y)\overline{Y_k^n(y)}d\sigma(y) \int_{\mathbb{S}^2} Y_k^n(x)\overline{Y_l^m(x)}d\sigma(x) \tag{28}$$

$$= \frac{4\pi}{2k+1} \sum_{n=-k}^{k} \widehat{f}(k, n)\delta_{n,m}\delta_{k,l} = \frac{4\pi}{2k+1}\widehat{f}(k, m)\delta_{k,l} \tag{29}$$

$$= \frac{4\pi}{2l+1}\widehat{f}(l, m)\delta_{k,l} = \sqrt{\frac{4\pi}{2l+1}}\widehat{f}(l, m)\widehat{P_k}(l, 0). \tag{30}$$

The last equality follows from (25). □

A reviewer of this paper has pointed out that a special case of the Funk-Hecke formula [12, (23) in Sec. 11.4] appears in this proof. The lemma implies that convolution with a zonal function maps the space of band-limited functions S_L into itself.

3 The Large Sieve Inequalities

3.1 L^p-Bounds for General Measures

Let us denote the space of zonal functions in $L^p(\mathbb{S}^2)$ that are supported in the spherical cap $\mathcal{C}_\delta(\eta)$ by $\mathcal{Z}_\delta^p$. Specifically, for $\delta \in [-1, 1]$, we set

$$\mathcal{Z}_\delta^p := \left\{g \in L^p(\mathbb{S}^2) : \ \mathrm{supp}(g) \subset [\delta, 1],\ g \text{ is zonal}\right\}.$$

The following lemma is used in our estimate of $\lambda_{S_L}^{(2)}(\Omega)$ given in Theorem 3.3. We adopt the notation $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{S}^2)}$.

Lemma 3.1 *Let μ be a positive σ-finite measure, and let $1 < p, q < \infty$ be conjugate exponents. If $g \in \mathcal{Z}_\delta^p \setminus \{0\}$, then*

$$\int_{\mathbb{S}^2} |f|^p d\mu \leqslant \sup_{h \in \mathcal{S}_L \setminus \{0\}} \frac{\|h\|_p^p \|g\|_q^p}{\|h * g\|_p^p} \cdot \|f\|_p^p \cdot \sup_{y \in \mathbb{S}^2} \mu(\mathcal{C}_\delta(y)), \qquad f \in \mathcal{S}_L. \quad (31)$$

Proof We may assume that convolution with g is invertible on $\mathcal{S}_L$. Otherwise, the first supremum in (31) is infinite. Since $\operatorname{supp}(g) \subset [\delta, 1]$, we have

$$g(\langle x, y\rangle) = g(\langle x, y\rangle) \cdot \chi_{\mathcal{C}_\delta(y)}(x), \qquad x, y \in \mathbb{S}^2.$$

If $f^* \in \mathcal{S}_L$ is a function such that $f = f^* * g$, then by Hölder's inequality we have

$$\begin{aligned} \int_{\mathbb{S}^2} |f|^p d\mu &= \int_{\mathbb{S}^2} \left| \int_{\mathbb{S}^2} f^*(y) g(\langle x, y\rangle) \chi_{\mathcal{C}_\delta(y)}(x) d\sigma(y) \right|^p d\mu(x) \\ &\leqslant \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} |f^*(y)|^p \chi_{\mathcal{C}_\delta(y)}(x) d\sigma(y) \left(\int_{\mathbb{S}^2} |g(\langle x, y\rangle)|^q d\sigma(y) \right)^{p/q} d\mu(x). \end{aligned} \quad (32)$$

From rotational invariance of the surface measure σ, we infer that

$$\left(\int_{\mathbb{S}^2} |g(\langle x, y\rangle)|^q d\sigma(y) \right)^{p/q} = \left(\int_{\mathbb{S}^2} |g(\langle \eta, y\rangle)|^q d\sigma(y) \right)^{p/q} = \|g\|_q^p, \qquad x \in \mathbb{S}^2.$$

Substituting this into (32) and changing the order of integration, we obtain

$$\begin{aligned} \int_{\mathbb{S}^2} |f|^p d\mu &\leqslant \|g\|_q^p \cdot \int_{\mathbb{S}^2} |f^*(y)|^p \ \mu(\mathcal{C}_\delta(y)) \ d\sigma(y) \\ &\leqslant \|g\|_q^p \cdot \|f^*\|_p^p \cdot \sup_{y \in \mathbb{S}^2} \mu(\mathcal{C}_\delta(y)) \\ &= \frac{\|f^*\|_p^p \|g\|_q^p}{\|f^* * g\|_p^p} \cdot \|f\|_p^p \cdot \sup_{y \in \mathbb{S}^2} \mu(\mathcal{C}_\delta(y)) \\ &\leqslant \sup_{h \in \mathcal{S}_L \setminus \{0\}} \frac{\|h\|_p^p \|g\|_q^p}{\|h * g\|_p^p} \cdot \|f\|_p^p \cdot \sup_{y \in \mathbb{S}^2} \mu(\mathcal{C}_\delta(y)). \end{aligned}$$

□

We denote the infimum over $g \in \mathcal{Z}_\delta^p \setminus \{0\}$ of the constants in (31) by

$$C_p(L, \delta) := \inf_{g \in \mathcal{Z}_\delta^p \setminus \{0\}} \sup_{h \in \mathcal{S}_L \setminus \{0\}} \frac{\|h\|_p^p \|g\|_q^p}{\|h * g\|_p^p}. \quad (33)$$

We note that the constant $C_p(L, \delta)$ is the optimal L^p-bound within this approach.

3.2 Concentration Estimates for $\lambda^{(2)}_{\mathcal{S}_L}(\Omega)$

In this section, we derive an explicit expression for $C_2(L,\delta)$, and analyze behavior of this quantity as $L \to \infty$. In Theorem 3.3, we give an upper bound on $\lambda^{(2)}_{\mathcal{S}_L}(\Omega)$ in terms of $C_2(L,\delta)$.

Theorem 3.2 *If $t_{L,L} \leqslant \delta < 1$, then the function $g_\delta := \chi_{\mathcal{C}_\delta(\eta)} \cdot P_L\big(\langle \cdot, \eta\rangle\big)$ is a minimizer for the extremal problem (33) defining $C_2(L,\delta)$, and the minimum is given by*

$$C_2(L,\delta) = \left(2\pi \int_\delta^1 P_L(t)^2 dt\right)^{-1}. \tag{34}$$

Proof First, we simplify the extremal problem (33). Let $g \in \mathcal{Z}^2_\delta \setminus \{0\}$. Using the convolution theorem (26) and Parseval's identity, we observe that

$$\begin{aligned}\sup_{h\in\mathcal{S}_L\setminus\{0\}} \frac{\|h\|_2^2\|g\|_2^2}{\|h * g\|_2^2} &= \sup_{h\in\mathcal{S}_L\setminus\{0\}} \|g\|_2^2\|h\|_2^2 \left(\sum_{l=0}^{L}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}|\widehat{h}(l,m)|^2 \cdot |\widehat{g}(l,0)|^2\right)^{-1} \\ &= \max_{0\leqslant l\leqslant L} \frac{2l+1}{4\pi}\frac{\|g\|_2^2}{|\widehat{g}(l,0)|^2}.\end{aligned} \tag{35}$$

We now show that the constant in (34) is attained by the function g_δ. From (22), we have

$$\sqrt{\frac{4\pi}{2l+1}}\widehat{g_\delta}(l,0) = 2\pi\int_0^{\arccos\delta} P_L(\cos\theta)P_l(\cos\theta)\sin\theta d\theta = 2\pi\int_\delta^1 P_L(t)P_l(t)dt.$$

Since $t_{L,L} \leqslant \delta < 1$, it follows from (16) that

$$\sqrt{\frac{4\pi}{2l+1}}\widehat{g_\delta}(l,0) = 2\pi\int_\delta^1 P_L(t)P_l(t)dt \geqslant 2\pi\int_\delta^1 P_L(t)^2 dx = \sqrt{\frac{4\pi}{2L+1}}\widehat{g_\delta}(L,0).$$

Consequently,

$$\begin{aligned}\max_{0\leqslant l\leqslant L} \frac{2l+1}{4\pi}\frac{\|g_\delta\|_2^2}{|\widehat{g_\delta}(l,0)|^2} &= \frac{2L+1}{4\pi}\frac{\|g_\delta\|_2^2}{|\widehat{g_\delta}(L,0)|^2} \\ &= 2\pi\int_\delta^1 P_L(t)^2dt \cdot \left(2\pi\int_\delta^1 P_L(t)^2 dt\right)^{-2} \\ &= \left(2\pi\int_\delta^1 P_L(t)^2dt\right)^{-1}.\end{aligned}$$

Finally, we demonstrate that the function g_δ is a minimizer of (35) in $\mathcal{Z}_\delta^2 \setminus \{0\}$. From the Cauchy-Schwarz inequality and (21), we obtain

$$\max_{0\leqslant l\leqslant L} \frac{2l+1}{4\pi}\frac{\|g\|_2^2}{|\widehat{g}(l,0)|^2} \geqslant \frac{2L+1}{4\pi}\frac{\|g\|_2^2}{|\widehat{g}(L,0)|^2} \geqslant \frac{2L+1}{4\pi}\frac{\|g\|_2^2}{\|g\|_2^2\cdot\|\chi_{\mathcal{C}_\delta(\eta)}\cdot Y_L^0\|_2^2}$$
$$= \left(2\pi\int_0^{\arccos\delta} P_L(\cos\theta)^2\sin\theta d\theta\right)^{-1}$$
$$= \left(2\pi\int_\delta^1 P_L(t)^2 dt\right)^{-1}.$$

□

We note that the extremal problem given by minimization of (35) has the same minimizer as the following problem:

Find a real valued function $g\in\mathcal{Z}_\delta^2$ such that $\widehat{g}(l,0)\geqslant\sqrt{2l+1},\quad l=0,\ldots,L,$ and whose norm $\|g\|_2$ is minimal.

First, observe that

$$\inf_{\substack{g\in\mathcal{Z}_\delta^2\\ \widehat{g}(l,0)\geqslant\sqrt{2l+1}}}\|g\|_2 \geqslant \inf_{\substack{g\in\mathcal{Z}_\delta^2\\ \widehat{g}(l,0)\geqslant\sqrt{2l+1}}}\max_{0\leqslant l\leqslant L}\sqrt{2l+1}\frac{\|g\|_2}{\widehat{g}(l,0)}$$
$$\geqslant \inf_{g\in\mathcal{Z}_\delta^2\setminus\{0\}}\max_{0\leqslant l\leqslant L}\sqrt{2l+1}\frac{\|g\|_2}{|\widehat{g}(l,0)|} = [4\pi C_2(L,\delta)]^{1/2}, \tag{36}$$

and that if $g^* = c\cdot g_\delta$ is normalized so that $\widehat{g^*}(L,0)=\sqrt{2L+1}$, then, in view of (16), $\widehat{g^*}(l,0)\geqslant\sqrt{2l+1}, l=0,\ldots,L$. Therefore, g^* is a minimizer as $\|g^*\|_2$ equals the right hand side of (36).

From this perspective, the problem resembles Beurling-Selberg's extremal problem [31], which plays a central role in the proof of Donoho-Logan's large sieve results for band-limited functions [10], and can be seen as a Fourier side counterpart of an extremal problem considered in [22, Theorem 4].

The following theorem contains our main result.

Theorem 3.3 *Let μ be a σ-finite measure, $\Omega\subset\mathbb{S}^2$ be measurable, and $t_{L,L}\leqslant\delta<1$. For $L=1,2,\ldots$ and every $f\in\mathcal{S}_L$, it holds*

$$\int_{\mathbb{S}^2}|f|^2 d\mu \leqslant \left(2\pi\int_\delta^1 P_L(t)^2 dt\right)^{-1}\cdot\|f\|_2^2\cdot\sup_{y\in\mathbb{S}^2}\mu(\mathcal{C}_\delta(y)). \tag{37}$$

Consequently,

$$\lambda_{\mathcal{S}_L}^{(2)}(\Omega)\leqslant B_L\cdot\rho(\Omega,L), \tag{38}$$

where

$$B_L := (1 - t_{L,L}) \left(\int_{t_{L,L}}^{1} P_L(t)^2 dt \right)^{-1}. \tag{39}$$

Proof Combining Lemma 3.1 and Theorem 3.2 gives (37). Taking $\mu = \chi_\Omega d\sigma$ in (37) and using (23) and (2), we obtain

$$\begin{aligned}\int_\Omega |f|^2 d\sigma &\leqslant \left(2\pi \int_\delta^1 P_L(t)^2 dt\right)^{-1} \cdot \|f\|_2^2 \cdot \sup_{y\in\mathbb{S}^2} |\Omega \cap \mathcal{C}_\delta(y)| \\ &\leqslant \left(2\pi \int_\delta^1 P_L(t)^2 dt\right)^{-1} \cdot \|f\|_2^2 \cdot \sup_{y\in\mathbb{S}^2} |\Omega \cap \mathcal{C}_{t_{L,L}}(y)| \cdot \frac{2\pi(1-t_{L,L})}{|\mathcal{C}_{t_{L,L}}(y)|} \\ &= (1-t_{L,L}) \left(\int_\delta^1 P_L(t)^2 dt\right)^{-1} \cdot \|f\|_2^2 \cdot \rho(\Omega, L),\end{aligned}$$

which implies (38). □

The behavior of B_L for large values of L is described in the following lemma.

Lemma 3.4

$$\lim_{L\to\infty} B_L = J_1(j_{0,1})^{-2} \approx 3.71038068570948, \tag{40}$$

where J_1 is the Bessel function of the first kind, and $j_{0,1}$ is the smallest positive zero of the Bessel function J_0.

Proof We express the integrand in (39) using Taylor's theorem with the remainder in the Lagrange form

$$\begin{aligned}B_L^{-1} &= (1-t_{L,L})^{-1} \int_{t_{L,L}}^1 P_L(t)^2 dt \\ &= \int_0^1 P_L\big(1 - s(1-t_{L,L})\big)^2 ds \\ &= \int_0^1 P_L\Big(1 - \frac{j_{0,1}^2}{2L^2} s + h_L s\Big)^2 ds \\ &= \int_0^1 \Big[P_L\Big(1 - \frac{j_{0,1}^2}{2L^2} s\Big)^2 + 2h_L s P_L(\xi_s) P_L'(\xi_s)\Big] ds,\end{aligned}$$

where $\xi_s \in \Big[1 - \frac{j_{0,1}^2}{2L^2}s, 1 - \frac{j_{0,1}^2}{2L^2}s + h_L s\Big]$, and $h_L = \mathcal{O}(L^{-3})$ in view of (17). It follows from (12) and (13) that $\|P_L\|_\infty \cdot \|P_L'\|_\infty = \mathcal{O}(L^2)$. From the Mehler-Heine formula

(18) and the dominated convergence theorem, we deduce that the integral converges to

$$\int_0^1 J_0(j_{0,1}\sqrt{s})^2 ds = \frac{2}{j_{0,1}^2}\int_0^{j_{0,1}} sJ_0(s)^2 ds$$
$$= \frac{s^2}{j_{0,1}^2}\left(J_0(s)^2 + J_1(s)^2\right)\Big|_0^{j_{0,1}} = J_1(j_{0,1})^2.$$

The anti-derivative of the function $sJ_0(s)^2$ is given in [15, 5.54.2]. □

3.3 Concentration Estimates for $\lambda_{\mathcal{S}_L}^{(p)}(\Omega)$, $1 < p < \infty$

Using interpolation and duality arguments, we can extend (38) to the case $1 < p < \infty$.

Theorem 3.5 *Let $\Omega \subset \mathbb{S}^2$ be measurable and $1 < p < \infty$. For $L = 1, 2, \ldots$, it holds*

$$\lambda_{\mathcal{S}_L}^{(p)}(\Omega) = \sup_{f \in \mathcal{S}_L \setminus \{0\}} \frac{\int_\Omega |f|^p d\sigma}{\int_{\mathbb{S}^2} |f|^p d\sigma} \leqslant \left(B_L \cdot \rho(\Omega, L)\right)^{\min(p-1,1)}.$$

Proof The operator $T_\Omega : \left(\mathcal{S}_L, \|\cdot\|_{L^r(\mathbb{S}^2)}\right) \to \left(\mathcal{S}_L, \|\cdot\|_{L^r(\mathbb{S}^2)}\right)$, $T_\Omega f := \chi_\Omega \cdot f$, is a contraction for every $1 < r < \infty$. Therefore, the Riesz–Thorin theorem implies that for $2 \leqslant p < \infty$

$$\|T_\Omega\|_p \leqslant \|T_\Omega\|_r^{1-\theta}\|T_\Omega\|_2^\theta \leqslant \|T_\Omega\|_2^\theta,$$

where $r > p$ and $\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{2}$. In the limit $r \to \infty$, we obtain $\|T_\Omega\|_p \leqslant \|T_\Omega\|_2^{\frac{2}{p}}$. Consequently,

$$\lambda_{\mathcal{S}_L}^{(p)}(\Omega) = \|T_\Omega\|_p^p \leqslant \|T_\Omega\|_2^2 = \lambda_{\mathcal{S}_L}^{(2)}(\Omega). \tag{41}$$

If $1 < p < 2$, we consider the adjoint operator $T_\Omega^* : \left(\mathcal{S}_L, \|\cdot\|_{L^q(\mathbb{S}^2)}\right) \to \left(\mathcal{S}_L, \|\cdot\|_{L^q(\mathbb{S}^2)}\right)$, $T_\Omega^* f := \chi_\Omega \cdot f$, $\frac{1}{p} + \frac{1}{q} = 1$. Since $2 < q < \infty$, we have

$$\lambda_{\mathcal{S}_L}^{(p)}(\Omega) = \|T_\Omega\|_p^p = \|T_\Omega^*\|_q^p = \left(\lambda_{\mathcal{S}_L}^{(q)}(\Omega)\right)^{\frac{p}{q}} \leqslant \left(\lambda_{\mathcal{S}_L}^{(2)}(\Omega)\right)^{\frac{p}{q}} = \left(\lambda_{\mathcal{S}_L}^{(2)}(\Omega)\right)^{p-1}. \tag{42}$$

The claim now follows from (41), (42) and (38). □

4 The Classical Large Sieve Inequality on $\mathbb{S}^2$

In this section, we study the case when the measure μ in Theorem 3.3 is a finite sum of Dirac delta distributions, i.e. $\mu = \sum_{k=1}^R \delta_{x_k}$. We derive an inequality analogous to the classical large sieve inequality for trigonometric polynomials (9), see [19,24]. To this end, let us assume that the points $x_1, \ldots, x_R$ are θ-separated on the sphere, i.e. $\langle x_k, x_l \rangle \leqslant \cos\theta$, $k \neq l$, for some $\theta \in (0, \pi]$. In other words, the angle between x_k and x_l is at least θ. We consider a spherical harmonics expansion with the maximum degree L

$$S := \sum_{l=0}^{L} \sum_{m=-l}^{l} a_l^m Y_l^m, \tag{43}$$

and intend to find a constant $D = D(\theta, L)$ such that

$$\sum_{k=1}^{R} |S(x_k)|^2 \leqslant D(\theta, L) \cdot \sum_{l=0}^{L} \sum_{m=-l}^{l} |a_l^m|^2. \tag{44}$$

From Theorem 3.3, we obtain the following spherical analogue of the classical large sieve principle.

Theorem 4.1 *If $\theta \in (0, \pi]$ and the points $x_1, \ldots, x_R \in \mathbb{S}^2$ are θ-separated, then (44) holds with the constant*

$$D(\theta, L) := \left(2\pi \int_{t_{L,L}}^{1} P_L(t)^2 dt\right)^{-1} \cdot \frac{1 - \cos\frac{\theta}{2} \cdot t_{L,L} + \sin\frac{\theta}{2} \cdot \sqrt{1 - t_{L,L}^2}}{1 - \cos\frac{\theta}{2}}. \tag{45}$$

Proof We apply Theorem 3.3 with $\delta = t_{L,L}$ and $f = S$, so that

$$\|f\|_2^2 = \|S\|_2^2 = \sum_{l=0}^{L} \sum_{m=-l}^{l} |a_l^m|^2. \tag{46}$$

It remains to estimate the last factor in (37), that is

$$\sup_{y \in \mathbb{S}^2} \mu(\mathcal{C}_{t_{L,L}}(y)) = \max_{y \in \mathbb{S}^2} \#\{X \cap \mathcal{C}_{t_{L,L}}(y)\}, \tag{47}$$

where $X := \{x_k\}_{k=1,\ldots,R}$. Since the points in X are θ-separated, the angle between every two distinct points in X is at least θ. Thus the interiors of the spherical caps $\mathcal{C}_{\cos\frac{\theta}{2}}(x_1), \ldots, \mathcal{C}_{\cos\frac{\theta}{2}}(x_R)$ with the polar angle $\frac{\theta}{2}$ are disjoint. Moreover, if $x_k \in \mathcal{C}_{t_{L,L}}(y)$, then $\mathcal{C}_{\cos\frac{\theta}{2}}(x_k) \subset \mathcal{C}_{\cos(\frac{\theta}{2}+\alpha)}(y)$, where $\alpha := \arccos(t_{L,L})$. Therefore, the number of points $x_1, \ldots, x_R$ lying in $\mathcal{C}_{t_{L,L}}(y)$ does not exceed the maximum number of spherical caps with the polar angle $\frac{\theta}{2}$ with disjoint interiors that are contained in a spherical cap with the polar angle $\frac{\theta}{2} + \alpha$. Comparing the combined areas of the spherical caps $\mathcal{C}_{\cos\frac{\theta}{2}}(x_1), \ldots, \mathcal{C}_{\cos\frac{\theta}{2}}(x_R)$ with the area of the spherical cap

$\mathcal{C}_{\cos(\frac{\theta}{2}+\alpha)}(y)$ and using (23), we obtain

$$\#\{X \cap \mathcal{C}_{t_{L,L}}(y)\} \leqslant \frac{|\mathcal{C}_{\cos(\frac{\theta}{2}+\alpha)}(y)|}{|\mathcal{C}_{\cos\frac{\theta}{2}}(\cdot)|} = \frac{2\pi(1-\cos(\frac{\theta}{2}+\alpha))}{2\pi(1-\cos\frac{\theta}{2})}. \tag{48}$$

Substituting the following equation

$$\cos\left(\frac{\theta}{2}+\alpha\right) = \cos\frac{\theta}{2}\cos\alpha - \sin\frac{\theta}{2}\sin\alpha = \cos\frac{\theta}{2}\cdot t_{L,L} - \sin\frac{\theta}{2}\cdot\sqrt{1-t_{L,L}^2}$$

into (48), and taking the maximum over $y \in \mathbb{S}^2$ yields

$$\max_{y\in\mathbb{S}^2} \#\{X \cap \mathcal{C}_{t_{L,L}}(y)\} \leqslant \frac{1-\cos\frac{\theta}{2}\cdot t_{L,L} + \sin\frac{\theta}{2}\cdot\sqrt{1-t_{L,L}^2}}{1-\cos\frac{\theta}{2}}. \tag{49}$$

Finally, (44) follows by combining (37), (45), (46), (47) and (49). □

We now discuss some basic properties of the expression appearing in (45). From (40) and (17), we infer that the following quantities are equivalent up to a constant

$$\left(2\pi\int_{t_{L,L}}^1 P_L(t)^2 dt\right)^{-1} \asymp (1-t_{L,L})^{-1} \asymp L^2.$$

The second factor in (45) is a decreasing function of $t_{L,L}$. Since $0 = t_{1,1} \leqslant t_{L,L} < 1$, we have

$$1 < \frac{1-\cos\frac{\theta}{2}\cdot t_{L,L} + \sin\frac{\theta}{2}\cdot\sqrt{1-t_{L,L}^2}}{1-\cos\frac{\theta}{2}} \leqslant \frac{1+\sin\frac{\theta}{2}}{1-\cos\frac{\theta}{2}}.$$

Consequently, for a fixed $\theta \in (0, \pi]$, it holds

$$D(\theta, L) \asymp L^2. \tag{50}$$

For $0 < \theta \leqslant \pi$, we have

$$2 \geqslant 1-\cos\frac{\theta}{2}\cdot t_{L,L} + \sin\frac{\theta}{2}\cdot\sqrt{1-t_{L,L}^2} \geqslant 1-t_{L,L} > 0$$

and

$$1-\cos\frac{\theta}{2} \asymp \theta^2 \asymp 1-\cos\theta.$$

Thus, for a fixed L, it holds

$$D(\theta, L) \asymp \frac{1}{1-\cos\theta}. \tag{51}$$

We end this section with a discussion on how close the bound in Theorem 4.1 is to being optimal. We derive two elementary lower bounds on the large sieve constants, and compare them with (45). First, let us assume that we take only one sample x_1 located at the north pole η, and that $a_l^m = \delta_{m,0}$, $|m| \leqslant l, l = 0, 1, \ldots$. Substituting (21) into (43), we obtain

$$S(\eta) = \sum_{l=0}^{L} Y_l^0(\eta) = \sum_{l=0}^{L} \sqrt{\frac{2l+1}{4\pi}}.$$

Consequently, the following quantities are equivalent up to a constant

$$|S(\eta)|^2 \asymp L^3 \asymp L^2 \sum_{l=0}^{L} \sum_{m=-l}^{l} |a_l^m|^2. \tag{52}$$

It follows from (50) and (52) that for a fixed θ, the bound $D(\theta, L)$ is optimal up to a constant factor.

It remains to analyze the behavior of $D(\theta, L)$ as a function of θ for a fixed L. Let $R_{max}(\theta)$ denote the maximum number of θ-separated points on $\mathbb{S}^2$. It is known [14, p. 121], [32, (24)] that

$$R_{max}(\theta) \geqslant \frac{2}{1 - \cos\theta}. \tag{53}$$

For a fixed θ, let $x_1, \ldots, x_{R_{max}(\theta)} \in \mathbb{S}^2$ be θ-separated, and $a_l^m = 0$, $|m| \leqslant l, l = 1, 2, \ldots$, and $a_0^0 = 1$. According to (21), we have

$$\sum_{k=1}^{R} |S(x_k)|^2 = \frac{R_{max}(\theta)}{4\pi}. \tag{54}$$

From (51), (53) and (54), we conclude that also for a fixed L, the bound $D(\theta, L)$ is within a constant factor from being optimal.

We note that the inequality (53) has a simple proof. If the points $x_1, \ldots, x_{R_{max}(\theta)}$ on $\mathbb{S}^2$ are θ-separated, then the union of the spherical caps $\mathcal{C}_{\cos\theta}(x_1), \ldots, \mathcal{C}_{\cos\theta}(x_{R_{max}(\theta)})$ covers the unit sphere. Otherwise, one could find an additional point on $\mathbb{S}^2$ that is θ-separated from the points $x_1, \ldots, x_{R_{max}(\theta)}$. Comparing the areas of the caps with that of the unit sphere, we obtain

$$R_{max}(\theta) \cdot 2\pi(1 - \cos\theta) \geqslant 4\pi, \tag{55}$$

which is equivalent to (53).

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