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Wave Propagation Speed on Fractals

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Abstract

We study the wave propagation speed problem on metric measure spaces, emphasizing on self-similar sets that are not post-critically finite. We prove that a sub-Gaussian lower heat kernel estimate leads to infinite propagation speed, extending a result of Lee (Infinite propagation speed for wave solutions on some p.c.f. fractals, https://archive.org/details/arxiv-1111.2938) to include bounded and unbounded generalized Sierpiński carpets, some fractal blowups, and certain iterated function systems with overlaps. We also formulate conditions under which a Gaussian upper heat kernel estimate leads to finite propagation speed, and apply this result to two classes of iterated function systems with overlaps, including those defining the classical infinite Bernoulli convolutions.

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1 Introduction

Strichartz [53] conjectured in 1999 that on certain fractals, such as the Sierpiński gasket, waves may propagate with infinite speed, due to the difference in time and Laplacian scalings (see [17]). This prediction shows that fractals could exhibit behaviors that differ fundamentally from classical smooth objects. Lee [42] recently proved that on a class of self-similar sets satisfying the post-critically finite (p.c.f.) condition, including the Sierpiński gasket, the conjecture is true. The first objective of this paper is to extend Lee's result to fractals that are non-p.c.f., such as generalized Sierpiński carpets. There are two main ingredients in Lee's proof, namely, sub-Gaussian heat kernel estimates and Kannai's transform. Using these we generalize Lee's theorem to locally compact metric measure spaces.

We refer the reader to Sect. 2 for the definitions of the unit, bounded, finite, and infinite propagation speed properties, abbreviated (UPS), (BPS), (FPS), and (IPS), respectively. The relationship between wave propagation speed and heat kernel estimates is well known. Cheeger et al. [15] obtained (UPS) for Laplacians defined on complete Riemannian manifolds and used it to study heat kernel estimates. Coulhon and Sikora [16,51] showed that (UPS) is equivalent to the Davies-Gaffney estimate [(see (5.1)], and obtained heat kernel estimates by assuming (UPS). We remark that in the literature (UPS) is called the *finite speed propagation property*. Our definition of finite propagation speed (FPS) [(see Definition 2.1(c)] is a weaker notion.

Let $\mathcal{H}$ be a Hilbert space and A be a non-negative self-adjoint operator on $\mathcal{H}$. The *wave equation* is defined as

$$\begin{cases} u_{tt}(t) = -Au(t), & t \geq 0, \\ u(0) = f, \quad u_t(0) = g. \end{cases} \tag{1.1}$$

The *heat equation* is defined as

$$\begin{cases} v_t(t) = -Av(t), \quad t \geq 0, \\ v(0) = f. \end{cases} \tag{1.2}$$

It is well known that each of these equations has a unique solution (see Definitions 2.2 and 2.3 for the definition of a solution).

Let (X, d) be a metric space, and μ be a Radon measure on (X, d). Let $C(X, d)$ denote the space of all real-valued continuous functions on X and $\|u\|_p$ denote the L^p-norm in $L^p(X, \mu)$ for $1 \leq p \leq \infty$.

Theorem 1.1 *(Lee's theorem for locally compact metric measure spaces.) Let* (X, d) *be a locally compact metric space,* μ *be a* σ*-finite Radon measure on* (X, d) *such that* $\operatorname{supp}(\mu) = X$*. Also, let* A *be a non-negative self-adjoint operator on* $L^2(X, \mu)$ *and* $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ *be the associated closed form. Assume the corresponding heat kernel* $p(t, x, y)$ *exists and there exist constants* $c_1 > 0$*,* $c_2 > 0$*,* $\epsilon > 0$*,* $\beta > 2$*, and* $\alpha \in \mathbb{R}$ *such that the heat kernel* $p(t, x, y)$ *satisfies*

$$p(t, x, y) \geq c_1 t^{-\alpha/2} \exp\Big(- c_2 \Big(\frac{d(x, y)}{t^{1/\beta}}\Big)^{\beta/(\beta-1)}\Big) \quad \text{for } \mu\text{-a.e. } x, y \in X \text{ and all } t \in (0, \epsilon). \tag{1.3}$$

Then the following conclusions hold:

(a) (X, d, μ, A) *satisfies (IPS).*
(b) *Let* $f \in \operatorname{dom}\mathcal{E}$ *be a non-negative and non-zero function. If* $\operatorname{dom}\mathcal{E} \subseteq C(X, d)$*, and there exists a constant* $C := C(f) > 0$ *such that* $\| \cos(t\sqrt{A}) f\|_\infty \leq C$ *for all* $t > 0$*, then for any* $x \in X$ *and any* $\delta > 0$*, there exists some* $t_0 \in (0, \delta)$ *such that* $\cos(t_0\sqrt{A}) f(x) > 0$.

Lower heat kernel estimates (1.3) have been obtained for several classes of fractals, including a class of p.c.f. fractals (see, e.g., [4]) and Sierpiński carpets (see, e.g., [10]). In general, the constant α is the spectral dimension and β is the walk dimension. We remark that for all $f \in \operatorname{dom}\mathcal{E}$, $\mathcal{E}\big(\cos(t\sqrt{A}) f, \cos(t\sqrt{A}) f\big) \leq \mathcal{E}(f, f)$ (see (4.2)) and thus $\cos(t\sqrt{A}) f \in \operatorname{dom}\mathcal{E}$ for all $t > 0$. Also, if $f \in \operatorname{dom} A$ and $g = 0$, then $\cos(t\sqrt{A}) f$ is the solution of the wave equation (1.1).

Lee [42] considered Laplacians A defined on p.c.f. fractals X with a regular harmonic structure. It follows from [32, Chapter 3] that the associated closed form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a resistance form with resistance metric $R^{1/2}$, which is compatible with the original topology (see [32, Chapter 2] for the definition of resistance form). In particular, $\operatorname{dom}\mathcal{E} \subseteq C(X, R^{1/2})$ and

$$|u(x) - u(y)|^2 \leq R(x, y)\mathcal{E}(u, u) \qquad \text{for all } x, y \in X \text{ and } u \in \operatorname{dom}\mathcal{E}; \tag{1.4}$$

consequently, $d_s < 2$ (see [37]). Also, functions of finite energy are locally bounded and so every single point has positive capacity (see [36, Chapter 9] for details). Using (1.4), we see that $\| \cos(t\sqrt{A}) f\|_\infty \leq C$ for all $t > 0$. Hence, the case considered in [42] satisfies all additional assumptions in Theorem 1.1(b). Also, Theorem 1.1 allows (X, d) to be unbounded, which is not considered in [42, Theorem 8]. Moreover, if (X, d) is bounded, the sub-Gaussian heat kernel estimate (1.3) is weaker than the one in [42, Theorem 8], which corresponds to the case $\alpha = 0$. Thus, Theorem 1.1 generalizes [42, Theorem 8] and we will see that it can be applied to certain non-p.c.f. and certain unbounded fractals.

A main motivation of this work is to study Strichartz wave propagation speed conjecture on fractals defined by iterated function systems (IFS) with overlaps together with the associated self-similar measures. Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a, b]$ and $H^1(a, b)$ be the usual Sobolev space on (a, b). Consider the bilinear form $\mathcal{E}$ defined as

$$\mathcal{E}(u, v) = \int_a^b u'(x)v'(x)\,dx \qquad \text{for all } u, v \in \operatorname{dom}\mathcal{E} := H^1(a, b). \tag{1.5}$$

It is well known that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a Dirichlet form on $L^2([a, b], \mu)$ (see [23]) and $\operatorname{dom}\mathcal{E} \subseteq C([a, b], d_{|\cdot|})$; here, and throughout rest of this paper, $d_{|\cdot|}$ denotes the Euclidean metric.

The first measure we study is the infinite Bernoulli convolution associated with the golden ratio. Let

$$S_0(x) = \rho x, \qquad S_1(x) = \rho x + (1 - \rho), \qquad \rho = \frac{\sqrt{5} - 1}{2}, \tag{1.6}$$

and let μ be the self-similar measure satisfying

$$\mu = \frac{1}{2}\mu \circ S_0^{-1} + \frac{1}{2}\mu \circ S_1^{-1}. \tag{1.7}$$

Clearly, $\operatorname{supp}(\mu) = [0, 1]$. We also study a family of convolutions of Cantor-type measures. Let

$$S_0(x) = \frac{1}{m}x, \qquad S_1(x) = \frac{1}{m}x + \frac{m-1}{m}, \tag{1.8}$$

where $m \geq 3$ is an integer. Let ν_m be the self-similar measure defined by the IFS (1.8) with probability weights $p_0 = p_1 = 1/2$. The m-fold convolution of ν_m, denoted μ_m, is the self-similar measure defined by the following IFS with overlaps (see [40,48]):

$$S_i(x) = \frac{1}{m}x + \frac{m-1}{m}i, \qquad i = 0, 1, \ldots, m,$$

together with probability weights

$$w_i := \frac{1}{2^m}\binom{m}{i}, \qquad i = 0, 1, \ldots, m.$$

That is,

$$\mu_m = \sum_{i=0}^{m} \frac{1}{2^m}\binom{m}{i}\mu_m \circ S_i^{-1}. \tag{1.9}$$

Note that $\operatorname{supp}(\mu_m) = [0, m]$.

Combining Theorem 1.1 with the sub-Gaussian heat kernel estimate obtained recently by Gu et al. [25], we have

Corollary 1.2 *Let μ be a positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) = [a, b]$, and A be the non-negative self-adjoint operator associated with $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ in (1.5).*

(a) *If* μ *is given by* (1.6) *and* (1.7)*, then* $([0,1], d_{|\cdot|}, \mu, A)$ *satisfies (IPS). Moreover, if* $0 \leq f \in \operatorname{dom}\mathcal{E}$ *is a non-zero function, then for any* $x \in [0,1]$ *and any* $\delta > 0$*, there exists some* $t_0 \in (0, \delta)$ *such that* $\cos(t_0\sqrt{A})f(x) > 0$.

(b) *If* $\mu := \mu_m$ *is given by* (1.9) *and* $m \geq 3$ *is an integer, then* $([0,m], d_{|\cdot|}, \mu_m, A)$ *satisfies (IPS). Moreover, If* $0 \leq f \in \operatorname{dom}\mathcal{E}$ *is a non-zero function, then for any* $x \in [0,m]$ *and any* $\delta > 0$*, there exists some* $t_0 \in (0,\delta)$ *such that* $\cos(t_0\sqrt{A})f(x) > 0$.

Theorem 1.1 allows us to prove (IPS) for certain fractal blowups, which are unbounded. We first describe fractal blowups (see, e.g., [26]). Let $\Sigma = \{1, \ldots, N\}$ for an integer $N \geq 2$ and $\{S_i\}_{i=1}^N$ be an IFS on $\mathbb{R}^n$. For any $m \geq 0$ and any word $\boldsymbol{i} = i_1 \cdots i_m \in \Sigma^m$, we use $|\boldsymbol{i}| = m$ to denote the *length* of $\boldsymbol{i}$, and $\boldsymbol{i}$ is the empty word if $|\boldsymbol{i}| = 0$. Denote by

$$S_{\boldsymbol{i}} := S_{i_1} \circ \cdots \circ S_{i_m}.$$

Definition 1.1 Let $N \geq 2$ and K be the self-similar set associated with an IFS $\{S_i\}_{i=1}^N$. Fix an infinite word $\theta = i_1 i_2 \cdots \in \Sigma^\infty$. For each $m \geq 1$, let

$$K^m := S^{-1}_{i_1 \cdots i_m}(K) := S^{-1}_{i_1} \circ \cdots \circ S^{-1}_{i_m}(K).$$

A *fractal blowup* K_∞ is defined as

$$K_\infty := \bigcup_{m=1}^{\infty} K^m. \tag{1.10}$$

Note that K_∞ is unbounded and determined by the choice of the infinite word θ.

Example 1.3 Let $S_i(x) = (x - a_i)/3 + a_i$ for $i = 1, 2, 3$, where $a_1 = 0$, $a_2 = 1/2$, $a_3 = 1$. Then the associated attractor $K = [0,1]$. Let K_∞ be the associated fractal blowup given by (1.10) with $\theta = 1313\cdots$.

We note that $K_\infty = S^{-1}_{1313\ldots}([0,1]) = \mathbb{R}$. Let μ be the self-similar measure defined by the IFS in Example 1.3. Kigami [33, Section 5] constructed a regular local Dirichlet form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ on $L^2(K,\mu)$. The wave equation defined by the operator A associated with $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ has recently been studied by Andrews et al. [3]. Let μ_∞ be the extension of μ on K_∞ as defined by Gu and Hu in [26, Section 4]. Beginning with $(\mathcal{E}, \operatorname{dom}\mathcal{E})$, Gu and Hu [26] constructed a regular local conservative Dirichlet form $(\widetilde{\mathcal{E}}, \operatorname{dom}\widetilde{\mathcal{E}})$ on $L^2(K_\infty, \mu_\infty)$.

Corollary 1.4 *Let* K_∞ *be defined as in Example* 1.3*,* μ *be the self-similar measure defined by the IFS in Example* 1.3 *together with the probability vector* $\{p_i\}_{i=1}^3$*, and* μ_∞ *be defined as in* [26, Section 4]*. Let* $\widetilde{A}$ *be the non-negative self-adjoint operators associated with the Dirichlet form* $(\widetilde{\mathcal{E}}, \operatorname{dom}\widetilde{\mathcal{E}})$ *in* [26]*. If* $p_1 = p_3 \neq p_2$*, then* $(K_\infty, d_{|\cdot|}, \mu_\infty, \widetilde{A})$ *satisfies (IPS).*

Theorem 1.1 also allows us to prove that waves propagate with infinite speed on generalized Sierpiński carpets, which are not p.c.f. self-similar sets. The definitions of these carpets, as well as the corresponding Laplacians, are given in Sect. 4. Generalized Sierpiński carpets in $\mathbb{R}^n$ have been studied in [6–8,10,11]. It is known that the spectral dimension d_s and the Hausdorff dimension d_f satisfy $d_s \leq d_f < n$ and thus $d_s < 2$ in $\mathbb{R}^2$. Generalized Sierpiński carpets with $n \geq 3$ and $d_s < 2$ can be found in [10,39].

Corollary 1.5 *Let F denote a generalized Sierpiński carpet, and $\widetilde{F}$ denote the corresponding unbounded Sierpiński carpet. Let A and $\widetilde{A}$ be the Laplacians on F and $\widetilde{F}$ respectively, as given in* [9,10] *or* [39]. *Then the following results hold.*

(a) *A and $\widetilde{A}$ satisfy (IPS).*
(b) *Let $(\mathcal{E}, \mathrm{dom}\,\mathcal{E})$ be the regular Dirichlet form on $L^2(F, \mu)$ associated with A, and $f \in \mathrm{dom}\,\mathcal{E}$ be a non-negative and non-zero function, where μ is given in Sect. 4. If $d_s < 2$, then for any $x \in F$ and any $\delta > 0$, there exists $t_0 \in (0, \delta)$ such that $\big(\cos(t_0\sqrt{A})f\big)(x) > 0$.*

We do not know whether Corollary 1.5(b) holds for $\widetilde{F}$ and $\widetilde{A}$. In this case, we are not able to verify inequality (4.7), since $\mu(\widetilde{F}) = +\infty$; as a result it is not clear whether the condition $\|\cos(t\sqrt{A})f\|_\infty \leq C$ in Theorem 1.1(b) holds.

For a class of p.c.f. fractals, including the Sierpiński gasket, Strichartz [54, Theorem 6.1] obtained two-sided sub-Gaussian heat kernel estimates for the product of these p.c.f. fractals, which are not p.c.f. Theorem 1.1 can be applied to these products.

In view of Lee's theorem, as well as its more general forms above, it is natural to ask whether a Gaussian upper heat kernel estimate will imply finite propagation speed. The second objective of this paper is to prove that, under suitable conditions, this is true.

Let (X, d, μ) be a metric measure space and A be a non-negative self-adjoint operator on $L^2(X, \mu)$. It is well known that if there exists a constant $c > 0$ such that

$$p(t, x, y) \leq c \exp\Big(-\frac{d(x, y)^2}{4t}\Big) \qquad \text{for all } x, y \in X \text{ and all } t > 0, \tag{1.11}$$

then the Davies–Gaffney estimate [(see (5.1)] holds. Thus, the Gaussian upper heat kernel estimate (1.11) implies (UPS). There is also an analogue of this for (BPS) (see Corollary 5.2). In order to prove our main result on finite propagation speed, we will first weaken the assumptions of these results and establish (BPS) (see Theorem 5.6), and then use strict locality to obtain (FPS) (see Theorem 1.6).

We are mainly interested in Laplacians defined by measures on a bounded subsets of $\mathbb{R}^n$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset of $\mathbb{R}^n$, μ be a positive finite Borel measure on $\mathbb{R}^n$ with $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. It is known that μ defines a Dirichlet Laplace operator Δ_μ, if the following *Poincaré inequality (or spectral gap inequality)* for a measure (MPI) holds: There exists a constant $C > 0$ such that for all $u \in C_c^\infty(\Omega)$,

$$\int_\Omega |u|^2\, d\mu \leq C \int_\Omega |\nabla u|^2\, dx \tag{1.12}$$

(see, e.g., [28,44,46]). Here $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions on Ω with compact support. We write $V \subset\subset \Omega$ if V is compactly contained in Ω, i.e., $\overline{V} \subset \Omega$ and $\overline{V}$ is compact. We call an open connected subset of $\mathbb{R}^n$ a *domain*.

In the following theorem, ρ stands for the intrinsic metric (see Definition 6.1).

Theorem 1.6 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. Assume that μ is equivalent to Lebesgue measure on $\overline{\Omega}$ with density $d\mu/dx = f \in L^\infty(\overline{\Omega}, \mu)$ and let $-\Delta_\mu$ be the Dirichlet Laplacian with respect to μ. Also, assume that for every $V \subset\subset \Omega$, there exists some constant $\varepsilon(V)$ such that $f \geq \varepsilon(V) > 0$ Lebesgue a.e. on V. Then $(\overline{\Omega}, \rho, \mu, -\Delta_\mu)$ satisfies (BPS) and $(\overline{\Omega}, d_{|\cdot|}, \mu, -\Delta_\mu)$ satisfies (FPS).*

Here we outline the main ideas of the proof of Theorem 1.6. First, the assumptions on f allow us to prove that under (MPI), the intrinsic metric ρ is topologically equivalent to $d_{|\cdot|}$ and hence the Dirichlet form in question is strictly local (see Definition 6.2). They also lead to the completeness property, the volume doubling property, and the strong Poincaré inequality (see Definition 6.3). Second, we invoke a theorem of Sturm [56] and establish a desired upper heat-kernel estimate with respect to the intrinsic metric, which leads to (BPS). Finally, we use strict locality to obtain (FPS) with respect to the Euclidean metric.

In Sect. 7, we apply Theorem 1.6 to two classes of self-similar measures on $\mathbb{R}$. Let μ be a self-similar measure defined by an IFS $\{S_i\}_{i=0}^N$ on $\mathbb{R}$. It is known that if $\operatorname{supp}(\mu)$ is not a singleton, then μ satisfies (MPI) (see, e.g., [28]).

The first family of measures we study is defined by the following IFS on $\mathbb{R}$:

$$S_i(x) = \frac{1}{2}x + \frac{i}{2}, \qquad i = 0, 1, \dots, N, \tag{1.13}$$

where $N \geq 3$. The second family consists of the well-known *infinite Bernoulli convolutions*, which are defined by the following class of IFSs on $\mathbb{R}$:

$$S_0(x) = rx, \qquad S_1(x) = rx + 1 - r, \qquad 0 < r < 1. \tag{1.14}$$

Theorem 1.7 *(a) Let μ be the self-similar measure defined by the IFS $\{S_i\}_{i=0}^N$ in (1.13) and probability weights $p_0 = \cdots = p_N = 1/(N+1)$. If $N \geq 3$ is odd, then $-\Delta_\mu$ satisfies (FPS).*

(b) Let μ be the self-similar measure defined by the IFS $\{S_0, S_1\}$ in (1.14) and probability weights $p_0 = p_1 = 1/2$. Assume μ is absolutely continuous with respect to Lebesgue measure and $r \in (2/3, 1)$. Then $-\Delta_\mu$ satisfies (FPS).

The rest of this paper is organized as follows. Section 2 summarizes some notation, definitions and results that will be needed throughout the paper. Section 3 is devoted to the proof of Theorem 1.1. In Sect. 4, we apply Theorem 1.1 to generalized Sierpiński carpets and unbounded Sierpiński carpets and prove Corollary 1.5. Section 5 studies (BPS) and (FPS) in general metric measure spaces. In Sect. 6, we prove Theorem 1.6. In Sect. 7, we provide examples of finite propagation speed, including infinite Bernoulli convolutions, and prove Theorem 1.7. Finally, we state some open questions and comments in Sect. 8.

2 Preliminaries

In this section, we summarize some notation, definitions, and preliminary results that will be used throughout this paper. Let (X, d) be a metric space. Let $d(U_1, U_2) := \inf\{d(x, y) : x \in U_1, y \in U_2\}$ denote the distance between $U_1, U_2 \subseteq X$, and write $d(x, U) := d(\{x\}, U)$. $B_d(x, r) := \{y \in X : d(x, y) < r\}$ denote an open ball with radius r and center x. In particular, if $d = d_{|\cdot|}$, then we write $B(x, r) := B_d(x, r)$.

Let $\mathcal{L}^n$ (and dx) be Lebesgue measure on $\mathbb{R}^n$. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be a bounded open subset, and let $\operatorname{diam}(U) := \sup\{|x - y| : x, y \in U\}$ denote the diameter of $U \subseteq \Omega$. Let $H^1(\Omega)$ be the Sobolev space with inner product

$$\langle u, v\rangle_{H^1(\Omega)} := \int_\Omega uv\, dx + \int_\Omega \nabla u \nabla v\, dx,$$

and let $H_0^1(\Omega)$ denote the completion of $C_c^\infty(\Omega)$ in the $H^1(\Omega)$ norm.

For the definitions of closed quadratic form, Dirichlet form, Markov property, as well as the strongly local, and regular properties of Dirichlet forms, we refer the reader to Fukushima et al. [23]. We also refer the reader to Yosida [59] for the definitions of semigroup and ultracontractivity.

2.1 Heat Kernel

Let X be a Hausdorff topological space. A positive Borel measure μ on X is called a *Radon measure* if it is (1) *inner regular*, i.e., for each measurable set A, $\mu(A) = \sup\{\mu(K) : K \subseteq A,\ K \text{ is compact}\}$ and (2) *locally finite*, i.e., each point in X has a neighborhood U such that $\mu(U) < \infty$.

Let (X, μ) be a measure space with a Radon measure μ, A be a non-negative self-adjoint operator on $L^2(X, \mu)$, and $\{T_t\}_{t>0}$ be the associated semigroup on $L^2(X, \mu)$, i.e., $T_t = e^{-tA}$. A non-negative measurable function $p(t, x, y)$ on $(0, \infty) \times X \times X$ is called the *heat kernel* of the semigroup $\{T_t\}_{t>0}$ (or of the operator A) if $p(t, x, y)$ is the integral kernel of the operator T_t, i.e., for any $t > 0$ and any $f \in L^2(X, \mu)$,

$$(T_t f)(x) = \int_X p(t, x, y) f(y)\, d\mu(y) \qquad \text{for } \mu\text{-a.e. } x \in X.$$

Heat kernel may not exist in general. However, it is known to exist in many spaces such as Euclidean spaces, certain Riemannian manifolds, and certain classes of fractals (see, for instance, [18] for a sufficient condition). If it exists then it is unique (up to a set of measure zero).

2.2 Wave Propagation Speed

Let (X, d, μ) be a metric measure space, i.e., μ is a Borel measure with respect to the topology defined by the metric d. For any measurable subset $U \subseteq X$, we denote $L^2(U, \mu|_U)$ simply by $L^2(U, \mu)$.

Definition 2.1 Let (X, d, μ) be a metric measure space and A be a non-negative self-adjoint operator on $L^2(X, \mu)$. Regarding solutions of the corresponding wave equation, we say (X, d, μ, A) (or simply A) has the

(a) unit propagation speed property (UPS) if there exists some $0 < s \leq 1$ such that

$$\big(\cos(t\sqrt{A}) f_1, f_2\big)_\mu = 0$$

for all $t \in (0, r/s)$, all open subsets $U_i \subseteq X$ with $r := d(U_1, U_2) > 0$, and all $f_i \in L^2(U_i, \mu)$, $i = 1, 2$, (cf. [51]);

(b) bounded propagation speed property (BPS) if there exists some $s > 0$ such that

$$\big(\cos(t\sqrt{A}) f_1, f_2\big)_\mu = 0$$

for all $t \in (0, r/s)$, all open subsets $U_i \subseteq X$ with $r := d(U_1, U_2) > 0$, and all $f_i \in L^2(U_i, \mu)$, $i = 1, 2$;

(c) finite propagation speed property (FPS) if for any open subsets $U_i \subseteq X$ $(i = 1, 2)$ with $r := d(U_1, U_2) > 0$, there exists some $s > 0$ (may depend on U_1, U_2) such that

$$\big(\cos(t\sqrt{A}) f_1, f_2\big)_\mu = 0$$

for all $0 < t < r/s$ and all $f_i \in L^2(U_i, \mu)$;

(d) infinite propagation speed property (IPS) if there exist open subsets $U_i \subseteq X$ $(i = 1, 2)$ with $d(U_1, U_2) > 0$ such that for any $s > 0$, there exist some $t \in (0, s)$ and $f_i \in L^2(U_i, \mu)$ satisfying

$$\big(\cos(t\sqrt{A}) f_1, f_2\big)_\mu \neq 0.$$

As mentioned in the introduction, our definition of (UPS) is equivalent to the definition of finite speed propagation property in the literature (see, e.g., [16]). From (BPS) one obtains (UPS) by a simple change of the metric d, and vice versa. It follows from Definition 2.1 that (UPS) implies (BPS), which in turn implies (FPS). (BPS) implies that wave propagation speed is less than s. (FPS) and (IPS) are negations of each other.

We say that two metric spaces (X, d_1) and (X, d_2) are *strongly equivalent* if there exist two positive constants c_1 and c_2 such that $c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y)$ for all $x, y \in X$. The following theorem compares wave propagation speeds in two different metrics; part (b) will be needed in the proof of Theorem 1.6.

Proposition 2.1 *Let (X, d_i, μ), $i = 1, 2$, be two metric measure spaces and let A be a non-negative self-adjoint operator on $L^2(X, \mu)$.*

(a) *If d_1 and d_2 are strongly equivalent and (X, d_1, μ, A) satisfies (BPS), then so does (X, d_2, μ, A).*

(b) *Assume that d_1 is topologically equivalent to d_2, and for all open subsets $U_1, U_2 \subseteq X$, $d_2(U_1, U_2) > 0$ implies $d_1(U_1, U_2) > 0$. If (X, d_1, μ, A) satisfies (FPS), then (X, d_2, μ, A) satisfies (FPS).*

Proof The proof of (a) is straightforward; we only prove (b). let $U_i \subseteq X (i = 1, 2)$ be two open subsets on the metric space (X, d_2) with $d_2(U_1, U_2) > 0$. Since d_1 is topologically equivalent to d_2, U_1 and U_2 are two open subsets on the metric space (X, d_1). Let $C := C(U_1, U_2) = d_1(U_1, U_2)/d_2(U_1, U_2)$. By assumption, there exists some constant $s > 0$, which may depend on U_i, such that $(\cos(t\sqrt{A}) f_1, f_2)_\mu = 0$ for all $0 < t < d_1(U_1, U_2)/s$ and all $f_i \in L^2(U_i, \mu)$. It follows that $(\cos(t\sqrt{A}) f_1, f_2)_\mu = 0$ for all $0 < t < C d_2(U_1, U_2)/s$ and all $f_i \in L^2(U_i, \mu)$. Hence, (X, d_2, μ, A) satisfies (FPS). □

Finally, we remark that the condition "$d_2(U_1, U_2) > 0$ implies $d_1(U_1, U_2) > 0$" in Proposition 2.1(b) need not hold in general. To see this, let $X = (0, 1)$ and define

$$d_1(x, y) := \min\left\{|x - y|, 1 - |x - y|\right\} \quad \text{and} \quad d_2(x, y) := |x - y| \quad \text{for } x, y \in X.$$

Then d_1 and d_2 are topologically equivalent. However, with $U_1 = (0, 1/4)$ and $U_2 = (3/4, 1)$, one has $d_2(U_1, U_2) = 1/2$ but $d_1(U_1, U_2) = 0$.

2.3 Wave and Heat Equations on Hilbert Spaces

Let $\mathcal{H}$ be a (real or complex) Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, A be a non-negative self-adjoint operator on $\mathcal{H}$ with domain dom A, and $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be the associated closed quadratic form on $\mathcal{H}$. Let $A = \int_0^\infty \lambda \, dE_\lambda$ be the unique spectral representation of A. Then

$$\operatorname{dom}\mathcal{E} = \left\{u \in \mathcal{H} : \int_0^\infty \lambda \, d(E_\lambda u, v) < \infty \text{ for any } v \in \mathcal{H}\right\} \tag{2.1}$$

and

$$\operatorname{dom} A = \left\{u \in \mathcal{H} : \int_0^\infty \lambda^2 \, d(E_\lambda u, v) < \infty \text{ for any } v \in \mathcal{H}\right\}. \tag{2.2}$$

Definition 2.2 A function $u : \mathbb{R} \to \mathcal{H}$ is called a solution of (1.1) if its second-order strong derivative with respect to t exists, $u(t) \in \operatorname{dom} A$ for any $t \in \mathbb{R}$, and equation (1.1) is satisfied.

The existence and uniqueness of solution of the abstract wave equation (1.1) is well known (see, e.g., [50]).

Theorem 2.2 *Let $\mathcal{H}$ be a complex Hilbert space and A be a non-negative self-adjoint operator on $\mathcal{H}$ with domain* dom A*. Then for any $f \in \operatorname{dom} A$ and $g \in \operatorname{dom}\sqrt{A}$, the initial value problem*

$$u_{tt}(t) = -Au(t), \quad u(0) = f, \quad u_t(0) = g,$$

has a unique solution $u : [0, \infty) \to \mathcal{H}$ *given by*

$$u(t) = \int_0^\infty \cos(t\sqrt{\lambda})\, dE_\lambda f + \int_0^\infty \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\, dE_\lambda g,$$

where $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ *is the spectral family associated with* A.

Definition 2.3 A function $v : [0, \infty) \to \mathcal{H}$ is called a solution of (1.2) if $v(t)$ is strongly continuous at $t = 0$, $v(t)$ is differentiable on $(0, \infty)$, $v(t) \in \operatorname{dom} A$ for any $t > 0$ and satisfies equation (1.2).

Let $\{T_t\}_{t>0}$ be the strongly continuous semigroup associated with A. By the definition of semigroup (see, e.g., [23]), T_t is contractive for all $t > 0$. It is well known that for any $f \in \mathcal{H}$, there exists a unique solution $v : [0, \infty) \to \mathcal{H}$ of the heat equation (1.2), given by $v(t) = T_t f$ (see, e.g., [32]).

The following relation between the wave equation (1.1) and the heat equation (1.2) is known as the Kannai transform [31]. We include a proof in the Appendix for completeness.

Lemma 2.3 *Let* $\mathcal{H}$ *be a separable Hilbert space and let* A *be a non-negative self-adjoint operator on* $\mathcal{H}$. *Then for any* $f \in \mathcal{H}$, *the function* $v(t)$ *defined by*

$$v(t) := \begin{cases} f, & t = 0, \\ \dfrac{1}{\sqrt{\pi t}} \displaystyle\int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big) \cos(s\sqrt{A}) f\, ds, & t \in (0, \infty), \end{cases} \tag{2.3}$$

is the solution of the heat equation (1.2) *with initial data* f.

In the rest of this section, we remark on the case when A has compact resolvent, even though the results are not needed in the paper. Let $(\varphi_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{H}$ consisting of the eigenfunctions of A such that $A\varphi_n = \lambda_n \varphi_n$ for $n \geq 1$, $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots$, and $\lim_{n\to\infty} \lambda_n = \infty$. The domains $\operatorname{dom} \mathcal{E}$ and $\operatorname{dom} A$ can be expressed by using eigenfunctions and eigenvalues as

$$\operatorname{dom} \mathcal{E} = \Big\{ \sum_{n=1}^\infty \alpha_n \varphi_n : \sum_{n=1}^\infty \alpha_n^2 \lambda_n < \infty \Big\}$$

and

$$\operatorname{dom} A = \Big\{ \sum_{n=1}^\infty \alpha_n \varphi_n : \sum_{n=1}^\infty \alpha_n^2 \lambda_n^2 < \infty \Big\}$$

(cf. (2.1) and (2.2)). Moreover, for $u = \sum_{n=1}^\infty \alpha_n \varphi_n \in \operatorname{dom} A$, $Au = \sum_{n=1}^\infty \alpha_n \lambda_n \varphi_n$.

In the wave equation (1.1), let

$$f = \sum_{n=1}^\infty \alpha_n \varphi_n \quad \text{and} \quad g = \sum_{n=1}^\infty \beta_n \varphi_n. \tag{2.4}$$

Let

$$
\begin{aligned}
u(t) &:= \sum_{n=1}^{\infty} \alpha_n \cos(t\sqrt{\lambda_n})\varphi_n + \sum_{n=1}^{\infty} \beta_n \frac{\sin(t\sqrt{\lambda_n})}{\sqrt{\lambda_n}}\varphi_n, \\
G(t) &:= -\sum_{n=1}^{\infty} \alpha_n \sqrt{\lambda_n} \sin(t\sqrt{\lambda_n})\varphi_n + \sum_{n=1}^{\infty} \beta_n \cos(t\sqrt{\lambda_n})\varphi_n, \\
K(t) &:= -\sum_{n=1}^{\infty} \alpha_n \lambda_n \cos(t\sqrt{\lambda_n})\varphi_n - \sum_{n=1}^{\infty} \beta_n \sqrt{\lambda_n} \sin(t\sqrt{\lambda_n})\varphi_n.
\end{aligned} \tag{2.5}
$$

Using Theorem 2.2, one can prove that for any $f \in \operatorname{dom} A$ and $g \in \operatorname{dom} \sqrt{A}$, $u(t)$ defined in (2.5) is the unique solution of the wave equation (1.1). Moreover, $u_t = G(t) \in \operatorname{dom} \sqrt{A}$ and $u_{tt} = K(t) \in \mathcal{H}$ for any $t \in \mathbb{R}$.

3 Proof of Theorem 1.1

We prove Theorem 1.1 in this section. Some key ideas of Lee [42] are used.

Proof of Theorem 1.1 Let U be a bounded open subset of X, and $f \in L^2(X, \mu)$ be non-negative on X and non-zero on U. Let $v(x,t)$ be the solution of the heat equation (1.2) with initial data f. Using (1.3), we have for μ-a.e. $x \in U$ and $t \in (0, \epsilon)$,

$$
\begin{aligned}
v(x,t) &= \int_X p(t,x,y) f(y)\, d\mu(y) \\
&\geq c_1 \int_X t^{-\alpha/2} \exp\Big(-c_2\Big(\frac{d(x,y)}{t^{1/\beta}}\Big)^{\beta/(\beta-1)}\Big) f(y)\, d\mu(y) \\
&\geq c_1 \int_U t^{-\alpha/2} \exp\Big(-c_2\Big(\frac{d(x,y)}{t^{1/\beta}}\Big)^{\beta/(\beta-1)}\Big) f(y)\, d\mu(y) \\
&\geq c_1 \|f\|_{L^1(U,\mu)} t^{-\alpha/2} \exp\Big(-\frac{C}{t^{\gamma}}\Big),
\end{aligned} \tag{3.1}
$$

where $C := c_2 \sup\{d(x,y)^{\beta/(\beta-1)} : x, y \in U\} < \infty$ and $\gamma := 1/(\beta-1)$.

(a) Suppose, on the contrary, that A satisfies (FPS). Let V_1, V_2 be two bounded open subsets of X such that $d(V_1, V_2) > 0$ and $\mu(V_2) < +\infty$, and $f \in L^2(V_1, \mu)$ be a non-negative and non-zero function, extended by zero to X. Then there exists some constant $\delta > 0$ such that

$$
\big(\cos(t\sqrt{A}) f, g\big)_\mu = 0 \qquad \text{for all } 0 < t < \delta \text{ and all } g \in L^2(V_2, \mu).
$$

It follows that $\cos(t\sqrt{A}) f(x) = 0$ for μ-a.e. $x \in V_2$ and all $t \in (0, \delta)$. Since $f \geq 0$, $v(x,t) = (T_t f)(x) \geq 0$ for all $t > 0$. By Lemma 2.3, for all $t \geq 0$,

$$
\begin{aligned}
\int_{V_2} |v(x,t)|\,d\mu = \int_{V_2} v(x,t)\,d\mu &= \frac{1}{\sqrt{\pi t}} \int_{V_2} \int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big) \cos(s\sqrt{A}) f\,ds\,d\mu \\
&= \frac{1}{\sqrt{\pi t}} \int_{V_2} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big) \cos(s\sqrt{A}) f\,ds\,d\mu \\
&= \frac{1}{\sqrt{\pi t}} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big) \int_{V_2} \cos(s\sqrt{A}) f\,d\mu\,ds \\
&\le \frac{1}{\sqrt{\pi t}} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big) \|\cos(s\sqrt{A}) f\|_2 \sqrt{\mu(V_2)}\,ds \\
&\le \frac{\sqrt{\mu(V_2)}}{\sqrt{\pi t}} \|f\|_2 \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big)\,ds \\
&\le \frac{C_1}{\sqrt{\pi t}} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big)\,ds, \qquad (3.2)
\end{aligned}
$$

where C_1 is a positive constant. Letting $\omega = (s-\delta)/\sqrt{4t}$ in (3.2), we see that

$$
\begin{aligned}
\int_{V_2} |v(x,t)|\,d\mu &\le \frac{2C_1}{\sqrt{\pi}} \int_0^\infty \exp\Big(-\big(\frac{\delta}{\sqrt{4t}}+\omega\big)^2\Big)\,d\omega \\
&= \frac{2C_1}{\sqrt{\pi}} \exp\Big(-\frac{\delta^2}{4t}\Big) \int_0^\infty \exp\Big(-\omega^2 - \frac{\delta}{\sqrt{t}}\omega\Big)\,d\omega \\
&= \frac{2C_1}{\sqrt{\pi}} \exp\Big(-\frac{\delta^2}{4t}\Big)\Big(\frac{\sqrt{t}}{C_2} + O\big(t^{3/2}\big)\Big) \qquad \text{as } t \to 0^+, \qquad (3.3)
\end{aligned}
$$

where C_2 is a positive constant. On the other hand, applying (3.1) with $U = V_1 \cup V_2$ and using the fact that $f = 0$ on V_2, we have

$$
\int_{V_2} |v(x,t)|\,d\mu \ge c_1 \mu(V_2) \|f\|_{L^1(V_1,\mu)} t^{-\alpha/2} \exp\Big(-\frac{C}{t^\gamma}\Big) \qquad \text{for all } t \in (0,\epsilon), \quad (3.4)
$$

where $C := c_2 \sup\{d(x,y)^{\beta/(\beta-1)} : x, y \in V_1 \cup V_2\} < \infty$. Combining (3.3) and (3.4) yields

$$
c_1 \mu(V_2) \|f\|_{L^1(V_1,\mu)} t^{-\alpha/2} \exp\Big(-\frac{C}{t^\gamma}\Big) \le \frac{2C_1}{\sqrt{\pi}} \exp\Big(-\frac{\delta^2}{4t}\Big)\Big(\frac{\sqrt{t}}{C_2} + O\big(t^{3/2}\big)\Big).
$$

As $\|f\|_{L^1(V_1,\mu)} > 0$, we get

$$
t^{-\alpha/2} \exp\Big(t^{-\gamma}\big(-C + \frac{\delta^2}{4t^{1-\gamma}}\big)\Big) \le C_3\Big(\frac{\sqrt{t}}{C_2} + O\big(t^{3/2}\big)\Big), \qquad (3.5)
$$

where C_3 is a positive constant. Letting $t \to 0^+$, the left side tends to ∞, while the right side tends to 0, a contradiction. Hence, A satisfies (IPS).

(b) We now assume, in addition, that $f \in \operatorname{dom} \mathcal{E}$. Fix any $x \in U$. Suppose, on the contrary, that there exists a constant $\delta > 0$ such that

$$
(\cos(t\sqrt{A}) f)(x) \le 0 \quad \text{for all } 0 < t < \delta. \qquad (3.6)
$$

It follows from Lemma 2.3 and the assumption $\|\cos(t\sqrt{A})f\|_\infty \le C$ for all $t > 0$ that

$$v(x,t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big)(\cos(s\sqrt{A})f)(x)\,ds$$
$$\le \frac{1}{\sqrt{\pi t}} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big)|(\cos(s\sqrt{A})f)(x)|\,ds$$
$$\le \frac{C}{\sqrt{\pi t}} \int_\delta^\infty \exp\Big(-\frac{s^2}{4t}\Big)\,ds \quad \text{for all } t \ge 0. \tag{3.7}$$

Letting $\omega = (s-\delta)/\sqrt{4t}$ in (3.7), we obtain, as in the proof of (a), that

$$v(x,t) \le \frac{2C}{\sqrt{\pi}} \exp\Big(-\frac{\delta^2}{4t}\Big)\Big(\frac{\sqrt{t}}{C_4} + O\big(t^{3/2}\big)\Big) \qquad \text{as } t \to 0^+, \tag{3.8}$$

where C_4 is a positive constant. Combining (3.1) and (3.8) yields

$$c_1\|f\|_{L^1(U,\mu)}t^{-\alpha/2}\exp\Big(-\frac{C}{t^\gamma}\Big) \le \frac{2C}{\sqrt{\pi}}\exp\Big(-\frac{\delta^2}{4t}\Big)\Big(\frac{\sqrt{t}}{C_4} + O\big(t^{3/2}\big)\Big),$$

which implies an analogue of (3.5) and a contradiction. Since U is arbitrary, the proof is complete. □

4 Fractals with Infinite Propagation Speed

4.1 Iterated Function Systems with Overlaps

Proof of Corollary 1.2 (a) Gu et al. [25, Theorem 1.2] obtained the following lower heat kernel estimate: there exist positive constants c_1, c_2 such that

$$p(t,x,y) \ge \frac{c_1}{V(x,t^{1/\beta})}\exp\left(-c_2\left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right), \tag{4.1}$$

for all $t \in (0,1)$ and $x, y \in [0,1]$, where $\beta > 2$, d_* is a metric on $[0,1]$ (see [25, Section 3]), and $V(x,t^{1/\beta}) := \mu(B_{d_*}(x,t^{1/\beta}))$. Notice that $V(x,t^{1/\beta}) \le \mu([0,1]) = 1$ for all $t \in (0,1)$. Thus the lower heat kernel estimate (1.3) holds with $\epsilon = 1$ and $\alpha = 0$. It follows from Theorem 1.1(a) that $([0,1], d_*, \mu, A)$ satisfies (IPS). Moreover, according to [25, Lemma 3.8],

$$|x-y|\mu([x,y]) \asymp d_*(x,y)^\beta,$$

This implies that d_* is topologically equivalent to $d_{|\cdot|}$ and that for any open subsets $U_1, U_2 \subseteq [0,1]$, $d_*(U_1, U_2) > 0$ if and only if $d_{|\cdot|}(U_1, U_2) > 0$. Hence, Proposition 2.1(b) implies that $([0,1], d_{|\cdot|}, \mu, A)$ has (IPS).

Let $f \in \operatorname{dom}\mathcal{E}$ be a non-negative and non-zero function. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family associated with A. Since

$$\begin{aligned}\mathcal{E}\big(\cos(t\sqrt{A})f, \cos(t\sqrt{A})f\big) &= \int_0^\infty \lambda \cos^2(t\sqrt{\lambda})\, d(E_\lambda f, f)\\ &\leq \int_0^\infty \lambda\, d(E_\lambda f, f) = \mathcal{E}(f, f),\end{aligned} \tag{4.2}$$

$\cos(t\sqrt{A})f \in \operatorname{dom}\mathcal{E}$. Note that $|u(x)-u(y)| \leq |x-y|^{1/2}\sqrt{\mathcal{E}(u,u)}$ for all $u \in \operatorname{dom}\mathcal{E}$. It follows that $\operatorname{dom}\mathcal{E} \subseteq C([0,1], d_{|\cdot|})$, and hence there exists a constant $C := C(f) > 0$ such that $\|\cos(t\sqrt{A})f\|_\infty \leq C$ for all $t > 0$. Hence, the assertion in (a) follows from Theorem 1.1(b).

(b) The following lower heat kernel estimate is obtained in [25, Theorem 1.3]:

$$p(t,x,y) \geq \frac{c_1}{V(x, t^{1/\beta})} \exp\left(-c_2\left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right),$$

for all $t \in (0,1)$ and $x, y \in [0, m]$, where $\beta > 2$, d_* is a metric on $[0, m]$ defined in [25, Section 4], and $V(x, t^{1/\beta}) := \mu_m(B_{d_*}(x, t^{1/\beta}))$. The rest of proof is similar to that of part (a). □

4.2 Fractal Blowups

In this subsection, we will apply our results on infinite propagation speed to a time changed Brownian motion on $\mathbb{R}$, which is constructed by blowing up a given self-similar set (see [26]).

Proof of Corollary 1.4 Gu and Hu [26] obtained the lower estimates of the heat kernel:

$$p(t,x,y) \geq \frac{c_1}{V(x, t^{1/\beta})} \exp\left(-c_2\left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right), \tag{4.3}$$

for all $t > 0$ and $x, y \in K_\infty$, where $\beta > 2$, d_* is a metric on K_∞ defined as in [26, Section 4], and $V(x, t^{1/\beta}) := \mu(B_{d_*}(x, t^{1/\beta}))$. Let ϵ be a positive constant such that $V(x, \epsilon^{1/\beta}) \leq 1$ for all $x \in K_\infty$. Then (1.3) holds. Moreover, by [26, Chapter 4], we have d_* is topologically equivalent to $d_{|\cdot|}$ and for any open subsets $U_1, U_2 \subseteq K_\infty$, $d_*(U_1, U_2) > 0$ if and only if $d_{|\cdot|}(U_1, U_2) > 0$. Similar to the proof of Corollary 1.2(a), the assertion follows from Theorem 1.1(a) and Proposition 2.1(b). □

4.3 Generalized Sierpiński Carpets

In this subsection, we illustrate Theorem 1.1 by using both the classes of bounded and unbounded generalized Sierpiński carpets. The following definition is given in [5,10,12].

Let $n \geq 2$, $F_0 = [0, 1]^n$, and let $l_F \in \mathbb{N}$ with $l_F \geq 3$ being fixed. For $k \in \mathbb{Z}$, let $\mathcal{Q}_k$ be the collection of closed cubes with side length l_F^{-k} and with vertices at $l_F^{-k}\mathbb{Z}^n$. For $E \subseteq \mathbb{R}^n$, let ∂E and E° be the boundary and interior of E respectively, and let

$$\mathcal{Q}_k(E) := \{Q \in \mathcal{Q}_k : Q^\circ \cap E \neq \emptyset\}. \tag{4.4}$$

For $Q \in \mathcal{Q}_k$, let Ψ_Q be the orientation preserving affine map (i.e., similitude with no rotation part) which maps F_0 onto Q. Define a decreasing sequence $\{F_k\}$ of closed subsets of F_0. Let m_F be an integer satisfying $1 \leq m_F \leq l_F^n$, and F_1 be the union of m_F distinct elements of $\mathcal{Q}_1(F_0)$. We impose the following conditions on F_1.

(H1) (*Symmetry*) F_1 is preserved by all the isometries of the unit cube F_0.
(H2) (*Connectedness*) F_1° is connected.
(H3) (*Non-diagonality*) Let $m \geq 1$ and $B \subseteq F_0$ be a cube of side length $2l_F^{-m}$, which is the union of 2^n distinct elements of $\mathcal{Q}_m$. Then if $\mathrm{int}(F_1 \cap B)$ is non-empty, it is connected.
(H4) (*Border included*) F_1 contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = \cdots = x_n = 0\}$.

One may think of F_1 as being derived from F_0 by removing the interiors of $l_F^n - m_F$ cubes in $\mathcal{Q}_1(F_0)$. Iterating this, we obtain a sequence $\{F_k\}$, where F_k is the union of m_F^k cubes in $\mathcal{Q}_k(F_0)$. Formally, we define

$$F_{k+1} = \bigcup_{Q \in \mathcal{Q}_k(F_k)} \Psi_Q(F_1) = \bigcup_{Q \in \mathcal{Q}_1(F_1)} \Psi_Q(F_k), \quad k \geq 1. \tag{4.5}$$

We call the set $F := \bigcap_{k=0}^{\infty} F_k$ a *generalized Sierpiński carpet (GSC).*

Example 4.1 (Sierpiński carpet, [35]) Let $p_1 = 0$, $p_2 = 1/2$, $p_3 = 1$, $p_4 = 1 + \sqrt{-1}/2$, $p_5 = 1 + \sqrt{-1}$, $p_6 = 1/2 + \sqrt{-1}$, $p_7 = \sqrt{-1}$ and $p_8 = \sqrt{-1}/2$. Define $S_i : \mathbb{C} \to \mathbb{C}$ as $S_i(z) = (z - p_i)/3 + p_i$ for $i \in \{1, \ldots, 8\}$. Then there exists a unique nonempty compact subset F, which satisfies $F = \bigcup_{i=1}^{8} S_i(F)$. F is called the standard Sierpiński carpet.

The standard Sierpiński carpet in the above Example is a GSC with $n = 2, l_F = 3$, $m_F = 8$ and with F_1 being obtained from F_0 by removing the middle cube.

We also consider a related set, which has a large-scale structure similar to the small-scale structure of F. Set $F_k := F_0$ for $k < 0$ and for $i \in \mathbb{Z}$, let

$$\widetilde{F}_i = \bigcup_{r=0}^{\infty} l^r F_{i+r},$$

and $\widetilde{F} = \bigcap_{i=0}^{\infty} \widetilde{F}_i$, is called an *unbounded generalized Sierpiński carpet*. Let $\mu_k(dx) = m_F^k 1_{F_k} dx$ and μ be the weak limit of the μ_k. Then μ is a constant multiple of the $\log m_F / \log l_F$-dimensional Hausdorff measure on $\widetilde{F}$.

For $n = 2$, Barlow and Bass [6] constructed a Brownian motion on the standard Sierpiński carpet, i.e., a strong Markov process with state space F that has continuous paths and is invariant under an appropriate class of transformations. They later extended the strong Markov process to unbounded Sierpiński carpets $\widetilde{F}$ and obtained upper and lower bounds for the transition densities $p(t, x, y)$ on $\widetilde{F}$ (see [9]). Subsequently, Kusuoka and Zhou [39] gave a different construction of a continuous strong Markov process on the standard Sierpiński carpet, which also has the invariance properties of the Brownian motion constructed in [6]. In [10], the results of [6,9] are extended to GSC and unbounded Sierpiński carpets embedded in $\mathbb{R}^n$ for $n \geq 3$. Furthermore, the following sub-Gaussian heat kernel estimates are obtained:

$$\begin{aligned} &\widetilde{c}_1 \cdot t^{-d_s/2} \exp\left(-\widetilde{c}_2 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) \\ &\leq p(t,x,y) \\ &\leq \widetilde{c}_3 \cdot t^{-d_s/2} \exp\left(-\widetilde{c}_4 \left(\frac{|x-y|^{d_w}}{t}\right)^{1/(d_w-1)}\right) \end{aligned} \tag{4.6}$$

for any $t > 0$ and any $x, y \in \widetilde{F}$, where $d_w := 2d_f/d_s > 2$ (see [10]). Barlow et al. [12] showed that, up to scalar multiples of the time parameter, there exists only one such Brownian motion on a generalized Sierpiński carpet. Since the Laplacian is the infinitesimal generator of the semigroup associated with the process, the Laplacian on GSC is uniquely defined.

Let A be the Laplacian in $L^2(F, \mu)$ associated with the process X_t constructed in [9, 10] or [39] with domain dom A. Write $\widetilde{X}_t$ for the extension of X_t to the corresponding unbounded Sierpiński carpet $\widetilde{F}$, and let $(\widetilde{A}, \operatorname{dom} \widetilde{A})$ be the associated Laplacian of $\widetilde{X}_t$ in $L^2(\widetilde{F}, \mu)$.

Proof of Corollary 1.5 (a) By virtue of (4.6), we can see that inequality (1.3) holds with $\epsilon = 1, \alpha = d_s$ and $\beta = d_w > 2$. It follows from Theorem 1.1 (a) that A and $\widetilde{A}$ satisfy (IPS).

(b) Let $\{T_t\}_{t>0}$ be the semigroup associated with $(\mathcal{E}, \operatorname{dom}\mathcal{E})$. Since the Nash inequality holds, i.e., there exists some constant $c_1 > 0$ such that

$$\|u\|_2^{2+4/d_s} \leq c_1 \mathcal{E}(u,u) \|u\|_1^{4/d_s}, \qquad u \in \operatorname{dom}\mathcal{E},$$

(see [10, Theorem 7.1]), we conclude that $\{T_t\}_{t>0}$ is ultracontractive and there exists $c > 0$ such that $\|T_t\|_{1\to\infty} \leq ct^{-d_s/2}$ for any $t \in (0, 1]$. Moreover, it follows from $\mu(F) < \infty$ and the assumption $d_s \in (0, 2)$ that there exists $M > 0$ such that

$$\|u\|_\infty^2 \leq M(\mathcal{E}(u,u) + \|u\|_{L^2(F,\mu)}^2) \qquad \text{for any } u \in \operatorname{dom}\mathcal{E}, \tag{4.7}$$

and that $\operatorname{dom}\mathcal{E} \subseteq C(F, d_{|\cdot|})$ (see [35, Theorem A.6]). Let $f \in \operatorname{dom}\mathcal{E}$ be a non-negative and non-zero function. As in (4.2), $\mathcal{E}\big(\cos(t\sqrt{A})f, \cos(t\sqrt{A})f\big) \le \mathcal{E}(f, f)$, $\cos(t\sqrt{A})f \in \operatorname{dom}\mathcal{E}$ and

$$\mathcal{E}\big(\cos(t\sqrt{A})f, \cos(t\sqrt{A})f\big) + \|\cos(t\sqrt{A})f\|^2_{L^2(F,\mu)} \le \mathcal{E}(f, f) + \|f\|^2_{L^2(F,\mu)},$$

which, together with (4.7), yields $\|\cos(t\sqrt{A})f\|_\infty \le M(\mathcal{E}(f, f) + \|f\|^2_{L^2(F,\mu)}) := C < \infty$ for all $t > 0$. Hence, Theorem 1.1(b) now implies that for any $x \in F$, and any $\delta > 0$, there is $t_0 \in (0, \delta)$ such that $\big(\cos(t_0\sqrt{A})f\big)(x) > 0$. □

5 Finite Propagation Speed

In this section, we let (X, d, μ) be a metric measure space, A be a non-negative self-adjoint operator on $L^2(X, \mu)$, and $T_t = \exp(-tA)$, $t > 0$, be the associated semigroup on $L^2(X, \mu)$.

Definition 5.1 Let (X, d, μ) be a metric measure space and A be a non-negative self-adjoint operator on $L^2(X, \mu)$. We say that (X, d, μ, A) (or simply A) satisfies the Davies-Gaffney estimate if

$$\big|(T_t f_1, f_2)_\mu\big| \le \|f_1\|_2\|f_2\|_2 \exp\big(-r^2/(4t)\big) \tag{5.1}$$

for $t > 0$, open subsets $U_i \subseteq X$ and $f_i \in L^2(U_i, \mu)$, $i = 1, 2$, where $r := d(U_1, U_2) > 0$.

It is well known that the Davies–Gaffney estimate holds for essentially all self-adjoint elliptic second-order differential operators (see [16,19,51]). The following theorem shows that the Davies-Gaffney estimate is equivalent to (UPS).

Theorem 5.1 [16, Theorem 3.4] *Assume that (X, d, μ) is a metric measure space and A is a non-negative self-adjoint operator on $L^2(X, \mu)$. Then (UPS) and the Davies-Gaffney estimate (5.1) are equivalent.*

The following remark is a direct consequence of Theorem 5.1.

Remark 5.2 Assume the hypotheses of Theorem 5.1. Then (X, d, μ, A) satisfies (BPS) with maximum propagation speed s if and only if $(X, (1/s)d, \mu, A)$ satisfies the Davies-Gaffney estimate (5.1).

An analogous relation holds for (FPS), as shown in the following theorem. It can be proved by modifying that of [16, Theorem 3.4]; we omit the details.

Theorem 5.3 *Assume the hypotheses of Theorem 5.1. Then (X, d, μ, A) satisfies (FPS) if and only if for any open subsets $U_i \subseteq X$, $i = 1, 2$, there exists a constant $c > 0$, which may depend on U_1, U_2, such that*

$$\big|(T_t f_1, f_2)_\mu\big| \le \|f_1\|_2\|f_2\|_2 \exp\big(-r^2/(ct)\big) \tag{5.2}$$

for $t > 0$ *and* $f_i \in L^2(U_i, \mu)$*, where* $r := d(U_1, U_2) > 0$*.*

We now study the relationship between upper heat kernel estimate and wave propagation speed. For any complex number $z \in \mathbb{C}$, we denote the real part of z by $\Re(z)$. Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re(z) > 0\}$ denote the right half of the complex plane. Define $T_z f := \exp(-zA)f$ for any $f \in L^2(X, \mu)$ and $z \in \mathbb{C}_+$. We note that

$$\|T_z f\|_2 \le \|f\|_2 \quad \text{for all } f \in L^2(X, \mu) \text{ and all } z \in \mathbb{C}_+. \tag{5.3}$$

In fact, if $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family associated with A, then for any $z \in \mathbb{C}_+$ and any $f \in L^2(X, \mu)$,

$$\begin{aligned} \|T_z f\|_2^2 &= \Big\| \int_0^\infty \exp(-z\lambda)\, dE_\lambda f \Big\|_2^2 = \int_0^\infty \big|\exp(-z\lambda)\big|^2 \, d\|E_\lambda f\|_2^2 \\ &= \int_0^\infty \exp\big(-\Re(z)\lambda\big)^2 \, d\|E_\lambda f\|_2^2 = \big\|T_{\Re(z)} f\big\|_2^2 \le \|f\|_2^2, \end{aligned}$$

where the last inequality is because the semigroup $\{T_t\}_{t>0}$ is contractive.

The following lemma is a slight modification of a similar one in [16]; the proof is the same.

Lemma 5.4 [16, Proposition 2.2] *Let* u *be an analytic function on* $\mathbb{C}_+$*. Assume that there exist positive numbers* c_1, γ *and a positive number* $c_2 = c_2(\gamma)$ *(which may depend on* γ*) such that*

$$|u(z)| \le c_1 \quad \text{for all } z \in \mathbb{C}_+, \text{ and } \quad |u(t)| \le c_2 \exp(-\gamma/t) \quad \text{for all } t > 0.$$

Then

$$|u(z)| \le c_1 \exp\big(-\Re(\gamma/z)\big) \quad \text{for all } z \in \mathbb{C}_+.$$

In the following lemma we modify a result in [16] by allowing the constant c to depend on f_i, in order to suit our purpose. We include a proof for completeness.

Lemma 5.5 [16, Lemma 3.1] *Let* (X, d, μ) *be a separable metric measure space. Assume that for any* $f_i \in L^2(X, \mu)$*,* $\operatorname{supp}(f_i) \subseteq \overline{B}_d(x_i, r_i)$*,* $i = 1, 2$*, and* $r := d(B_d(x_1, r_1), B_d(x_2, r_2)) > 0$*, there exists a constant* $c := c(f_1, f_2, B_d(x_1, r_1), B_d(x_2, r_2)) > 0$ *such that*

$$\big|(T_t f_1, f_2)_\mu\big| \le c\|f_1\|_2\|f_2\|_2 \exp\big(-r^2/(4t)\big) \quad \text{for } t > 0. \tag{5.4}$$

Then the Davies-Gaffney estimate (5.1) holds.

Proof Let $f_i \in L^2(X, \mu)$, $\operatorname{supp}(f_i) \subseteq \overline{B}_d(x_i, r_i)$, $i = 1, 2$, and $r := d(B_d(x_1, r_1), B_d(x_2, r_2)) > 0$. Combining (5.3), (5.4) and Lemma 5.4, we have

$$\big|u(z)\big| := \big|(T_z f_1, f_2)_\mu\big| \le \|f_1\|_2\|f_2\|_2 \exp\big(-r^2\Re(1/(4z)\big) \quad \text{for } z \in \mathbb{C}_+.$$

In particular,

$$\big|(T_t f_1, f_2)_\mu\big| \le \|f_1\|_2 \|f_2\|_2 \exp\big(-r^2/(4t)\big) \quad \text{for } t>0. \tag{5.5}$$

Now let U_1, U_2 be arbitrary open subsets of X such that $r := d(U_1, U_2) > 0$. Let $f := \sum_{i=1}^k f_i$, where for all $1 \le i \le k$, $f_i \in L^2(B_d(x_i, r_i), \mu)$, $B_d(x_i, r_i) \subset U_1$, and $f_{i_1} f_{i_2} = 0$ in $L^2(X, \mu)$ for all $1 \le i_1 < i_2 \le k$. Notice that $\|f\|_2^2 = \sum_{i=1}^k \|f_i\|_2^2$. Similarly, let $g := \sum_{j=1}^\ell g_j$, where $g_j \in L^2(B_d(y_j, s_j), \mu)$, $B_d(y_j, s_j) \subset U_2$ for all $1 \le j \le \ell$, and $g_{j_1} g_{j_2} = 0$ in $L^2(X, \mu)$ for all $1 \le j_1 < j_2 \le \ell$. It is obvious that $d(B_d(x_i, r_i), B_d(y_j, s_j)) \ge d(U_1, U_2) = r$. Thus (5.5) implies that for $t > 0$,

$$\begin{aligned}\big|(T_t f, g)_\mu\big| &\le \sum_{i=1}^k \sum_{j=1}^\ell \big|(T_t f_i, g_j)_\mu\big| \le \sum_{i=1}^k \sum_{j=1}^\ell \|f_i\|_2 \|g_j\|_2 \exp\Big(-\frac{r^2}{4t}\Big)\\ &= \Big(\sum_{i=1}^k \|f_i\|_2\Big)\Big(\sum_{j=1}^\ell \|g_j\|_2\Big) \exp\Big(-\frac{r^2}{4t}\Big)\\ &\le \sqrt{k\ell}\,\|f\|_2 \|g\|_2 \exp\Big(-\frac{r^2}{4t}\Big).\end{aligned}$$

Combining this with (5.3) and Lemma 5.4 gives

$$\big|(T_t f, g)_\mu\big| \le \|f\|_2 \|g\|_2 \exp\big(-r^2/(4t)\big) \qquad \text{for } t>0.$$

To finish the proof of the lemma, it suffices to note that, since (X, d) is a separable metric space, the space of all possible finite linear combinations of functions h of the form $\operatorname{supp}(h) \subset \overline{B}_d(x, r)$ and $B_d(x, r) \subseteq U$ is dense in $L^2(U, \mu)$. Moreover, if $h := \sum_{i=1}^m h_i$ and $h_i \in L^2(B_d(x_i, r_i), \mu)$ for all $1 \le i \le m$, then there exist functions $\widetilde{h}_i \in L^2(B_d(x_i, r_i), \mu)$ such that $h := \sum_{i=1}^m \widetilde{h}_i$ and $\widetilde{h}_{i_1} \widetilde{h}_{i_2} = 0$ in $L^2(X, \mu)$ for all $1 \le i_1 < i_2 \le k$. □

Theorem 5.6 *Let (X, d) be a locally compact separable metric space, μ be a Radon measure on (X, d), A be a non-negative self-adjoint operator on $L^2(X, \mu)$, and $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be the closed quadratic form on $L^2(X, \mu)$ associated with A. Assume that the heat kernel $p(t, x, y)$ of $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ exists and that there exist positive constants c_1, c_2, γ such that for any $Y \subset\subset X$, there exist positive constants $\widetilde{c}_1 := \widetilde{c}_1(c_1, Y)$, $\widetilde{c}_2 := \widetilde{c}_2(c_2, Y)$ and $\delta := \delta(Y) > 0$ satisfying*

$$p(t, x, y) \le \widetilde{c}_1 \exp\Big(-\frac{d(x, y)^2}{c_1 t}\Big) \quad \textit{for all } t > \delta \textit{ and } x, y \in Y, \tag{5.6}$$

and

$$p(t, x, y) \le \widetilde{c}_2 t^{-\gamma} \exp\Big(-\frac{d(x, y)^2}{c_2 t}\Big) \quad \textit{for } 0 < t \le \delta \textit{ and } x, y \in Y. \tag{5.7}$$

Then (X, d, μ, A) satisfies (BPS).

Proof Let $\{T_t\}_{t>0}$ be the semigroup on $L^2(X, \mu)$ associated with A. Let $f_i \in L^2(X, \mu)$ with $\mathrm{supp}(f_i) \subseteq \overline{B}_d(x_i, r_i)$, $i = 1, 2$, and with $r := d(B_d(x_1, r_1), B_d(x_2, r_2)) > 0$. For any $x \in \mathrm{supp}(f_1)$ and $y \in \mathrm{supp}(f_2)$, $d(x, y)^2 \geq r^2$ and thus for $t > 0$ and $i = 1, 2$,

$$\exp\Big(-\frac{d(x, y)^2}{c_i t}\Big) \leq \exp\Big(-\frac{r^2}{c_i t}\Big). \tag{5.8}$$

Choose a subset $Y \subset\subset X$ such that $\mathrm{supp}(f_1) \cup \mathrm{supp}(f_2) \subset Y$. Then

$$\begin{aligned}\big(T_t f_1, f_2\big)_\mu &= \int_X (T_t f_1)(y) f_2(y)\, d\mu(y) = \int_{X\times X} p(t, y, x) f_1(x) f_2(y)\, d\mu(x) d\mu(y)\\ &= \int_{\mathrm{supp}(f_1)\times\mathrm{supp}(f_2)} p(t, x, y) f_1(x) f_2(y)\, d\mu(x) d\mu(y),\end{aligned} \tag{5.9}$$

where the symmetry of $p(t, x, y)$ is used in the last equality. Together with (5.6) and (5.8), this implies, for $t > \delta$,

$$\begin{aligned}|(T_t f_1, f_2)_\mu| &\leq \widetilde{c}_1 \int_{\mathrm{supp} f_1\times\mathrm{supp} f_2} \exp\Big(-\frac{r^2}{c_1 t}\Big)\big|f_1(x) f_2(y)\big|\, d\mu(x) d\mu(y)\\ &\leq \widetilde{c}_1 \Big(\mu(\mathrm{supp}(f_1))\mu(\mathrm{supp}(f_2))\Big)^{1/2} \|f_1\|_2 \|f_2\|_2 \exp\Big(-\frac{r^2}{c_1 t}\Big).\end{aligned} \tag{5.10}$$

Similarly, for $0 < t \leq \delta$, combining (5.7), (5.8), and (5.9) yields

$$\begin{aligned}|(T_t f_1, f_2)_\mu| &\leq \widetilde{c}_2 t^{-\gamma} \int_{\mathrm{supp} f_1\times\mathrm{supp} f_2} \exp\Big(-\frac{r^2}{c_2 t}\Big)\big|f_1(x) f_2(y)\big|\, d\mu(y) d\mu(x)\\ &\leq \widetilde{c}_2 \Big(\mu(\mathrm{supp}(f_1))\mu(\mathrm{supp}(f_2))\Big)^{1/2} \|f_1\|_2 \|f_2\|_2 t^{-\gamma} \exp\Big(-\frac{r^2}{c_2 t}\Big).\end{aligned} \tag{5.11}$$

Since $\lim_{t\to 0^+} t^{-\gamma} \exp(-c/t) = 0$ for any constant $c > 0$, there exists some constant $\widetilde{c}_3 := \widetilde{c}_3(c_2, r, \gamma, \delta) > 0$ such that for $0 < t \leq \delta$,

$$t^{-\gamma} \exp\Big(-\frac{r^2}{c_2 t}\Big) \leq \widetilde{c}_3 \exp\Big(-\frac{r^2}{(c_2+1)t}\Big).$$

Together with (5.11), this implies that for $0 < t \leq \delta$,

$$|(T_t f_1, f_2)_\mu| \leq \widetilde{c}_2 \widetilde{c}_3 \Big(\mu(\mathrm{supp}(f_1))\mu(\mathrm{supp}(f_2))\Big)^{1/2} \|f_1\|_2 \|f_2\|_2 \exp\Big(-\frac{r^2}{(c_2+1)t}\Big). \tag{5.12}$$

It follows from (5.10) and (5.12) that there exist constants $c_3 := c_3(c_1, c_2) > 0$ (independent on Y) and $\widetilde{c}_4 := \widetilde{c}_4(\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3) > 0$ such that

$$\left|(T_t f_1, f_2)_\mu\right| \leq \widetilde{c}_4 \|f_1\|_2 \|f_2\|_2 \exp\Big(-\frac{r^2}{c_3 t}\Big) \quad \text{for } t > 0. \tag{5.13}$$

Thus, using Lemma 5.5, we see that $(X, (2/\sqrt{c_3})d, \mu, A)$ satisfies the Davies-Gaffney estimate. Hence, the assertion now follows by using Remark 5.2. □

6 Proof of Theorem 1.6

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be a bounded open subset, and μ be a positive finite Borel measure on $\mathbb{R}^n$ with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. For convenience, we summarize the definition of the Dirichlet Laplacian with respect to a measure μ; details can be found in [28]. We further suppose μ satisfies (MPI) (see (1.12)), which implies that each equivalence class $u \in H_0^1(\Omega)$ contains a unique (in $L^2(\Omega, \mu)$ sense) member $\hat{u}$ that belongs to $L^2(\Omega, \mu)$ and satisfies both conditions below:

(1) there exists a sequence $\{u_n\}$ in $C_c^\infty(\Omega)$ such that $u_n \to \hat{u}$ in $H_0^1(\Omega)$ and $u_n \to \hat{u}$ in $L^2(\Omega, \mu)$;
(2) $\hat{u}$ satisfies inequality (1.12).

We call $\hat{u}$ the $L^2(\Omega, \mu)$-representative of u. Define a mapping $\iota : H_0^1(\Omega) \to L^2(\Omega, \mu)$ by

$$\iota(u) = \hat{u}.$$

ι is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(\Omega)$ defined as

$$\mathcal{N} := \big\{u \in H_0^1(\Omega) : \|\iota(u)\|_2 = 0\big\}.$$

Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(\Omega)$. Then $\iota : \mathcal{N}^\perp \to L^2(\Omega, \mu)$ is injective. Unless explicitly stated otherwise, we will denote the $L^2(\Omega, \mu)$-representative $\hat{u}$ simply by u and identify $\iota(\mathcal{N}^\perp)$ with $\mathcal{N}^\perp$.

Consider a non-negative bilinear form $\mathcal{E}(\cdot, \cdot)$ on $L^2(\Omega, \mu)$ given by

$$\mathcal{E}(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx \tag{6.1}$$

with domain $\operatorname{dom} \mathcal{E} = \mathcal{N}^\perp$. (MPI) implies that $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ is a closed quadratic form on $L^2(\Omega, \mu)$. Hence there exists a unique non-negative self-adjoint operator A on $L^2(\Omega, \mu)$ such that

$$\mathcal{E}(u, v) = \big(A^{1/2}u, A^{1/2}v\big) \quad \text{and} \quad \operatorname{dom} \mathcal{E} = \operatorname{dom}(A^{1/2}).$$

We write $\Delta_\mu = -A$, and call it the *(Dirichlet) Laplacian* with respect to μ.

As can be seen above, it is often difficult to describe $\operatorname{dom}\mathcal{E}$ precisely. However, if μ is equivalent to Lebesgue measure on $\overline{\Omega}$, we have the following result; the proof is omitted.

Proposition 6.1 *Use the notation above, and assume that μ satisfies (MPI). Assume μ is equivalent to Lebesgue measure on $\overline{\Omega}$. Then* $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$. *Moreover,* $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ *is a regular, strongly local Dirichlet form on $L^2(\Omega, \mu)$ with $C_c^\infty(\Omega)$ being a core.*

We remark that if $n = 1$ and $\Omega = (a, b)$, it suffices to assume in Proposition 6.1 that $\operatorname{supp}(\mu) = [a, b]$ (see [14]).

Hereafter, assume that μ satisfies (MPI) and μ is equivalent to Lebesgue measure on $\overline{\Omega}$, and let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be defined as in (6.1). Then $\mu(\partial\Omega) = 0$ and so $L^2(\Omega, \mu)$ can be identified with $L^2(\overline{\Omega}, \mu)$. It follows from Proposition 6.1 that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a regular, strongly local Dirichlet form on $L^2(\overline{\Omega}, \mu)$ with domain $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$. We denote by $\nu \ll \mu$ if ν is absolutely continuous with respect to μ.

Definition 6.1 Use the notation and hypotheses in the above paragraph. A pseudo metric ρ on $\overline{\Omega}$ is defined by

$$\rho(x, y) = \sup\big\{u(x) - u(y) : u \in C^1(\overline{\Omega}),\, |\nabla u|^2 dx \ll \mu \text{ with density} \\ \leq 1\ \mu\text{-a.e. on } \overline{\Omega}\big\}, \tag{6.2}$$

called the intrinsic (or Carathéodory) metric.

Note that $|\nabla u|^2 dx$ is called the *energy measure*. The intrinsic metric is a generalization of the classical notion

$$\rho(x, y) = \sup\big\{u(x) - u(y) : u \in C^1,\ |\nabla u| \leq 1\big\};$$

those induced by strongly local regular Dirichlet forms were studied by Biroli, Mosco, Sturm, and others (see [13,18,21,27,29,45,56,57] and the references therein).

Definition 6.2 [1,2] A strongly local Dirichlet form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ on $L^2(\overline{\Omega}, \mu)$ is said to be strictly local if it is regular and if ρ (defined by (6.2)) is a metric on $\overline{\Omega}$ whose topology coincides with the original one.

We remark that the property "strictly local" is also called *strongly regular* (see, e.g., [56]). Following [56], we state and discuss several properties of the associated Dirichlet form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ on U.

Definition 6.3 Assume $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a strictly local Dirichlet form on $L^2(\overline{\Omega}, \mu)$. Fix an arbitrary subset $U \subset \overline{\Omega}$.

(1) *Completeness property (C)*: For any ball $B_\rho(x, 2r) \subset U$, the closed ball $\overline{B}_\rho(x, r)$ is complete (or, equivalently, compact) on the metric space $(\overline{\Omega}, \rho)$, where $\overline{B}_\rho(x, r) := \{y \in \overline{\Omega} : \rho(x, y) \leq r\}$.

(2) *Doubling property (VD)*: There exists a constant $N := N(U)$ such that for all balls $B_\rho(x, 2r) \subset U$,

$$\mu\big(B_\rho(x, 2r)\big) \le N\mu\big(B_\rho(x, r)\big). \tag{6.3}$$

(3) *Strong Poincaré inequality (SPI)*: There exists a constant $C_P := C_P(U)$ such that for all balls $B_\rho(x, r) \subseteq U$ and all $u \in \operatorname{dom}\mathcal{E}$,

$$\int_{B_\rho(x,r)} |u - u_{B_\rho(x,r),\mu}|^2 \, d\mu \le C_P \cdot r^2 \int_{B_\rho(x,r)} |\nabla u|^2 \, dx, \tag{6.4}$$

where $u_{B_\rho(x,r),\mu} := \int_{B_\rho(x,r)} u \, d\mu / \mu(B_\rho(x, r))$.

Theorem 6.2 [56, Theorem 4.1] *Let X be a locally compact separable Hausdorff space and μ be a Radon measure with $\operatorname{supp}(\mu) = X$. Assume $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a strictly local Dirichlet form on $L^2(X, \mu)$. Assume (C), (VD), and (SPI) are simultaneously satisfied on the open set $Y \subset X$. Then for every $\epsilon > 0$, there exists a constant $C > 0$, depending only on ϵ, $N = N(Y)$ and $c_P = c_P(Y)$ (in (6.3) and (6.4) respectively), such that the following estimate holds for all $x, y \in Y$ and $t > 0$:*

$$p(t, x, y) \le C\mu\big(B_\rho(x, \sqrt{\tau})\big)^{-1/2}\mu\big(B_\rho(y, \sqrt{\tau})\big)^{-1/2} \exp\Big(-\frac{\rho(x, y)^2}{(4 + \epsilon)t}\Big),$$

where $\tau = \inf\{t, R^2\}$ with $R := \inf\{\rho(x, X \setminus Y), \rho(y, X \setminus Y)\}$ ($R := +\infty$ if $X = Y$).

In fact, Theorem 6.2 is a special case of Theorem 4.1 in [56] with $\mathcal{E}_t \equiv \mathcal{E}$, $\kappa = 1$ and $p(t, y, s, x) = p(t - s, y, x)$. The following lemma is needed in the proof of Theorem 1.6.

Lemma 6.3 *Assume the hypotheses of Theorem 1.6, and let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be defined as in (6.1). Then*

(a) *ρ is a metric on $\overline{\Omega}$ and is topologically equivalent to $d_{|\cdot|}$.*
(b) *for any $V \subset\subset \Omega$, there exists $c(V) > 0$ such that*

$$c(V)|x - y| \le \rho(x, y) \qquad \textit{for any } x, y \in V. \tag{6.5}$$

(c) *for any open subsets $U, V \subseteq \overline{\Omega}$, $\rho(U, V) = 0$ if and only if $d_{|\cdot|}(U, V) = 0$.*

Proof We first note that μ satisfies (MPI). We use the method in [57, Theorem 4.1].

(a) Assume $u \in C^1(\overline{\Omega})$ with $(|\nabla u|^2 dx)/d\mu = |\nabla u|^2/f \le 1$ on $\overline{\Omega}$. Thus for $x \in \overline{\Omega}$,

$$\big|\nabla u(x)\big|^2 \le \|f\|_\infty. \tag{6.6}$$

If $n = 1$, then $\rho(x, y) \le \|f\|_\infty^{1/2}|x - y|$ for all $x, y \in \overline{\Omega}$. For $n \ge 2$, we use an argument in [57]. We fix arbitrary $x, y \in \overline{\Omega}$. Without loss of generality, let $x = (0, \dots, 0)$ and

$y = (R, 0, \ldots, 0)$. Let $C_\epsilon = [0, R] \times B'(0, \epsilon) = \{w = (r, w') \in \mathbb{R}^n : 0 \le r \le R, |w'| < \epsilon\}$. Since u is continuous, by Lebesgue's density theorem,

$$u(x) - u(y) = \lim_{\epsilon \to 0^+} \frac{1}{\mathcal{L}^{n-1}(B'(0,\epsilon))} \int_{B'(0,\epsilon)} \big(u(0, w') - u(R, w')\big)\, dw'.$$

By using a similar argument as that in [57, Theorem 4.1], we get

$$u(x) - u(y) = -\lim_{\epsilon \to 0^+} \frac{1}{\mathcal{L}^{n-1}(B'(0,\epsilon))} \int_{C_\epsilon} \frac{\partial}{\partial x_1} u(w)\, dw. \tag{6.7}$$

Combining (6.7) and (6.6) yields

$$\begin{aligned} \big|u(x) - u(y)\big| &\le \lim_{\epsilon \to 0} \frac{1}{\mathcal{L}^{n-1}(B'(0,\epsilon))} \int_{C_\epsilon} \Big|\frac{\partial}{\partial x_1} u(w)\Big|\, dw \\ &\le \lim_{\epsilon \to 0} \frac{1}{\mathcal{L}^{n-1}(B'(0,\epsilon))} \int_{C_\epsilon} \|f\|_\infty^{1/2}\, dw = \|f\|_\infty^{1/2} R = \|f\|_\infty^{1/2} |x - y|. \end{aligned}$$

In other words, for all those $x, y \in \Omega$ that can be connected by a straight line in Ω, we have $|u(x) - u(y)| \le \|f\|_\infty^{1/2} |x - y|$ and hence

$$\rho(x, y) \le \|f\|_\infty^{1/2} |x - y|. \tag{6.8}$$

On the other hand, let $\{x_n\} \subseteq \overline{\Omega}$. Assume that $\rho(x_n, x) \to 0$ as $n \to \infty$. Since $(\overline{\Omega}, d_{|\cdot|})$ is compact, there exist a subsequence $\{x_{n_k}\}_k \subseteq \{x_n\}$ and $x^* \in \overline{\Omega}$ such that $|x_{n_k} - x^*| \to 0$ as $k \to \infty$. It follows from (6.8) that $x^* = x$. This implies that $|x_n - x| \to 0$ as $n \to \infty$. Combining this with (6.8) proves that ρ is a metric and is topologically equivalent to the Euclidean metric.

(b) Fix any $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ and define a map $g_w : \overline{\Omega} \to \mathbb{R}$ by

$$g_w(x) := w \cdot x = \sum_{i=1}^n w_i x_i.$$

Then $g_w \in C^\infty(\overline{\Omega})$ with $\nabla g_w = w$. Fix any open subset $V \subset\subset \Omega$. Let U be an open subset such that $V \subset\subset U \subset\subset \Omega$. Then there exists some $h_w \in C_c^\infty(\overline{\Omega})$ such that

$$h_w = g_w \quad \text{on } V, \quad |\nabla h_w| \le c(U, V, w) =: c \quad \text{on } U, \text{ and} \quad h_w = 0 \text{ on } \overline{\Omega} \setminus U.$$

Define $v(x) := (\sqrt{\varepsilon(U)}/c) h_w(x)$ for $x \in \overline{\Omega}$, where $\varepsilon(U)$ is the constant in Theorem 1.6. Then $|\nabla v(x)|^2 = \varepsilon(U) \big|\nabla h_w\big|^2 / c^2 \le \varepsilon(U) \le f(x)$ on U, and $|\nabla v(x)|^2 = 0 \le f(x)$ on $\overline{\Omega} \setminus U$. Hence $|\nabla v|^2 dx \ll \mu$. Fix any distinct $y, y' \in V$ and choose $w = (y - y')/|y - y'|$. Then

$$v(y) - v(y') = \frac{\sqrt{\varepsilon(U)}}{c} \big(g_w(y) - g_w(y')\big) = \frac{\sqrt{\varepsilon(U)}}{c} |y - y'|.$$

By the definition of ρ, $(\sqrt{\varepsilon(U)}/c)|y-y'| \le \rho(y,y')$. Since y, y' are arbitrary, the desired inequality holds.

(c) Fix any two open subsets $U, V \subseteq \overline{\Omega}$. Assume $\rho(U,V) > 0$. It follows from (6.8) that $d_{|\cdot|}(U,V) > 0$. On the other hand, if $d_{|\cdot|}(U,V) > 0$, then $d_{|\cdot|}(\overline{U},\overline{V}) > 0$. In particular, $\overline{U} \cap \overline{V} = \emptyset$. By compactness, $\rho(\overline{U},\overline{V}) > 0$ and consequently $\rho(U,V) > 0$, which completes the proof. □

Proof of Theorem 1.6 We note that $\operatorname{supp}(\mu) = \overline{\Omega}$. Proposition 6.1 and Lemma 6.3 imply $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a strictly local Dirichlet form on $L^2(\overline{\Omega},\mu)$ with domain $\operatorname{dom}\mathcal{E} = H_0^1(\Omega)$.

Fix any $V \subset\subset \Omega$. Since ρ is topologically equivalent to $d_{|\cdot|}$, $\overline{B}_\rho(x,r)$ is closed in $(\overline{\Omega}, d_{|\cdot|})$ for any ball $B_\rho(x,2r) \subset V$. Thus $\overline{B}_\rho(x,r)$ is compact in $(\overline{\Omega}, d_{|\cdot|})$ and hence $\overline{B}_\rho(x,r)$ is compact in $(\overline{\Omega},\rho)$. Therefore, Property (C) holds on V. Lemma 6.3 implies that there exists $c_1 := c_1(V) > 0$ such that for any $x, y \in V$,

$$c_1|x-y| \le \rho(x,y) \le \|f\|_\infty^{1/2}|x-y|.$$

It follows that $B(x, r/\|f\|_\infty^{1/2}) \subseteq B_\rho(x,r) \subseteq B(x,r/c_1)$ for any ball $B_\rho(x,r) \subseteq V$, and thus there exist positive constants c_2 and $c_3 := c_3(V)$ such that

$$c_2 r^n = \mathcal{L}^n(B(x,r/\|f\|_\infty^{1/2})) \le \mathcal{L}^n(B_\rho(x,r)) \le \mathcal{L}^n(B(x,r/c_1)) = c_3 r^n.$$

Hence,

$$\begin{aligned} c_2\varepsilon(V)r^n \le \varepsilon(V)\mathcal{L}^n(B_\rho(x,r)) &\le \mu(B_\rho(x,r)) \\ &\le \|f\|_\infty \mathcal{L}^n(B_\rho(x,r)) \le c_3\|f\|_\infty r^n, \end{aligned} \tag{6.9}$$

which implies that there exists $N := N(V) > 0$ such that for all balls $B_\rho(x,2r) \subset V$, $\mu(B_\rho(x,2r)) \le N\mu(B_\rho(x,r))$. Thus (VD) holds on V. Property (SPI) follows from the following inequality (see [29, Theorem 2.1]) and the assumption $f \in L^\infty(\overline{\Omega},\mu)$:

$$\int_{B_\rho(x,r)} (u - u_{B_\rho(x,r)})^2\,dy \le c_1 r^2 \int_{B_\rho(x,r)} |\nabla u|^2\,dy$$

for all $u \in C^\infty(\overline{B}_\rho(x,r))$, where $u_{B_\rho(x,r)} := \int_{B_\rho(x,r)} u\,dy/\mathcal{L}^n(B_\rho(x,r))$. Hence, Theorem 6.2 implies that the heat kernel $p(t,x,y)$ of $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ exists; moreover, for every $\epsilon > 0$, there exists a constant $c_4 := c_4(\epsilon, V)$ such that the following estimate holds for all $x, y \in V$ and $t > 0$:

$$\begin{aligned} p(t,x,y) \le c_4 \mu\big(B_\rho(x,\sqrt{\tau_V})\big)^{-1/2} \cdot \mu\big(B_\rho(y,\sqrt{\tau_V})\big)^{-1/2} \\ \cdot \exp\Big(-\frac{\rho(x,y)^2}{(4+\epsilon)t}\Big), \end{aligned} \tag{6.10}$$

where $\tau_V := \inf\{t, R^2\}$ with $R := \inf\{\rho(x,\overline{\Omega}\setminus V), \rho(y,\overline{\Omega}\setminus V)\}$ ($R := +\infty$ if $\overline{\Omega} = V$). It is easy to see that $B_\rho(x,\sqrt{\tau_V}) \subseteq V$ and $B_\rho(y,\sqrt{\tau_V}) \subseteq V$. Using (6.9), there exists $c_5 := c_5(V)$ such that

$$\mu\big(B_\rho(x, \sqrt{\tau_V})\big)^{-1/2} \cdot \mu\big(B_\rho(y, \sqrt{\tau_V})\big)^{-1/2} \le c_5 \tau_V^{-n/2}.$$

Together with (6.10), this implies there exists $c_6 := c_6(\epsilon, V) > 0$ such that

$$p(t, x, y) \le c_6 \tau_V^{-n/2} \exp\Big(-\frac{\rho(x, y)^2}{(4+\epsilon)t}\Big) \qquad \text{for all } x, y \in V \text{ and all } t > 0.$$

Choose a set U such that $V \subset\subset U \subset\subset \Omega$. We observe that there exists $c_7 := c_7(\epsilon, V) > 0$ such that

$$p(t, x, y) \le c_7 \tau_U^{-n/2} \exp\Big(-\frac{\rho(x, y)^2}{(4+\epsilon)t}\Big) \qquad \text{for all } x, y \in U \text{ and all } t > 0.$$

Let $\delta := \rho^2(V, \overline{\Omega} \setminus U) > 0$. Then $\tau_U = t$ for $0 < t \le \delta$ and $x, y \in V$; $\tau_U \ge \delta$ for $t > \delta$ and $x, y \in V$. Thus for $x, y \in V$ and $0 < t \le \delta$,

$$p(t, x, y) \le c_7 t^{-n/2} \exp\Big(-\frac{\rho(x, y)^2}{(4+\epsilon)t}\Big),$$

while for $x, y \in V$ and $t > \delta$,

$$p(t, x, y) \le c_7 \delta^{-n/2} \exp\Big(-\frac{\rho(x, y)^2}{(4+\epsilon)t}\Big).$$

By Theorem 5.6 and Proposition 2.1, $(\overline{\Omega}, \rho, \mu, -\Delta_\mu)$ has (BPS) and thus $(\overline{\Omega}, d_{|\cdot|}, \mu, -\Delta_\mu)$ has (FPS). □

7 Self-similar Measures with Overlaps

In this section, we apply Theorem 1.6 to self-similar measures on $\mathbb{R}$. Let μ be the self-similar measure defined by the IFS of contractive similitudes of the form

$$S_i(x) = \rho_i x + b_i, \qquad i = 0, 1, \dots, N, \tag{7.1}$$

and probability vector $\{p_i\}_{i=0}^N$, where for each i, $0 < \rho_i < 1$ and $b_i \in \mathbb{R}$. Let K be the associated attractor (or self-similar set). Assume μ is absolutely continuous with respect to Lebesgue measure with density $d\mu/dx =: f(x)$. Thus μ is equivalent to Lebesgue measure on K (see, e.g., Peres et al. [49]). Moreover, $f(x) = \lim_{r\to 0} \mu(B(x, r))/(2r)$ for Lebesgue a.e. $x \in \mathbb{R}$. Hence,

$$f(x) = \sum_{i=0}^N \frac{p_i}{\rho_i} f \circ S_i^{-1}(x) \qquad \text{for Lebesgue a.e. } x \in \mathbb{R}. \tag{7.2}$$

It follows that $f = d\mu/dx$ if and only if f satisfies (7.2) and $\int_K f\,dx = 1$.

7.1 A Family of Scaling Functions

In this subsection, we consider the self-similar measure μ defined by the IFS $\{S_i\}_{i=0}^N$ in (1.13) and probability weights $p_0 = \cdots = p_N = 1/(N+1)$. It is known (see, e.g., [41,47]) that if N is odd, then μ is absolutely continuous with respect to Lebesgue measure with density $f \in L^2(\mathbb{R})$.

Define $\widetilde{f}$ by

$$\widetilde{f}(x) := \begin{cases} x^{\log_2(N+1)-1}, & x \in [0,1], \\ 0, & x \in (-\infty, 0), \end{cases} \tag{7.3}$$

and

$$\widetilde{f}(x) = \frac{N+1}{2}\widetilde{f}\Big(\frac{x}{2}\Big) - \sum_{i=1}^{N} \widetilde{f}(x-i) \quad \text{for all } x \in \mathbb{R}. \tag{7.4}$$

The following proposition shows that the definitions in (7.3) and (7.4) are compatible and $\widetilde{f}$ is well defined.

Proposition 7.1 *Let μ be the self-similar measure defined by the IFS $\{S_i\}_{i=0}^N$ in (1.13) and probability weights $p_0 = \cdots = p_N = 1/(N+1)$. Assume $N \geq 3$ is an odd integer, and let $f := d\mu/dx$ be the density of μ. Then*

(a) *$\widetilde{f}$ is well defined and $f = c\widetilde{f}(x)$, where $c^{-1} := \int_0^N \widetilde{f}\,dx$.*
(b) *f is continuous, bounded on $\mathbb{R}$, and positive on $(0, N)$.*

Proof (a) We first notice that for $x \in (-\infty, 1]$, $\widetilde{f}(x)$ defined by (7.3) satisfies (7.4). In fact,

$$\widetilde{f}(x) = 0 = \frac{N+1}{2}\widetilde{f}\Big(\frac{x}{2}\Big) - \sum_{i=1}^{N}\widetilde{f}(x-i) \quad \text{for any } x \in (-\infty, 0)$$

and for any $x \in [0, 1]$,

$$\widetilde{f}(x) = x^{\log_2(N+1)-1} = \frac{N+1}{2}\Big(\frac{x}{2}\Big)^{\log_2(N+1)-1} = \frac{N+1}{2}\widetilde{f}\Big(\frac{x}{2}\Big) - \sum_{i=1}^{N}\widetilde{f}(x-i).$$

Next, we show that for $x \in (1, +\infty)$, the value $\widetilde{f}(x)$ is uniquely defined. For $x \in (1, 2]$, we have $x/2 \in (0, 1]$ and $x - i \in (-\infty, 1]$ for any $i = 1, \ldots, N$. Combining (7.3) and (7.4), we see for $x \in (1, 2]$, $\widetilde{f}(x)$ is uniquely defined as

$$\widetilde{f}(x) = \frac{N+1}{2}\widetilde{f}\Big(\frac{x}{2}\Big) - \sum_{i=1}^{N}\widetilde{f}(x-i) = x^{\log_2(N+1)-1} - (x-1)^{\log_2(N+1)-1}.$$

By induction, for any $x \in (1, +\infty)$, the value $\widetilde{f}(x)$ is uniquely defined, proving that $\widetilde{f}$ is well defined. To complete the proof of (a), notice that by (7.4),

$$\widetilde{f}(x) = \frac{2}{N+1} \sum_{i=0}^{N} \widetilde{f}(2x - i) \tag{7.5}$$

for any $x \in \mathbb{R}$, i.e., $\widetilde{f}(x)$ satisfies (7.2). Thus $f = c\widetilde{f}(x)$, where $c^{-1} := \int_0^N \widetilde{f}\,dx$.

(b) By (a), it suffices to show that $\widetilde{f}$ is continuous, bounded on $\mathbb{R}$, and positive on $(0, N)$. Clearly, $\widetilde{f}$ is continuous on $(-\infty, 1)$. Next, assume that $\widetilde{f}$ is continuous on some interval of the form $[0, q]$ with $q \geq 1/2$. Since $x/2, x - i \in (-\infty, q]$ for all $x \in [q, q + 1/2]$ and $i = 1, \ldots, N$, (7.4) implies the continuity of $\widetilde{f}$ on $[q, q + 1/2]$. By induction, $\widetilde{f}$ is continuous on $\mathbb{R}$. Since $\widetilde{f}$ is symmetric about $x = N/2$, $\widetilde{f}(x) = 0$ for all $x \in (-\infty, 0] \cup [N, +\infty)$. It follows that $\widetilde{f}$ is bounded on $\mathbb{R}$. As $\widetilde{f} = (1/c)f$, we also conclude that $\widetilde{f}(x) \geq 0$ for all $x \in \mathbb{R}$.

Finally, to show that $\widetilde{f}$ is positive on $(0, N)$, we first observe from definition that $\widetilde{f}(x) > 0$ for all $x \in (0, 1]$. Since $\widetilde{f}$ is symmetric about $x = N/2$, $\widetilde{f}(x) > 0$ for all $x \in [N-1, N)$. For $x \in (1, N-1)$, let $j \in \{0, 1, \ldots, N\}$ such that $2x - j \in (0, 1] \cup [N-1, N)$. Since $\widetilde{f}(x) \geq 0$ for all $x \in \mathbb{R}$, (7.5) implies that $\widetilde{f}(x) \geq 2/(N+1)\,\widetilde{f}(2x - j) > 0$, which completes the proof. □

When $N = 3$, we can derive an explicit formula for the density f (Fig. 1).

Corollary 7.2 *For the case $N = 3$ in Proposition 7.1, the density of μ is*

$$f(x) = \begin{cases} x/2, & 0 \leq x < 1, \\ 1/2, & 1 \leq x \leq 2, \\ (3-x)/2, & 2 < x \leq 3, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof By Proposition 7.1 and the symmetry of $\widetilde{f}$ about $x = 3/2$, $\widetilde{f}$ is defined by $\widetilde{f}(x) = 2\widetilde{f}(x/2) - \sum_{i=1}^{3} \widetilde{f}(x - i)$ for any $x \in \mathbb{R}$ and

$$\widetilde{f}(x) = \begin{cases} x, & x \in [0, 1], \\ 3 - x, & x \in [2, 3], \\ 0, & x \in (-\infty, 0) \cup (3, \infty). \end{cases}$$

For any $x \in (1, 2)$, $\widetilde{f}(x) = 2\widetilde{f}(x/2) - \widetilde{f}(x-1) - \widetilde{f}(x-2) - \widetilde{f}(x-3) = 1$. Thus $c = 2$, which completes the proof. □

7.2 Infinite Bernoulli Convolutions

In this subsection we study the infinite Bernoulli convolutions μ defined by the IFS in (1.14). It is known that for $0 < r < 1/2$, μ is a Cantor-type measure with Hausdorff

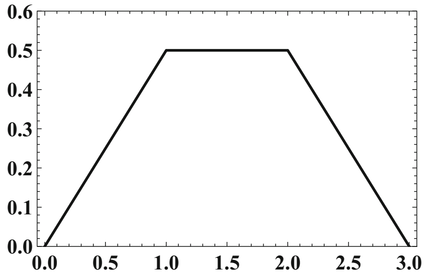


Fig. 1 Density of the measure μ in Corollary 7.2

dimension $\ln 2/\ln r$. If $r = 1/2$, μ is the restriction of Lebesgue measure on $[0, 1]$. We are mainly interested in the case $1/2 < r < 1$. Erdős [20] proved that if r^{-1} is a Pisot number, then μ is singular. On the other hand, Wintner [58] proved that μ is absolutely continuous for $r = 2^{-1/k}$, for $k \geq 1$, and Garsia [24] found a family of algebraic integers with the corresponding μ being absolutely continuous. Solomyak [52] proved that for Lebesgue a.e. $r \in (1/2, 1)$, μ is absolutely continuous; in particular, for Lebesgue a.e. $r \in (1/\sqrt{2}, 1)$, μ has bounded density. Feng and Wang [22] constructed a family of non-Pisot type Bernoulli convolutions such that their density functions, if exist, are not in L^2. Mauldin and Simon [43] proved that Bernoulli convolutions are either singular or equivalent to Lebesgue measure. It follows that absolutely continuous Bernoulli convolutions are equivalent to Lebesgue measure.

For $2/3 \leq r < 1$, define $\widetilde{f}$ by the following dilation equation:

$$\widetilde{f}(x) = \begin{cases} x^{-\log_r 2 - 1}, & x \in [0, r^{-1} - 1], \\ 0, & x \in (-\infty, 0), \end{cases} \tag{7.6}$$

and

$$\widetilde{f}(x) = 2r\widetilde{f}(rx) - \widetilde{f}(x + 1 - r^{-1}), \qquad x \in \mathbb{R}. \tag{7.7}$$

Note that the condition $2/3 \leq r < 1$ implies that

$$1 - r < r^{-1} - 1 \leq 1/2 \leq 2 - r^{-1} < r \quad \text{ and } \quad 1 - r \leq r^2 < 2 - r^{-1}. \tag{7.8}$$

The following proposition shows that the definitions in (7.6) and (7.7) are compatible, and $\widetilde{f}$ is well defined. We remark that Jordan et al. [30] showed that for Lebesgue a.e. $r \in (1/\sqrt{2}, 1)$, the density is continuous on $\mathbb{R}$ and positive on $(0, 1)$. Proposition 7.3 below enlarges the interval on which absolutely continuous measures are known to have positive density on (0,1); moreover, it gives an explicit expression for the density on part of the domain.

Proposition 7.3 *Let μ be a self-similar measure defined by an IFS in the family* (1.14)*, together with probability weights $p_0 = p_1 = 1/2$. Assume $r \in [2/3, 1)$ and μ is absolutely continuous with respect to Lebesgue measure with density f. Let $\widetilde{f}$ be defined as in* (7.6) *and* (7.7)*. Then*

(a) $\widetilde{f}$ *is well defined, and* $f(x) = c\widetilde{f}(x)$*, where* $c^{-1} := \int_0^N \widetilde{f}\,dx$*;*
(b) f *is continuous and bounded on* $\mathbb{R}$*, and is positive on* $(0, 1)$.

Proof (a) The proof of part (a) is similar to that of Proposition 7.1(a); we omit the details.

(b) By definition, $\widetilde{f}$ is continuous on $(-\infty, r^{-1} - 1)$. Assume that $\widetilde{f}$ is continuous on $(-\infty, r_0]$, where $0 < r_0 < 1$. Since

$$r_0r^{-1} + 1 - r^{-1} - r_0 = r_0(r^{-1} - 1) + (1 - r^{-1}) = (r_0 - 1)(r^{-1} - 1) < 0,$$

we have $r_0r^{-1} + 1 - r^{-1} < r_0$. Thus, for any $x \in [r_0, r_0r^{-1}]$,

$$\begin{aligned} rx \in [r_0r, r_0] \subset (0, r_0] \quad \text{and} \quad x + 1 - r^{-1} &\in [r_0 + 1 - r^{-1}, r_0r^{-1} + 1 - r^{-1}] \\ &\subseteq (-\infty, r_0]. \end{aligned}$$

(7.7) implies that $\widetilde{f}$ is continuous on $[r_0, r_0r^{-1}]$. By induction, $\widetilde{f}$ is continuous on $\mathbb{R}$. Since $\widetilde{f}$ is symmetric about $x = 1/2$, $\widetilde{f}(x) = 0$ for all $x \in (-\infty, 0] \cup [1, +\infty)$. It follows that $\widetilde{f}$ is bounded on $\mathbb{R}$.

To show the positivity of $\widetilde{f}$ on $(0, 1)$, we first notice that by definition, $\widetilde{f}(x) > 0$ for any $x \in (0, r^{-1} - 1]$. Since $\widetilde{f}$ is symmetric about $x = 1/2$, $\widetilde{f}(x) > 0$ for all $x \in [2 - r^{-1}, 1)$. Fix $x \in (r^{-1} - 1, 2 - r^{-1})$. (7.8) implies that there exists some $m \in \mathbb{N}$ such that $r^{-m}x \in [2 - r^{-1}, 1)$. Rewrite (7.7) as

$$\widetilde{f}(x) = (2r)^{-1}\widetilde{f}(r^{-1}x) + (2r)^{-1}\widetilde{f}(r^{-1}x + 1 - r^{-1}),$$

and using the fact that $\widetilde{f}$ is non-negative on $\mathbb{R}$, we have $\widetilde{f}(x) \geq (2r)^{-m}\widetilde{f}(r^{-m}x) > 0$, which completes the proof. □

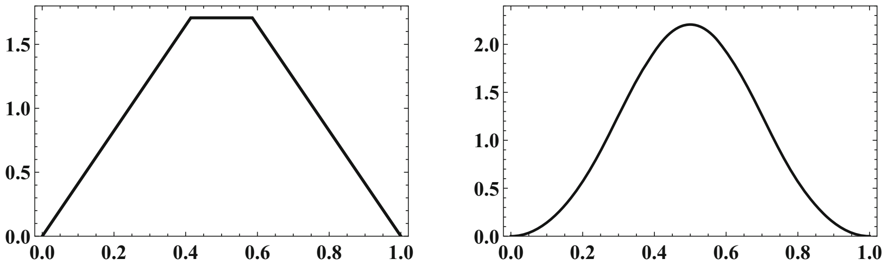
For $r = 2^{-1/k} \in [2/3, 1)$, $k = 2, 3, \ldots$, μ is absolutely continuous with respect to Lebesgue measure [58]. In [55], a numerical method is described to compute the density of μ with $r = 1/\sqrt{2}$. Here we give explicit formulas for f when $r = 1/\sqrt{2}$ and $r = 1/\sqrt[3]{2}$ (see Fig. 2).

Example 7.4 For the case $r = 1/\sqrt{2}$ in Proposition 7.3,

$$f(x) = \begin{cases} (3/\sqrt{2} + 2)x, & 0 \leq x < \sqrt{2} - 1, \\ 1 + 1/\sqrt{2}, & \sqrt{2} - 1 \leq x \leq 2 - \sqrt{2}, \\ -(3/\sqrt{2} + 2)(x - 1), & 2 - \sqrt{2} < x \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof By Proposition 7.3 and symmetry, $\widetilde{f}$ is defined by

$$\widetilde{f}(x) = \sqrt{2}\widetilde{f}(x/\sqrt{2}) - \widetilde{f}(x + 1 - \sqrt{2});$$



(a) $n = 2$ **(b)** $n = 3$

Fig. 2 Density of the Bernoulli convolution with $r = 1/\sqrt[n]{2}$, $n = 2, 3$

moreover,

$$\widetilde{f}(x) = \begin{cases} x, & x \in [0, \sqrt{2} - 1], \\ 1 - x, & x \in [2 - \sqrt{2}, 1], \\ 0, & x \in (-\infty, 0) \cup (1, +\infty). \end{cases}$$

Hence for any $x \in (\sqrt{2} - 1, 2 - \sqrt{2})$,

$$\widetilde{f}(x) = \sqrt{2}\widetilde{f}(x/\sqrt{2}) - \widetilde{f}(x + 1 - \sqrt{2}) = \sqrt{2}(x/\sqrt{2}) - (x + 1 - \sqrt{2}) = \sqrt{2} - 1.$$

Thus $c = (\int_0^1 \widetilde{f}\, dx)^{-1} = 3/\sqrt{2} + 2$, which gives the formula for f. □

To state the next example, we introduce the following abbreviations: Let

$$a := r^{-1}, \qquad \alpha_i := a^{i-1}(a - 1), \quad i = 1, 2, 3, 4.$$

Example 7.5 For the case $r = 1/\sqrt[3]{2}$ in Proposition 7.3, $f = c\widetilde{f}$, where $\widetilde{f}$ is given by

$$\widetilde{f}(x) = \begin{cases} x^2, & 0 \le x < \alpha_1, \\ 2(a-1)x - (a-1)^2, & \alpha_1 \le x < \alpha_2, \\ -x^2 + 2(a^2 - 1)x - (a-1)^2(a^2+1), & \alpha_2 \le x < \alpha_3, \\ -2x^2 + 2x - (a+1)^2(a-1)^2, & \alpha_3 \le x < 1 - \alpha_3, \\ -x^2 + 2(2 - a^2)x, & 1 - \alpha_3 \le x < 1 - \alpha_2, \\ -2(a-1)x - (a-1)(a-3), & 1 - \alpha_2 \le x < 1 - \alpha_1, \\ (1-x)^2, & 1 - \alpha_1 \le x \le 1, \\ 0, & \text{otherwise}. \end{cases} \tag{7.9}$$

and $c^{-1} = \int_0^1 \widetilde{f}\, dx$.

Proof By Proposition 7.3 and symmetry, $\widetilde{f}(x)$ is defined by

$$\widetilde{f}(x) = 2a^{-1}\widetilde{f}(x/a) - \widetilde{f}(x + 1 - a); \tag{7.10}$$

moreover,

$$\widetilde{f}(x) = \begin{cases} x^2, & x \in [0, \alpha_1], \\ (1-x)^2, & x \in [1-\alpha_1, 1], \\ 0, & x \in (-\infty, 0) \cup (1, +\infty). \end{cases}$$

For $x \in [\alpha_1, \alpha_2)$, $x/a, x+1-a \in [0, \alpha_1]$ and thus by (7.10),

$$\widetilde{f}(x) = 2(a-1)x - (a-1)^2.$$

Similarly, for $2 \leq i \leq 3$ and $x \in [\alpha_i, \alpha_{i+1})$, we notice that $x/a \in [\alpha_{i-1}, \alpha_i)$ and $x+1-a \in [0, \alpha_1]$ and thus (7.10) implies

$$\widetilde{f}(x) = \begin{cases} -x^2 + 2(a^2-1)x - (a-1)^2(a^2+1), & x \in [\alpha_2, \alpha_3), \\ -2x^2 + 2x - (a+1)^2(a-1)^2, & x \in [\alpha_3, \alpha_4). \end{cases}$$

Since $1/2 < \alpha_4 < 1 - \alpha_3$, for any $x \in [\alpha_4, 1-\alpha_3)$, $1 - x \in (\alpha_3, 1-\alpha_4] \subseteq [\alpha_3, \alpha_4)$ and thus $\widetilde{f}(x) = \widetilde{f}(1-x) = -2x^2 + 2x - (a+1)^2(a-1)^2$. Hence, (7.9) holds, by using the symmetry of $\widetilde{f}$. □

It is known (see [30]) that for $r \in (1/2, (\sqrt{5}-1)/2)$, there exist infinitely many $x \in (0, 1)$ such that $\lim_{r \to 0^+} \mu(B(x, r))/(2r) = 0$. If the corresponding measure is absolutely continuous with a continuous density, the following remark finds an explicit family of zeros of the density.

Remark 7.6 Let μ be a self-similar measure defined by an IFS in (1.14) together with probability weights $p_0 = p_1 = 1/2$. Assume that μ is absolutely continuous with respect to Lebesgue measure with continuous density f and $r \in (1/2, (\sqrt{5}-1)/2)$. Then $f(r^m/(r+1)) = 0$ for $m \geq 0$.

Proof For all $x \in [0, 1]$, $f(x) = (2r)^{-1} f(r^{-1}x) + (2r)^{-1} f(r^{-1}x + 1 - r^{-1})$. Since $S_2^{-1}(x) = r^{-1}x + 1 - r^{-1} \leq 0$ for $x \in [0, 1-r]$, $f(x) = (2r)^{-1} f(r^{-1}x)$ and thus

$$f(x) = 2rf(rx) \qquad \text{for all } x \in [0, r^{-1} - 1]. \tag{7.11}$$

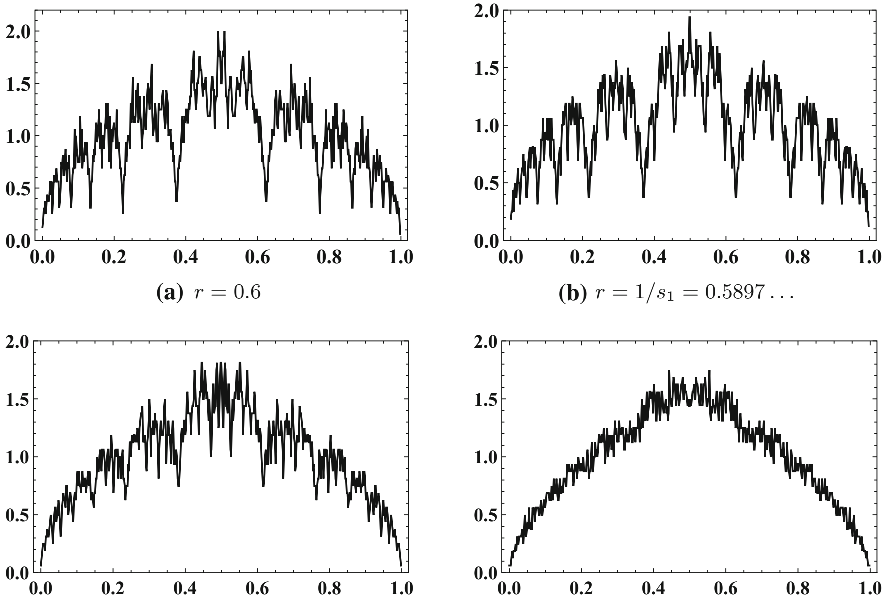
The inequality

$$1/(r+1) - (r^{-1} - 1) = (r^2 + r - 1)/(r(r+1)) < 0, \quad r \in (1/2, (\sqrt{5}-1)/2),$$

implies $1/(1+r) \in [0, r^{-1} - 1]$. Since f is symmetric about $x = 1/2$, (7.11) implies that

$$f\big(1/(1+r)\big) = 2rf\big(r/(1+r)\big) = 2rf\big(1 - r/(1+r)\big) = 2rf\big(1/(1+r)\big).$$

It follows that $f(1/(1+r)) = 0$ and hence $f(r^m/(1+r)) = 0$ for $m \geq 0$. □



(a) $r = 0.6$ **(b)** $r = 1/s_1 = 0.5897\ldots$ **(c)** $r = 0.62$ **(d)** $r = 1/s_2 = 0.6477\ldots$

Fig. 3 Numerical approximations to the densities of some infinite Bernoulli convolutions with contraction ratio r

Figure 3 shows numerical approximations to the densities f for two numbers r in the interval $(1/2, (\sqrt{5}-1)/2)$ and two in the interval $((\sqrt{5}-1)/2, 2/3)$. The numbers s_1 and s_2 are the solutions in $(1, 2)$ of the equations $x^3 - x^2 - 2 = 0$ and $x^3 - 2x^2 + 2x - 2 = 0$ respectively; the corresponding measures are shown by Garsia [24] to be absolutely continuous. It is unknown whether the measures in (a) and (c) are absolutely continuous or singular. According to Proposition 7.3, the density function in (d) is positive on $(0, 1)$, while according to [30] (see also Remark 7.6), the one in (b) has countably infinitely many zeros in $(0, 1)$.

Proof of Theorem 1.7 Again, (MPI) holds since μ is supported on $\mathbb{R}$.

(a) Since N is odd, μ is equivalent to Lebesgue measure on $[0, N]$. By Proposition 7.1, f is continuous and bounded on $[0, N]$ and $f(x) > 0$ on $(0, N)$. Theorem 1.6 now implies $-\Delta_\mu$ satisfies (FPS).

(b) Similar to that of (a). Use Proposition 7.3 instead. □

8 Comments and Open Questions

It is of interest to determine wave propagation speed for other rationally ramified fractals (see, [35, Definition 1.5.10]) and other non-p.c.f. fractals such as the diamond fractal (see [38]). The condition rationally ramified is by definition weaker than the condition finitely ramified. A finitely ramified fractal can be disconnected by removing a finite number of points. For example, the Sierpinski carpet is rationally ramified, but not finitely ramified, and the Sierpinski gasket is not only rationally ramified fractal, but also finitely ramified.

We do not know whether the condition $f \geq \varepsilon(V) > 0$ in Theorem 1.6 can be improved. In view of Propositions 7.3 and 7.6, it is of interest to know whether the density function of those absolutely continuous infinite Bernoulli convolutions with $r \in ((\sqrt{5}-1)/2, 2/3)$ has a zero in $(0, 1)$, and whether (FPS) or (IPS) holds.

Our result on finite propagation speed can also be applied to the Sierpiński gasket equipped with the Kusuoka measure and the so-called harmonic geodesic metric, as two-sided Gaussian heat kernel estimates have been obtained by Kigami [34].

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Appendix: Proof of Lemma 2.3

Proof of Lemma 2.3 Since $T_t f := \exp(-tA)f$ is the unique solution of the heat equation (1.2) with initial data $f \in \mathcal{H}$, it suffices to show that $v(t) = T_t f$. For any $t > 0$ and $f \in \mathcal{H}$,

$$\begin{aligned} \frac{1}{\sqrt{\pi t}} \int_0^\infty \int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big) \cos(s\sqrt{\lambda})\, ds\, dE_\lambda f &= \int_0^\infty \exp(-\lambda t)\, dE_\lambda f \\ &= \exp(-tA) f = T_t f, \end{aligned} \tag{A.1}$$

where $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ is the spectral family associated with A. By using a result concerning Bochner's integral (see, e.g., [59, Section V.5, Corollary 2]) and Fubini's Theorem, we obtain, for any $t > 0$ and $w \in \mathcal{H}$,

$$\begin{aligned} \big(v(t), w\big) &= \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big) \big(\cos(s\sqrt{A}) f, w\big)\, ds && \text{(Bochner)} \\ &= \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big) \int_0^\infty \cos(s\sqrt{\lambda})\, d(E_\lambda f, w)\, ds \\ &= \frac{1}{\sqrt{\pi t}} \int_0^\infty \int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big) \cos(s\sqrt{\lambda})\, ds\, d(E_\lambda f, w) && \text{(Fubini)} \\ &= \Big(\frac{1}{\sqrt{\pi t}} \int_0^\infty \int_0^\infty \exp\Big(-\frac{s^2}{4t}\Big) \cos(s\sqrt{\lambda})\, ds\, dE_\lambda f, w\Big) \\ &= \big(T_t f, w\big), && \text{(by (A.1))} \end{aligned}$$

which completes the proof. □

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