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Two Theorems on Convergence of Schrödinger Means

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Abstract
Localization and convergence almost everywhere of Schrödinger means are studied.

Keywords Schrödinger equation · Convergence · Localizaton · Sobolev spaces

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1 Introduction

For $f \in L^2(\mathbb{R}^n)$, $n \geqslant 1$ and $a > 1$ we set

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx, \quad \xi \in \mathbb{R}^n,$$

and

$$S_t f(x) = \int_{\mathbb{R}^n} e^{i\xi\cdot x} e^{it|\xi|^a} \widehat{f}(\xi)d\xi, \quad x \in \mathbb{R}^n, \quad t \geqslant 0.$$

For $a = 2$ and f belonging to the Schwartz class $\mathscr{S}(\mathbb{R}^n)$ we set $u(x,t) = S_t f(x)/(2\pi)^n$. It then follows that $u(x,0) = f(x)$ and u satisfies the Schrödinger equation $i\partial u/\partial t = \Delta u$.

We introduce Sobolev spaces $H_s = H_s(\mathbb{R}^n)$ by setting

$$H_s = \{f \in \mathscr{S}'; \|f\|_{H_s} < \infty\}, \quad s \in \mathbb{R},$$

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where

$$\|f\|_{H_s} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

In the case n = 1 it is well-known (see Sjölin [7] and Vega [9] and in the case $a = 2$ Carleson [3] and Dahlberg and Kenig [4]) that

$$\lim_{t\to 0} (2\pi)^{-n} S_t f(x) = f(x) \tag{1}$$

almost everywhere if $f \in H_{1/4}$. Also it is known that $H_{1/4}$ can not be replaced by H_s if $s < 1/4$.

Assuming $n \geqslant 2$ and $a = 2$ Bourgain [1] has proved that (1) holds almost everywhere if $f \in H_s$ and $s > 1/2 - 1/4n$. On the other hand Bourgain [2] has proved that $s \geqslant n/2(n+1)$ is necessary for convergence for $a = 2$ and all $n \geqslant 2$. In the case $n = 2$ and $a = 2$, Du, Guth, and Li [5] proved that the condition $s > 1/3$ is sufficient. Recently Du and Zhang [6] proved that the condition $s > n/2(n+1)$ is sufficient for $a = 2$ and all $n \geqslant 3$.

In the case $a > 1, n = 2$, it is known that (1) holds almost everywhere if $f \in H_{1/2}$ and in the case $a > 1, n \geqslant 3$, convergence has been proved for $f \in H_s$ with $s > 1/2$ (see [7] and [9]).

If $f \in L^2(\mathbb{R}^n)$ then $(2\pi)^{-n} S_t f \to f$ in L^2 as $t \to 0$. It follows that there exists a sequence $(t_k)_1^\infty$ satisfying

$$1 > t_1 > t_2 > t_3 > \cdots > 0 \text{ and } \lim_{k\to\infty} t_k = 0 \tag{2}$$

such that

$$\lim_{k\to\infty} (2\pi)^{-n} S_{t_k} f(x) = f(x)$$

almost everywhere.

We shall here study the problem of deciding for which sequences $(t_k)_1^\infty$ one has

$$\lim_{k\to\infty} (2\pi)^{-n} S_{t_k} f(x) = f(x)$$

almost everywhere if $f \in H_s$. We have the following result.

Theorem 1 *Assume $n \geqslant 1$ and $a > 1$ and $s > 0$. We assume that (2) holds and that $\sum_{k=1}^\infty t_k^{2s/a} < \infty$ and $f \in H_s(\mathbb{R}^n)$. Then*

$$\lim_{k\to\infty} (2\pi)^{-n} S_{t_k} f(x) = f(x)$$

for almost every x in $\mathbb{R}^n$.

Now assume $n = 1$, $a > 1$, and $0 \leqslant s < 1/4$. In Sjölin [8] we studied the problem if there is localization or localization almost everywhere for the above operators S_t and the functions $f \in H_s$ with compact support, that is, do we have

$$\lim_{t\to 0} S_t f(x) = 0$$

everywhere or almost everywhere in $\mathbb{R}^n \backslash (\operatorname{supp} f)$?

It is proved in [8] that there is no localization or localization almost everywhere of this type for $0 \leqslant s < 1/4$. In fact the following theorem was proved in Sjölin [8].

Theorem A *There exist two disjoint compact intervals I and J in $\mathbb{R}$ and a function f which belongs to H_s for all $s < 1/4$, with the properties that $(\operatorname{supp} f) \subset I$ and for every $x \in J$ one does not have*

$$\lim_{t\to 0} S_t f(x) = 0.$$

Let ω be a continuous and decreasing function on $[0, \infty)$. Assume that $\omega(t) \to 0$ as $t \to \infty$. Set

$$H_\omega = \{f \in \mathscr{S}'; \|f\|_{H_\omega} < \infty\}$$

where

$$\|f\|_{H_\omega} = \left(\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 (1+\xi^2)^{1/4} \omega(|\xi|) d\xi \right)^{1/2}$$

We have the following result.

Theorem 2 *The function f in theorem A can be chosen so that $f \in H_\omega$.*

Theorem 2 shows that the sufficient condition $f \in H_{1/4}$ for convergence almost everywhere and localization almost everywhere of Schrödinger means is very sharp. In the case a = 2 Theorem 2 was obtained in 2009 (unpublished). After proving Theorem 2 we shall use Theorem 1 to make a remark on the Schrödinger means $S_t f(x)$ for the function f which was constructed in [8] to prove Theorem A.

2 Proofs

In the proof of Theorem 1 we shall need the following lemma.

Lemma 1 *Assume $n \geqslant 1$, $a > 1$, $0 < s < 1$, and $0 < \delta < 1$. Set*

$$m(\xi) = \frac{e^{i\delta|\xi|^a} - 1}{(1+|\xi|^2)^{s/2}}, \quad \xi \in \mathbb{R}^n.$$

Then one has

$$\|m\|_\infty \leqslant C\delta^{s/a}$$

where the constant C does not depend on δ, and $\|m\|_\infty$ denotes the norm of m in $L^\infty(\mathbb{R}^n)$.

Proof of Lemma 1. We shall write $A \lesssim B$ if there is a constant C such that $A \leqslant CB$.

In the case $|\xi| \geqslant \delta^{-1/a}$ one has

$$|\xi|^s \geqslant \delta^{-s/a} \text{ and } |m(\xi)| \lesssim \frac{1}{|\xi|^s} \leqslant \delta^{s/a}.$$

Then assume $0 \leqslant |\xi| \leqslant 1$. We obtain

$$|m(\xi)| \lesssim \delta|\xi|^a \leqslant \delta \leqslant \delta^{s/a}.$$

In the remaining case $1 < |\xi| < \delta^{-1/a}$ one obtains

$$|m(\xi)| \lesssim \frac{\delta|\xi|^a}{|\xi|^s} = \delta|\xi|^{a-s} \lesssim \delta\delta^{-(a-s)/a} = \delta\delta^{-1+s/a} = \delta^{s/a}$$

and the proof of Lemma 1 is complete. □

We shall then give the proof of Theorem 1.

Proof of Theorem 1. We may assume $0 < s < 1$. We set

$$h_k(x) = (2\pi)^{-n} S_{t_k} f(x) - f(x), \quad x \in \mathbb{R}^n, \text{ for } k = 1, 2, 3, \ldots$$

We have $f \in H_s$ and we define g by taking

$$\widehat{g}(\xi) = \widehat{f}(\xi)(1 + |\xi|^2)^{s/2}.$$

It then follows that $g \in L^2(\mathbb{R}^n)$.

We have

$$S_{t_k} f(x) = \int e^{ix\cdot\xi} e^{it_k|\xi|^a} (1 + |\xi|^2)^{-s/2} \widehat{g}(\xi) d\xi$$

and

$$f(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} (1 + |\xi|^2)^{-s/2} \widehat{g}(\xi) d\xi.$$

Hence

$$h_k(x) = (2\pi)^{-n} \int e^{ix\cdot\xi}(e^{it_k|\xi|^a} - 1)(1+|\xi|^2)^{-s/2}\widehat{g}(\xi)d\xi$$
$$= (2\pi)^{-n} \int e^{ix\cdot\xi} m(\xi)\widehat{g}(\xi)d\xi,$$

where

$$m(\xi) = (e^{it_k|\xi|^a} - 1)(1+|\xi|^2)^{-s/2}.$$

According to Lemma 1 we have $\|m\|_\infty \lesssim t_k^{s/a}$ and applying the Plancherel theorem we obtain

$$\|h_k\|_2^2 = c\int |m(\xi)\widehat{g}(\xi)|^2 d\xi \lesssim \|m\|_\infty^2 \int |\widehat{g}(\xi)|^2 d\xi \lesssim t_k^{2s/a}\|f\|_{H_s}^2.$$

It follows that

$$\sum_1^\infty \int |h_k|^2 dx \lesssim (\sum_1^\infty t_k^{2s/a})\|f\|_{H_s}^2 < \infty$$

and applying the theorem on monotone convergence one also obtains

$$\int \left(\sum_1^\infty |h_k|^2\right) dx < \infty.$$

We conclude that $\sum_1^\infty |h_k|^2$ is convergent almost everywhere and hence $\lim_{k\to\infty} h_k(x) = 0$ and

$$\lim_{k\to\infty} (2\pi)^{-n} S_{t_k} f(x) = f(x)$$

almost everywhere. □

Now assume $n = 1$ and $a > 1$. We set

$$m(\xi) = e^{i|\xi|^a}, \quad \xi \in \mathbb{R},$$

and let K denote the Fourier transform of m so that $K \in \mathscr{S}'(\mathbb{R})$. According to Sjölin [8] p.142, $K \in C^\infty(\mathbb{R})$ and there exists a number $\alpha \geqslant 0$ such that

$$|K(x)| \lesssim 1 + |x|^\alpha \text{ for } x \in \mathbb{R}$$

For $t > 0$ it is then clear that

$$e^{it|\xi|^a} = m(t^{1/a}\xi)$$

has the Fourier transform

$$K_t(y) = t^{-1/a} K(t^{-1/a} y).$$

It follows that $S_t f(x) = K_t \star f(x)$ for $f \in L^2(\mathbb{R}^m)$ with compact support. We let $\check{g}$ denote the inverse Fourier transform of g and choose $g \in \mathscr{S}(\mathbb{R})$ such that supp$\check{g} \subset (-1, 1)$ and $\check{g}(0) \neq 0$. We set

$$f_v(x) = e^{-ix/v^2} \check{g}(x/v), \quad 0 < v < 1, \quad x \in \mathbb{R}.$$

According to [7], p.143, one has $\widehat{f_v}(\xi) = vg(v\xi + 1/v)$ and

$$\| f_v \|_{H_s} \lesssim v^{1/2-2s} \text{ for } 0 < s < 1/4.$$

We shall state three lemmas from [8].

Lemma 2 *There exist positive numbers c_0, δ and v_0 such that*

$$|S_{xv^{2a-2}/a} f_v(x)| \geqslant c_0$$

for $0 < v < v_0$ *and* $0 < x < \delta$.

In the remaining part of this paper δ and v_0 are given by Lemma 2. We may also assume that $\delta < 1$.

Lemma 3 *For* $0 < v < \min(v_0, \delta/4)$, $0 < t < 1$, *and* $\delta/2 < x < \delta$ *one has*

$$|S_t f_v(x)| \lesssim \frac{v}{t^\gamma}$$

where $\gamma = (1 + \alpha)/a > 0$.

Lemma 4 *For* $0 < v < \min(v_0, \delta/4)$, $0 < t < 1$, *and* $\delta/2 < x < \delta$ *one has*

$$|S_t f_v(x)| \lesssim \frac{t}{v^\beta}$$

where $\beta = 2a$.

We shall use these lemmas to prove Theorem 2.

Proof of Theorem 2. Now let v_1 satisfy $0 < v_1 < \min(v_0, \delta/4)$ and set $\epsilon_k = 2^{-k}$, $k = 1, 2, 3, \ldots$

We also set $\mu = \max((2a-2)\gamma, \beta/(2a-2))$ and choose v_k, $k = 2, 3, 4, \ldots$, such that $0 < v_k \leqslant \epsilon_k v_{k-1}^\mu$ and

$$\sum_{k=1}^{\infty} \sqrt{\omega(1/v_k^{1/2})} < \infty.$$

We then set $f = \sum_{k=1}^{\infty} f_{v_k}$ and shall prove that $f \in H_\omega$.

Arguing as in [8, pp. 145–147], it follows from Lemmas 2, 3, and 4 that with $t_k(x) = x v_k^{2a-2}/a$ one has

$$|S_{t_k(x)} f(x)| \geqslant c_0 > 0$$

for $\delta/2 < x < \delta$ and $k \geqslant k_0$. Hence we do not have $\lim_{t\to 0} S_t f(x) = 0$ in the interval $(\delta/2, \delta)$. Taking $I = [-v_1, v_1]$ and $J \subset (\delta/2, \delta)$ we have supp$f \subset I$ and for every $x \in J$ one does not have $\lim_{t\to 0} S_t f(x) = 0$. Thus Theorem 2 follows. It remains to prove that $f \in H_\omega$.

We have

$$\|f_v\|_{H_\omega}^2 = \int |\widehat{f_v}(\xi)|^2 (1+\xi^2)^{1/4} \omega(|\xi|) d\xi \lesssim I_1 + I_2,$$

where

$$I_1 = \int_{-1}^{1} |\widehat{f_v}(\xi)|^2 d\xi \leqslant C v^2$$

and

$$I_2 = \int |\widehat{f_v}(\xi)|^2 |\xi|^{1/2} \omega(|\xi|) d\xi.$$

It follows that

$$\begin{aligned} I_2 &= \int v^2 |g(v\xi + 1/v)|^2 |\xi|^{1/2} \omega(|\xi|) d\xi \\ &= \int v^{1/2} |g(\eta + 1/v)|^2 |\eta|^{1/2} \omega(\frac{|\eta|}{v}) d\eta = \\ &= v^{1/2} \int |g(\xi)|^2 |\xi - 1/v|^{1/2} \omega(\frac{|\xi - 1/v|}{v}) \xi \leqslant C v^{1/2} \\ &\times \int_{|\xi - 1/v| \leqslant v^{1/2}} |g(\xi)|^2 v^{1/4} d\xi \\ &+ C v^{1/2} \int_{|\xi - 1/v| \geqslant v^{1/2}} |g(\xi)|^2 (|\xi|^{1/2} + v^{-1/2}) \omega(v^{-1/2}) d\xi \\ &\leqslant C v^{3/4} + C\omega(v^{-1/2}). \end{aligned}$$

Hence

$$\|f_v\|_{H_\omega}^2 \lesssim v^{3/4} + \omega(v^{-1/2}), \quad 0 < v < 1,$$

and

$$\|f_v\|_{H_\omega} \lesssim v^{3/8} + \sqrt{\omega(v^{-1/2})}.$$

We have $f = \sum_1^\infty f_{v_k}$ and it follows that

$$\|f\|_{H_\omega} \leqslant \sum_1^\infty \|f_{v_k}\|_{H_\omega} \lesssim \sum_1^\infty v_k^{3/8} + \sum_1^\infty \sqrt{\omega(v_k^{-1/2})} < \infty$$

since $v_k \leqslant \epsilon_k$.

We conclude that $f \in H_\omega$ and the proof of Theorem 2 is complete. □

Remark 1 In Sjölin [8] the function f in Theorem A is given by the formula

$$f = \sum_1^\infty f_{v_k},$$

where v_k is defined by taking $0 < v_1 < \min(v_0, \delta/4)$ and $v_k = \epsilon_k v_{k-1}^\mu$ for $k = 2, 3, 4, \ldots$ Here $\epsilon_k = 2^{-k}$ and $\mu > 0$ is given in the proof of Theorem 2. Also let the intervals I and J be defined as in the proof of Theorem 2. We then set $t_k(x) = x v_k^{2a-2}/a$ for $x \in J$ and $k = 1, 2, 3, \ldots$

It is proved in [8] that for every $x_0 \in J$

$$\text{one does not have } \lim_{k\to\infty} S_{t_k(x_0)} f(x_0) = 0. \tag{3}$$

We now fix $x_0 \in J$ and shall use Theorem 1 to prove that although (3) holds one also has

$$\lim_{k\to\infty} S_{t_k(x_0)} f(x) = 0 \text{ for almost every } x \in J. \tag{4}$$

We have $v_k < \epsilon_k$ and it follows that

$$0 < t_k(x_0) \leqslant \epsilon_k^{2a-2}$$

and

$$\sum_1^\infty (t_k(x_0))^{2s/a} \leqslant \sum_1^\infty 2^{-k(2a-2)2s/a} < \infty$$

for $0 < s < 1/4$. Also $f \in H_s$ for $0 < s < 1/4$ and (4) follows from an application of Theorem 1.

Remark 2 In the case $a = 2$ one has $\mu = 2$ and $v_k = \epsilon_k v_{k-1}^2$, and we also have $0 < v_1 < 1/4$. It can be proved that it follows that

$$v_k = 4 \cdot 2^{k-d2^k}$$

where d is a constant and $d > 2$.

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