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On the Fourier Transforms of Nonlinear Self-similar Measures

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Abstract

In-homogeneous self-similar measures can be viewed as special cases of nonlinear self-similar measures. In this paper, we study the asymptotic behaviour of the Fourier transforms of nonlinear self-similar measures. Some typical examples are exhibited, and we show that the Fourier transforms of those measures are usually localized, i.e., the Fourier transforms decay rapidly at ∞. We also discuss the infinity lower Fourier dimension of in-homogeneous self-similar measures and obtain its non-trivial bounds. The result confirms Conjecture 2.3 in Olsen and Snigireva (Math Proc Camb Philos Soc 144:465–493, 2008).

Keywords Nonlinear self-similar measure · In-homogenous self-similar measure · Fourier transform · Infinity lower Fourier dimension

Mathematics Subject Classification Primary 42A38 · Secondary 28A80

1 Introduction

Let $S_j : \mathbb{R}^d \to \mathbb{R}^d$ for $j = 1, \ldots, N$ be contracting similarities and let $(p_1, \ldots, p_N)$ be a probability vector. Then there exits a unique probability measure μ on $\mathbb{R}^d$ such that

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} \tag{1.1}$$

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by Hutchinson [9]. The measure μ is called the *self-similar measure* with respect to the iterated function system (IFS) $\{S_j\}_{j=1}^N$ and probability vector $(p_1, \ldots, p_N)$. Self-similar measures have been studied intensively for the past 30 years and many literatures investigated various aspects of them (see [6] and references therein). There are many generalizations of self-similar measures, for example self-affine measures, self-conformal measures, and statistical self-similar measures. All of them paly an important role in fractal geometry.

It is natural to write Eq. (1.1) as

$$\mu - \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} = 0.$$

This viewpoint suggests to investigate the equation with nonlinear or in-homogeneous term. In [4], Clickenstein and Strichartz studied nonlinear self-similar measures on $\mathbb{R}^d$ which satisfy the equation (involving convolutions)

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} + \sum_{j=1}^{M} q_j (\mu * \mu) \circ T_j^{-1},$$

where $\{S_j\}_{j=1}^N$ and $\{T_j\}_{j=1}^M$ are two classes of contracting similarities, nonnegative numbers p_j's and q_j's satisfy the equation $\sum_{j=1}^N p_j + \sum_{j=1}^M q_j = 1$, and Lip $T_j < \frac{1}{2}$ for all j. Under condition that nonlinear self-similar measures are not degenerate, they investigated the decay rate at ∞ of the Fourier transforms of these measures. Specifically speaking, they found that such measures are usually absolutely continuous, and the density has regularity properties that get stronger as the the linear terms get smaller. In [14], Olsen and Snigireva studied in-homogenous self-similar measures on $\mathbb{R}^d$ satisfying the equation

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} + p\,\nu,$$

where ν is a Borel probability measure on $\mathbb{R}^d$ with compact support, $\{S_j\}_{j=1}^N$ is a class of contracting similarities, and $(p_1, \ldots, p_N, p)$ is a probability vector. They discussed the Fourier transforms of the measures whose properties are affected by the in-homogenous term ν to a large extent. More precisely, they obtained the lower bounds for infinity lower Fourier dimension and 2'nd lower Fourier dimension of the in-homogenous self-similar measures. Besides, other aspects of in-homogenous self-similar measures are also investigated by many authors, for example L^q spectra and Rényi dimensions [11–13], lower and upper quantization dimensions [15,17,18], and box dimensions of in-homogeneous self-similar sets [1,7].

In this paper, we generalize the definition of nonlinear self-similar measures and then use Fourier transform methods to study such measures.

We begin by introducing some notations that will be used throughout this work. Let $(\mathbb{R}^d, \rho)$ be the d-dimensional Euclidean space with Euclidean metric. The open ball centered at x with radius r is denoted by $B(x, r) := \{y \in \mathbb{R}^d : \rho(x, y) < r\}$, and $\overline{B}(x, r)$ is its closure. We denote the set of all nonempty compact sets of $\mathbb{R}^d$ by $\mathcal{H}(\mathbb{R}^d)$. Let $\mathcal{P}(\mathbb{R}^d)$ denote the set of all Borel probability measures on $\mathbb{R}^d$ with compact support. For $\varphi \in \mathcal{P}(\mathbb{R}^d)$, the Fourier transform of φ is defined by

$$\hat{\varphi}(x) = \int e^{ix\cdot y} d\varphi(y).$$

Here $x \cdot y = \sum_{i=1}^{d} x_i y_i$ for $x, y \in \mathbb{R}^d$. The infinity lower Fourier dimension of φ is defined by

$$\underline{\Delta}_\infty(\varphi) = \liminf_{R\to\infty} \frac{\log \sup_{|x|\geq R} |\hat{\varphi}(x)|}{-\log R}.$$

Remark For $\varphi \in \mathcal{P}(\mathbb{R}^d)$, if $\underline{\Delta}_\infty(\varphi) > C > 0$, then we could find some $R > 0$ such that $|\hat{\varphi}(x)| \leq |x|^{-C}$ for all $|x| \geq R$. Alternatively we could find some constant $D \geq 1$ such that $|\hat{\varphi}(x)| \leq D|x|^{-C}$ for all x. On the other hand, if there are $C' > 0$, $D' \geq 1$ such that $|\hat{\varphi}(x)| \leq D'|x|^{-C'}$ for all x, we have $\underline{\Delta}_\infty(\varphi) \geq C'$.

For $D \in \mathcal{H}(\mathbb{R}^d)$, let $\mathcal{P}(D)$ denote the set of probability measures on D. For $\varphi, \psi \in \mathcal{P}(D)$, we set

$$L_D(\varphi, \psi) := \sup\Big\{\Big|\int_D f d\varphi - \int_D f d\psi\Big| : f \in \mathrm{Lip}_1(D)\Big\},$$

where $\mathrm{Lip}_1(D) := \{f : D \to \mathbb{R} : |f(x) - f(y)| \leq \rho(x, y)$ for any $x, y \in D\}$. L_D is called Hutchinson metric. Note that $(\mathcal{P}(D), L_D)$ is a complete metric space (refer to [2, Chapter 9, Theorem 5.1]). We denote the Hausdorff metric of $\mathcal{H}(\mathbb{R}^d)$ by d_H. For $E, F \in \mathcal{H}(\mathbb{R}^d)$, it is well-known that

$$d_H(E, F) = \sup\{\rho(x, F), \rho(y, E) : x \in E, y \in F\},$$

and $(\mathcal{H}(\mathbb{R}^d), d_H)$ is also a complete metric space (refer to [2, Chapter 2, Theorem 7.1]).

We say that a map $\phi : \mathcal{H}(\mathbb{R}^d) \to \mathcal{H}(\mathbb{R}^d)$ is a *monotone transformation* if $\phi(A) \subset \phi(B)$ for $A, B \in \mathcal{H}(\mathbb{R}^d)$ with $A \subset B$.

Definition 1 For a transformation $\Phi : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$, we say that Φ satisfies condition **(H)** if there exists a monotone transformation $\phi : \mathcal{H}(\mathbb{R}^d) \to \mathcal{H}(\mathbb{R}^d)$ such that for any $\varphi \in \mathcal{P}(\mathbb{R}^d)$, $\mathrm{spt}\,\Phi(\varphi) = \phi(\mathrm{spt}\,\varphi)$, where $\mathrm{spt}\,\varphi$ is the support of φ.

Remark For any $D \in \mathcal{H}(\mathbb{R}^d)$, there exists a Borel probability measure φ such that $\mathrm{spt}\,\varphi = D$. Hence if Φ satisfies condition **(H)**, the monotone transformation ϕ is unique. Because of this fact, we also say ϕ is determined by Φ.

Definition 2 Let $\{S_j\}_{j=1}^N$ be a class of contracting similarities on $\mathbb{R}^d$ such that

$$S_j(x) = r_j R_j x + a_j, \quad j = 1, \dots, N,$$

where R_j is an orthogonal matrix, $0 < r_j < 1$, and $a_j \in \mathbb{R}^d$. Let $\Phi : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ be a transformation satisfying condition **(H)** and $(p_1, \dots, p_N, p)$ a probability vector. Assume ϕ is determined by Φ. We obtain a nonlinear self-similar identity

$$\mu = \sum_{j=1}^N p_j \mu \circ S_j^{-1} + p\,\Phi(\mu). \tag{1.2}$$

If the solutions of this equation exist, we call them nonlinear self-similar measures. If $\Phi(\mu) \equiv \nu$ for some $\nu \in \mathcal{P}(\mathbb{R}^d)$, the solution of the equation exists and is unique, which was called in-homogeneous self-similar measure (refer to [3,13,14]).

We consider nonlinear self-similar measures in Sect. 2. At the beginning of the section, we give a necessary condition and a sufficient condition for the existence of nonlinear self-similar measures. The proofs are based on Banach's fixed-point theorem and rather standard, but to the best of our knowledge, no literature supplied complete and clear proofs for existence of nonlinear self-similar measures. After that we shall exhibit some typical nonlinear self-similar measures, and study the decay rate of the Fourier transforms of them. We find that the Fourier transforms of these measures are usually localized and that the decay rate at ∞ is dependent on the probability vector and the contraction ratios of contracting similarities. Moreover, let t be the positive constant such that $\sum_{j=1}^N p_j r_j^{-t} = 1$, we prove that the infinity lower Fourier dimension of these measures is not less than t. Afterwards we give a example to show that the lower bound t is the best possible.

In Sect. 3, we study the infinity lower Fourier dimension of in-homogeneous self-similar measures. The main result is Theorem 7. We obtain the lower and upper bounds for infinity lower Fourier dimension, which are dependent on the infinity lower Fourier dimension of the in-homogeneous term. The result improves the works of Olsen and Snigireva (see [14, Theorem 2.1]). In [14, Conjecture 2.3] the authors proposed the following conjecture:

Conjecture 1 *For equation* (1.2), *we assume* $\Phi(\mu) \equiv \nu$ *for some compactly supported probability measure* ν *and let* μ *be the in-homogeneous self-similar measure that satisfies this equation. Let* t *be the positive constant that satisfies* $\sum_{j=1}^N p_j r_j^{-t} = 1$. *Then, for all choices of* $r_1, \dots, r_N$ *we have*

$$\underline{\Delta}_\infty(\mu) \geq \min(t\,,\, \underline{\Delta}_\infty(\nu)).$$

Our result also confirms this conjecture.

2 Nonlinear Self-similar Measures

2.1 Existence of Nonlinear Self-similar Measures

Now, we consider the existence of nonlinear self-similar measures. First of all, we claim that if μ is the solution of Eq. (1.2), the support sptμ of μ must satisfy the following condition.

Lemma 1 *If μ is a solution of Eq. (1.2), then*

$$\operatorname{spt}\mu = \bigcup_{j=1}^{N} S_j(\operatorname{spt}\mu) \cup \phi(\operatorname{spt}\mu).$$

Proof It is easy to see that

$$\mu \circ S_j^{-1}\left(\bigcup_{j=1}^{N} S_j(\operatorname{spt}\mu) \cup \phi(\operatorname{spt}\mu)\right) = 1 \quad \text{for all } j,$$
$$\Phi(\mu)\left(\bigcup_{j=1}^{N} S_j(\operatorname{spt}\mu) \cup \phi(\operatorname{spt}\mu)\right) = 1.$$

Hence $\operatorname{spt}\mu \subset \bigcup_{j=1}^{N} S_j(\operatorname{spt}\mu) \cup \phi(\operatorname{spt}\mu)$.

On the other hand, since $\mu(\operatorname{spt}\mu) = 1$ and $(p_1, \ldots, p_N, p)$ is a probability vector, we have

$$\Phi(\mu)(\operatorname{spt}\mu) = 1 \quad \text{and} \quad \mu \circ S_j^{-1}(\operatorname{spt}\mu) = 1 \quad \text{for all } j.$$

Hence

$$\phi(\operatorname{spt}\mu) \subset \operatorname{spt}\mu \quad \text{and} \quad S_j(\operatorname{spt}\mu) \subset \operatorname{spt}\mu \quad \text{for all } j.$$

That is $\bigcup_{j=1}^{N} S_j(\operatorname{spt}\mu) \cup \phi(\operatorname{spt}\mu) \subset \operatorname{spt}\mu$. We complete the proof. □

Lemma 1 gives a necessary condition for the existence of solutions of Eq. (1.2), i.e., there exists $K \in \mathcal{H}(\mathbb{R}^d)$ such that $K = \bigcup_{j=1}^{N} S_j(K) \cup \phi(K)$. We give an example to illustrate this.

Example 1 We define $\Phi : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by

$$\Phi(\varphi) = (\varphi * \varphi * \varphi) \circ T^{-1}, \quad \varphi \in \mathcal{P}(\mathbb{R}),$$

where $T(x) = \frac{x}{2}$. Assume that $S(x) = \frac{x}{2} + 1$. We obtain a nonlinear self-similar identity

$$\mu = \frac{1}{2}\mu \circ S^{-1} + \frac{1}{2}(\mu * \mu * \mu) \circ T^{-1}. \tag{2.1}$$

If the above equation has a solution, we could find $K \in \mathcal{H}(\mathbb{R})$ such that

$$K = \left(\frac{1}{2}K + 1\right) \cup \frac{1}{2}(K + K + K).$$

Since $K \setminus \{0\} \neq \emptyset$, we assume $x_1 \in K \setminus \{0\}$. One can verify that $(3/2)^n x_1 \in K$ for all $n \in \mathbb{N}$. But K is a compact set, which is a contradiction. Thus Eq. (2.1) has no solution.

Now we present a sufficient condition for the existence of the solutions of Eq. (1.2). In what follows, for $D \in \mathcal{H}(\mathbb{R}^d)$ and $\psi \in \mathcal{P}(D)$, we define the measure $\psi_D \in \mathcal{P}(\mathbb{R}^d)$ by

$$\psi_D(E) = \psi(E \cap D) \quad \text{for Borel sets } E. \tag{2.2}$$

If $\varphi \in \mathcal{P}(\mathbb{R}^d)$ and $\operatorname{spt}\varphi \subset D$, φ can be regarded as a member of $\mathcal{P}(D)$ of course.

Theorem 1 *Assume that there exists $K \in \mathcal{H}(\mathbb{R}^d)$ such that $K = \bigcup_{j=1}^N S_j(K) \cup \phi(K)$. If for any $f \in \operatorname{Lip}_1(K)$ and any $\varphi, \psi \in \mathcal{P}(K)$, we have*

$$\Big|\int_K f d\Phi(\varphi_K) - \int_K f d\Phi(\psi_K)\Big| \leq L_K(\varphi, \psi).$$

Then there exists a unique measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfying Eq. (1.2) with $\operatorname{spt}\mu \subset K$. Moreover, if the set $K \in \mathcal{H}(\mathbb{R}^d)$ satisfying $K = \bigcup_{j=1}^N S_j(K) \cup \phi(K)$ is unique, then the solution of Eq. (1.2) is unique too.

Proof Define a map $\mathcal{M} : \mathcal{P}(K) \to \mathcal{P}(\mathbb{R}^d)$ by

$$\mathcal{M}(\varphi) = \sum_{j=1}^N p_j \varphi_K \circ S_j^{-1} + p\Phi(\varphi_K).$$

Note that $\operatorname{spt}\varphi_K = \operatorname{spt}\varphi \subset K$, we obtain that $\bigcup_{j=1}^N S_j(\operatorname{spt}\varphi_K) \cup \phi(\operatorname{spt}\varphi_K) \subset K$. But

$$\mathcal{M}(\varphi)\Big(\bigcup_{j=1}^N S_j(\operatorname{spt}\varphi_K) \cup \phi(\operatorname{spt}\varphi_K)\Big) = 1,$$

hence $\operatorname{spt}\mathcal{M}(\varphi) \subset K$. Thus it is proper to regard $\mathcal{M}$ as a map from $\mathcal{P}(K)$ to itself.

Now we show $\mathcal{M}$ is a contraction from $\mathcal{P}(K)$ to itself with respect to the metric L_K. For any $\varphi, \psi \in \mathcal{P}(K)$ and for any $f \in \operatorname{Lip}_1(K)$, we have

$$\Big|\int_K f d\mathcal{M}(\varphi) - \int_K f d\mathcal{M}(\psi)\Big|$$

$$\leq \sum_{j=1}^{N} p_j \Big| \int_K f d\varphi_K \circ S_j^{-1} - \int_K f d\psi_K \circ S_j^{-1} \Big| + p \Big| \int_K f d\Phi(\varphi_K) - \int_K f d\Phi(\psi_K) \Big|$$

$$\leq \sum_{j=1}^{N} p_j \Big| \int_{S_j^{-1}(K)} f \circ S_j d\varphi_K - \int_{S_j^{-1}(K)} f \circ S_j d\psi_K \Big| + pL_K(\varphi, \psi)$$

$$= \sum_{j=1}^{N} p_j r_j \Big| \int_K \frac{1}{r_j} f \circ S_j d\varphi - \int_K \frac{1}{r_j} f \circ S_j d\psi \Big| + pL_K(\varphi, \psi)$$

$$\leq \big(\sum_{j=1}^{N} p_j r_j + p \big) L_K(\varphi, \psi).$$

By the arbitrariness of f, we have

$$L_K\big(\mathcal{M}(\varphi), \mathcal{M}(\psi)\big) \leq \big(\sum_{j=1}^{N} p_j r_j + p \big) L_K(\varphi, \psi) < \left(\sum_{j=1}^{N} p_j + p \right) L_K(\varphi, \psi) = L_K(\varphi, \psi).$$

By Banach's fixed-point theorem, there exists a unique $\omega \in \mathcal{P}(K)$ such that $\mathcal{M}(\omega) = \omega$. Set $\mu = \omega_K$. μ is actually a solution of Eq. (1.2).

If $\nu_1 \in \mathcal{P}(\mathbb{R}^d)$ satisfies Eq. (1.2) and spt $\nu_1 \subset K$, there is a unique $\lambda_1 \in \mathcal{P}(K)$ such that $\nu_1 = (\lambda_1)_K$. $\mathcal{M}(\lambda_1) = \lambda_1$ obviously. Still by Banach's fixed-point theorem, $\lambda_1 = \omega$, that is $\nu_1 = \mu$.

Now assume the set $K \in \mathcal{H}(\mathbb{R}^d)$ satisfying $K = \bigcup_{j=1}^{N} S_j(K) \cup \phi(K)$ is unique. If $\nu_2 \in \mathcal{P}(\mathbb{R}^d)$ satisfies Eq. (1.2), we have spt $\nu_2 = \bigcup_{j=1}^{N} S_j(\text{spt}\, \nu_2) \cup \phi(\text{spt}\, \nu_2)$ by Lemma 1. Thus spt $\nu_2 = K$. By the above argument, we have $\nu_2 = \mu$ of course. The proof is complete. □

Inspired by Theorem 1, Eq. (1.2) may have more than one solution. We give an example to illustrate this.

Example 2 We define $\Phi : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by

$$\Phi(\varphi) = \begin{cases} \varphi \circ S^{-1} & \text{if } \ 0 \notin \text{spt}\, \varphi, \\ \frac{1}{2}\varphi \circ S^{-1} + \frac{1}{2}\delta_0 & \text{if } \ 0 \in \text{spt}\, \varphi. \end{cases}$$

where $S(x) = \frac{x}{3} + \frac{1}{3}$ and δ_0 is the Dirac measure supported at $\{0\}$. ϕ is determined by Φ and we have

$$\phi(E) = \begin{cases} S(E) & \text{if } \ 0 \notin E, \\ S(E) \cup \{0\} & \text{if } \ 0 \in E. \end{cases}$$

We obtain a nonlinear self-similar identity

$$\mu = \frac{1}{2}\mu \circ S^{-1} + \frac{1}{2}\Phi(\mu). \tag{2.3}$$

Since there exist two compact set that satisfy $K = S(K) \cup \phi(K)$, i.e., $K = \{1/2\}$ or K is the unique compact set satisfying $K = \left(\frac{1}{3}K + \frac{1}{3}\right) \cup \{0\}$, the uniqueness of the solution may break down. Actually, Eq. (2.3) has two solutions, one of which is $\delta_{\frac{1}{2}}$ and the other is the unique Borel probability measure satisfying $\mu = \frac{3}{4}\mu \circ S^{-1} + \frac{1}{4}\delta_0$.

2.2 Some Typical Examples

In what follows, we focus on some typical nonlinear self-similar measures. Let $\{S_i\}_{i=1}^N$, $\{T_j\}_{j=1}^M$ and $\{Q_k\}_{k=1}^M$ be three classes of contracting similarities on $\mathbb{R}^d$ with

$$S_i(x) = r_i R_i x + a_i, \quad T_j(x) = \rho_j P_j x + b_j \quad \text{and} \quad Q_k(x) = \gamma_k O_k x + c_k,$$

where R_i, P_j, O_k are orthogonal matrices, $r_i, \rho_j, \gamma_k \in (0,1)$, and $a_i, b_j, c_k \in \mathbb{R}^d$. Let $(p_1, \dots, p_N, q_1, \dots, q_M)$ be a probability vector. Assume ν_j is a Borel probability measure with compact support C_j for all j. Set $p = \sum_{j=1}^M q_j$. We could define three transformations from $\mathcal{P}(\mathbb{R}^d)$ to itself:

$$\Phi_1 : \varphi \mapsto \sum_{j=1}^M \frac{q_j}{p}(\varphi \circ T_j^{-1}) * (\varphi \circ Q_j^{-1}), \quad \Phi_2 : \varphi \mapsto \sum_{j=1}^M \frac{q_j}{p}(\varphi \circ T_j^{-1}) * \nu_j,$$

$$\Phi_3 : \varphi \mapsto \sum_{j=1}^M \frac{q_j}{p}(\underbrace{\varphi * \cdots * \varphi}_{k_j \text{ times}}) \circ T_j^{-1}.$$

In addition, we assume $\rho_j + \gamma_j < 1$ for all j when talking about Φ_1, and assume $\rho_j < \frac{1}{k_j} \le \frac{1}{2}$ for all j when talking about Φ_3. Suppose ϕ_i is determined by Φ_i. For $D \in \mathcal{H}(\mathbb{R}^d)$, it is easy to obtain that $\phi_1(D) = \bigcup_{j=1}^M \big(T_j(D) + Q_j(D)\big)$, $\phi_2(D) = \bigcup_{j=1}^M \big(T_j(D) + C_j\big)$, and $\phi_3(D) = \bigcup_{j=1}^M T_j\big(\underbrace{D + \cdots + D}_{k_j \text{ times}}\big)$.

By Definition 2, we obtain three nonlinear self-similar equations, that is

$$\mu = \sum_{j=1}^N p_j \mu \circ S_j^{-1} + \sum_{j=1}^M q_j (\mu \circ T_j^{-1}) * (\mu \circ Q_j^{-1}), \tag{2.4}$$

$$\mu = \sum_{j=1}^N p_j \mu \circ S_j^{-1} + \sum_{j=1}^M q_j (\mu \circ T_j^{-1}) * \nu_j, \tag{2.5}$$

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} + \sum_{j=1}^{M} q_j (\underbrace{\mu * \cdots * \mu}_{k_j \text{ times}}) \circ T_j^{-1}. \tag{2.6}$$

The existence and uniqueness of solutions of Eqs. (2.4–2.6) is stated below.

Lemma 2 *There exists a unique $K_i \in \mathcal{H}(\mathbb{R}^d)$ such that $K_i = \bigcup_{j=1}^{N} S_j(K_i) \cup \phi_i(K_i)$ for $i = 1, 2, 3$.*

Proof For each i, we define a map $\mathcal{N}_i : \mathcal{H}(\mathbb{R}^d) \to \mathcal{H}(\mathbb{R}^d)$ by $\mathcal{N}_i(E) := \bigcup_{j=1}^{N} S_j(E) \cup \phi_i(E)$. It is easy to see that $\mathcal{N}_i$ is a contraction mapping from $\mathcal{H}(\mathbb{R}^d)$ to itself. By Banach's fixed-point theorem, there is a unique nonempty compact set K_i such that $K_i = \bigcup_{j=1}^{N} S_j(K_i) \cup \phi_i(K_i)$. We complete the proof. □

Theorem 2 *Equations (2.4–2.6) all have a unique solution and we denote them by μ_1, μ_2, μ_3 respectively. Moreover,* spt $\mu_i = K_i$ *for $i = 1, 2, 3$.*

Proof We just prove this theorem for $i = 1$ while you could use very similar arguments to prove the rest. For $j = 1, 2, \ldots, M$, $f \in \mathrm{Lip}_1(K_1)$ and $\varphi \in \mathcal{P}(K_1)$, by Fubini's theorem, we have

$$\int_{K_1} f d(\varphi_{K_1} \circ T_j^{-1}) * (\varphi_{K_1} \circ Q_j^{-1}) = \int f(T_j(x) + Q_j(y)) d\varphi_{K_1} \times \varphi_{K_1}(x, y).$$

We define

$$h_j(y) = \frac{1}{\gamma_j} \int_{K_1} f(T_j(x) + Q_j(y)) d\varphi_{K_1}(x) \quad \text{and}$$
$$g_j(x) = \frac{1}{\rho_j} \int_{K_1} f(T_j(x) + Q_j(y)) d\psi_{K_1}(y).$$

Note that h_j, g_j are members of $\mathrm{Lip}_1(K_1)$. For $\varphi, \psi \in \mathcal{P}(K_1)$, still by Fubini's theorem, we obtain

$$\begin{aligned} &\Big| \int f(T_j(x) + Q_j(y)) d\varphi_{K_1} \times \varphi_{K_1}(x, y) \\ &\quad - \int f(T_j(x) + Q_j(y)) d\psi_{K_1} \times \psi_{K_1}(x, y) \Big| \\ &\le \gamma_j \Big| \int h_j(y) d\varphi_{K_1}(y) - \int h_j(y) d\psi_{K_1}(y) \Big| + \rho_j \Big| \int g_j(x) d\varphi_{K_1}(x) \\ &\quad - \int g_j(x) d\psi_{K_1}(x) \Big| \\ &\le (\gamma_j + \rho_j) L_{K_1}(\varphi, \psi). \end{aligned}$$

In conclude, we have

$$\Big| \int_{K_1} f d\Phi_1(\varphi_{K_1}) - \int_{K_1} f d\Phi_1(\psi_{K_1}) \Big| \le L_{K_1}(\varphi, \psi).$$

Then the result is an easy consequence of Theorem 1. ◻

Now we discuss the decay rate at ∞ of $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3$. Observe that the Fourier transform versions of Eqs. (2.4–2.6) are

$$\hat{\mu}_1(x) = \sum_{j=1}^{N} p_j e^{ix\cdot a_j} \hat{\mu}_1(r_j R_j^* x) + \sum_{j=1}^{M} q_j e^{ix\cdot(b_j+c_j)} \hat{\mu}_1(\rho_j P_j^* x)\hat{\mu}_1(\gamma_j O_j^* x) \tag{2.7}$$

$$\hat{\mu}_2(x) = \sum_{j=1}^{N} p_j e^{ix\cdot a_j} \hat{\mu}_2(r_j R_j^* x) + \sum_{j=1}^{M} q_j e^{ix\cdot b_j} \hat{\mu}_2(\rho_j P_j^* x)\hat{\nu}_j(x), \tag{2.8}$$

$$\hat{\mu}_3(x) = \sum_{j=1}^{N} p_j e^{ix\cdot a_j} \hat{\mu}_3(r_j R_j^* x) + \sum_{j=1}^{M} q_j e^{ix\cdot b_j} \hat{\mu}_3(\rho_j P_j^* x)^{k_j}. \tag{2.9}$$

We also assume μ_1 and μ_3 are not degenerate in the rest of this subsection. For $\varphi \in \mathcal{P}(\mathbb{R}^d)$, if there exists an affine hyperplane $\{x : x\cdot\omega_1 = l_1\}$ for ω_1 a unit vector such that $\operatorname{spt}\varphi \subset \{x : x\cdot\omega_1 = l_1\}$, we say that φ is degenerate. Otherwise φ is not degenerate.

Lemma 3 *If μ_1 is not degenerate, then $\hat{\mu}_1$ vanishes at infinity. The same holds true for $\hat{\mu}_3$.*

Remark The conclusion of Lemma 3 is similar to [4, Lemmas 2.1, 3.1], but our proof is different and simpler.

Proof We just prove this lemma for $\hat{\mu}_1$. Firstly, we prove $|\hat{\mu}_1(x)| < 1$ for $x \neq 0$. Suppose $|\hat{\mu}_1(x_0)| = 1$ for some $x_0 \neq 0$. Since $\sum_{j=1}^N p_j + \sum_{j=1}^M q_j = 1$, we have $|\hat{\mu}_1(r_j R_j^* x_0)| = 1$ for some j. By iteration we obtain a sequence of points $x_l = (r_j R_j^*)^l x_0$ tending to zero with $|\hat{\mu}_1(x_l)| = 1$. Because μ_1 is not degenerate, for any x_l, $l \in \mathbb{Z}_+$, there exist $\xi_l, \eta_l \in \operatorname{spt}\mu_1$ such that $x_l\cdot(\xi_l - \eta_l) \neq 0$. Since $\operatorname{spt}\mu_1$ is compact and the sequence of points x_l tends to zero, there exists $k \in \mathbb{Z}_+$ such that $0 < |x_k\cdot(\xi_k - \eta_k)| < \pi$. For some sufficiently small $\delta > 0$, if $y \in \overline{B}(\xi_k, \delta) =: B_1$, $x \in \overline{B}(\eta_k, \delta) =: B_2$, we have $|x_k\cdot(y - x)| \in (0, \pi)$. Note that

$$1 = \int e^{ix_k\cdot(x-y)} d\mu_1\times\mu_1(x,y) = \int_{B_2\times B_1} e^{ix_k\cdot(x-y)} d\mu_1\times\mu_1(x,y) + \int_{\mathbb{R}^d\times\mathbb{R}^d - B_2\times B_1} e^{ix_k\cdot(x-y)} d\mu_1\times\mu_1(x,y).$$

Thus

$$\int_{B_2\times B_1} e^{ix_k\cdot(x-y)} d\mu_1\times\mu_1(x,y) = \mu_1\big(\overline{B}(\eta_k,\delta)\big)\mu_1\big(\overline{B}(\xi_k,\delta)\big) > 0.$$

Hence

$$\int_{B_2\times B_1}\cos\big(x_k\cdot(x-y)\big)d\mu_1\times\mu_1(x,y)=\mu_1\big(\overline{B}(\eta_k,\delta)\big)\mu_1\big(\overline{B}(\xi_k,\delta)\big),$$

which is a contradiction.

We have proved $|\hat{\mu}_1(x)|<1$ for $x\neq 0$. Thus there exists $c<1$ such that

$$|\hat{\mu}_1(x)|\leq c\quad\text{for } 1\leq|x|\leq A, \tag{2.10}$$

where A is chosen larger than all $\frac{1}{r_j}, \frac{1}{\rho_j}, \frac{1}{\gamma_j}$. Take a number $B>1$ such that B is less than all $\frac{1}{r_j}, \frac{1}{\rho_j}, \frac{1}{\gamma_j}$. Thus for $A\leq|x|\leq BA$, by Eqs. (2.7) and (2.10) we have $|\hat{\mu}_1(x)|\leq\sum_{j=1}^N p_jc+\sum_{j=1}^M q_jc^2\leq c$. Similarly, we can prove that Eq. (2.10) holds for all $A\leq|x|\leq B^nA$, where $n=1,2,\ldots$. That is, Eq. (2.10) holds for all $|x|\geq 1$. Since each time we apply Eq. (2.10) we square the values of $\hat{\mu}_1(x)$ in the second sum, so $|\hat{\mu}_1(x)|\leq(\sum_{j=1}^N p_j)c+(\sum_{j=1}^M q_j)c^2=(1-\varepsilon)c$ for $\varepsilon=(1-c)\sum_{j=1}^M q_j$ provided $|x|\geq A$. By iterating this argument we obtain $|\hat{\mu}_1(x)|\leq(1-\varepsilon)^kc$ if $|x|\geq A^k$. □

Theorem 3 *Let t denote the positive constant satisfying the equation $\sum_{j=1}^N p_jr_j^{-t}=1$.*

(i) *For any positive constant $\varepsilon<t$, there exists a positive constant c such that*

$$|\hat{\mu}_1(x)|\leq c|x|^{-\varepsilon}. \tag{2.11}$$

The same holds true for $\hat{\mu}_3$.

(ii) *Assume $\lim_{|x|\to\infty}|\hat{\nu}_j(x)|=0$ for all j. Then for any positive constant $\varepsilon<t$, there exists a positive constant c such that*

$$|\hat{\mu}_2(x)|\leq c|x|^{-\varepsilon}.$$

Proof We just prove this theorem for $\hat{\mu}_1$. Choose a positive δ such that

$$\sum_{j=1}^N p_jr_j^{-\varepsilon}+\delta\sum_{j=1}^M q_j\rho_j^{-\varepsilon}<1. \tag{2.12}$$

Then by Lemma 3, we can find B such that

$$|\hat{\mu}_1(x)|\leq\delta\quad\text{for } |x|\geq B.$$

We choose a constant A such that A is larger than all $\frac{1}{\gamma_j}$ and then we choose a c such that Eq. (2.11) holds for $|x|\leq BA$. Set $\rho=\max\{r_j,\rho_j\}$.

If $BA \le |x| \le \rho^{-1}BA$, then $|r_j R_j^* x|, |\rho_j P_j^* x| \le BA$ and $|\gamma_j O_j^* x| \ge B$. Thus we obtain

$$|\hat{\mu}_1(x)| \le \sum_{j=1}^{N} p_j c|r_j x|^{-\varepsilon} + \sum_{j=1}^{M} q_j \delta c|\rho_j x|^{-\varepsilon} \le c|x|^{-\varepsilon}.$$

By iterating this argument we obtain Eq. (2.11) holds for $BA \le |x| \le \rho^{-k}BA$ for all k, hence it holds for all x. □

Theorem 3 implies that $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3$ decay rapidly at ∞ and they all have a decay rate of $O(|x|^{-\varepsilon})$, where $0 < \varepsilon < t$. In other words, $\underline{\Delta}_\infty(\mu_j) \ge t$ for $j = 1, 2, 3$. A simple example in next subsection shows t is the best possible. Next theorem shows if we add some restrictions on contracting similarities, they will have a decay rate of $O(|x|^{-t})$.

Theorem 4 *For Eqs. (2.4–2.6), we assume $S_i(x) = rRx + a_i$ and $T_j(x) = rRx + b_j$ in addition. Let $t = \log\left(\sum_{j=1}^{N} p_j\right)/\log r$. Then*

(i) $|\hat{\mu}_i(x)| = O(|x|^{-t})$ *as* $|x| \to \infty$ *for* $i = 1, 3$.
(ii) *If* $\underline{\Delta}_\infty(\nu_j) > 0$ *all* j, *then* $|\hat{\mu}_2(x)| = O(|x|^{-t})$ *as* $|x| \to \infty$.

Proof Set $q = \sum_{j=1}^{N} p_j$, $p_j^{'} = \frac{p_j}{q}$ and $q_j^{'} = \frac{q_j}{q}$. For $i = 1, 2, 3$, we define

$$g_i(x) = \sum_{j=1}^{N} p_j^{'} e^{ix\cdot a_j} + \sum_{j=1}^{M} q_j^{'} e^{ix\cdot b_j} h_{i,j}(x),$$

where $h_{1,j}(x) = e^{ix\cdot c_j}\hat{\mu}_1(\gamma_j O_j^* x)$, $h_{2,j}(x) = \hat{\nu}_j(x)$ and $h_{3,j}(x) = \hat{\mu}_3(rR^*x)^{k_j-1}$. By Theorem 3 and Assumption (ii) of Theorem 4, there exist $s > 0$ and $D_j > 0$ such that for $i = 1, 2, 3$, we have $|h_{i,j}(x)| \le D_j|x|^{-s}$ for $x \in \mathbb{R}^d$. Hence for $x \in \mathbb{R}^d$, $|g_i(x)| \le 1 + \left(\sum_{j=1}^{M} q_j^{'} D_j\right)|x|^{-s} =: 1 + D|x|^{-s}$. Take Fourier transforms of Eqs. (2.4–2.6), we obtain $\hat{\mu}_i(x) = qg_i(x)\hat{\mu}_i(rR^*x)$ for $i = 1, 2, 3$. Iterating these functions, we have

$$\hat{\mu}_i(x) = q^n \left(\prod_{j=0}^{n-1} g_i\left(r^j (R^*)^j x\right)\right) \hat{\mu}_i\left(r^n (R^*)^n x\right). \tag{2.13}$$

For any x with $|x| > \frac{1}{r}$, there exists $n(x) \in \mathbb{N}$ such that $|r^{n(x)}x| \in [1, r^{-1}]$. If we write $|\hat{\mu}_i(x)| = \left|\hat{\mu}_i(r^{-n(x)}r^{n(x)}x)\right|$, then

$$|\hat{\mu}_i(x)| = q^{n(x)} \left(\prod_{j=0}^{n(x)-1} \left|g_i(r^j (R^*)^j r^{-n(x)} r^{n(x)} x)\right|\right) \left|\hat{\mu}_i((R^*)^{n(x)} r^{n(x)} x)\right|$$

$$
\begin{aligned}
&\le r^{tn(x)} \prod_{k=1}^{n(x)} \left| g_i(r^{-k}(R^*)^{n(x)-k} r^{n(x)} x) \right| \\
&\le r^{tn(x)} \prod_{k=1}^{n(x)} (1 + Dr^{sk}) \le r^{tn(x)} \widetilde{D} \\
&\le \widetilde{D} r^{tn(x)} r^{-t} |r^{n(x)} x|^{-t} = \widetilde{D} r^{-t} |x|^{-t},
\end{aligned}
$$

where $\widetilde{D} := \Pi_{k=1}^{\infty}(1 + Dr^{sk}) < +\infty$. The proof is complete. □

2.3 A Simple Example

In this subsection we shall construct a nonlinear self-similar measure μ_4 under the hypothesis of Theorem 4 and compute the exact decay rate of $\hat{\mu}_4$. Let $\mu_4 \in \mathcal{P}(\mathbb{R})$ be the solution of following identity

$$
\mu = \frac{1}{4}\mu \circ S_1^{-1} + \frac{1}{8}\mu \circ S_2^{-1} + \frac{1}{8}\mu \circ S_3^{-1} + \frac{1}{2}(\mu \circ T^{-1}) * \nu. \tag{2.14}
$$

Here $S_1(x) = \rho x$, $S_2(x) = \rho x + 2\pi$, $S_3(x) = \rho x - 2\pi$ and $T(x) = \rho x$ for $0 < \rho < 1$. $\nu \in \mathcal{P}(\mathbb{R})$ with $\underline{\Delta}_\infty(\nu) > 0$. Before stating our results it is useful to introduce the following terminology and notations.

Definition 3 An algebraic integer $\beta > 1$ is called a PV-number if all its conjugate roots have modulus strictly less than 1

For any real number a, we denote $\|a\| = \min\{|a-n| : n \in \mathbb{Z}\}$. If k is the integer nearest to a, we write $a = k + \langle a \rangle$ (If $\|a\| = \frac{1}{2}$, we let $\langle a \rangle > 0$). It is well known that if ρ^{-1} is not a PV-number, then for any $s \in \mathbb{R} - \{0\}$, we have $\sum_{j=1}^{\infty} \|s\rho^{-j}\|^2 = \infty$. Otherwise if ρ^{-1} is a PV-number, then for every integer k, we have $\sum_{j=1}^{\infty} \|k\rho^{-j}\|^2 < \infty$. For more details about PV-number, we refer to [16]. For $j \in \mathbb{Z}$, we define $d_j(x) = \frac{1}{2}(1 + \cos(2\pi\rho^j x))$. In this example, note that $t = -\frac{\log 2}{\log \rho}$.

The Fourier transform version of Eq. (2.14) is

$$
\begin{aligned}
\hat{\mu}_4(x) &= \frac{1}{4}\hat{\mu}_4(\rho x) + \frac{1}{4}\cos(2\pi x)\hat{\mu}_4(\rho x) + \frac{1}{2}\hat{\nu}(x)\hat{\mu}_4(\rho x) \\
&= \frac{1}{2}\big(d_0(x) + \hat{\nu}(x)\big)\hat{\mu}_4(\rho x).
\end{aligned}
$$

Iterating the above equation, we get

$$
\hat{\mu}_4(x) = \frac{1}{2^n}\left(\prod_{j=0}^{n-1}\big(d_j(x) + \hat{\nu}(\rho^j x)\big)\right)\hat{\mu}_4(\rho^n x). \tag{2.15}
$$

Replacing x by $\rho^{-n}x$, we get

$$\hat{\mu}_4(\rho^{-n}x) = \frac{1}{2^n}\left(\prod_{j=1}^{n}\big(d_{-j}(x)+\hat{\nu}(\rho^{-j}x)\big)\right)\hat{\mu}_4(x). \tag{2.16}$$

Next lemma derives some properties of $\{d_j(x)\}_{j\in\mathbb{Z}}$, and we omit the simple proof here.

Lemma 4 *For any integer j and $x \in \mathbb{R}$, we have*

$$(2-\sqrt{2})\langle\rho^{-j}x\rangle^2 \le 1 - d_{-j}(x) \le \pi^2\langle\rho^{-j}x\rangle^2. \tag{2.17}$$

Theorem 5 *If ρ^{-1} is not a PV-number, then $|\hat{\mu}_4(x)| = o(|x|^{-t})$ as $|x| \to \infty$.*

We need the following lemma in order to prove Theorem 5.

Lemma 5 *If ρ^{-1} is not a PV-number, then for any $x \neq 0$, $\lim_{n\to\infty} 2^n\hat{\mu}_4(\rho^{-n}x) = 0$.*

Proof We fix a $x \neq 0$ and take $s \in \big(0, \underline{\Delta}_\infty(\nu)\big)$. Then there exists $C \ge 1$ such that $|\hat{\nu}(\rho^{-j}x)| \le C\rho^{sj}$ for all $j \in \mathbb{N}$. Since ρ^{-1} is not a PV-number, by Eq. (2.17), we have

$$\sum_{j=1}^{\infty} 1 - d_{-j}(x) \ge (2-\sqrt{2})\sum_{j=1}^{\infty}\langle\rho^{-j}x\rangle^2 = \infty. \tag{2.18}$$

Take strictly increasing positive integer sequence $\{a_k\}_{k=1}^m$ for $m = \infty$ or is finite, where $\{a_k\}_{k=1}^m$ includes all the numbers such that $d_{-a_k}(x) + |\hat{\nu}(\rho^{-a_k}x)| \ge 1$. Note that

$$\prod_{k=1}^{m}\left|d_{-a_k}(x)+\hat{\nu}(\rho^{-a_k}x)\right| \le \prod_{k=1}^{m}(1+C\rho^{sa_k}) < \infty \tag{2.19}$$

and

$$\sum_{k=1}^{m} 1 - d_{-a_k}(x) \le \sum_{k=1}^{m}|\hat{\nu}(\rho^{-a_k}x)| \le \sum_{k=1}^{m} C\rho^{sa_k} < \infty. \tag{2.20}$$

Let $\{b_k\}_{k=1}^\infty$ be the strictly increasing positive integer sequence such that $\{b_k\}_{k=1}^\infty \cup \{a_k\}_{k=1}^m = \mathbb{N}$ and $\{b_k\}_{k=1}^\infty \cap \{a_k\}_{k=1}^m = \emptyset$. By Eqs. (2.18) and (2.20), we have

$$\sum_{k=1}^{\infty} 1 - \left|d_{-b_k}(x)+\hat{\nu}(\rho^{-b_k}x)\right| \ge \sum_{k=1}^{\infty}\big(1 - d_{-b_k}(x) - |\hat{\nu}(\rho^{-b_k}x)|\big) = \infty.$$

Since $d_{-b_k}(x) + |\hat{\nu}(\rho^{-b_k} x)| < 1$, we obtain that

$$\prod_{k=1}^{\infty} \left(d_{-b_k}(x) + \hat{\nu}(\rho^{-b_k} x) \right) = 0. \tag{2.21}$$

Here we use the fact that if positive sequence $\{d_k\}_{k=1}^{\infty}$ has the property $d_k < 1$, then $\sum_{k=1}^{\infty}(1 - d_k) = \infty$ implies that $\Pi_{k=1}^{\infty} d_k = 0$.

Combining Eqs. (2.19) and (2.21), we have

$$\prod_{j=1}^{\infty} \left(d_{-j}(x) + \hat{\nu}(\rho^{-j} x) \right) = 0.$$

Hence by Eq. (2.16), we have

$$\lim_{n\to\infty} 2^n \hat{\mu}_4(\rho^{-n} x) = \lim_{n\to\infty} \left(\prod_{j=1}^{n} \left(d_{-j}(x) + \hat{\nu}(\rho^{-j} x) \right) \right) \hat{\mu}_4(x) = 0.$$

The proof is complete. □

Proof of Theorem 5 Assume the conclusion is false. Without loss of generality, we assume there are $\delta > 0$ and a strictly increasing positive sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = +\infty$ such that $|x_n|^t |\hat{\mu}_4(x_n)| \geq \delta$. Fix some $s \in \left(0, \underline{\Delta}_{\infty}(\nu)\right)$. Then there exists $C' > 0$ such that $|\hat{\nu}(x)| \leq C' |x|^{-s}$ for $x \in \mathbb{R}$. Take a constant C such that $C > \prod_{k=1}^{\infty}(1 + C' \rho^{-s} \rho^{ks})$.

For every x_n, there exists $k_n \in \mathbb{Z}$ such that $\rho^{k_n} x_n \in [\rho, 1]$. Without loss of generality, we assume $\lim_{n\to\infty} \rho^{k_n} x_n = x_0 \in [\rho, 1]$. By Lemma 5, for any $\varepsilon \in (0, \frac{\delta}{C})$, there exists $n_0 \in \mathbb{N}$ such that $2^{n_0} |\hat{\mu}_4(\rho^{-n_0} x_0)| < \varepsilon$. Since $2^{n_0} |\hat{\mu}_4(\rho^{-n_0} x)|$ is a continuous function, if n is sufficiently large, we have

$$2^{n_0} |\hat{\mu}_4(\rho^{-n_0} \rho^{k_n} x_n)| < \varepsilon.$$

If $k_n > n_0$ and n is sufficiently large, we write $2^{k_n} |\hat{\mu}_4(x_n)| = 2^{k_n} |\hat{\mu}_4(\rho^{-k_n} \rho^{k_n} x_n)|$, thus

$$\begin{aligned} 2^{k_n} |\hat{\mu}_4(x_n)| &= 2^{n_0} |\hat{\mu}_4(\rho^{-n_0} \rho^{k_n} x_n)| \times \left| \prod_{k=n_0+1}^{k_n} \left(d_{-k}(\rho^{k_n} x_n) + \hat{\nu}(\rho^{-k} \rho^{k_n} x_n) \right) \right| \\ &\leq 2^{n_0} C |\hat{\mu}_4(\rho^{-n_0} \rho^{k_n} x_n)| < \delta. \end{aligned}$$

By previous assumption, we also have

$$|x_n|^t |\hat{\mu}_4(x_n)| = |\rho^{-k_n} \rho^{k_n} x_n|^t |\hat{\mu}_4(x_n)| = 2^{k_n} |\hat{\mu}_4(x_n)| |\rho^{k_n} x_n|^t \geq \delta.$$

Hence $2^{k_n}|\hat{\mu}_4(x_n)| \geq \delta$, which is a contradiction. □

Assume ρ^{-1} is a PV-number. By Eq. (2.17) it follows that

$$\sum_{j=1}^{\infty} 1 - d_{-j}(1) \leq \sum_{j=1}^{\infty} \pi^2 \langle \rho^{-j} \rangle^2 < \infty.$$

Thus $\lim_{j\to\infty} d_{-j}(1) = 1$, and we may a find positive integer M such that $d_{-j}(1) > \frac{1}{2}$ for all $j \geq M$. Now we fix some $\nu = \nu(\rho^{-1}, M) \in \mathcal{P}(\mathbb{R})$ in the rest of this subsection. Let $f(x) = \frac{1}{2\pi} e^{-\frac{x^2}{4\pi}}$. Then $\omega := f dx$ is a Borel probability measure and $\hat{\omega}(x) = e^{-\pi x^2}$. Let $\phi(x) : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with the following properties:

1. $\phi(x) = 1$ if $|x| \leq 1$; 2. $\phi(x) = 0$ if $|x| \geq 2$; 3. $0 \leq \phi(x) \leq 1$.

Define $\phi_k(x) = \phi(\frac{x}{k})$ for $k \in \mathbb{N}$. Set $c_k = (\int \phi_k(x) f(x) dx)^{-1}$ for $k \in \mathbb{N}$ and $c_0 = (\int_{-1}^{1} f(x) dx)^{-1}$. Note that $1 < c_k \leq c_0$ and that $\lim_{k\to\infty} c_k = 1$. We define $\nu_k := c_k \phi_k f dx$. Then $\nu_k \in \mathcal{P}(\mathbb{R})$ with $\underline{\Delta}_\infty(\nu_k) = \infty$ and $\hat{\nu}_k \in C^\infty$. A very standard argument shows $\hat{\nu}_k(x)$ uniformly converges to $e^{-\pi x^2}$ as $k \to \infty$. So we could find some $k \in \mathbb{N}$ such that $\|\hat{\nu}_k - \hat{\omega}\|_\infty < \frac{1}{2} e^{-\pi \rho^{-2M}}$. Set $\nu = \nu_k$ which is the measure we need.

Theorem 6 *Assume ρ^{-1} is a PV-number and $\nu = \nu(\rho^{-1}, M)$ is the measure constructed above, then $\lim_{n\to\infty} 2^n |\hat{\mu}_4(\rho^{-n})| > 0$. Hence $|\hat{\mu}_4(x)| = O(|x|^{-t})$ and t is the best possible.*

We need the following lemma in order to prove Theorem 6.

Lemma 6 *Assume ρ^{-1} is a PV-number and $\nu = \nu(\rho^{-1}, M)$ is the measure constructed above, then*

(i) $d_j(1) + \hat{\nu}(\rho^j) \neq 0$ *for all* $j \in \mathbb{Z}$;
(ii) $\hat{\mu}_4(1) \neq 0$;
(iii) $\prod_{j=1}^{\infty} |d_{-j}(1) + \hat{\nu}(\rho^{-j})| > 0$.

Proof (i) If $j \geq -M$, note that $|\hat{\nu}(\rho^j) - e^{-\pi \rho^{2j}}| < \frac{1}{2} e^{-\pi \rho^{-2M}}$, we have $\mathrm{Re}(\hat{\nu}(\rho^j)) > \frac{1}{2} e^{-\pi \rho^{-2M}}$. Since $d_j(1) \geq 0$, we have $d_j(1) + \hat{\nu}(\rho^j) \neq 0$.

If $j \leq -M$, $|d_j(1) + \hat{\nu}(\rho^j)| \geq |d_j(1)| - |\hat{\nu}(\rho^j)| > \frac{1}{2} - \frac{3}{2} e^{-\pi \rho^{-2M}} > 0$.
(ii) We define $G(x) = \frac{1}{2}\left(\frac{1+\cos 2\pi x}{2} + \hat{\nu}(x)\right)$. It is obvious that $G \in C^1$. Thus there exists $L > 0$ such that

$$|G(\rho^j) - 1| = \left| \frac{1}{2}(d_j(1) + \hat{\nu}(\rho^j)) - 1 \right| \leq L \rho^j \quad \text{for } j \geq 0.$$

By Eq. (2.15) we have $\hat{\mu}_4(1) = \prod_{j=0}^{\infty} G(\rho^j)$, where $0 < |G(\rho^j)| \leq 1$. But

$$\sum_{j=0}^{\infty} 1 - |G(\rho^j)| \leq \sum_{j=0}^{\infty} |G(\rho^j) - 1| \leq \sum_{j=0}^{\infty} L\rho^j < \infty,$$

thus $|\hat{\mu}_4(1)| > 0$.

(iii) Note that

$$\begin{aligned}\sum_{j=1}^{\infty} \Big|\log |d_{-j}(1) + \hat{\nu}(\rho^{-j})|\Big| &= \sum_{j=1}^{\infty} \Big|\log(1 + |d_{-j}(1) + \hat{\nu}(\rho^{-j})| - 1)\Big| \\ &\lesssim \sum_{j=1}^{\infty} \Big||d_{-j}(1) + \hat{\nu}(\rho^{-j})| - 1\Big| \\ &\leq \sum_{j=1}^{\infty} |1 - d_{-j}(1)| + |\hat{\nu}(\rho^{-j})| \\ &< \infty.\end{aligned}$$

Hence $\prod_{j=1}^{\infty} |d_{-j}(1) + \hat{\nu}(\rho^{-j})| \in (0, +\infty)$. □

Proof of Theorem 6 By Eq. (2.16) we have

$$\lim_{n\to\infty} 2^n |\hat{\mu}_4(\rho^{-n})| = \lim_{n\to\infty} \left| \prod_{j=1}^{n} \left(d_{-j}(1) + \hat{\nu}(\rho^{-j})\right) \right| |\hat{\mu}_4(1)| > 0.$$

The proof is complete. □

3 Infinity Lower Fourier Dimension of In-homogenous Self-similar Measures

In the section, we fix an in-homogenous self-similar measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfying

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} + p\nu, \tag{3.1}$$

where ν is a Borel probability measure on $\mathbb{R}^d$ with compact support V. $S_j(x) = r_j R_j x + a_j$, where $0 < r_j < 1$, R_j is an orthogonal matrix and $a_j \in \mathbb{R}^d$, for all j. The support of μ satisfies the equation

$$\operatorname{spt} \mu = \bigcup_{j=1}^{N} S_j(\operatorname{spt} \mu) \cup V.$$

Olsen and Snigireva derived a connection between $\underline{\Delta}_\infty(\mu)$ and $\underline{\Delta}_\infty(\nu)$ (see [14, Theorem 2.1]). Now we continue their work with following theorem.

Theorem 7 *Let t be the positive constant satisfying the equation $\sum_{j=1}^N p_j r_j^{-t} = 1$. Then*

(1) *If $\underline{\Delta}_\infty(\nu) \le t$, then $\underline{\Delta}_\infty(\mu) = \underline{\Delta}_\infty(\nu)$;*
(2) *If $\underline{\Delta}_\infty(\nu) > t$, then $t \le \underline{\Delta}_\infty(\mu) \le \underline{\Delta}_\infty(\nu)$.*

Proof (1) Firstly, we prove $\underline{\Delta}_\infty(\nu) \ge \underline{\Delta}_\infty(\mu)$. Taking Fourier transforms in both sides of Eq. (3.1), we have

$$\hat{\mu}(x) = \sum_{j=1}^N p_j e^{ix\cdot a_j} \hat{\mu}(r_j R_j^* x) + p\hat{\nu}(x). \tag{3.2}$$

We could assume $\underline{\Delta}_\infty(\mu) > 0$. For any $\varepsilon \in \big(0, \underline{\Delta}_\infty(\mu)\big)$, there exits $C_1 \ge 1$ such that $|\hat{\mu}(x)| \le C_1 |x|^{-(\underline{\Delta}_\infty(\mu)-\varepsilon)}$ for all x. Besides, by Eq. (3.2), for all $|x| \ge 1$, we have

$$\begin{aligned} |\hat{\nu}(x)| &= \frac{1}{p}\Big|\hat{\mu}(x) - \sum_{j=1}^N p_j e^{ix\cdot a_j}\hat{\mu}(r_j R_j^* x)\Big| \\ &\le \frac{C_1}{p}\left(1 + \sum_{j=1}^N p_j r_j^{-(\underline{\Delta}_\infty(\mu)-\varepsilon)}\right)|x|^{-(\underline{\Delta}_\infty(\mu)-\varepsilon)}. \end{aligned}$$

Thus $\underline{\Delta}_\infty(\nu) \ge \underline{\Delta}_\infty(\mu) - \varepsilon$. By arbitrariness of ε, we have the desired result.

(2) If $\underline{\Delta}_\infty(\nu) \le t$, for proving $\underline{\Delta}_\infty(\mu) \ge \underline{\Delta}_\infty(\nu)$, we may assume $\underline{\Delta}_\infty(\nu) > 0$. For any $\varepsilon \in \big(0, \underline{\Delta}_\infty(\nu)\big)$, there exists $C_2 \ge 1$ such that $|\hat{\nu}(x)| \le C_2|x|^{-(\underline{\Delta}_\infty(\nu)-\frac{\varepsilon}{2})}$ for all x. Obviously, we have

$$\lim_{|x|\to\infty} |x|^{\underline{\Delta}_\infty(\nu)-\varepsilon}|\hat{\nu}(x)| = 0. \tag{3.3}$$

We define $A = \sum_{j=1}^N p_j r_j^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)}$. Note that $A < 1$. Take any $\delta \in \big(0, A(1-A)\big)$, there exists $R_0 > 0$ such that $p|x|^{\underline{\Delta}\infty(\nu)-\varepsilon}|\hat{\nu}(x)| < \delta$ for $|x| \ge R_0$. For any $R > R_0$, we define

$$M(R) := \max_{R_0 \le |x| \le R}\{|x|^{\underline{\Delta}_\infty(\nu)-\varepsilon}|\hat{\mu}(x)|\}.$$

We claim that for all $R > R_0$,

$$M(R) \le M(R_0 r_{\min}^{-1}) + 1.$$

Here $r_{\min} = \min\{r_1, \ldots, r_N\}$.

We prove this by contradiction. Suppose that there is $R > R_0$ such that $M(R) > M(R_0 r_{\min}^{-1}) + 1 > 1$. Then $R > R_0 r_{\min}^{-1}$ obviously, and there is at least one point

$y \in \{x : R_0 r_{\min}^{-1} \leq |x| \leq R\}$ such that $M(R) = |y|^{\underline{\Delta}_\infty(\nu)-\varepsilon}|\hat{\mu}(y)|$. By Eq. (3.2), we have

$$|y|^{\underline{\Delta}_\infty(\nu)-\varepsilon}\hat{\mu}(y) = \sum_{j=1}^{N} p_j r_j^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)} e^{iy\cdot a_j}|r_j R_j^* y|^{\underline{\Delta}_\infty(\nu)-\varepsilon}\hat{\mu}(r_j R_j^* y) + p|y|^{\underline{\Delta}_\infty(\nu)-\varepsilon}\hat{\nu}(y).$$

Let $\widetilde{M}(R) = \max\{|r_j R_j^* y|^{\underline{\Delta}_\infty(\nu)-\varepsilon}|\hat{\mu}(r_j R_j^* y)| : j = 1, \cdots, N\}$, we have

$$\left(\sum_{j=1}^{N} p_j r_j^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)}\right)\widetilde{M}(R) + \delta \geq M(R). \tag{3.4}$$

Since $y \in \{x : R_0 r_{\min}^{-1} \leq |x| \leq R\}$, thus

$$R_0 \leq r_j r_{\min}^{-1} R_0 \leq |r_j R_j^* y| = r_j|y| \leq R$$

for all j. Then we have $M(R) \geq \widetilde{M}(R)$. Since $M(R) > 1$ and $\delta < A(1-A)$, we have

$$A(1-A) + AM(R) < M(R).$$

From inequality (3.4), we have

$$M(R) > A(1-A) + AM(R) > \delta + A\widetilde{M}(R) \geq M(R),$$

which is a contradiction.

Thus for any $R > R_0$, $M(R) \leq M(R_0 r_{\min}^{-1}) + 1$. That is for any $R > R_0$, we have

$$\max_{R_0 \leq |x| \leq R}\{|x|^{\underline{\Delta}_\infty(\nu)-\varepsilon}|\hat{\mu}(x)|\} \leq \max_{R_0 \leq |x| \leq R_0 r_{\min}^{-1}}\{|x|^{\underline{\Delta}_\infty(\nu)-\varepsilon}|\hat{\mu}(x)|\} + 1.$$

Now take $C_3 = \max_{|x| \leq R_0 r_{\min}^{-1}}\{|x|^{\underline{\Delta}_\infty(\nu)-\varepsilon}|\hat{\mu}(x)|\} + 1$, we have

$$|\hat{\mu}(x)| \leq C_3|x|^{-(\underline{\Delta}_\infty(\nu)-\varepsilon)} \quad \text{for all } x, \tag{3.5}$$

i.e., $\underline{\Delta}_\infty(\mu) \geq \underline{\Delta}_\infty(\nu) - \varepsilon$. By arbitrariness of ε, we obtain the desired result.

(3) If $\underline{\Delta}_\infty(\nu) > t$, we need to prove $\underline{\Delta}_\infty(\mu) \geq t$. In [14], the authors gave a very detailed proof and we omit the details here. □

According to Theorem 7, if $\underline{\Delta}_\infty(\nu) \geq t$, we have $\underline{\Delta}_\infty(\mu) \geq t$. Theorem 6 also shows the lower bound t is optimal.

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References

1. Baker, S., Fraser, J.M., Máthé, Á.: Inhomogeneous self-similar sets with overlaps. Ergodic Theory Dyn. Syst. **3**(9), 1–18 (2019)
2. Barnsley, M.: Fractals Everywhere. Academic Press, Boston (1988)
3. Barnsley, M.: Existence and uniqueness of orbital measures. Preprint
4. Clickenstein, D., Strichartz, R.: Nonlinear self-similar measures and their Fourier transforms. Indiana Univ. Math. J. **45**, 205–220 (1996)
5. Falconer, K.: Fract. Geom. Wiley, Chichester (1990)
6. Falconer, K.: Techniques in Fractal Geometry. Wiley, Chichester (1997)
7. Fraser, J.M.: Inhomogeneous self-similar sets and box dimensions. Stud. Math. **213**, 133–156 (2012)
8. Hu, T.Y.: Asymptotic behavior of Fourier transforms of self-similar measures. Proc. Am. Math. Soc. **129**, 1713–1720 (2000)
9. Hutchinson, J.E.: Fractals and self-similarity. Indiana Univ. Math. J. **30**, 713–747 (1981)
10. Lau, K. S.: Self-Similarity, L^q-Spectrum and multifractal formalism. In: Fractal Geometry and Stochastics, Progr. Probab., **37**, Birkhäuser, Basel, (1995), pp. 55–90
11. Liszka, P.: On inhomogeneous self-similar measures and their L^q spectra. Ann. Polon. Math. **109**(1), 75–92 (2013)
12. Liszka, P.: The L^q spectra and Rényi dimension of generalized inhomogeneous self-similar measures. Central Eur. J. Math. **12**(9), 1305–1319 (2014)
13. Olsen, L., Snigireva, N.: L^q spectra and Rényi dimensions of in-homogeneous self-similar measures. Nonlinearity **20**, 151–175 (2007)
14. Olsen, L., Snigireva, N.: In-homogenous self-similar measures and their Fourier transforms. Math. Proc. Camb. Philos. Soc. **144**, 465–493 (2008)
15. Roychowdhury, M.K.: Quantization dimension estimate of inhomogeneous self-similar measures. Bull. Pol. Acad. Sci. Math. **61**, 35–45 (2013)
16. Salem, R.: Algebraic Numbers and Fourier Analysis. Health, Boston (1963)
17. Zhu, S.: The quantization for in-homogeneous self-similar measures with in-homogeneous open set condition. Int. J. Math. **26**, 1–23 (2015)
18. Zhu, S.: Asymptotics of the quantization errors for in-homogeneous self-similar measures supported on self-similar sets. Sci. China Math. **59**, 337–350 (2016)