Journal of Fourier Analysis and Applications (2019) 25:1958–1986
https://doi.org/10.1007/s00041-019-09662-4

Paley–Wiener Theorems on the Siegel Upper Half-Space

Nicola Arcozzi[1] · **Alessandro Monguzzi**[2] · **Marco M. Peloso**[2] · **Maura Salvatori**[2]

Received: 27 October 2017 / Revised: 24 November 2018 / Published online: 14 February 2019

Abstract
In this paper we study spaces of holomorphic functions on the Siegel upper half-space $\mathcal{U}$ and prove Paley–Wiener type theorems for such spaces. The boundary of $\mathcal{U}$ can be identified with the Heisenberg group $\mathbb{H}_n$. Using the group Fourier transform on $\mathbb{H}_n$, Ogden and Vagi (Adv Math 33(1):31–92, 1979) proved a Paley–Wiener theorem for the Hardy space $H^2(\mathcal{U})$. We consider a scale of Hilbert spaces on $\mathcal{U}$ that includes the Hardy space, the weighted Bergman spaces, the weighted Dirichlet spaces, and in particular the Drury–Arveson space, and the Dirichlet space $\mathcal{D}$. For each of these spaces, we prove a Paley–Wiener theorem, some structure theorems, and provide some applications. In particular we prove that the norm of the Dirichlet space modulo constants $\dot{\mathcal{D}}$ is the unique Hilbert space norm that is invariant under the action of the group of automorphisms of $\mathcal{U}$.

Keywords Siegel upper half-space · Holomorphic function spaces · Reproducing kernel Hilbert space · Drury–Arveson · Dirichlet · Hardy · Bergman spaces

Mathematics Subject Classification 30H99 · 46E22 · 30C15 · 30C40

1 Introduction and Statement of the Main Results

Let $\mathbb{C}_+$ be the upper half-plane $\{z = x + iy \in \mathbb{C} : \ y > 0\}$. Let $H^2(\mathbb{C}_+)$ denote the Hardy space, that is the space of holomorphic functions in $\mathbb{C}_+$ such that

---

Dedicated to Fulvio Ricci, for his teaching, guidance and friendship.

Communicated by Yura Lyubarskii.

Authors are partially supported by the 2015 PRIN grant *Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis* of the Italian Ministry of Education (MIUR). N. Arcozzi is partially supported by the grants INDAM-GNAMPA 2017 “Operatori e disuguaglianze integrali in spazi con simmetrie”. All authors are members of INDAM.

✉ Marco M. Peloso
marco.peloso@unimi.it

Extended author information available on the last page of the article

$$\|f\|^2_{H^2(\mathbb{C}_+)} := \sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^2\, dx < +\infty.$$

The classical Paley–Wiener theorem [17] says that, given $f \in H^2(\mathbb{C}_+)$ there exists $g \in L^2(0,+\infty)$ such that

$$f(z) = \frac{1}{2\pi} \int_0^{+\infty} e^{iz\xi} g(\xi)\, d\xi\,, \tag{1}$$

and

$$\|f\|^2_{H^2(\mathbb{C}_+)} = \frac{1}{2\pi} \|g\|^2_{L^2(0,+\infty)}. \tag{2}$$

Conversely, given any $g \in L^2(0,+\infty)$, defining f as in (1), we have that $f \in H^2(\mathbb{C}_+)$ and (2) holds. Since then, the Fourier transform on the real line has appeared as a fundamental tool in modern complex analysis and in the theory of holomorphic function spaces in $\mathbb{C}_+$. We mention, for instance, the regularity of projection operators, the boundary behavior and growth conditions of holomorphic functions, the boundedness and compactness of Hankel and Toeplitz operators, just to name some of the most important, see e.g. [19,23]. The Paley–Wiener theorem has been extended to other Hilbert function spaces on $\mathbb{C}_+$: the weighted Bergman spaces (see e.g. [6,8,11]), and more recently to the Dirichlet space [13,14].

The domain $\mathbb{C}_+$ is biholomorphic equivalent to the unit disk in the plane. In principle, it is possible to transfer analogous results from the unit disk to $\mathbb{C}_+$. However, often, it is far more natural to study a problem directly on the unbounded domain $\mathbb{C}_+$.

In this paper we wish to extend the approach described above to $\mathbb{C}^{n+1}$, where we always assume that $n \geq 1$.

The Siegel upper-half space is the domain in $\mathbb{C}^{n+1}$

$$\mathcal{U} = \left\{ \zeta = (\zeta', \zeta_{n+1}) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im} \zeta_{n+1} > \tfrac{1}{4}|\zeta'|^2 \right\}\,,$$

and we denote by $\varrho(\zeta) = \operatorname{Im}\zeta_{n+1} - \frac{1}{4}|\zeta'|^2$ its defining function. The domain $\mathcal{U}$ is biholomorphic to the unit ball B in $\mathbb{C}^{n+1}$ via the (multi-dimensional) Cayley transform $\mathcal{C} : B \to \mathcal{U}$,

$$\mathcal{C}(\omega) = \left(\frac{2\omega'}{1-\omega_{n+1}}, i\frac{1+\omega_{n+1}}{1-\omega_{n+1}} \right).$$

The boundary $\partial\mathcal{U}$ of $\mathcal{U}$ can be endowed with the structure of a nilpotent Lie group, namely the Heisenberg group $\mathbb{H}_n$. Thus, it is possible to use the group Fourier transform on $\partial\mathcal{U}$ to characterize the boundary values of holomorphic functions on $\mathcal{U}$ in various Hilbert function spaces, and therefore to take a first step in the program outlined in the case of $\mathbb{C}_+$. In particular we prove Paley–Wiener type theorems for functions in weighted Bergman and Dirichlet spaces, and on the Dirichlet space. In fact, we show that the latter one is the unique Hilbert space modulo constants that is invariant under the group of automorphisms of $\mathcal{U}$. We now describe the content of the present paper in greater details.

The boundary $\partial\mathcal{U}$ is characterized by the points of $\mathbb{C}^{n+1}$ such that $\varrho(\zeta) = 0$, that is, $\zeta = (\zeta', t + \frac{i}{4}|\zeta'|^2)$, with $t \in \mathbb{R}$. We introduce a parametrization of $\mathcal{U}$ by means of a foliation of copies of the boundary. We set $\mathbf{U} = \mathbb{C}^n \times \mathbb{R} \times (0, +\infty)$. Given $\zeta = (\zeta', \zeta_{n+1}) \in \mathcal{U}$, we define $\Psi(\zeta', \zeta_{n+1}) = (z, t, h) \in \mathbf{U}$ by

$$\begin{cases} z = \zeta' \\ t = \operatorname{Re} \zeta_{n+1} \\ h = \operatorname{Im} \zeta_{n+1} - \frac{1}{4}|\zeta'|^2. \end{cases} \tag{3}$$

Then, $\Psi : \overline{\mathcal{U}} \to \overline{\mathbf{U}}$ is a C^∞-diffeomorphism, and Ψ^{-1} is given by

$$\Psi^{-1}(z, t, h) = \big(z, t + i\tfrac{1}{4}|z|^2 + ih\big) =: (\zeta', \zeta_{n+1}). \tag{4}$$

Notice that $h = \varrho(\zeta', \zeta_{n+1})$. When $h = 0$, we write $[z, t]$ in place of $(z, t, 0)$. The points on the boundary act on $\overline{\mathcal{U}}$ as biholomorphic maps in the following way. For $[z, t] \in \partial\mathcal{U}$, we define

$$\Phi_{[z,t]}(\omega', \omega_{n+1}) = \big(\omega' + z, \omega_{n+1} + t + i\tfrac{1}{4}|z|^2 + \tfrac{i}{2}\omega' \cdot \bar{z}\big), \tag{5}$$

where $\omega' \cdot \bar{z} = \sum_{j=1}^n \omega_j \bar{z}_j$ denotes the hermitian inner product in $\mathbb{C}^n$. Notice that

$$\varrho\Big(\Phi_{[z,t]}(\omega', \omega_{n+1})\Big) = \varrho(\omega', \omega_{n+1}),$$

that is, the maps $\Phi_{[z,t]}$ preserve the defining function ϱ. In particular, for $(\omega', \omega_{n+1}) \in \partial\mathcal{U}$ and $[w, s] = \Psi(\omega', \omega_{n+1})$, by (5) we have

$$\begin{aligned} \Phi_{[z,t]}\big((\omega', \omega_{n+1})\big) &= \Phi_{[z,t]}\Big(\Psi^{-1}(w, s, 0)\Big) = \Phi_{[z,t]}\big(w, s + i\tfrac{1}{4}|w|^2\big) \\ &= \big(w + z, s + \tfrac{i}{4}|w|^2 + t + \tfrac{i}{4}|z|^2 + \tfrac{i}{2} w \cdot \bar{z}\big) \\ &= \big[w + z, s + t - \tfrac{1}{2}\operatorname{Im}(w \cdot \bar{z})\big] \\ &=: [w, s][z, t]. \end{aligned} \tag{6}$$

Therefore, it is possible to introduce a group structure on $\partial\mathcal{U}$ itself.

Definition The Heisenberg group $\mathbb{H}_n$ is the set $\mathbb{C}^n \times \mathbb{R}$ endowed with product

$$[w, s][z, t] = \big[w + z, s + t - \tfrac{1}{2}\operatorname{Im}(w \cdot \bar{z})\big].$$

The Heisenberg group $\mathbb{H}_n$ is a nilpotent Lie group of step 2, and the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ coincides with both the right and left Haar measure on $\mathbb{H}_n$. In other words, the Lebesgue measure is both right and left translation invariant.

If x is a vector of the Euclidean space $\mathbb{R}^d$, we denote by dx the Lebesgue measure in $\mathbb{R}^d$. Notice that, since $|\det \operatorname{Jac}\Psi| = 1$, for F integrable on $\mathcal{U}$, setting $\widetilde{F} = F \circ \Psi^{-1}$ and $\widetilde{F}_h[z, t] := \widetilde{F}(z, t, h)$, we have

$$\int_{\mathcal{U}} F(\zeta)\, d\zeta = \int_{\mathbf{U}} \widetilde{F}(z, t, h)\, dz dt dh = \int_0^{+\infty} \int_{\mathbb{H}_n} \widetilde{F}_h[z, t]\, dz dt dh.$$

We now introduce the Hilbert function spaces object of our study.

Definition For $\nu > -1$, we consider the *weighted Bergman spaces* A^2_ν

$$A^2_\nu = \Big\{ F \in \mathrm{Hol}(\mathcal{U}) : \|F\|^2_{A^2_\nu} := \int_{\mathcal{U}} |F(\zeta)|^2 \rho(\zeta)^\nu \, d\zeta = \int_{\mathbf{U}} |\widetilde{F}(z,t,h)|^2 \, h^\nu \, dz dt dh < +\infty \Big\}.$$

For $-n-2 < \nu < -1$ and m a positive integer such that $2m+\nu > -1$, the *weighted Dirichlet spaces* are defined as follows

$$\mathcal{D}_{\nu,(m)} = \Big\{ F \in \mathrm{Hol}(\mathcal{U}) : \quad \text{(i)} \lim_{|\zeta'|\le R,\, \mathrm{Im}\,\zeta_{n+1}\to+\infty} F(\zeta) = 0\,; \quad \text{(ii)} \int_{\mathcal{U}} |\rho^m(\zeta)\partial^m_{\zeta_{n+1}} F(\zeta)|^2 \, \rho^\nu(\zeta) d\zeta < +\infty \Big\}. \tag{7}$$

For F as above, we define the norm on $\mathcal{D}_{\nu,(m)}$ as

$$\|F\|^2_{\mathcal{D}_{\nu,(m)}} = \int_{\mathcal{U}} |\rho^m(\zeta)\partial^m_{\zeta_{n+1}} F(\zeta)|^2 \, \rho^\nu(\zeta) d\zeta.$$

Finally, for $\nu = -n-2$ and $2m > n+1$, we define the *Dirichlet space* $\mathcal{D}_{(m)}$ as

$$\mathcal{D}_{(m)} = \Big\{ F \in \mathrm{Hol}(\mathcal{U}) : \quad \text{(i)} \lim_{|\zeta'|\le R,\, \mathrm{Im}\,\zeta_{n+1}\to+\infty} \partial_{\zeta_j} F(\zeta) = 0 \text{ for } j = 1,\dots,n+1; \quad \text{(ii)} \int_{\mathcal{U}} |\rho^m(\zeta)\partial^m_{\zeta_{n+1}} F(\zeta)|^2 \, \rho^{-n-2}(\zeta) d\zeta < +\infty \Big\}, \tag{8}$$

with norm given by

$$\|F\|^2_{\mathcal{D}_{(m)}} = \|\partial^m_{\zeta_{n+1}} F\|^2_{A^2_{2m-n-2}} + |F(\mathbf{i})|^2 \tag{9}$$

where $\mathbf{i} = (0', i) \in \mathcal{U}$.

The case $\nu = -1$ corresponds to the classical *Hardy space* H^2, defined as

$$H^2 = \Big\{ F \in \mathrm{Hol}(\mathcal{U}) : \|F\|^2_{H^2} := \sup_{h>0} \int_{\mathbb{H}_n} |\widetilde{F}_h[z,t]|^2 \, dz dt < +\infty \Big\}. \tag{10}$$

We point out that for ν fixed, when $2m+\nu > -1$, the spaces $\mathcal{D}_{\nu,(m)}$ all coincide, with the same norms up to a positive constant multiple (see Theorem 2). Thus, when the choice of the norm is unambiguous, we simply denote them by $\mathcal{D}_\nu$. Analogously, the spaces $\mathcal{D}_{(m)}$ do not depend on the choice of the integer m (see Theorem 3), and we denote them by $\mathcal{D}$. Moreover, we will show that the norm of $\mathcal{D}$ modulo constants is invariant under the automorphism group; hence it is legitimate to call the space $\mathcal{D}$ the *Dirichlet space* on the Siegel upper half-space.

Our main technical tool is the Fourier transform on the Heisenberg group. For this, and other basic facts concerning the Heisenberg group, we refer the reader to [12] and [20].

Let $\lambda \in \mathbb{R} \setminus \{0\}$. We set

$$\mathcal{F}^\lambda = \left\{ F \in \mathrm{Hol}(\mathbb{C}^n) : \left(\frac{|\lambda|}{2\pi}\right)^n \int_{\mathbb{C}^n} |F(z)|^2 e^{-\frac{\lambda}{2}|z|^2} dz < +\infty \right\} \tag{11}$$

when $\lambda > 0$, and $\mathcal{F}^\lambda = \mathcal{F}^{|\lambda|}$ when $\lambda < 0$, and call this space the Fock space. We present further properties of such space in Sect. 2.2.

For $\lambda \in \mathbb{R}\setminus\{0\}$ and $[z, t] \in \mathbb{H}_n$, the Bargmann representation $\sigma_\lambda[z, t]$ is the operator acting on $\mathcal{F}^\lambda$ given by,

$$\sigma_\lambda[z, t]F(w) = e^{i\lambda t - \frac{\lambda}{2} w\cdot\overline{z} - \frac{\lambda}{4}|z|^2} F(w + z) \tag{12}$$

if $\lambda > 0$, and, if $\lambda < 0$, as $\sigma_\lambda[z, t] = \sigma_{-\lambda}[\overline{z}, -t]$, that is,

$$\sigma_\lambda[z, t]F(w) = e^{i\lambda t + \frac{\lambda}{2} w\cdot z + \frac{\lambda}{4}|z|^2} F(w + \bar{z}). \tag{13}$$

If $f \in L^1(\mathbb{H}_n)$, for $\lambda \in \mathbb{R} \setminus \{0\}$, $\sigma_\lambda(f)$ is the operator acting on $\mathcal{F}^\lambda$ as

$$\sigma_\lambda(f)F(w) = \int_{\mathbb{H}_n} f[z, t]\sigma_\lambda[z, t]F(w)\, dzdt.$$

Before stating our main results, we recall a result proved by Ogden and Vagi [16], that extends the classical Paley–Wiener theorem for the Hardy space, from the upper half-plane to the case of $\mathcal{U}$. We point out however, that Ogden and Vagi proved their main result in the case of Siegel domains of type II. It would certainly be of interest to extend our results to the latter more general class of domains.

Theorem [16] Let $F \in H^2$. Then, there exists $\widetilde{F}_0 \in L^2(\mathbb{H}_n)$ such that $\widetilde{F}_h \to \widetilde{F}_0$ in $L^2(\mathbb{H}_n)$, as $h \to 0^+$. Moreover, the function $\widetilde{F}_0$ is such that

(i) $\|F\|_{H^2} = \|\widetilde{F}_0\|_{L^2(\mathbb{H}_n)}$;
(ii) $\sigma_\lambda(\widetilde{F}_0) = 0$ when $\lambda > 0$;
(iii) $\mathrm{ran}\,\big(\sigma_\lambda(\widetilde{F}_0)\big) \subseteq \mathrm{span}\{1\}$ for $\lambda < 0$.

Conversely, if $f \in L^2(\mathbb{H}_n)$ is such that (ii) and (iii) are satisfied, then setting

$$F(\zeta) = \widetilde{F}_h[z, t] = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 e^{h\lambda}\, \mathrm{tr}\, \big(\sigma_\lambda(f)\sigma_\lambda[z, t]^*\big)\, |\lambda|^n d\lambda\,, \tag{14}$$

then $F \in H^2$ is such that $\widetilde{F}_0 = f$ and (i)–(iii) hold.

We also need the following

Definition For $\nu \in \mathbb{R}$ we define the space $\mathcal{L}^2_\nu$ as the space of functions τ on $\mathbb{R} \setminus \{0\}$ such that:

(i) $\tau(\lambda) \in \mathrm{HS}(\mathcal{F}^\lambda)$ for every λ, i.e., $\tau(\lambda) : \mathcal{F}^\lambda \to \mathcal{F}^\lambda$ is a Hilbert–Schmidt operator;
(ii) $\tau(\lambda) = 0$ for $\lambda > 0$;
(iii) $\mathrm{ran}(\tau(\lambda)) \subseteq \mathrm{span}\{1\}$;
(iv) $\|\tau\|^2_{\mathcal{L}^2_\nu} := \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 \|\tau(\lambda)\|^2_{\mathrm{HS}} |\lambda|^{n-(\nu+1)} d\lambda < +\infty$, where $\|\cdot\|_{\mathrm{HS}} := \|\cdot\|_{\mathrm{HS}(\mathcal{F}^\lambda)}$.

(15)

Our first main result is the following Paley–Wiener type theorem for the weighted Bergman spaces A^2_ν.

Theorem 1 Let $\nu > -1$ be fixed. Given $F \in A^2_\nu$, there exists $\tau \in \mathcal{L}^2_\nu$ such that, for $\zeta \in \mathcal{U}$,

$$F(\zeta) = \widetilde{F}_h[z,t] = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 e^{h\lambda} \,\mathrm{tr}\left(\tau(\lambda)\sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda\,, \tag{16}$$

and

$$\|F\|^2_{A^2_\nu} = \frac{\Gamma(\nu+1)}{2^{\nu+1}} \|\tau\|^2_{\mathcal{L}^2_\nu}. \tag{17}$$

Conversely, given $\tau \in \mathcal{L}^2_\nu$, let F be defined as in (16). Then $F \in A^2_\nu$ and (17) holds.

Next we consider the case of weighted Dirichlet spaces.

Theorem 2 Let $-(n+2) < \nu < -1$, and let $m > -\frac{\nu+1}{2}$. Let $F \in \mathcal{D}_{\nu,(m)}$. Then, there exists $\tau \in \mathcal{L}^2_\nu$ such that, for $\zeta \in \mathcal{U}$,

$$F(\zeta) = \widetilde{F}_h[z,t] = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 e^{h\lambda} \,\mathrm{tr}\left(\tau(\lambda)\sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda\,, \tag{18}$$

and

$$\|F\|^2_{\mathcal{D}_{\nu,(m)}} = \frac{\Gamma(2m+\nu+1)}{2^{2m+\nu+1}} \|\tau\|^2_{\mathcal{L}^2_\nu}. \tag{19}$$

Conversely, given $\tau \in \mathcal{L}^2_\nu$, let F be defined as in (18). Then $F \in \mathcal{D}_{\nu,(m)}$ and (19) holds.

Therefore, for each $m > -\frac{\nu+1}{2}$, the spaces $\mathcal{D}_{\nu,(m)}$ all coincide and their norms satisfy (19).

Hence, if no confusion arises, we simply write $\mathcal{D}_\nu$ in place of $\mathcal{D}_{\nu,(m)}$.

In the case $\nu = -n-1$, $\mathcal{D}_\nu$ is called the *Drury–Arveson space* and we denote it by DA. The Drury–Arveson space on the unit ball B has drawn a great deal of interest in the recent years, see [2–4,7,9,22,25,26], and references therein, to name a few. When

$n \geq 1$, DA plays a role similar to the one played by the Hardy space on the unit disk, and for this reason it is sometimes denoted as H^2_{n+1}. If f is holomorphic on B, $f(\zeta) = \sum_{|\alpha|\geq 0} a_\alpha \zeta^\alpha$, the norm in $\mathrm{DA}(B)$ is given by

$$\|f\|^2_{\mathrm{DA}(B)} = \sum_{|\alpha|\geq 0} \frac{\alpha!}{|\alpha|!}|a_\alpha|^2.$$

However, to the best of our knowledge, no integral representation of this norm has been found. In this paper we provide such a description, see Theorem 6.1.

The last main result is the following.

Theorem 3 Let $m > \frac{n+1}{2}$ be fixed. Let $F \in \mathcal{D}_{(m)}$. Then, there exists $\tau \in \mathcal{L}^2_{-n-2}$ such that, for $\zeta \in \mathcal{U}$,

$$F(\zeta) = \widetilde{F}_h[z,t] = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{0} \operatorname{tr}\Big(\tau(\lambda)\big(e^{\lambda h}\sigma_\lambda[z,t]^* - e^{\lambda}\sigma_\lambda(0,0)^*\big)\Big)\,|\lambda|^n d\lambda + c\,, \tag{20}$$

where $c = F(\mathbf{i})$, and

$$\|F\|^2_{\mathcal{D}_{(m)}} = \frac{\Gamma(2m-n-1)}{2^{2m-n-1}}\|\tau\|^2_{\mathcal{L}^2_{-n-2}} + |F(\mathbf{i})|^2. \tag{21}$$

Conversely, given $\tau \in \mathcal{L}^2_{-n-2}$, let F be defined as in (20). Then $F \in \mathcal{D}_{(m)}$, $c = F(\mathbf{i})$ and (21) holds.

Therefore, for each $m > \frac{n+1}{2}$, the spaces $\mathcal{D}_{(m)}$ all coincide and their norms satisfy (21).

Hence, if no confusion arises, we simply write $\mathcal{D}$ in place of $\mathcal{D}_{(m)}$. We shall also denote by $\dot{\mathcal{D}}$ the quotient space $\mathcal{D}/\mathbb{C}$, endowed with any of the norms $\|\partial^m_{\zeta_{n+1}} F\|_{A^2_{2m-n-2}}$. We are going to show that $\dot{\mathcal{D}}$ is the unique Hilbert space of functions modulo constants that is invariant under composition with automorphisms, see Theorem 5.5. We would like to point out that, when $\nu = -n-2$, even given the integrability condition of the derivative of sufficiently high order m, it is not possible to find an anti-derivative of order m that vanishes as $\operatorname{Im}\zeta_{n+1} \to +\infty$. Hence, the decay property in (8) is required on the gradient of the function and not on the function itself. We will comment and make more remarks in Sects. 4 and 5.

Beside their intrinsic interest, there are several reasons to study Paley–Wiener type theorems. All the spaces we are considering are Hilbert spaces, in particular with a reproducing kernel, and it is possible to define the same scale of space with $p \neq 2$. These spaces are classical Besov–Sobolev spaces; for the case of the unit ball, see, e.g., [5,26,29]. The boundary behavior of functions in the weighted Bergman spaces $A^p_\nu(\mathcal{U})$ with $\nu > -1$ was studied by Feldman [10], following the case of the upper half-plane in $\mathbb{C}_+$ obtained by Ricci and Taibleson [21]. Among other results, we provide the explicit expression of the reproducing kernels for all these spaces. These kernel are also the integral kernels for the corresponding orthogonal projections. It would be of interest to study the regularity properties of such projections on the scale of the appropriate homogeneous Sobolev spaces. As potential application of our results, we also mention the theory of invariant subspaces, in the spirit of [15], [8], e.g., that deal with this question in the 1-dimensional setting of the upper half-plane. Furthermore, we point out that the the invariance of the norm of the Dirichlet space under the composition with the automorphisms is much easier to prove in the setting of the Siegel half-plane than in the unit ball—cfr. (2) in Theorem 5.5 and [29, Theorem 6.15].

The paper is organized as follows. Section 2 is a preliminary section where we recall some standard results on the Siegel half-space, the Heisenberg group and the Hardy space on $\mathcal{U}$. In Sects. 3, 4 and 5 the weighted Bergman spaces, the weighted Dirichlet spaces and the Dirichlet space are studied respectively. In Sect. 6 we provide the integral norm of the Drury–Arveson space on the unit ball, and then we conclude with some final remarks and possible future directions of research.

2 Preliminaries

In this part we recall some well-known facts that will be used in what follows.

2.1 More on the Heisenberg Group and the the Siegel Upper Half-Space

The following lemma is well known, see e.g. [27, 7.5.18]; we thank F. Ricci for pointing this reference to us.

Lemma 2.1 The group $\mathrm{Aut}(\mathcal{U})$ of biholomorphic self-maps of $\mathcal{U}$ is given by

$$\mathrm{Aut}(\mathcal{U}) = \bigcup_{\gamma\in\{\mathrm{Id},\upsilon\}} (MAN)\gamma(MAN)\,,$$

where

(i) $N = \left\{\Phi_{[z,t]} : [z,t] \in \mathbb{H}_n\right\}$ (the subgroup of Heisenberg translations);
(ii) $A = \left\{D_\delta : \delta > 0,\ D_\delta(\zeta',\zeta_{n+1}) = (\delta\zeta',\delta^2\zeta_{n+1})\right\}$ (the subgroup of non-isotropic dilations);
(iii) $M = \left\{U \in U(n) : \Phi_U(\zeta',\zeta_{n+1}) = (U\zeta',\zeta_{n+1})\right\}$ (the subgroup of unitary transformations in $\mathbb{C}^n$);
(iv) $\upsilon(\zeta) = \left(\frac{i\zeta'}{\zeta_{n+1}}, -\frac{1}{\zeta_{n+1}}\right)$ (the inversion map).

On $\mathbb{H}_n$ we define a *homogeneous* norm by setting

$$|[z,t]|_{\mathbb{H}_n} := \left(\tfrac{1}{16}|z|^4 + t^2\right)^{1/4}.$$

This norm satisfies the following properties:

- $|[z,t]|_{\mathbb{H}_n} \geq 0$ and it is 0 if and only if $[z,t] = (0,0)$;
- $\big|[z,t][w,s]\big| \leq |[z,t]| + |[w,s]|$;
- $|D_\delta[z,t]|_{\mathbb{H}_n} = \delta|[z,t]|_{\mathbb{H}_n}$.

The topology induced by the metric $d_{\mathbb{H}_n}\big([z,t],[w,s]\big) = |[z,t][w,s]^{-1}|_{\mathbb{H}_n}$ is equivalent to the Euclidean topology of $\mathbb{C}^n \times \mathbb{R}$. We also set

$$B\big([z,t],r\big) = \Big\{[w,s] \in \mathbb{H}_n : |[w,s][z,t]^{-1}|_{\mathbb{H}_n} < r\Big\}.$$

We recall that a holomorphic function F satisfies the mean value property $F(\zeta) = \frac{1}{|Q|}\int_Q F(\omega)\,d\omega$, where $Q = Q(\zeta, R)$ denotes the polydisk $\{\omega : |\omega_j - \zeta_j| < R_j\}$ of polyradius R, contained in the region of holomorphy of F, and $|Q|$ is its Lebesgue measure. We will also consider a metric on $\mathcal{U}$, which is somehow conformally invariant. If $\Psi(\zeta', \zeta_{n+1}) = (z,t,h)$ as in (3), we set

$$\begin{aligned} P((z,t,h),r) &= B\big([z,t],r\big) \times \big\{k : |h-k| < r^2\big\} \quad \text{and} \\ P(\zeta,r) &= \Psi^{-1}\big(P((z,t,h),r)\big). \end{aligned}$$

Then, we have

- $|P(\zeta,r)| = c_n r^{2n+4}$;
- $\Phi_{[w,s]}\big(P(\zeta,r)\big) = P\big(\Phi_{[w,s]}(\zeta),r\big)$;
- $D_\delta\big(P(\zeta,r)\big) = P\big(D_\delta(\zeta), r\delta\big)$.

It is elementary to see that a holomorphic function F satisfies the submean value property

$$|F(\zeta)| \le \frac{C}{|P(\zeta,r)|}\int_{|h-k|<r^2}\int_{B([z,t],r)} |\widetilde{F}(w,s,k)|\,dwdsdk.$$

2.2 The Fock Space and the Fourier Transform on the Heisenberg Group

Recall that the Fock space $\mathcal{F}^\lambda$ is defined in (11) and thus has inner product

$$\langle f, g\rangle_{\mathcal{F}^\lambda} = \left(\frac{|\lambda|}{2\pi}\right)^n \int_{\mathbb{C}^n} f(z)\overline{g(z)}\, e^{-\frac{\lambda}{2}|z|^2} dz.$$

Observe that $\big(\frac{|\lambda|}{2\pi}\big)^n e^{-\frac{|\lambda|}{2}|z|^2} dz$ is a probability measure, and that the normalized monomials $\big\{z^\alpha/\|z^\alpha\|_{\mathcal{F}^\lambda}\big\}$, form a complete orthonormal basis, and

$$\|z^\alpha\|^2_{\mathcal{F}^\lambda} = \alpha!\left(\frac{2}{|\lambda|}\right)^{|\alpha|}.$$

Moreover, $\mathcal{F}^\lambda$ is a reproducing kernel Hilbert space, with reproducing kernel $e^{\frac{|\lambda|}{2}z\bar{w}}$, [12].

Introducing real coordinates on $\mathbb{H}_n$, $z_j = x_j + iy_j$, $j = 1,\dots,n$, then $\mathbb{H}_n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, and a basis for the left-invariant vector fields is $\{X_1,\dots,X_n,Y_1,\dots,Y_n,T\}$, where

$$X_j = \partial_{x_j} - \tfrac{1}{2}y_j\partial_t\,,\ Y_j = \partial_{y_j} + \tfrac{1}{2}x_j\partial_t\,,\ T = \partial_t.$$

A basis for the complexified vector fields is $\{Z_1,\dots,Z_n,\bar Z_1,\dots,\bar Z_n,T\}$, where

$$Z_j=\tfrac12(X_j-iY_j)=\partial_{z_j}-\tfrac{i}{4}\bar z_j\partial_t,\ \bar Z_j=\tfrac12(X_j+iY_j)=\partial_{\bar z_j}+\tfrac{i}{4}z_j\partial_t,\\ j=1,\dots,n\,,\ T\,,$$

with non-trivial commutation rules

$$[Z_j,\bar Z_j]=\tfrac{i}{2}T,\ j=1,\dots,n.$$

We denote by $Z_j^{(R)}$ and $\bar Z_j^{(R)}$, resp., $j=1,\dots,n$, the right-invariant vector fields that coincide with Z_j and $\bar Z_j$, resp., at the origin. It turns out that

$$Z_j^{(R)}=\partial_{z_j}+\tfrac{i}{4}\bar z_j\partial_t,\ \bar Z_j^{(R)}=\partial_{\bar z_j}-\tfrac{i}{4}z_j\partial_t\,,\ j=1,\dots,n.$$

Then, the differentials of the Bargmann representations, that are defined in (12) and (13), can be computed to give, in particular:

(i) for all $\lambda\neq0$, $d\sigma_\lambda(T)=i\lambda$;
(ii) for $\lambda>0$, $d\sigma_\lambda(\overline Z_j^{(R)})=-\frac{\lambda}{2}w_j$;
(iii) for $\lambda<0$, $d\sigma_\lambda(\overline Z_j^{(R)})=\partial_{w_j}$;

see [12]. It is important to recall that, with our choice of normalization of the Fourier transform, if $f,g\in L^1(\mathbb H_n)$, $\sigma_\lambda(f*g)=\sigma_\lambda(f)\sigma_\lambda(g)$, so that

$$\sigma_\lambda(V^{(L)}f)=-\sigma_\lambda(f)d\sigma_\lambda(V^{(L)})\quad\text{and}\quad\sigma_\lambda(V^{(R)}f)=-d\sigma_\lambda(V^{(R)})\sigma_\lambda(f)\,,$$

where $V^{(L)}$ and $V^{(R)}$ denote a left-invariant and a right-invariant vector field, respectively.

If $f\in L^2(\mathbb H_n)$, we have Plancherel's formula

$$\|f\|^2_{L^2(\mathbb H_n)}=\frac{1}{(2\pi)^{n+1}}\int_{\mathbb R}\|\sigma_\lambda(f)\|^2_{\mathrm{HS}}|\lambda|^n\,d\lambda\,,$$

and, if $f\in L^1\cap L^2(\mathbb H_n)$ the inversion formula

$$f[z,t]=\frac{1}{(2\pi)^{n+1}}\int_{\mathbb R}\operatorname{tr}\big(\sigma_\lambda(f)\sigma_\lambda[z,t]^*\big)|\lambda|^n\,d\lambda. \tag{22}$$

2.3 The Cauchy–Riemann Equations and the Hardy Space

We consider now functions that are holomorphic in $\mathcal U$. For $F\in\mathrm{Hol}(\mathcal U)$, recalling (4), we write $\widetilde F=F\circ\Psi^{-1}$, so that

$$F(\zeta',\zeta_{n+1})=\widetilde F\Big(\zeta',\tfrac{\zeta_{n+1}+\overline\zeta_{n+1}}{2};\tfrac{\zeta_{n+1}-\overline\zeta_{n+1}}{2i}-\tfrac14\zeta'\cdot\bar\zeta'\Big).$$

The equation $\partial_{\overline{\zeta}_{n+1}} F = 0$ now reads

$$\begin{aligned} 0 = \partial_{\overline{\zeta}_{n+1}} F(\zeta', \zeta_{n+1}) &= \tfrac{1}{2} \partial_t \widetilde{F}(z,t,h) - \tfrac{1}{2i} \partial_h \widetilde{F}(z,t,h) \\ &= \tfrac{1}{2}\big(\partial_t \widetilde{F} + i \partial_h \widetilde{F}\big)(z,t,h) , \end{aligned}$$

that is,

$$i \partial_t \widetilde{F}_h = \partial_h \widetilde{F}_h . \tag{23}$$

The remaining Cauchy–Riemann equations $\partial_{\overline{\zeta}_j} F = 0$, $j = 1, \dots, n$, respectively, become

$$0 = \partial_{\overline{\zeta}_j} F(\zeta', \zeta_{n+1}) = \big(\partial_{\overline{z}_j} - \tfrac{i}{4} z_j \partial_t\big) \widetilde{F}_h[z,t] = \overline{Z}_j^{(R)} \widetilde{F}_h[z,t]. \tag{24}$$

If we also have that $\widetilde{F}_h \in L^1(\mathbb{H}_n)$, using (24) we obtain that

$$0 = \sigma_\lambda\big(\overline{Z}_j^{(R)} \widetilde{F}_h\big) = -d\sigma_\lambda\big(\overline{Z}_j^{(R)}\big)\sigma_\lambda\big(\widetilde{F}_h\big) ,$$

for $j = 1, \dots, n$. These imply that $\operatorname{ran}\big(\sigma_\lambda(\widetilde{F}_h)\big) \subseteq \ker d\sigma_\lambda\big(\overline{Z}_j^{(R)}\big)$, so that, by (ii) and (iii) in Sect. 2.2, it follows that

- $\sigma_\lambda(\widetilde{F}_h) = 0$ for $\lambda > 0$;
- $\operatorname{ran}\big(\sigma_\lambda(\widetilde{F}_h)\big) \subseteq \operatorname{span}\{1\}$.

We learnt this argument from [20].

We recall that H^2 defined in (10) is a reproducing kernel Hilbert space, whose inner product can be realized by the L^2-inner product of the boundary values, that is,

$$\langle F, G \rangle_{H^2} = \int_{\mathbb{H}_n} \widetilde{F}_0[z,t] \overline{\widetilde{G}_0[z,t]} \, dz dt .$$

The reproducing kernel, which is called the *Szegö* kernel, is given by

$$S(\omega, \zeta) = \frac{n!}{(4\pi)^{n+1}} \Big(\frac{\omega_{n+1} - \overline{\zeta}_{n+1}}{2i} - \tfrac{1}{4} \omega' \cdot \overline{\zeta}' \Big)^{-(n+1)} ,$$

or, equivalently,

$$\widetilde{S}_{(z,t,h),k}[w,s] = \frac{n!}{(2\pi)^{n+1}} \Big(h + k - i\big(s - t + \tfrac{1}{2} \operatorname{Im}(w \cdot \overline{z})\big) + \tfrac{1}{4} |w - z|^2 \Big)^{-(n+1)} ,$$

see e.g. [24].

3 The Weighted Bergman Spaces

The next result will be used repeatedly throughout the remainder of the paper. We recall that the spaces $\mathcal{L}^2_\nu$ where defined in (15).

Lemma 3.1 Let $\nu > -n-2$ and $\tau \in \mathcal{L}^2_\nu$. For $(z,t,h) = \Psi(\zeta)$ with $\zeta \in \mathcal{U}$ define

$$\widetilde{F}(z,t,h) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{0} e^{\lambda h} \operatorname{tr}\left(\tau(\lambda)\sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda.$$

Then, F is holomorphic in $\mathcal{U}$.

Proof We first show that the integral defining F converges absolutely. Consider the orthonormal basis $\{e_\alpha\}$ of $\mathcal{F}^\lambda$, where $e_\alpha(z) = z^\alpha/\|z^\alpha\|_{\mathcal{F}^\lambda}$, α a multiindex. As $\tau(\lambda)$ and $\sigma_\lambda[z,t]$ are operators on $\mathcal{F}^\lambda$, we compute

$$\begin{aligned}\operatorname{tr}\left(\tau(\lambda)\sigma_\lambda[z,t]^*\right) &= \operatorname{tr}\left(\sigma_\lambda[z,t]^*\tau(\lambda)\right) = \sum_\alpha \langle \tau(\lambda)e_\alpha,\ \sigma_\lambda[z,t]e_\alpha\rangle_{\mathcal{F}^\lambda} \\ &= \sum_\alpha \langle \tau(\lambda)e_\alpha,\ P_0\sigma_\lambda[z,t]e_\alpha\rangle_{\mathcal{F}^\lambda} = \operatorname{tr}\left(\tau(\lambda)P_0\sigma_\lambda[z,t]^*\right), \qquad (25)\end{aligned}$$

where P_0 denotes the orthogonal projection onto the subspace generated by e_0, since $\operatorname{ran}(\tau(\lambda)) \subseteq \operatorname{span}\{e_0\}$. Therefore,

$$\left|\operatorname{tr}(\tau(\lambda)\sigma_\lambda[z,t]^*)\right| \le \|\tau(\lambda)\|_{\mathrm{HS}}\|P_0\sigma_\lambda[z,t]\|_{\mathrm{HS}}.$$

Since $\lambda < 0$, we have that

$$\begin{aligned}P_0\sigma_\lambda[z,t]e_\alpha &= \langle \sigma_\lambda[z,t]e_\alpha, e_0\rangle_{\mathcal{F}^\lambda} e_0 \\ &= \left(e^{i\lambda t + \frac{\lambda}{4}|z^2|}\left(\frac{|\lambda|}{2\pi}\right)^n \int_{\mathbb{C}^n} \frac{e^{\frac{\lambda}{2}w\cdot z}(\overline{z}+w)^\alpha}{\|w^\alpha\|_{\mathcal{F}^\lambda}} e^{-\frac{|\lambda|}{2}|w|^2} dw\right)e_0 \\ &= \left(\frac{1}{\sqrt{\alpha!}}\left(\frac{|\lambda|}{2}\right)^{|\alpha|/2} e^{i\lambda t + \frac{\lambda}{4}|z|^2}\overline{z}^\alpha\right)e_0. \qquad (26)\end{aligned}$$

Therefore,

$$\|P_0\sigma_\lambda[z,t]\|^2_{\mathrm{HS}} = e^{\frac{\lambda}{2}|z|^2}\sum_\alpha \frac{1}{\alpha!}\left(\frac{|\lambda|}{2}\right)^{|\alpha|}|z^\alpha|^2 = 1\,,$$

so that

$$\begin{aligned}\int_{-\infty}^{0} e^{\lambda h}\left|\operatorname{tr}(\tau(\lambda)\sigma_\lambda[z,t]^*)\right| |\lambda|^n d\lambda &\le \int_{-\infty}^{0} e^{\lambda h}\|\tau(\lambda)\|_{\mathrm{HS}}\,|\lambda|^n d\lambda \\ &\le \|\tau\|_{\mathcal{L}^2_\nu}\left(\int_{-\infty}^{0} e^{2\lambda h}|\lambda|^{n+\nu+1}d\lambda\right)^{1/2}, \qquad (27)\end{aligned}$$

which is finite since $\nu > -n-2$. This inequality also shows that the integral is locally uniformly bounded in $(z,t,h) \in \mathbf{U}$.

In order to show that F is holomorphic in $\mathcal{U}$, by the previous estimate, it suffices to show that the integrand $\widetilde{J}(z,t,h) = e^{h\lambda}\operatorname{tr}\left(\tau(\lambda)\sigma_\lambda[z,t]^*\right)$ satisfies Eqs. (23) and (24). Indeed, using (25) and (26) we have

$$\widetilde{J}(z,t,h) = e^{h\lambda}\,\mathrm{tr}\left(\tau(\lambda)\sigma_\lambda[z,t]^*\right)$$
$$= e^{h\lambda - i\lambda t + \frac{\lambda}{4}|z|^2} \sum_\alpha \frac{1}{\sqrt{\alpha!}}\left(\frac{|\lambda|}{2}\right)^{|\alpha|/2} z^\alpha \langle \tau(\lambda)e_\alpha,\ e_0\rangle_{\mathcal{F}^\lambda}.$$

Hence,

$$(\partial_t + i\partial_h)\widetilde{J}(z,t,h) = (\partial_{\overline{z}_j} - \tfrac{i}{4} z_j \partial_t)\widetilde{J}(z,t,h) = 0.$$

The conclusion follows. □

We now turn to the Bergman spaces. We begin with the elementary observation that if $\Phi_{[w,s]}$, Φ_U and D_δ are as in Lemma 2.1, and $F \in A^2_\nu$, then $F \circ \Phi_{[w,s]}$ and $F \circ \Phi_U$ have the same norm as F, while $\|F \circ D_\delta\|_{A^2_\nu} = \delta^{-(2n+4)/2}\|F\|_{A^2_\nu}$. We set $F_{(\varepsilon)} = F(\cdot + \varepsilon \mathbf{i})$.

Proposition 3.2 Let $\nu > -1$. The following properties hold.

(i) There exists a constant $C > 0$ such that for all $\zeta \in \mathcal{U}$, $\varepsilon > 0$ and $F \in A^2_\nu$,

$$|F(\zeta + \varepsilon\mathbf{i})| \le C\varepsilon^{-(n+2+\nu)/2}\|F\|_{A^2_\nu}.$$

As a consequence, A^2_ν is a reproducing kernel Hilbert space.

(ii) There exists a constant $C > 0$ such that for all $\varepsilon > 0$ and $F \in A^2_\nu$

$$\|\widetilde{F}_{(\varepsilon),h}\|_{L^2(\mathbb{H}_n)} \le C\varepsilon^{-(1+\nu)/2}\|F\|_{A^2_\nu}.$$

In particular, $F_{(\varepsilon)} \in H^2$ and $\|F_{(\varepsilon)}\|_{H^2} \le C\varepsilon^{-(1+\nu)/2}\|F\|_{A^2_\nu}$.

Proof We begin by observing that if $\zeta = h\mathbf{i}$, then $P(h\mathbf{i}, r)$ is comparable to $P = P(r) = \{(w,s,k)\ : |w| < r,\ |s| < r^2,\ |h-k| < r^2\}$. We have that

$$|F(h\mathbf{i})|^2 \le C\frac{1}{|P(r)|}\int_P |F(\omega)|^2\,d\omega$$
$$\le Ch^{-(n+2)}\int_{|h-k|<r^2}\int_{B([0,0],r)} |\widetilde{F}(w,s,k)|^2\,dwds\,dk.$$

For a generic $\zeta = \Psi^{-1}(z,t,h)$, we have $\zeta = \Phi_{[z,t]}(h\mathbf{i})$, and $F(\zeta) = \big(F \circ \Phi_{[z,t]}\big)(h\mathbf{i})$, so that

$$|F(\zeta)|^2 \le Ch^{-(n+2)}\int_{|h-k|<r^2}\int_{B([0,0],r)} |(F\circ\Phi_{[z,t]})\widetilde{\ }(w,s,k)|^2\,dwds\,dk$$
$$= Ch^{-(n+2)}\int_{|h-k|<r^2}\int_{B([z,t]),r)} |\widetilde{F}(w,s,k)|^2\,dwds\,dk$$
$$= Ch^{-(n+2)}\int_{P((z,t,h),r)} |\widetilde{F}(w,s,k)|^2\,dwds\,dk. \tag{28}$$

Given $\varepsilon > 0$, we apply (28) to $F(\zeta + \varepsilon \mathbf{i}) = \widetilde{F}_{(h)}(z, t, \varepsilon)$ and obtain

$$|F(\zeta + \varepsilon \mathbf{i})|^2 = |\widetilde{F}_{(h)}(z, t, \varepsilon)|^2 \le C\varepsilon^{-(n+2)} \int_{P((z,t,\varepsilon),\sqrt{\varepsilon/2})} |\widetilde{F}_{(h)}(w, s, k)|^2 \, dw ds \, dk$$
$$\le C\varepsilon^{-(n+2+\nu)} \int_{P((z,t,\varepsilon),\sqrt{\varepsilon/2})} |\widetilde{F}_{(h)}(w, s, k)|^2 \, dw ds \, k^\nu dk.$$

Since $\|F_{(h)}\|_{A^2_\nu} \le \|F\|_{A^2_\nu}$, this proves (i).

Next, if $F \in A^2_\nu$ and $\varepsilon > 0$,

$$|\widetilde{F}_{h+\varepsilon}[z, t]|^2 = |F(\zeta + \varepsilon \mathbf{i})|^2$$
$$\le C\varepsilon^{-(n+2+\nu)} \int_{P((z,t,\varepsilon),\sqrt{\varepsilon/2})} |\widetilde{F}_{(h)}(w, s, k)|^2 \, dw ds \, k^\nu dk$$
$$\le C\varepsilon^{-(n+2+\nu)} \int_{\varepsilon/2}^{3\varepsilon/2} \int_{B([z,t],\sqrt{\varepsilon/2})} |\widetilde{F}_{h+k}[w, s]|^2 \, dw ds \, k^\nu dk.$$

Therefore,

$$\|\widetilde{F}_{(\varepsilon),h}\|^2_{L^2(\mathbb{H}_n)}$$
$$\le C\varepsilon^{-(n+2+\nu)} \int_{\mathbb{H}_n} \int_{\varepsilon/2}^{3\varepsilon/2} \int_{B([z,t],\sqrt{\varepsilon/2})} |\widetilde{F}_{h+k}[w, s]|^2 \, dw ds \, k^\nu dk \, dz dt$$
$$= C\varepsilon^{-(n+2+\nu)} \int_{B([0,0],\varepsilon)} \int_{\varepsilon/2}^{3\varepsilon/2} \int_{\mathbb{H}_n} |\widetilde{F}_{h+k}\big([w, s][z, t]\big)|^2 \, dz dt \, k^\nu dk \, dw ds$$
$$\le C\varepsilon^{-(1+\nu)} \|F\|^2_{A^2_\nu}.$$

Moreover,

$$\|\widetilde{F}_\varepsilon\|_{L^2(\mathbb{H}_n)} \le \|F_{(\varepsilon)}\|_{H^2} = \sup_{h>0} \|\widetilde{F}_{\varepsilon+h}\|_{L^2(\mathbb{H}_n)} \le C\varepsilon^{-2(1+\nu)/p} \|F\|_{A^2_\nu}.$$

This proves (ii). □

With a standard argument, see, for instance, [6, Remark 1.16], it is possible to prove the following result.

Proposition 3.3 If $\nu \le -1$, then $A^2_\nu = \{0\}$.

We now have all the ingredients to prove our first main result.

Proof of Theorem 1 Let $F \in A^2_\nu$ and $\varepsilon > 0$. Then, $F_{(\varepsilon)} \in H^2$ so that $\widetilde{F}_{(\varepsilon),h} \in L^2(\mathbb{H}_n)$ for $h > 0$, and $\sigma_\lambda(\widetilde{F}_{\varepsilon+h}) = \sigma_\lambda(\widetilde{F}_{(\varepsilon),h}) = 0$ if $\lambda > 0$. Moreover, by (14) and the inversion formula (22), we have that $\sigma_\lambda(\widetilde{F}_{(\varepsilon),h}) = e^{h\lambda}(\widetilde{F}_{(\varepsilon),0})$ if $\lambda < 0$. Therefore, if $F \in A^2_\nu$, $\varepsilon, h > 0$:

- $\widetilde{F}_h \in L^2(\mathbb{H}_n)$ for all $h > 0$;

- $\sigma_\lambda(\widetilde{F}_h) = 0$ if $\lambda > 0$;
- $\operatorname{ran}(\sigma_\lambda(\widetilde{F}_h)) \subseteq \operatorname{span}\{1\}$ if $\lambda < 0$;
- $\sigma_\lambda(\widetilde{F}_{\varepsilon+h}) = e^{h\lambda}\sigma(\widetilde{F}_\varepsilon)$.

Since $F_{(\varepsilon)} \in H^2$, by the Ogden–Vagi Theorem, there exists $g_\varepsilon \in L^2(\mathbb{H}_n)$ such that $\sigma_\lambda(g_\varepsilon) = 0$ if $\lambda > 0$, $\operatorname{ran}(\sigma_\lambda(g_\varepsilon)) \subseteq \operatorname{span}\{1\}$ if $\lambda < 0$, and

$$F(\zeta + \varepsilon\mathbf{i}) = \widetilde{F}_{(\varepsilon),h}[z,t] = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{0} e^{h\lambda} \operatorname{tr}\left(\sigma_\lambda(g_\varepsilon)\sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda\,,$$

where $\Psi(\zeta) = (z,t,h)$. Switching the roles of h and ε and arguing as above, there exists $g_h \in L^2(\mathbb{H}_n)$ such that $\sigma_\lambda(g_h) = 0$ if $\lambda > 0$, $\operatorname{ran}(\sigma_\lambda(g_h)) \subseteq \operatorname{span}\{1\}$ if $\lambda < 0$, and

$$F(\zeta + \varepsilon\mathbf{i}) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{0} e^{\varepsilon\lambda} \operatorname{tr}\left(\sigma_\lambda(g_h)\sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda.$$

These equalities imply that $e^{h\lambda}\sigma_\lambda(g_\varepsilon) = e^{\varepsilon\lambda}\sigma_\lambda(g_h)$ for all $\varepsilon, h > 0$, that is, for every $\lambda < 0$

$$\operatorname{HS}(\mathcal{F}^\lambda) \ni e^{-\varepsilon\lambda}\sigma_\lambda(g_\varepsilon) =: \tau(\lambda)$$

is well defined, i.e. independent of ε, with $\tau(\lambda) = 0$ if $\lambda > 0$ and $\operatorname{ran}(\tau(\lambda)) \subseteq \operatorname{span}\{1\}$. Hence,

$$F(\zeta + \varepsilon\mathbf{i}) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{0} e^{(h+\varepsilon)\lambda} \operatorname{tr}\left(\tau(\lambda)\sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda\,,$$

that is,

$$F(\zeta) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{0} e^{h\lambda} \operatorname{tr}\left(\tau(\lambda)\sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda.$$

In particular, $\sigma_\lambda(\widetilde{F}_h) = e^{h\lambda}\tau(\lambda)$. Therefore,

$$\begin{aligned}
\|F\|^2_{A^2_\nu} &= \int_0^{+\infty} \int_{\mathbb{H}_n} |\widetilde{F}_h[z,t]|^2\, dz dt\, h^\nu dh \\
&= \frac{1}{(2\pi)^{n+1}} \int_0^{+\infty} \int_{-\infty}^{0} \|\sigma_\lambda(\widetilde{F}_h)\|^2_{\mathrm{HS}}\, |\lambda|^n d\lambda\, h^\nu dh \\
&= \frac{1}{(2\pi)^{n+1}} \int_0^{+\infty} \|\tau(-\lambda)\|^2_{\mathrm{HS}} \int_0^{+\infty} e^{-2\lambda h} h^\nu dh\, \lambda^n d\lambda \\
&= \frac{\Gamma(\nu+1)}{2^{\nu+1}(2\pi)^{n+1}} \int_0^{+\infty} \|\tau(-\lambda)\|^2_{\mathrm{HS}} \lambda^{n-1-\nu}\, d\lambda \\
&= \frac{\Gamma(\nu+1)}{2^{\nu+1}} \|\tau\|^2_{\mathcal{L}^2_\nu}. \qquad (29)
\end{aligned}$$

Conversely, let $\tau \in \mathcal{L}^2_\nu$ and F defined by (16). By Lemma 3.1 we have that $F \in \mathrm{Hol}(\mathcal{U})$. Moreover, $\widetilde{F}_h \in L^2(\mathbb{H}_n)$ for every $h > 0$, since $\nu > -1$ and by Plancherel's formula

$$\|\widetilde{F}_h\|^2_{L^2(\mathbb{H}_n)} = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 \|e^{\lambda h}\tau(\lambda)\|^2_{\mathrm{HS}} |\lambda|^n d\lambda \le C_h \|\tau\|^2_{\mathcal{L}^2_\nu}.$$

Moreover, $\sigma_\lambda(\widetilde{F}_h) = e^{h\lambda}\tau(\lambda)$. Hence, identities (29) hold true, and (17) follows. □

An immediate consequence is the following result.

Corollary 3.4 Let $\nu > -1$ and $F \in A^2_\nu$. For $\varepsilon > 0$, let $F_{(\varepsilon)}(\zeta) = F(\zeta + \varepsilon\mathbf{i})$. Then, F is holomorphic in a neighborhood of $\mathcal{U}$ and $F_{(\varepsilon)} \to F$ in A^2_ν as $\varepsilon \to 0^+$.

Remark 3.5 It is well known that the reproducing kernel for A^2_ν is the kernel function, called the *weighted Bergman kernel*,

$$K_\nu(\omega, \zeta) = \gamma_{n,\nu}\left(\frac{\omega_{n+1} - \overline{\zeta}_{n+1}}{2i} - \tfrac{1}{4}\omega' \cdot \overline{\zeta'}\right)^{-(n+2+\nu)},$$

where $\gamma_{n,\nu} = \frac{1}{(4\pi)^{n+1}}\frac{\Gamma(n+2+\nu)}{\Gamma(\nu+1)}$. This fact can be obtained from the expression of the kernel of the corresponding weighted Bergman space on the unit ball, by means of the transformation rule for the Bergman kernel, or as a corollary of the Paley–Wiener theorem, using the same techniques we will use in Corollary 4.3.

4 The Weighted Dirichlet Spaces

Recall that the weighted Dirichlet spaces $\mathcal{D}_{\nu,(m)}$ are defined in (7). Note that condition (i) in (7) means that

$$\lim_{\mathrm{Im}\,\zeta_{n+1}\to+\infty} \sup_{|\zeta'|\le R} |F(\zeta)| = 0.$$

An analogous remark holds for (i) in (8) as well. We begin with an elementary lemma.

Lemma 4.1 The following properties hold true.

(i) Let $a, b \in \mathbb{R}$. If $a > -1$ and $b > 0$, then there exists $C_0 > 0$ such that

$$I(\zeta) = \int_{\mathcal{U}} \frac{\rho^a(\omega)}{\left|\frac{\zeta_{n+1}-\bar{\omega}_{n+1}}{2i} - \frac{1}{4}\zeta' \cdot \bar{\omega}'\right|^{a+b+n+2}} d\omega = C_0 \frac{1}{(\mathrm{Im}\,\zeta_{n+1} - \frac{1}{4}|\zeta'|^2)^b}.$$

If $a \le -1$ or $b \le 0$, then the above integral equals $+\infty$.

(ii) The spaces $\mathcal{D}_{\nu,(m)}$ are reproducing kernel Hilbert spaces.

Proof (i) This is an elementary calculation. We provide the details for sake of completeness. We observe that, if $(z, t, h) = \Psi(\zeta', \zeta_{n+1})$ and $(w, s, k) = \Psi(\omega', \omega_{n+1})$, we have

$$\begin{aligned}&\left|\tfrac{1}{2i}\big(\zeta_{n+1} - \bar{\omega}_{n+1}\big) - \tfrac{1}{4}\zeta' \cdot \bar{\omega}'\right|^2 \\ &\quad = \tfrac{1}{4}\big(h + k + \tfrac{1}{2}|z - w|^2\big)^2 + \tfrac{1}{4}\big((s - t) - \tfrac{1}{2}\operatorname{Im} z \cdot \bar{w}\big)^2.\end{aligned}$$

Then, by the standard translation invariance of the Lebesgue measure in $\mathbb{R}$ and $\mathbb{C}^n$, and integration in polar coordinates in $\mathbb{C}^n$, we have

$$\begin{aligned}&\widetilde{I}(z, t, h) \\ &= 2^{a+b+n+2}\int_{\mathbf{U}} \frac{k^a}{\big(\big(h + k + \frac{1}{4}|w|^2\big)^2 + s^2\big)^{(a+b+n+2)/2}}\, ds\, dw\, dk \\ &= \frac{2^{a+b+n+3}\pi^{n+1}}{n!}\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} \frac{k^a r^{2n-1}}{\big(\big(h + k + \frac{1}{4}r^2\big)^2 + s^2\big)^{(a+b+n+2)/2}}\, ds\, dr\, dk \\ &= \frac{2^{a+b+n+3}\pi^{n+1}}{n!} C_1 \int_0^{+\infty}\int_0^{+\infty} \frac{k^a r^{2n-1}}{\big(h + k + \frac{1}{4}r^2\big)^{a+b+n+1}}\, dr\, dk \\ &= \frac{2^{a+b+3n+3}\pi^{n+1}}{n!} C_1 C_2 \int_0^{+\infty} \frac{k^a}{(h + k)^{a+b+1}}\, dk \\ &= \frac{C_0}{h^b},\end{aligned}$$

as we wished to show.

(ii) Let $F \in \mathcal{D}_{\nu,(m)}$. Then, $\partial^m_{\zeta_{n+1}} F \in A^2_{2m+\nu}$. For $\zeta \in \mathcal{U}$ we define

$$G(\zeta) = c\int_{\mathcal{U}} \frac{\partial^m_{\zeta_{n+1}} F(\omega)}{\big(\frac{\zeta_{n+1} - \bar{\omega}_{n+1}}{2i} - \frac{1}{4}\zeta' \cdot \bar{\omega}'\big)^{n+2+\nu+m}}\, \rho(\omega)^{2m+\nu}\, d\omega\,, \tag{30}$$

where c is a constant to be chosen later. Then, (i) and Cauchy–Schwarz's inequality give that

$$|G(\zeta)| \le C_0 h^{-(n+2+\nu)/2}\, \|\partial^m_{\zeta_{n+1}} F\|_{A^2_{2m+\nu}}\,, \tag{31}$$

where C_0 is as in (i), and $h = \operatorname{Im}\zeta_{n+1} - \frac{1}{4}|\zeta'|^2$. Arguing as above, it is easy to see that G is holomorphic in $\mathcal{U}$, and that we can differentiate under the integral sign m times to obtain that, using Remark 3.5, for a suitable constant c, $\partial^m_{\zeta_{n+1}} G = \partial^m_{\zeta_{n+1}} F$. Therefore, $(F - G)(\zeta) = \sum_{j=0}^{m-1} g_j(\zeta')\zeta^j_{n+1}$, where the g_j's are entire functions in $\mathbb{C}^n$. By (31) it also follows that $\lim_{|\zeta'| \le R,\, \operatorname{Im}\zeta_{n+1} \to +\infty} G(\zeta) = 0$. Therefore, for each ζ' fixed, the polynomial $F(\zeta', \cdot) - G(\zeta', \cdot)$ tends to 0 as $\operatorname{Im}\zeta_{n+1} \to +\infty$. This implies that the g_j's are identically 0; hence $G = F$.

Now, (31) with G replaced by F gives that the point evaluations are bounded on $\mathcal{D}_{\nu,(m)}$, and also implies uniform estimates on compact subsets of $\mathcal{U}$. An elementary argument shows that $\mathcal{D}_{\nu,(m)}$ is complete; hence a reproducing kernel Hilbert space. □

We set

$$\mathcal{H}_m = \big\{ F \in \mathrm{Hol}(\overline{\mathcal{U}}) : \partial_\zeta^\alpha F \in H^2,\ |\alpha| \le m \big\}. \tag{32}$$

Lemma 4.2 Let $-(n+2) < \nu < -1$, and let $m > -\frac{\nu+1}{2}$. Then, $\mathcal{D}_{\nu,(m)} \cap \mathcal{H}_m$ is dense in $\mathcal{D}_{\nu,(m)}$.

Proof Let $F \in \mathcal{D}_{\nu,(m)}$. For $\varepsilon, \delta > 0$, $q > 0$ to be selected later, and $\zeta \in \mathcal{U}$, we define

$$\begin{aligned} &G_{(\varepsilon,\delta)}(\zeta) \\ &\quad = c \int_{\mathcal{U}} \frac{1}{\big(-\varepsilon i \omega_{n+1} + 1\big)^q} \cdot \frac{\partial^m_{\zeta_{n+1}} F(\omega)}{\big(\frac{\zeta_{n+1}+i\delta-\bar\omega_{n+1}}{2i} - \frac14 \zeta' \cdot \bar\omega'\big)^{n+2+\nu+m}} \, \rho(\omega)^{2m+\nu} d\omega\,, \end{aligned}$$

where c is as in (30). Recall that $\partial^m_{\zeta_{n+1}} F \in A^2_{2m+\nu}$. Observe that the factor $\big(-\varepsilon i \omega_{n+1} + 1\big)^{-q}$ is bounded on $\overline{\mathcal{U}}$. Then, the same argument as in Lemma 4.1 (ii) and the dominated convergence theorem give that $G_{(\varepsilon,\delta)}$ are holomorphic and, by (30), converge uniformly on compact subsets to F, as $\varepsilon, \delta \to 0^+$. Thus, we need to show: (a) that $G_{(\varepsilon,\delta)} \in \mathcal{H}_m$; and (b) that converge to F in $\mathcal{D}_{\nu,(m)}$.

Let $\alpha = (\alpha', \alpha_{n+1})$ be a multiindex, $|\alpha| \le m$. Then, differentiating under the integral sign, for a suitable constant c', we have

$$\begin{aligned} &\partial^\alpha_\zeta G_{(\varepsilon,\delta)}(\zeta) \\ &\quad = c' \int_{\mathcal{U}} \frac{\overline{\omega}'^{\alpha'}}{\big(-\varepsilon i \omega_{n+1} + 1\big)^q} \cdot \frac{\partial^m_{\zeta_{n+1}} F(\omega)}{\big(\frac{\zeta_{n+1}+i\delta-\bar\omega_{n+1}}{2i} - \frac14 \zeta' \cdot \bar\omega'\big)^{n+2+\nu+m+|\alpha|}} \, \rho(\omega)^{2m+\nu} \, d\omega \\ &\quad =: \int_{\mathcal{U}} K(\zeta,\omega) \partial^m_{\zeta_{n+1}} F(\omega) \rho(\omega)^{2m+\nu} \, d\omega. \end{aligned}$$

Letting $(z,t,h) = \Psi(\zeta', \zeta_{n+1})$, $(w,s,k) = \Psi(\omega', \omega_{n+1})$ and writing $\widetilde{K} = K\big(\Psi(\cdot), \Psi(\cdot)\big)$, we see that

$$\begin{aligned} &|\widetilde{K}\big((z,t,h),(w,s,k)\big)| \\ &\quad \le C \frac{1}{\big(\varepsilon(k + \frac14 |w|^2 + |s|) + 1\big)^{q-|\alpha'|/2}} \\ &\qquad \times \frac{1}{\big(h + \delta + k + \frac12 |z-w|^2 + \big|(s-t) - \frac12 \,\mathrm{Im}\, z \cdot \bar w\big|\big)^{n+2+\nu+m+|\alpha|}} \\ &\quad =: C \widetilde{L}\big((z,t,h),(w,s,k)\big). \end{aligned}$$

Using Cauchy–Schwarz's inequality, for $q > 0$ sufficiently large, we have

$$\begin{aligned} &\int_{\mathbb{H}_n} \big| \big(\partial^\alpha_\zeta G_{(\varepsilon,\delta)}\big)^{\sim}_h [z,t] \big|^2 \, dz dt \\ &\quad \le \int_{\mathbb{H}_n} \bigg(\int_{\mathbf{U}} \widetilde{L}\big((z,t,h),(w,s,k)\big) |\partial^m_k \widetilde{F}(w,s,k)| \, ds dw \, k^{2m+\nu} dk \bigg)^2 dz dt \end{aligned}$$

$$
\begin{aligned}
&\le \left(\int_{\mathbf{U}} \frac{1}{\big(\varepsilon(k+\frac{1}{4}|w|^2+|s|)+1\big)^{2q-|\alpha'|}}\, ds dw\, k^{2m+\nu} dk \right) \\
&\quad \times \int_{\mathbb{H}_n} \int_{\mathbf{U}} \frac{|\partial_k^m \widetilde{F}(w,s,k)|^2}{\big(h+\delta+k+\frac{1}{2}|z-w|^2+\big|(s-t)-\frac{1}{2}\operatorname{Im} z\cdot\bar{w}\big|\big)^{2(n+2+\nu+m+|\alpha|)}} \\
&\quad \times ds dw\, k^{2m+\nu} dk dz dt \\
&\le C \int_{\mathbf{U}} |\partial_k^m \widetilde{F}(w,s,k)|^2 \int_{\mathbb{H}_n} \frac{1}{\big(h+\delta+k+\frac{1}{2}|z|^2+|t|\big)^{2(n+2+\nu+m+|\alpha|)}} \\
&\quad \times dz dt ds dw\, k^{2m+\nu} dk. \qquad (33)
\end{aligned}
$$

By Lemma 4.1 (i) it follows that $\|\partial_\zeta^\alpha G_{(\varepsilon,\delta)}\|_{H^2} \le C\|F\|^2_{\mathcal{D}_{\nu,(m)}}$, for $|\alpha| \le m$, and when $|\alpha| = m$, also that

$$
\begin{aligned}
&\int_0^{+\infty} \int_{\mathbb{H}_n} \big|\big(\partial_\zeta^\alpha G_{(\varepsilon,\delta)}\big)^{\sim}_h[z,t]\big|^2 dz dt\, h^{2\nu+m} dh \\
&\le C \int_{\mathbf{U}} |\partial_k^m \widetilde{F}(w,s,k)|^2 \\
&\quad \times \left(\int_{\mathbf{U}} \frac{1}{\big(h+\delta+k+\frac{1}{2}|z|^2+|t|\big)^{2(n+2+\nu+2m)}}\, dz dt\, h^{2\nu+m} dh \right) ds dw\, k^{2m+\nu} dk.
\end{aligned}
$$

This implies that $\partial^m_{\zeta_{n+1}} G_{(\varepsilon,\delta)} \in A^2_{2m+\nu}$. It is also easy to see that $\lim_{|z|\le R,\, h\to+\infty} \widetilde{G}_{(\varepsilon,\delta)}(z,t,h) = 0$. Therefore, $G_{(\varepsilon,\delta)} \in \mathcal{H}_m$, i.e. the conclusion (a) follows. Now, it is elementary to show (b). Indeed, since $\big(-\varepsilon i\omega_{n+1}+1\big)^{-q}\partial^m_{\zeta_{n+1}} F \in A^2_{2m+\nu}$, we have that

$$
\begin{aligned}
&\partial^m_{\zeta_{n+1}} G_{(\varepsilon,\delta)}(\zeta) \\
&= c \int_{\mathcal{U}} \frac{\partial^m_{\zeta_{n+1}} F(\omega)}{\big(-\varepsilon i\omega_{n+1}+1\big)^q} \cdot \frac{1}{\big(\frac{\zeta_{n+1}+i\delta-\bar{\omega}_{n+1}}{2i} - \frac{1}{4}\zeta'\cdot\bar{\omega}'\big)^{n+2+2m+\nu}}\, \rho(\omega)^{2m+\nu} d\omega \\
&= \frac{\partial^m_{\zeta_{n+1}} F(\zeta+\delta\mathbf{i})}{\big(-\varepsilon i\zeta_{n+1}+\varepsilon\delta+1\big)^q}.
\end{aligned}
$$

A simple application of the dominated convergence theorem together with Corollary 3.4 give the desired conclusion. □

Proof of Theorem 2 Let $F \in \mathcal{D}_{\nu,(m)} \cap \mathcal{H}_m$, with $m > -\frac{\nu+1}{2}$. Observe that, since $\partial^j_{\zeta_{n+1}} F \in A^2_{2m+\nu} \cap H^2$ for $j = 0,\dots,m$, we have that $\partial_h^j \widetilde{F}_h \in L^2(\mathbb{H}_n)$ for all $h > 0$, and $j = 0,\dots,m$. This easily implies that, for $F \in \mathcal{H}_m$, setting $\tau(\lambda) = \sigma_\lambda(\widetilde{F}_0)$,

$$
\sigma_\lambda\big(\partial_h^m \widetilde{F}_h\big) = \partial_h^m \sigma_\lambda\big(\widetilde{F}_h\big) = \partial_h^m\big(e^{h\lambda}\tau(\lambda)\big) = \lambda^m e^{h\lambda}\tau(\lambda).
$$

Observe that with this choice of τ, formula (18) holds for F. Moreover,

$$
\begin{aligned}
\|F\|^2_{\mathcal{D}_{\nu,(m)}} &= \int_{\mathcal{U}} |\rho^m(\zeta)\partial^m_{\zeta_{n+1}} F(\zeta)|^2 \rho^\nu(\zeta) d\zeta \\
&= \int_0^{+\infty} \int_{\mathbb{H}_n} \left|h^m \partial_h^m \widetilde{F}_h[z,t]\right|^2 dz dt\, h^\nu dh \\
&= \frac{1}{(2\pi)^{n+1}} \int_0^{+\infty} \int_{-\infty}^0 \left\|\sigma_\lambda\left(\partial_h^m \widetilde{F}_h\right)\right\|^2_{\mathrm{HS}} |\lambda|^n d\lambda\, h^{2m+\nu}\, dh \\
&= \frac{1}{(2\pi)^{n+1}} \int_0^{+\infty} \int_{-\infty}^0 e^{2h\lambda} \|\tau(\lambda)\|^2_{\mathrm{HS}} |\lambda|^{2m+n} d\lambda\, h^{2m+\nu}\, dh \\
&= \frac{1}{(2\pi)^{n+1}} \int_0^{+\infty} \|\tau(-\lambda)\|^2_{\mathrm{HS}} |\lambda|^{2m+n} \int_0^{+\infty} e^{-2h\lambda} h^{2m+\nu}\, dh d\lambda \\
&= \frac{1}{(2\pi)^{n+1}} \frac{\Gamma(2m+\nu+1)}{2^{2m+\nu+1}} \int_0^{+\infty} \|\tau(-\lambda)\|^2_{\mathrm{HS}} |\lambda|^{n-(\nu+1)}\, d\lambda \\
&= \frac{\Gamma(2m+\nu+1)}{2^{2m+\nu+1}} \|\tau\|^2_{\mathcal{L}^2_\nu}.
\end{aligned}
$$

Suppose now that $F \in \mathcal{D}_{\nu,(m)}$ and let $\{F_N\}$ be a sequence in $\mathcal{D}_{\nu,(m)} \cap \mathcal{H}_m$, $F_N \to F$ in $\mathcal{D}_{\nu,(m)}$. Then, F_N converges to F also uniformly on compact subsets. Let $\tau_N \in \mathcal{L}^2_\nu$ be such that $(\widetilde{F}_N)_h[z,t] = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 e^{h\lambda} \operatorname{tr}\left(\tau_N(\lambda)\sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda$, and let $\tau = \lim_{N\to+\infty} \tau_N$ in $\mathcal{L}^2_\nu$. Then,

$$
\begin{aligned}
\widetilde{F}_h[z,t] &= \lim_{N\to+\infty} \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 e^{h\lambda} \operatorname{tr}\left(\tau_N(\lambda)\sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda \\
&= \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 e^{h\lambda} \operatorname{tr}\left(\tau(\lambda)\sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda\,,
\end{aligned}
$$

by applying estimate (27). This proves (18), and (19) follows as well.

Conversely, let F be given by (18). Then, Lemma 3.1 gives that F is holomorphic in $\mathcal{U}$. Plancherel's formula now gives (19).

Finally, we observe that by (19) it follows easily that the spaces $\mathcal{D}_{\nu,(m)}$ do not depend on the choice of the integer m and their norms coincide, up to a multiplicative constant. □

Corollary 4.3 Let $-(n+2) < \nu < -1$. Let m be a positive integer, $m > -(\nu+1)/2$. Then, there exists a constant $\gamma_{\nu,n,m}$ such that the reproducing kernel K_ν, expressed with respect to the inner product in $\mathcal{D}_{\nu,(m)}$ is given by

$$
K_\nu(\omega,\zeta) = \gamma_{n,m,\nu} \left(\frac{\omega_{n+1} - \overline{\zeta}_{n+1}}{2i} - \tfrac14 \omega' \cdot \overline{\zeta'} \right)^{-(n+2+\nu)},
$$

where $\gamma_{n,m,\nu} = \frac{4^m}{(4\pi)^{n+1}} \frac{\Gamma(n+2+\nu)}{\Gamma(2m+\nu+1)}$.

Proof We use the inversion formula and the polarized identity coming from the Paley–Wiener type Theorem 2. For $F \in \mathcal{D}_{\nu,(m)}$, let τ_F denote the element of $\mathcal{L}^2_\nu$ such that

$$\widetilde{F}(z,t,h) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 e^{h\lambda} \operatorname{tr}\left(\tau_F(\lambda) P_0 \sigma_\lambda[z,t]^*\right) |\lambda|^n d\lambda\,,$$

where P_0 denotes the orthogonal projection onto the subspace generated by e_0. Moreover, by the reproducing formula for $\mathcal{D}_{\nu,(m)}$, (19), writing $K_\nu(\zeta,\cdot) = K_\zeta$ and $(z,t,h) = \Psi(\zeta)$, we have

$$\begin{aligned}\widetilde{F}(z,t,h) &= F(\zeta) = \langle F, K_\zeta\rangle_{\mathcal{D}_{\nu,(m)}} \\ &= \frac{\Gamma(2m+\nu+1)}{2^{2m+\nu+1}(2\pi)^{n+1}} \int_{-\infty}^0 \operatorname{tr}\left(\tau_F(\lambda)\tau_{K_\zeta}(\lambda)^*\right) |\lambda|^{n-\nu-1} d\lambda.\end{aligned}$$

Since these two equalities hold for all $\tau \in \mathcal{L}^2_\nu$, it follows that

$$\tau_{K_\zeta}(\lambda) = \frac{2^{2m+\nu+1}}{\Gamma(2m+\nu+1)} |\lambda|^{\nu+1} P_0 \sigma_\lambda[z,t].$$

Therefore, using (26) and writing $C = \frac{2^{2m+\nu+1}}{\Gamma(2m+\nu+1)(2\pi)^{n+1}}$, we have

$$\begin{aligned}&\widetilde{K}_{(z,t,h)}(w,s,k) \\ &= C \int_{-\infty}^0 e^{(h+k)\lambda} \operatorname{tr}\left(P_0\sigma_\lambda[z,t] P_0 \sigma_\lambda[w,s]^*\right) |\lambda|^{n+\nu+1} d\lambda \\ &= C \int_{-\infty}^0 e^{(h+k)\lambda} \sum_\alpha \frac{1}{\alpha!}\left(\frac{|\lambda|}{2}\right)^{|\alpha|} e^{i\lambda(t-s)+\frac{\lambda}{4}|z|^2+\frac{\lambda}{4}|w|^2} w^\alpha \bar{z}^\alpha\, |\lambda|^{n+\nu+1} d\lambda \\ &= C \int_0^{+\infty} e^{-\lambda(h+k-i(s-t+\frac{1}{2}\operatorname{Im}(w\cdot\bar{z}))+\frac{1}{4}|w-z|^2)}\, \lambda^{n+\nu+1} d\lambda \\ &= C\,\Gamma(n+2+\nu)\Big(h+k+\tfrac{1}{4}|w-z|^2 - i(s-t+\tfrac{1}{2}\operatorname{Im}(w\cdot\bar{z}))\Big)^{-(n+2+\nu)},\end{aligned}$$

that is,

$$K_\nu(\omega,\zeta) = \frac{4^m}{(4\pi)^{n+1}} \frac{\Gamma(n+2+\nu)}{\Gamma(2m+\nu+1)} \left(\frac{\omega_{n+1}-\overline{\zeta}_{n+1}}{2i} - \tfrac{1}{4}\omega'\cdot\overline{\zeta'}\right)^{-(n+2+\nu)},$$

as we wished to show. □

5 The Dirichlet Space

In this section we prove Theorem 3 and we provide justification of the name Dirichlet space for the space $\mathcal{D}_{(m)}$.

In order to simplify some formulas, we introduce the notation

$$Q(\omega,\zeta)=\frac{\omega_{n+1}-\overline{\zeta}_{n+1}}{2i}-\tfrac{1}{4}\omega'\cdot\overline{\zeta}',$$

whereas we remind the reader that $\mathbf{i}$ denotes the point $(0,i)\in\mathcal{U}$.

We start proving a couple of lemmas which are the analogue of Lemmas 4.1 and 4.2.

Lemma 5.1 The following properties hold true.

(i) Let $m>\frac{n+1}{2}$ be fixed. Then, there exists a constant $C>0$ such that

$$I(\zeta)=\int_{\mathcal{U}}\left|\frac{1}{Q^m(\zeta,\omega)}-\frac{1}{Q^m(\mathbf{i},\omega)}\right|^2\rho^{2m-n-2}(\omega)\,d\omega\le C\frac{(1+|\zeta|^2)^{2m+1}}{(\operatorname{Im}\zeta_{n+1}-\frac{1}{4}|\zeta'|^2)}.$$

(ii) The spaces $\mathcal{D}_{(m)}$ are reproducing kernel Hilbert spaces.

Proof (i) Given any $\zeta\in\mathcal{U}$ there exists a constant $C>0$, such that

$$\big|Q(\mathbf{i},\omega)-Q(\zeta,\omega)\big|=\left|\frac{i-\zeta_{n+1}}{2i}+\frac{1}{4}\zeta'\cdot\overline{\omega'}\right|\le C(1+|\zeta|)(1+|\omega'|),$$

so that,

$$\left|\frac{1}{Q^m(\zeta,\omega)}-\frac{1}{Q^m(\mathbf{i},\omega)}\right|^2\le C\frac{(1+|\zeta|)^2(1+|\omega'|)^2}{\big|Q^m(\zeta,\omega)Q^m(\mathbf{i},\omega)\big|}\sum_{j=0}^{m-1}\big|Q^j(\zeta,\omega)Q^{m-1-j}(\mathbf{i},\omega)\big|^2.$$

Thus, in order to conclude the proof, it is enough to estimate the integral

$$I_j(\zeta)=\int_{\mathcal{U}}(1+|\omega'|)^2\left|\frac{Q^j(\zeta,\omega)Q^{m-1-j}(\mathbf{i},\omega)}{Q^m(\zeta,\omega)Q^m(\mathbf{i},\omega)}\right|^2\rho^{2m-n-2}(\omega)\,d\omega.$$

Observing that

$$\left|\frac{Q(\zeta,\omega)}{Q(\mathbf{i},\omega)}\right|\le C(1+|\zeta|^2),$$

it holds that

$$\begin{aligned}I_j(\zeta)&=\int_{\mathcal{U}}\frac{(1+|\omega'|)^2}{|Q(\mathbf{i},\omega)|}\left|\frac{Q^j(\zeta,\omega)Q^{m-1-j}(\mathbf{i},\omega)}{Q^m(\zeta,\omega)Q^{m-\frac{1}{2}}(\mathbf{i},\omega)}\right|^2\rho^{2m-n-2}(\omega)\,d\omega\\&\le C(1+|\zeta|^2)^{1+2j}\int_{\mathcal{U}}\frac{\rho^{2m-n-2}(\omega)}{|Q(\zeta,\omega)|^{2m+1}}\,d\omega\\&\le C\frac{(1+|\zeta|^2)^{1+2j}}{(\operatorname{Im}\zeta_{n+1}-\frac{1}{4}|\zeta'|^2)},\end{aligned}$$

where the last inequality follows from (i) in Lemma 4.1. Thus,

$$I(\zeta) \le C(1+|\zeta|^2)^2 \sum_{j=0}^{m-1} I_j(\zeta) \le C \frac{(1+|\zeta|^2)^{2m+1}}{(\operatorname{Im}\zeta_{n+1} - \frac14|\zeta'|^2)},$$

as we wished to prove.

(ii) Let $F \in \mathcal{D}_{(m)}$. Then $\partial^m_{\zeta_{n+1}} F \in A^2_{2m-n-2}$. For $\zeta \in \mathcal{U}$ we define

$$G(\zeta) = c \int_{\mathcal{U}} \partial^m_{\zeta_{n+1}} F(\omega) \left[\frac{1}{Q^m(\zeta,\omega)} - \frac{1}{Q^m(\mathbf{i},\omega)} \right] \rho(\omega)^{2m-n-2}\, d\omega, \tag{34}$$

where c is a suitable constant to be chosen later. Then, (i) and Cauchy–Schwarz's inequality guarantee that G is well-defined, in particular,

$$|G(\zeta)| \le C \frac{(1+|\zeta|^2)^{m+\frac12}}{(\operatorname{Im}\zeta_{n+1} - \frac14|\zeta'|^2)^{\frac12}} \|\partial^m_{\zeta_{n+1}} F\|_{A^2_{2m-n-2}} \le C \frac{(1+|\zeta|^2)^{m+\frac12}}{(\operatorname{Im}\zeta_{n+1} - \frac14|\zeta'|^2)^{\frac12}} \|F\|_{\mathcal{D}_{(m)}}. \tag{35}$$

Arguing as in the proof of Lemma 4.1 we obtain that $G \in \mathcal{D}_{(m)}$ and, for a suitable choice of the constant c, $\partial^m_{\zeta_{n+1}} G = \partial^m_{\zeta_{n+1}} F$. Therefore, for each ζ' fixed, we obtain that $(F-G)(\zeta) = \sum_{j=0}^{m-1} g_j(\zeta')\zeta^j_{n+1}$, where the g_j's are entire functions in $\mathbb{C}^n$. Since both G and F belong to $\mathcal{D}_{(m)}$, it follows that $F(\zeta) - G(\zeta) = F(\mathbf{i})$. This fact and (34) give an integral representation for any function $F \in \mathcal{D}_{(m)}$ and that (9) is a norm. Finally, this integral representation and (35) show that the point evaluations are bounded on $\mathcal{D}_{(m)}$, and the fact that $\mathcal{D}_{(m)}$ is a reproducing kernel Hilbert space follows as in Lemma 4.1. □

Lemma 5.2 Let $m > \frac{n+1}{2}$ and let $\mathcal{H}_m$ be as in (32) and let $\mathcal{D}_{(m)}(\mathbf{i})$ be the closed subspace of $\mathcal{D}_{(m)}$ of functions that vanish in $\mathbf{i}$. Then, $\mathcal{D}_{(m)}(\mathbf{i}) \cap \mathcal{H}_m$ is dense in $\mathcal{D}_{(m)}(\mathbf{i})$.

Proof Let $F \in \mathcal{D}_{(m)}(\mathbf{i})$. For $\varepsilon, \delta > 0$, and $q > 0$ to be selected later, and $\zeta \in \mathcal{U}$ we define

$$G_{(\varepsilon,\delta)}(\zeta) = c \int_{\mathcal{U}} \frac{\partial^m_{\zeta_{n+1}} F(\omega)}{(1-\varepsilon i\omega_{n+1})^q} \left[\frac{1}{Q^m(\zeta+\delta\mathbf{i},\omega)} - \frac{1}{Q^m((1+\delta)\mathbf{i},\omega)} \right] \rho(\omega)^{2m-n-2}\, d\omega,$$

where $c > 0$ is as in (34). From Lemma 5.1 and the dominated convergence theorem we deduce that the functions $G_{(\varepsilon,\delta)}$ are holomorphic in $\overline{\mathcal{U}}$ and converge to F uniformly on compact subsets, as $\varepsilon, \delta \to 0^+$.

We now show that $G_{(\varepsilon,\delta)} \in \mathcal{H}_m$. Let $\alpha = (\alpha', \alpha_{n+1})$ be a multiindex, $\alpha \le m$. Then, for a suitable constant c',

$$\partial^\alpha_\zeta G_{(\varepsilon,\delta)}(\zeta) = c' \int_{\mathcal{U}} \frac{\overline{\omega}'^{\alpha'}}{\big(-\varepsilon i\omega_{n+1}+1\big)^q} \cdot \frac{\partial^m_{\zeta_{n+1}} F(\omega)}{Q^{m+|\alpha|}(\zeta,\omega)} \rho(\omega)^{2m-n-2}\, d\omega$$
$$=: \int_{\mathcal{U}} K(\zeta,\omega) \partial^m_{\zeta_{n+1}} F(\omega) \rho(\omega)^{2m-n-2}\, d\omega.$$

Arguing as in the proof of Lemma 4.2 and selecting q sufficiently large, we obtain that $\partial_\zeta^\alpha G_{(\varepsilon,\delta)} \in H^2$ for $|\alpha| \leq m$, $\partial_{\zeta_{n+1}}^m G_{(\varepsilon,\delta)} \in A^2_{2m-n-2}$ and $\lim_{|\zeta'|\leq R,\, \mathrm{Im}\, \zeta_{n+1}\to+\infty} \partial_{\zeta_j} G_{(\varepsilon,\delta)}(\zeta) = 0$, for $j = 1,\dots,n+1$. Thus, $G_{(\varepsilon,\delta)}$ belongs to $\mathcal{D}_{(m)} \cap \mathcal{H}_m$. The convergence of $G_{(\varepsilon,\delta)}$ to F in $\mathcal{D}_{(m)}$ follows with the same argument as in the proof of Lemma 4.2. The proof is therefore complete. □

In order to prove Theorem 3 we need the analogue of Lemma 3.1, in the case $\nu = -n-2$.

Lemma 5.3 Let $\tau \in \mathcal{L}^2_{-n-2}$. For $(z,t,h) = \Psi(\zeta)$ with $\zeta \in \mathcal{U}$ define

$$\widetilde{F}(z,t,h) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 \mathrm{tr}\Big(\tau(\lambda)\big(e^{\lambda h}\sigma_\lambda[z,t]^* - e^\lambda \sigma_\lambda[0,0]^*\big)\Big)\,|\lambda|^n\,d\lambda.$$

Then, F is holomorphic in $\mathcal{U}$.

Proof Firstly we show that F is defined by an absolutely convergent integral. From (25) and (26) we obtain that

$$\begin{aligned}
&\Big|\mathrm{tr}\Big(\tau(\lambda)\big(e^{\lambda h}\sigma_\lambda[z,t]^* - e^\lambda \sigma_\lambda[0,0]^*\big)\Big)\Big| \\
&\quad\leq \Big|\big\langle \tau(\lambda)e_0,\, (e^{\lambda(h+it+\frac{|z|^2}{4})} - e^\lambda)e_0\big\rangle_{\mathcal{F}^\lambda} + \sum_{\alpha\neq 0}\langle \tau(\lambda)e_\alpha,\ P_0\sigma_\lambda[z,t]e_\alpha\rangle_{\mathcal{F}^\lambda}\Big| \\
&\quad\leq \|\tau(\lambda)\|_{\mathrm{HS}}\left(\Big|e^{\lambda(h+it+\frac{|z|^2}{4})} - e^\lambda\Big| + \Big(1 - e^{\frac{\lambda}{2}|z|^2}\Big)^{\frac12}\right).
\end{aligned}$$

Therefore, using (27) we have

$$\begin{aligned}
&\int_{-\infty}^0 \mathrm{tr}\Big(\tau(\lambda)\big(e^{\lambda h}\sigma_\lambda[z,t]^* - e^\lambda \sigma_\lambda[0,0]^*\big)\Big)\,|\lambda|^n\,d\lambda \\
&\quad\leq \|\tau\|_{\mathcal{L}^2_{-n-2}}\left(\int_{-1}^0 \left(\Big|e^{\lambda(h+it+\frac{|z|^2}{4})} - e^\lambda\Big| + \Big(1 - e^{\frac{\lambda}{2}|z|^2}\Big)^{\frac12}\right)^2 |\lambda|^{-1}\,d\lambda\right)^{1/2} \\
&\qquad + \|\tau\|_{\mathcal{L}^2_{-n-2}}\left(\int_{-\infty}^{-1}\Big(e^{\lambda h} + e^\lambda\Big)^2 |\lambda|^{-1}\,d\lambda\right)^{1/2} \\
&\quad< \infty.
\end{aligned}$$

These two last inequalities also show that $\widetilde{F}(z,t,h)$ is locally uniformly bounded in $(z,t,h) \in \mathbf{U}$. The holomorphicity of F follows arguing as in the proof of Lemma 3.1. □

We can now prove Theorem 3.

Proof of Theorem 3. We first assume that $F \in \mathcal{D}_{(m)}(\mathbf{i})$. Then, from Lemma 5.2 and a minor modification of the proof of Theorem 2, we obtain (20) and (21). If $F \in \mathcal{D}_{(m)}$ does not vanish in $\mathbf{i}$, we apply the proof to $F - F(\mathbf{i})$ and we are done.

Conversely, let F be given by (20). Then, Lemma 5.3 guarantees the holomorphicity of F in $\mathcal{U}$ and Plancherel's formula gives (21). □

Corollary 5.4 Let m be a positive integer, $m > (n+1)/2$. Then, the reproducing kernel K, expressed with respect to the inner product in $\mathcal{D}_{(m)}$ is given by

$$K(\omega,\zeta) = 1 + \gamma_{n,m} \log \frac{Q(\omega,\mathbf{i})Q(\mathbf{i},\zeta)}{Q(\omega,\zeta)} ,$$

where $\gamma_{n,m} = \frac{2^{2m-n}}{\Gamma(2m-n-1)(2\pi)^{n+1}}$.

Proof Let τ_F denotes the element of $\mathcal{L}^2_{-n-2}$ such that

$$\begin{aligned}\widetilde{F}(z,t,h) = &\frac{1}{(2\pi)^{n+1}} \int_{-\infty}^0 \operatorname{tr}\Big(\tau_F(\lambda)P_0\big(e^{\lambda h}\sigma_\lambda[z,t]^* - e^{\lambda}\sigma_\lambda[0,0]^*\big)\Big)\,|\lambda|^n d\lambda \\ &+\widetilde{F}(0,0,1).\end{aligned}$$

Also, by the definition of reproducing kernel, (21), and $(z,t,h) = \Psi(\zeta)$, and writing $K(\zeta,\cdot) = K_\zeta$, we have

$$\begin{aligned}\widetilde{F}(z,t,h) &= F(\zeta) = \langle F, K_\zeta\rangle_{\mathcal{D}_{(m)}} \\ &= \frac{\Gamma(2m-n-1)}{2^{2m-n-1}(2\pi)^{n+1}} \int_{-\infty}^0 \operatorname{tr}\big(\tau_F(\lambda)\tau_{K_\zeta}(\lambda)^*\big)\,|\lambda|^{2n+1} d\lambda \;+ F(\mathbf{i})\overline{K_\zeta(\mathbf{i})}.\end{aligned}$$

Since these two equalities hold for all $\tau \in \mathcal{L}^2_{-n-2}$ we conclude that

$$K_\zeta(\mathbf{i}) = 1 \quad\text{and}\quad \tau_{K_\zeta}(\lambda) = C|\lambda|^{-n-1}P_0\big(e^{\lambda h}\sigma_\lambda[z,t] - e^{\lambda}\sigma_\lambda[0,0]\big)\,,$$

where $C = \frac{2^{2m-n-1}}{\Gamma(2m-n-1)}$. Thus, from Lemma 5.3, we obtain

$$\begin{aligned}&\widetilde{K}_{(z,t,h)}(w,s,k) \\ &\quad= \frac{C}{(2\pi)^{n+1}} \int_{-\infty}^0 \operatorname{tr}\Big(P_0\left(e^{\lambda h}\sigma_\lambda[z,t] - e^{\lambda}\sigma_\lambda[0,0]\right) P_0\left(e^{\lambda k}\sigma_\lambda[w,s]^* - e^{\lambda}\sigma_\lambda[0,0]^*\right)\Big) \\ &\quad\times |\lambda|^{-1} d\lambda + 1.\end{aligned}$$

Exploiting (26) the conclusion follows. □

We conclude the section providing the justification for referring to the space $\mathcal{D}_{(m)}$ as the Dirichlet space on the Siegel half-space. We denote by $\dot{\mathcal{D}}(B)$ the Dirichlet space modulo the constant functions on the unit ball $B \subseteq \mathbb{C}^{n+1}$. If f is holomorphic on B, $f(\zeta) = \sum_{|\alpha|\geq 0} a_\alpha \zeta^\alpha$, the norm in $\dot{\mathcal{D}}(B)$ is given by

$$\|f\|^2_{\dot{\mathcal{D}}(B)} = \sum_{|\alpha|\geq 0} |\alpha| \frac{\alpha!}{|\alpha|!} |a_\alpha|^2 ,$$

see [28]. Then, the reproducing kernel of $\dot{\mathcal{D}}(B)$ is given by

$$K^B(\omega,\zeta) = \frac{(n+1)!}{\pi^{n+1}} \log \frac{1}{1-\omega\cdot\overline{\zeta}}\,,$$

see, e.g. [18] or [29].

Theorem 5.5 Let m be a positive integer, $m > (n+1)/2$ and denote by $\dot{\mathcal{D}}_{(m)}$ the space $\mathcal{D}_{(m)}/\mathbb{C}$, that is, the space $\mathcal{D}_{(m)}$ modulo the constant functions. Then:

(1) The space $\dot{\mathcal{D}}_{(m)}$, identified with $\mathcal{D}_{(m)}(\mathbf{i})$ and endowed with the norm

$$\|F\|_{\dot{\mathcal{D}}_{(m)}} = \|\partial^m_{\zeta_{n+1}} F\|_{A^2_{2m-n-2}}\,,$$

is a Hilbert space with reproducing kernel

$$K(\omega,\zeta) = \gamma_{n,m} \log \frac{Q(\omega,\mathbf{i})Q(\mathbf{i},\zeta)}{Q(\omega,\zeta)}\,, \tag{36}$$

where $\gamma_{n,m}$ is as in Corollary 5.4.

(2) For every $\varphi \in \mathrm{Aut}(\mathcal{U})$ and every $F \in \dot{\mathcal{D}}_{(m)}$ it holds

$$\|F\circ\varphi\|_{\dot{\mathcal{D}}_{(m)}} = \|F\|_{\dot{\mathcal{D}}_{(m)}}. \tag{37}$$

(3) The space $\dot{\mathcal{D}}_{(m)}$ is isometrically equivalent to $\dot{\mathcal{D}}(B)$, the Dirichlet space modulo the constant functions on the unit ball $B \subset \mathbb{C}^{n+1}$. In particular, the space $\dot{\mathcal{D}}_{(m)}$ is the unique Hilbert space of holomorphic functions on $\mathcal{U}$ satisfying property (37).

Proof The proof of (1) is straightforward. We now prove (2). It is enough to prove it for a $\varphi \in \mathrm{Aut}(\mathcal{U})$ of type (i), (ii), (iii) and (iv) described in Lemma 2.1. If φ falls in the cases (i), (ii) or (iii), then it is immediate to obtain (37) by direct computations. If φ is of type (iv) we observe that

$$(K_\omega\circ\varphi)(\zeta) = K(\varphi(\zeta),\omega) = K(\zeta,\varphi(\omega)) = K_{\varphi(\omega)}(\zeta).$$

In particular,

$$K(\varphi(\zeta),\varphi(\omega)) = K(\zeta,\omega).$$

Thus, if $\omega_1,\dots,\omega_N \in \mathcal{U}$ and $F(\zeta) = \sum_{k=1}^N \alpha_k K(\zeta,\omega_k)$, we have

$$\begin{aligned}\|F\circ\varphi\|^2_{\dot{\mathcal{D}}_{(m)}} &= \Big\|\sum_{k=1}^N \alpha_k K(\varphi(\cdot),\omega_k)\Big\|^2_{\dot{\mathcal{D}}_{(m)}} = \Big\|\sum_{k=1}^N \alpha_k K(\cdot,\varphi(\omega_k))\Big\|^2_{\dot{\mathcal{D}}_{(m)}}\\ &= \sum_{j,k=1}^N \overline{\alpha_j}\alpha_k K(\varphi(\omega_j),\varphi(\omega_k)) = \sum_{j,k=1}^N \overline{\alpha_j}\alpha_k K(\omega_j,\omega_k)\\ &= \|F\|^2_{\dot{\mathcal{D}}_{(m)}}\,,\end{aligned}$$

as we wished to show. Since the functions of the form $\sum_{k=1}^{N} \alpha_k K(\zeta, \omega_k)$ are dense in $\dot{\mathcal{D}}_{(m)}$ the conclusion for a generic $F \in \dot{\mathcal{D}}_{(m)}$ follows.

At last, (3) follows from the following observation. Let K^B and $K^{\mathcal{U}}$ denote the reproducing kernel of $\dot{D}(B)$ and $\dot{\mathcal{D}}_{(m)}$ respectively. Then,

$$K^B(\omega, \zeta) = \frac{(n+1)!}{\pi^{n+1}\gamma_{n,m}} K^{\mathcal{U}}(\mathcal{C}(\omega), \mathcal{C}(\zeta)),$$

where $\mathcal{C}$ denote the Cayley transform and $\gamma_{n,m}$ is as in (36). From this it is easy to deduce that the map

$$F \mapsto \frac{(n+1)!}{\pi^{n+1}\gamma_{n,m}} (F \circ \mathcal{C})$$

is a surjective isometry from $\dot{\mathcal{D}}_{(m)}$ onto $\dot{\mathcal{D}}(B)$ as we wished to show. Hence, the uniqueness of $\dot{\mathcal{D}}_{(m)}$ follows from the analogous result for the space $\dot{\mathcal{D}}(B)$, see [1, 28]. □

6 The Drury–Arveson Norm on the Unit Ball and Final Remarks

Following [18], we set

$$\mathscr{R}_0 = \mathrm{Id}, \qquad \mathscr{R}_k = \Big(\mathrm{Id} + \frac{R}{k}\Big)\mathscr{R}_{k-1} \quad \text{for } k = 1, 2, \dots,$$

where R denotes the radial derivative $Rf(\zeta) = \sum_{j=1}^{n+1} \zeta_j \partial_{\zeta_j} f(\zeta)$. Then, we have the following result on the exact norm in $\mathrm{DA}(B)$.

Theorem 6.1 If $f \in \mathrm{DA}(B)$, then

$$\|f\|^2_{\mathrm{DA}(B)} = n \frac{n!}{\pi^{n+1}} \int_B \frac{(1-|\zeta|^2)^{n-1}}{|\zeta|^{2n}} \big|\mathscr{R}_n f(\zeta)\big|^2 d\zeta.$$

This is an elementary computation that follows from the identity $\mathscr{R}_n z^\alpha = \frac{(n+|\alpha|)!}{n!|\alpha|!} z^\alpha$, for every multiindex α.

We believe this work raises some interesting questions. We first mention the characterization of the Carleson measures and of the multiplier algebra for the scale of spaces studied in this work. Moreover, these spaces depend on the parameter ν where $\nu \geq -n-2$. It would be interesting to study the spaces corresponding to the values $\nu < -n-2$. Furthermore, we would like to study the analogous Banach spaces, whose underlying norm is the L^p-norm, with $p \neq 2$. Finally, it would be interesting to extend the results in this paper to the more general setting of Siegel domains of type II . We plan to come back to these problems in future works.

References

1. Arazy, J., Fisher, S.D.: The uniqueness of the Dirichlet space among Möbius-invariant Hilbert spaces. Ill. J. Math. **29**(3), 449–462 (1985)

2. Arcozzi, N., Rochberg, R., Sawyer, E.: Carleson measures for the Drury–Arveson Hardy space and other Besov–Sobolev spaces on complex balls. Adv. Math. **218**(4), 1107–1180 (2008)
3. Arcozzi, N., Rochberg, R., Sawyer, E.: Two variations on the Drury–Arveson space. In: Hilbert Spaces of Analytic Functions, CRM Proceedings and Lecture Notes, vol. 51, pp. 41–58. American Mathematical Society, Providence (2010)
4. Arveson, W.: Subalgebras of C^*-algebras. II. Acta Math. **128**(3–4), 271–308 (1972)
5. Beatrous, F., Burbea, J.: Holomorphic Sobolev spaces on the ball. Dissertationes Math. (Rozprawy Mat.), 276:60 (1989)
6. Békollé, D., Bonami, A., Garrigós, G., Nana, C., Peloso, M. M., Ricci, F.: Lecture notes on Bergman projectors in tube domains over cones: an analytic and geometric viewpoint. IMHOTEP J. Afr. Math. Pures Appl. **5**:Exp. I, front matter + ii + 75 (2004)
7. Costea, Ş., Sawyer, E., Wick, B.D.: The corona theorem for the Drury–Arveson Hardy space and other holomorphic Besov–Sobolev spaces on the unit ball in $\mathbb{C}^n$. Anal. PDE **4**(4), 499–550 (2011)
8. Duren, P., Gallardo-Gutiérrez, E.A., Montes-Rodriguez, A.: A Paley–Wiener theorem for Bergman spaces with application to invariant subspaces. Bull. Lond. Math. Soc. **39**(3), 459–466 (2007)
9. Drury, S.W.: A generalization of von Neumann's inequality to the complex ball. Proc. Am. Math. Soc. **68**(3), 300–304 (1978)
10. Feldman, M.: Mean oscillation, weighted Bergman spaces, and Besov spaces on the Heisenberg group and atomic decompositions. J. Math. Anal. Appl. **158**(2), 376–395 (1991)
11. Faraut, J., Korányi, A.: Analysis on Symmetric Cones. Oxford Mathematical Monographs. The Clarendon Press, New York (1994)
12. Folland, G.B.: Harmonic Analysis in Phase Space. Annals of Mathematics Studies, vol. 122. Princeton University Press, Princeton (1989)
13. Kucik, A.S.: Carleson measures for Hilbert spaces of analytic functions on the complex half-plane. J. Math. Anal. Appl. **445**(1), 476–497 (2017)
14. Kucik, A.S.: Multipliers of Hilbert spaces of analytic functions on the complex half-plane. Oper. Matrices **11**(2), 435–453 (2017)
15. Lax, P.D.: Translation invariant spaces. Acta Math. **101**, 163–178 (1959)
16. Ogden, R.D., Vági, S.: Harmonic analysis of a nilpotent group and function theory of Siegel domains of type II. Adv. Math. **33**(1), 31–92 (1979)
17. Paley, R.E.A.C., Wiener, N.: Fourier Transforms in the Complex Domain, vol. 19. American Mathematical Society, Providence (1987[1934]). Reprint of the 1934 original
18. Peloso, M.M.: Möbius invariant spaces on the unit ball. Mich. Math. J. **39**(3), 509–536 (1992)
19. Poltoratski, A.: Toeplitz approach to problems of the uncertainty principle, CBMS Regional Conference Series in Mathematics, vol. 121. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence (2015)
20. Ricci, F.: Harmonic Analysis on the Heisenberg Group. Unpublished notes, Cortona, pp. ii+67 (1992)
21. Ricci, F., Taibleson, M.: Boundary values of harmonic functions in mixed norm spaces and their atomic structure. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10**(1), 1–54 (1983)
22. Richter, S., Sunkes, J.: Hankel operators, invariant subspaces, and cyclic vectors in the Drury–Arveson space. Proc. Am. Math. Soc. **144**(6), 2575–2586 (2016)
23. Seip, K.: Interpolation and Sampling in Spaces of Analytic Functions. University Lecture Series, vol. 33. American Mathematical Society, Providence (2004)
24. Stein, E.M.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, vol. 43. Princeton University Press, Princeton (1993). With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III
25. Tchoundja, E.: Carleson measures for the generalized Bergman spaces via a $T(1)$-type theorem. Ark. Mat. **46**(2), 377–406 (2008)
26. Volberg, A., Wick, B.D.: Bergman-type singular integral operators and the characterization of Carleson measures for Besov–Sobolev spaces and the complex ball. Am. J. Math. **134**(4), 949–992 (2012)
27. Wallach, N.R.: Harmonic Analysis on Homogeneous Spaces. Pure and Applied Mathematics, vol. 19. Marcel Dekker, Inc., New York (1973)
28. Zhu, K.: Möbius invariant Hilbert spaces of holomorphic functions in the unit ball of C^n. Trans. Am. Math. Soc. **323**(2), 823–842 (1991)
29. Zhu, K.: Spaces of Holomorphic Functions in the Unit Ball. Graduate Texts in Mathematics, vol. 226. Springer, New York (2005)

Affiliations

Nicola Arcozzi[1] · Alessandro Monguzzi[2] · Marco M. Peloso[2] · Maura Salvatori[2]

Nicola Arcozzi
nicola.arcozzi@unibo.it

Alessandro Monguzzi
alessandro.monguzzi@unimi.it

Maura Salvatori
maura.salvatori@unimi.it

1 Dipartimento di Matematica, Alma Mater Studorium Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy

2 Dipartimento di Matematica, Università degli Studi di Milano, Via C. Saldini 50, 20133 Milan, Italy