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Phaseless Sampling and Reconstruction of Real-Valued Signals in Shift-Invariant Spaces

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Abstract

Sampling in shift-invariant spaces is a realistic model for signals with smooth spectrum. In this paper, we consider phaseless sampling and reconstruction of real-valued signals in a high-dimensional shift-invariant space from their magnitude measurements on the whole Euclidean space and from their phaseless samples taken on a discrete set with finite sampling density. The determination of a signal in a shift-invariant space, up to a sign, by its magnitude measurements on the whole Euclidean space has been shown in the literature to be equivalent to its nonseparability. In this paper, we introduce an undirected graph associated with the signal in a shift-invariant space and use connectivity of the graph to characterize nonseparability of the signal. Under the local complement property assumption on a shift-invariant space, we find a discrete set with finite sampling density such that nonseparable signals in the shift-invariant space can be reconstructed in a stable way from their phaseless samples taken on that set. In this paper, we also propose a reconstruction algorithm which provides an approximation to the original signal when its noisy phaseless samples are available only. Finally, numerical simulations are performed to demonstrate the robustness of the proposed algorithm to reconstruct box spline signals from their noisy phaseless samples.

Keywords Shift invariant spaces · Phaseless sampling and reconstruction · Undirected graph · Sampling density · Local complementary property · Reconstruction algorithm

Mathematics Subject Classification 94A12 · 94A20 · 42C15

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1 Introduction

In this paper, we consider the phaseless sampling and reconstruction problem of whether a real-valued signal f on $\mathbb{R}^d$ is determined, up to a sign, by its magnitude measurements $|f(x)|$ on $\mathbb{R}^d$ or on a discrete subset $X \subset \mathbb{R}^d$. The above problem is inherently ill-posed and it can be solved only if we have some extra information about the signal f.

The additional knowledge about the signals in this paper is that they live in a shift-invariant space

$$V(\phi) := \left\{ \sum_{k \in \mathbb{Z}^d} c(k)\phi(x-k) : \ c(k) \in \mathbb{R} \text{ for all } k \in \mathbb{Z}^d \right\} \tag{1.1}$$

generated by a real-valued continuous function ϕ with compact support. Shift-invariant spaces have been used in wavelet analysis and approximation theory, and sampling in shift-invariant spaces is a realistic model for signals with smooth spectrum, see [4,6,11,22,32] and references therein. Typical examples of shift-invariant spaces include those generated by refinable functions [21,36] and box splines [24,48,49].

The phaseless sampling and reconstruction problem of one-dimensional signals in shift-invariant spaces has been studied in [16,39,40,43,47]. Thakur proved in [47] that one-dimensional real-valued signals in a Paley–Wiener space, the shift-invariant space generated by the sinc function $\frac{\sin \pi t}{\pi t}$, could be reconstructed from their phaseless samples taken at more than twice the Nyquist rate. Reconstruction of one-dimensional signals in a shift-invariant space was studied in [43] when frequency magnitude measurements are available. Unlike signals in the Paley–Wiener space, not all signals in a shift-invariant space generated by a compactly supported function are determined, up to a sign, by their magnitude measurements on the whole line. In [16], the set of signals that can be determined by their magnitude measurements on the real line $\mathbb{R}$ is fully characterized, and a fast algorithm is proposed to reconstruct signals in a shift-invariant space from their phaseless samples taken on a discrete set with finite sampling density. Up to our knowledge, there is no literature available on the phaseless sampling and reconstruction of high-dimensional signals in a shift-invariant space, which is the core of this paper.

The phaseless sampling and reconstruction of signals in a shift-invariant space is an infinite-dimensional phase retrieval problem, which has received considerable attention in recent years [1–3,12,16,27,37,39,40,43,47]. In most of literatures mentioned above, the phase retrieval problem considers determining *all* signals in an infinite-dimensional linear space, up to a global phase, from magnitudes of their frame measurements, which has been characterized by the complement property for the frame. In our consideration of phaseless sampling and reconstruction, the set of signals in a shift-invariant space $V(\phi)$ that are determined, up to a sign, by their magnitudes on the whole Euclidean space is a true nonconvex subset of the entire space $V(\phi)$. So we consider the problem of whether a *particular* signal in the space is determined, up to a sign, by its magnitudes on the whole Euclidean space, which is characterized by its nonseparability.

The paper is organized as follows. An introductory problem about phaseless sampling and reconstruction in the shift-invariant space $V(\phi)$ is to decide when a real-valued signal f is determined, up to a sign, by its magnitudes $|f(x)|, x \in \mathbb{R}^d$. An equivalence has been provided in [16], see Theorem 2.1 in Sect. 2, that the signal f must be nonseparable, i.e., the signal f is not the sum of two nonzero signals in $V(\phi)$ with their supports being essentially disjoint. A natural question that arises is how to determine the nonseparability of a given signal in a shift-invariant space. In Sect. 2, we introduce an undirected graph $\mathcal{G}_f$ for the signal f in a shift-invariant space $V(\phi)$ and use connectivity of the graph $\mathcal{G}_f$ to characterize nonseparability of the signal f, i.e., when it is determined, up to a sign, by the magnitude measurements $|f(x)|, x \in \mathbb{R}^d$, see Theorems 2.4 and 2.6.

In Sect. 3, we consider the preparatory problem whether a signal in a shift-invariant space $V(\phi)$ is determined, up to a sign, by its phaseless samples taken on a discrete set with finite sampling density. In Theorem 3.1, we find a finite set $\Gamma \subset (0, 1)^d$ such that magnitudes $|f(x)|, x \in \mathbb{R}^d$, of any nonseparable signal $f \in V(\phi)$ are determined by their phaseless samples $|f(y)|, y \in \Gamma + \mathbb{Z}^d$. However, the above result does not yield an algorithm to reconstruct a nonseparable signal from its phaseless samples taken on the shift-invariant set $\Gamma + \mathbb{Z}^d$. To deal with phaseless reconstruction, we introduce local complement property for the shift-invariant space $V(\phi)$, which is similar to the complement property for frames in Hilbert/Banach spaces [3,7,9,12,16], see Definition 3.2 and Appendix A. Under the assumption that the shift-invariant space $V(\phi)$ has the local complement property, we are able to find a shift-invariant set $\Gamma + \mathbb{Z}^d$ with finite sampling density on which any nonseparable signal $f \in V(\phi)$ can be recovered, up to a sign, from its phaseless samples $|f(y)|, y \in \Gamma + \mathbb{Z}^d$, see Theorems 3.3 and 3.5 for constructions of the sampling set $\Gamma + \mathbb{Z}^d$ and Sect. 5 for a robust algorithm with linear computational complexity.

The study of stability is pivotal in phaseless sampling and reconstruction. Thakur provided a truncation error estimate in [47] to reconstruct bandlimited signals from their truncated phaseless samples with more than twice of the Nyquist rate; Cahill et al. found in [12] that phase retrievability of signals in an infinite-dimensional Hilbert space from magnitudes of their frame measurements is not uniformly stable; Alaifari et al. discussed the stable reconstruction of ϵ-concentrated signals in [2] from their magnitudes of Gabor measurements; and Grohs and Rathmair showed in [27] that stability of the Gabor phase reconstruction is bounded by the reciprocal of the Cheeger constant of a flat metric. In Theorem 4.1, we establish a stable reconstruction of nonseparable signals in a shift-invariant space from noisy phaseless samples taken on a discrete set with finite sampling density. The above stable reconstruction implies the nonexistence of resonance phenomenon when the noise level is far below the minimal magnitude of amplitude vector of the original signal, see Remark 4.2.

A fundamental problem in phaseless sampling and reconstruction is to design efficient and robust algorithms for signal reconstruction in a noisy environment. Based on the approach in Theorem 4.1, we propose an algorithm to reconstruct nonseparable signals in $V(\phi)$ from their noisy phaseless samples on a sampling set with finite density. The computational complexity of the proposed algorithm depends almost linearly on the support length of the original nonseparable signal. The reader may refer to [13,14,25,28,29,42] and references therein for various algorithms to reconstruct finite-dimensional signals from magnitudes of their frame measurements. The implementation and performance of the proposed algorithm to recover box spline signals from their noisy phaseless samples are given in Sect. 5.

Proofs are collected in Sect. 6. The local complement property for a locally finite-dimensional space is discussed in Appendix A.

Notation Denote the cardinality of a set E by $\#E$, the characteristic function on a set E by χ_E, and the closed ball in $\mathbb{R}^d$ with center x and radius $R \ge 0$ by $B(x, R)$. For $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$, $y = (y_1, \ldots, y_d)^T \in \mathbb{R}^d$ and $k = (k_1, \ldots, k_d)^T \in \mathbb{Z}_+^d$, we define the partial order $x \le y$ if $x_i \le y_i$, $1 \le i \le d$, and the power $x^k = \prod_{i=1}^d x_i^{k_i}$. For $t \in \mathbb{R}$, we set $\mathbf{t} = (t, \ldots, t)^T$. For positive quantities A and B, we use the notation $A = O(B)$ to represent $A \le CB$ for some absolute constant C.

2 Phase Retrievability, Nonseparability and Connectivity

Phase retrieval plays important roles in signal/speech/image processing, see [7,8,13–15,26,30,35,42] and references therein for historical remarks and recent advances. The phase retrievability of a real-valued signal on $\mathbb{R}^d$ is whether it is determined, up to a sign, by its magnitude measurements on $\mathbb{R}^d$. It is characterized in [16] as follows.

Theorem 2.1 *Let ϕ be a real-valued continuous function with compact support, and $V(\phi)$ be the shift-invariant space in (1.1) generated by ϕ. Then a signal $f \in V(\phi)$ is determined, up to a sign, by its magnitude measurements $|f(x)|$, $x \in \mathbb{R}^d$, if and only if f is nonseparable, i.e., there does not exist nonzero signals f_1 and f_2 in $V(\phi)$ such that*

$$f = f_1 + f_2 \quad \text{and} \quad f_1 f_2 = 0. \tag{2.1}$$

The question that arises is how to determine the nonseparability of a signal in a shift-invariant space. To answer the above question, we first recall the notion of global linear independence of a compactly supported function [10,32,41].

Definition 2.2 Let ϕ be a nonzero function with compact support. We say that ϕ has *global linear independence* if the correspondence

$$c := (c(k))_{k\in\mathbb{Z}^d} \longmapsto \sum_{k\in\mathbb{Z}^d} c(k)\phi(\cdot - k) =: f \in V(\phi) \tag{2.2}$$

between an amplitude vector c and a signal f in the shift-invariant space $V(\phi)$ is one-to-one.

In this paper, we always assume that the generator ϕ of the shift-invariant space $V(\phi)$ has global linear independence. For $d = 1$, the nonseparability of a signal $f = \sum_{k\in\mathbb{Z}} c(k)\phi(\cdot - k)$ in a shift-invariant space $V(\phi)$ is characterized in [16] that its amplitude vector $c := (c(k))_{k\in\mathbb{Z}}$ does not have consecutive zeros. However, there is no corresponding notion of consecutive zeros in the high-dimensional setting ($d \ge 2$). To characterize the nonseparability of signals on $\mathbb{R}^d$, $d \ge 1$, we introduce an undirected graph for a signal in the shift-invariant space $V(\phi)$.

Definition 2.3 For any $f(x) = \sum_{k\in\mathbb{Z}^d} c(k)\phi(x-k) \in V(\phi)$, define an *undirected graph*

$$\mathcal{G}_f := (V_f, E_f), \tag{2.3}$$

where the vertex set

$$V_f = \{k \in \mathbb{Z}^d : c(k) \neq 0\}$$

contains supports of the amplitude vector of the signal f, and

$$E_f = \left\{(k,k') \in V_f \times V_f : \ k \neq k' \text{ and } \phi(x-k)\phi(x-k') \neq 0 \text{ for some } x \in \mathbb{R}^d\right\}$$

is the edge set associated with the signal f.

The graph $\mathcal{G}_f$ in (2.3) is well-defined for any signal f in the shift-invariant space $V(\phi)$ as the generator ϕ has the global linear independence. Moreover,

$$(k,k') \in E_f \text{ if and only if } k-k' \in \Lambda_\phi, \tag{2.4}$$

where Λ_ϕ contains all $k \in \mathbb{Z}^d$ such that

$$S_k := \{x \in \mathbb{R}^d : \phi(x)\phi(x-k) \neq 0\} \neq \emptyset. \tag{2.5}$$

In the following theorem, we show that connectivity of the graph $\mathcal{G}_f$ is a necessary condition for the nonseparability of the signal $f \in V(\phi)$.

Theorem 2.4 *Let ϕ be a compactly supported continuous function on $\mathbb{R}^d$ with global linear independence, and $V(\phi)$ be the shift-invariant space* (1.1) *generated by ϕ. If $f \in V(\phi)$ is nonseparable, then the graph $\mathcal{G}_f$ in* (2.3) *is connected.*

Before stating sufficiency for the connectivity of the graph $\mathcal{G}_f$, we recall the notion of local linear independence on an open set, cf. Definition 2.2.

Definition 2.5 Let ϕ be a nonzero function with compact support and A be an open set. We say that ϕ has *local linear independence on A* if $\sum_{k\in\mathbb{Z}^d} c(k)\phi(x-k) = 0$ for all $x \in A$ implies that $c(k) = 0$ for all $k \in \mathbb{Z}^d$ satisfying $\phi(x-k) \not\equiv 0$ on A.

The global linear independence of a compactly supported function ϕ can be interpreted as its local linear $\mathbb{R}^d$ [10,46]. Define

$$\Phi_A(x) := \big(\phi(x-k)\big)_{k\in K_A},\ x\in A, \tag{2.6}$$

and

$$K_A := \{k\in\mathbb{Z}^d:\ \phi(\cdot-k)\not\equiv 0 \text{ on } A\}. \tag{2.7}$$

One may verify that ϕ has local linear independence on a bounded open set A if and only if the dimension of the linear space spanned by $\Phi_A(x)$, $x\in A$, is the cardinality of the set K_A. The above characterization can be used to verify the local linear independence on a bounded open set, especially when ϕ has an explicit expression. For instance, one may verify that the generator ϕ_0 in Example 2.7 below has local linear independence on $(0, 1)$, but it is locally linearly dependent on $(0, 1/2)$ and $(1/2, 1)$.

In the following theorem, we show that the converse in Theorem 2.4 is true if ϕ has local linear independence on all bounded open sets.

Theorem 2.6 *Let ϕ be a compactly supported continuous function on $\mathbb{R}^d$ with local linear independence on all bounded open sets, and f be a signal in the shift-invariant space $V(\phi)$. If the graph $\mathcal{G}_f$ in (2.3) is connected, then f is nonseparable.*

The conclusion in Theorem 2.6 follows from Theorem 3.3 in Sect. 3 and Proposition A.6 in Appendix A, see Sect. 6.4 for the detailed argument. We remark that the connectivity of the graph $\mathcal{G}_f$ in Theorem 2.6 is not sufficient for the signal f to be nonseparable if the local linear independence assumption on the generator ϕ is dropped.

Example 2.7 Define $\phi_0(t) = h(4t-1) + h(4t-3) + h(4t-5) - h(4t-7)$, where $h(t) = \max(1-|t|, 0)$ is the hat function supported on $[-1, 1]$. One may easily verify that ϕ_0 is a continuous function having global linear independence. Set

$$f_1(t) = \sum_{k\in\mathbb{Z}} \phi_0(t-k) \ \text{ and } \ f_2(t) = \sum_{k\in\mathbb{Z}} (-1)^k \phi_0(t-k).$$

Then f_1 and f_2 are nonzero signals in $V(\phi_0)$ supported on $[0, 1/2]+\mathbb{Z}$ and $[1/2, 1]+\mathbb{Z}$ respectively, and $f_1(t)f_2(t) = 0$ for all $t\in\mathbb{R}$. Hence $f_1\pm 2f_2$ have the same magnitude measurements $|f_1| + 2|f_2|$ on the real line but they are different, even up to a sign, i.e., $f_1+2f_2 \not\equiv \pm(f_1-2f_2)$. On the other hand, one may verify that their associated graphs $\mathcal{G}_{f_1\pm 2f_2}$ are connected.

Consider a continuous solution ϕ of a refinement equation

$$\phi(x) = \sum_{n=0}^{N} a(n)\phi(2x-n) \ \text{ and } \ \int_{\mathbb{R}} \phi(x)dx = 1 \tag{2.8}$$

with global linear independence, where $\sum_{n=0}^{N} a(n) = 2$ and $N \ge 1$ [21,36]. The B-spline B_N of order N, which is obtained by convolving the characteristic function $\chi_{[0,1)}$ on the unit interval N times, satisfies the above refinement equation [48,49]. The function ϕ in (2.8) has support $[0, N]$ and it has local linear independence on any bounded open set if and only if it has global linear independence [18,33,34,38,44]. Therefore we have the following result for wavelet signals by Theorems 2.4 and 2.6 , which is also established in [16] with a different approach.

Corollary 2.8 *Let ϕ satisfy the refinement equation (2.8) and have global linear independence. Then $f \in V(\phi)$ is nonseparable if and only if the graph $\mathcal{G}_f$ in (2.3) is connected.*

Given a matrix $\Xi \in \mathbb{Z}^{d\times s}$ of full rank d, define the box spline M_Ξ by

$$\int_{\mathbb{R}^d} g(x)M_\Xi(x)dx = \int_{[0,1)^s} g(\Xi y)dy, \quad g \in L^2(\mathbb{R}^d). \tag{2.9}$$

It is known that the box spline M_Ξ has local linear independence on any bounded open set if and only if it has global linear independence if and only if all $d \times d$ submatrices of Ξ have determinants being either 0 or ± 1 [19,20,23,31]. The reader may refer to [24] for more properties and applications of box splines. Therefore, as applications of Theorems 2.4 and 2.6 , we have the following result for box spline signals.

Corollary 2.9 *Let $\Xi \in \mathbb{Z}^{d\times s}$ be a matrix of full rank d such that all of its $d \times d$ submatrices have determinants being either* 0 *or* ± 1*. Then $f \in V(M_\Xi)$ is nonseparable if and only if the graph $\mathcal{G}_f$ in (2.3) is connected.*

3 Phaseless Sampling and Reconstruction

In this section, we consider the problem of whether a signal in the shift-invariant space $V(\phi)$ is determined, up to a sign, by its phaseless samples taken on a discrete set with finite sampling density. Here the sampling density of a discrete set $X \subset \mathbb{R}^d$ is defined by

$$D(X) := \lim_{R\to+\infty, x\in\mathbb{R}^d} \frac{\#(X \cap B(x, R))}{R^d} \tag{3.1}$$

[4,17,45]. One may easily verify that a shift-invariant set $X = \Gamma + \mathbb{Z}^d$ generated by a finite set $\Gamma \subset [0, 1)^d$ has sampling density $\#\Gamma$. In Theorem 3.1, we show that for a shift-invariant space $V(\phi)$ generated by a compactly supported function ϕ, there exists a shift-invariant set $\Gamma + \mathbb{Z}^d$ with finite density such that any nonseparable signal $f \in V(\phi)$ can be determined, up to a sign, from the phaseless samples $|f(y)|$, $y \in \Gamma + \mathbb{Z}^d$ taken on that set. However, the above unique determination does not lead to a reconstruction algorithm. To design efficient and robust algorithms to reconstruct signals in a shift-invariant space $V(\phi)$ from their phaseless samples, we require in this paper that the shift-invariant space $V(\phi)$ has local complement property on some open sets. Under the local complement property for the shift-invariant space $V(\phi)$, we provide two methods in Theorems 3.3 and 3.5 to construct finite sets Γ such that any

nonseparable signal f in the shift-invariant space $V(\phi)$ can be reconstructed from its phaseless samples $|f(y)|$, $y \in \Gamma + \mathbb{Z}^d$. Applying Theorems 3.3 and 3.5 to box splines of tensor-product type, we obtain two phaseless sampling sets in Corollaries 3.7 and 3.8 .

To determine a signal, up to a sign, by its phaseless samples taken on a discrete set, a necessary condition is that the signal is nonseparable (hence phase retrievable). In the next theorem, we show that the above requirement is also sufficient.

Theorem 3.1 *Let ϕ be a compactly supported continuous function and $V(\phi)$ be the shift-invariant space in (1.1) generated by ϕ. Then there exists a finite set $\Gamma \subset (0,1)^d$ such that any nonseparable signal $f \in V(\phi)$ is determined, up to a sign, by its phaseless samples $|f(\gamma)|$, $\gamma \in \Gamma + \mathbb{Z}^d$, on the set $\Gamma + \mathbb{Z}^d$ with finite sampling density.*

The proof of Theorem 3.1 is given in Sect. 6.2, where the finite set $\Gamma \subset (0,1)^d$ is chosen so that magnitude measurements $|f(x)|^2$, $x \in [0,1)^d$, of any signal $f \in V(\phi)$ is determined by $|f(\gamma)|^2$, $\gamma \in \Gamma$. However, it does not provide an algorithm to reconstruct a nonseparable signal from its phaseless samples taken on the shift-invariant set $\Gamma + \mathbb{Z}^d$. To design stable and efficient algorithms to reconstruct signals in a shift-invariant space $V(\phi)$ from their phaseless samples, we require in this paper that the shift-invariant space $V(\phi)$ has local complement property on some open sets.

Definition 3.2 We say that the shift-invariant space $V(\phi)$ has local complement property on a set A if for any $A' \subset A$, there does not exist $f, g \in V(\phi)$ such that $f, g \not\equiv 0$ on A, but $f(x) = 0$ for all $x \in A'$ and $g(y) = 0$ for all $y \in A \backslash A'$.

The local complement property on $\mathbb{R}^d$ is the complement property in [16] for ideal sampling functionals on $V(\phi)$, cf. the complement property for frames in Hilbert/Banach spaces [3,7,9,12]. Local complement property is closely related to local phase retrievability. In fact, following the argument in [16], the shift-invariant space $V(\phi)$ has the local complement property on A if and only if all signals in $V(\phi)$ are *local phase retrievable* on A, i.e., for any $f, g \in V(\phi)$ satisfying $|g(x)| = |f(x)|, x \in A$, there exists $\delta \in \{-1, 1\}$ such that $g(x) = \delta f(x)$ for all $x \in A$. More discussions on the local complement property will be given in Appendix A.

Under the assumption that the shift-invariant space $V(\phi)$ has local complement property, a shift-invariant set $\Gamma + \mathbb{Z}^d$ can be constructed so that any nonseparable signal f can be reconstructed, up to a sign, from its phaseless samples $|f(y)|$, $y \in \Gamma + \mathbb{Z}^d$, via a reconstruction algorithm presented in Sect. 5.

Theorem 3.3 *Let $A_1, \ldots, A_M$ be bounded open sets, ϕ be a compactly supported continuous function, $V(\phi)$ be the shift-invariant space in (1.1) generated by ϕ, and let S_k, $k \in \mathbb{Z}^d$, be as in (2.5). Assume that ϕ has local linear independence on A_m, $1 \le m \le M$, the shift-invariant space $V(\phi)$ has local complement property on A_m, $1 \le m \le M$, and*

$$S_k \cap \left(\cup_{m=1}^{M} (A_m + \mathbb{Z}^d) \right) \neq \emptyset \tag{3.2}$$

for all $k \in \mathbb{Z}^d$ with $S_k \neq \emptyset$. Then we can construct finite sets $\Gamma_m \subset A_m$, $1 \le m \le M$, explicitly such that the following statements are equivalent for any signal $f \in V(\phi)$:

(i) *The signal f is nonseparable.*
(ii) *The graph $\mathcal{G}_f$ in (2.3) is connected.*
(iii) *The signal f can be reconstructed, up to a sign, from its phaseless samples $|f(y)|$, $y \in \Gamma + \mathbb{Z}^d$, where*

$$\Gamma = \cup_{m=1}^{M} \Gamma_m. \tag{3.3}$$

The implication (i) $\Longrightarrow$ (ii) has been established in Theorem 2.4 and the implication (iii) $\Longrightarrow$ (i) is obvious. Write $f = \sum_{k\in\mathbb{Z}^d} c(k)\phi(\cdot - k)$. To prove (ii) $\Longrightarrow$ (iii), we first determine $c(k)$, $k \in K_{A_m} + l$, up to a sign $\delta_{l,m} \in \{-1, 1\}$, by phaseless samples $|f(\gamma + l)|$, $\gamma \in \Gamma_m$, $1 \le m \le M$, and then we use the connectivity of the graph $\mathcal{G}_f$ to adjust phases $\delta_{l,m}$, $1 \le m \le M$, $l \in \mathbb{Z}^d$. Finally we sew those pieces together to recover amplitudes $c(k)$, $k \in \mathbb{Z}^d$, and the signal f, up to a sign, in a noiseless environment, see Sect. 6.3 for detailed argument.

Definition 3.4 We say that $\mathcal{M} = \{a_m \in \mathbb{R}^d, 1 \le m \le M\}$ is a *phase retrievable frame* for $\mathbb{R}^d$ if any vector $x \in \mathbb{R}^d$ is determined, up to a sign, by its measurements $|\langle x, a_m\rangle|$, $a_m \in \mathcal{M}$, and that $\mathcal{M}$ is a *minimal phase retrieval frame* for $\mathbb{R}^d$ if any true subset of $\mathcal{M}$ is not a phase retrievable frame.

Given a compactly supported function ϕ and a bounded open set A, let

$$W_A \text{ be the linear space spanned by } \Phi_A(x)(\Phi_A(x))^T,\ x \in A, \tag{3.4}$$

where Φ_A is given in (2.6). In the proof of Theorem 3.3 given in Sect. 6.3, the finite sets $\Gamma_m \subset A_m$, $1 \le m \le M$, are so chosen that outer products $\Phi_{A_m}(\gamma)(\Phi_{A_m}(\gamma))^T$, $\gamma \in \Gamma_m$, form a basis of the linear space W_{A_m}. This together with the local complement property on A_m, $1 \le m \le M$, for the shift-invariant space $V(\phi)$ implies that $\Phi_{A_m}(\gamma)$, $\gamma \in \Gamma_m$, is a phase retrievable frame for $\mathbb{R}^{\#K_{A_m}}$. After careful examination of the proof of Theorem 3.3, we can apply the pare-down technique to the above phase retrievable frame and find a subset $\Gamma' \subset \Gamma$ such that nonseparable signals in the shift-invariant space $V(\phi)$ can be reconstructed from their phaseless samples taken on $\Gamma' + \mathbb{Z}^d$, which has smaller sampling density than the set $\Gamma + \mathbb{Z}^d$ has.

Theorem 3.5 *Let A_m, $1 \le m \le M$, and ϕ be as in Theorem 3.3, and let $\Gamma'_m \subset A_m$, $1 \le m \le M$, be chosen so that $\Phi_{A_m}(\gamma')$, $\gamma' \in \Gamma'_m$, are phase retrievable frames for $\mathbb{R}^{\#K_{A_m}}$. Then any nonseparable signal $f \in V(\phi)$ can be reconstructed, up to a sign, from its phaseless samples $|f(y)|$, $y \in \Gamma' + \mathbb{Z}^d$, where*

$$\Gamma' = \cup_{m=1}^{M} \Gamma'_m. \tag{3.5}$$

Let Γ be the phaseless sampling set either in Theorem 3.3 or in Theorem 3.5, a phaseless reconstruction algorithm is proposed in Sect. 5. We remark that the phase retrievable frame property for the sampling set Γ' in Theorem 3.5, $\Phi_A(\gamma')$, $\gamma' \in \Gamma'$

may not imply that their outer products $\Phi_A(\gamma')(\Phi_A(\gamma'))^T$, $\gamma' \in \Gamma'$, form a basis (or a spanning set) of W_A in (3.4), as shown in the following example.

Example 3.6 Let

$$\phi_1(x) = \begin{cases} x^3/2 & \text{if } 0 \le x < 1 \\ -x^3 + 3x^2 - 2x + 1/2 & \text{if } 1 \le x < 2 \\ x^3/2 - 3x^2 + 5x - 3/2 & \text{if } 2 \le x < 3 \\ 0 & \text{otherwise,} \end{cases}$$

and set $\Phi_1(x) = (\phi_1(x), \phi_1(x+1), \phi_1(x+2))^T$, $0 \le x < 1$. Then

$$\Phi_1(x) = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} x + \frac{1}{2}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} x^3$$

and

$$\Phi_1(x)\Phi_1(x)^T = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} x^2$$
$$+ \frac{1}{4}\begin{pmatrix} 0 & 1 & 1 \\ 1 & -4 & -1 \\ 1 & -1 & 2 \end{pmatrix} x^3 + \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ 1 & -4 & 3 \\ -1 & 3 & -2 \end{pmatrix} x^4 + \frac{1}{4}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} x^6.$$

Therefore the space spanned by $\Phi_1(x)$, $0 < x < 1$, is $\mathbb{R}^3$, and the space $W_{(0,1)}$ spanned by $\Phi_1(x)\Phi_1(x)^T$, $0 < x < 1$, is the 6-dimensional linear space of symmetric matrices of size 3×3. On the other hand, any 3×3 square submatrices of the 3×5 matrix

$$\left(\Phi_1(0)\ \Phi_1\left(\frac{1}{5}\right)\ \Phi_1\left(\frac{2}{5}\right)\ \Phi_1\left(\frac{3}{5}\right)\ \Phi_1\left(\frac{4}{5}\right)\right) = \frac{1}{250}\begin{pmatrix} 0 & 1 & 8 & 27 & 64 \\ 125 & 173 & 209 & 221 & 197 \\ 125 & 76 & 33 & 2 & -11 \end{pmatrix}$$

is nonsingular, which implies that $\Phi_1(m/5)$, $0 \le m \le 4$, form a phase retrievable frame for $\mathbb{R}^3$, but their outer products do not form a spanning set of the 6-dimensional space $W_{(0,1)}$.

We finish this section with explicit construction of finite sets Γ in Theorems 3.3 and 3.5 for the shift-invariant space generated by a box spline of tensor-product type, cf. Sects. 5.1 and 5.2. Take $\mathbf{N} = (N_1, \ldots, N_d)^T$ with $N_i \ge 2$, $1 \le i \le d$, and let B_{N_i} be the B-spline of order N_i. Define the box spline function of tensor-product type

$$B_{\mathbf{N}}(x) := B_{N_1}(x_1) \times \cdots \times B_{N_d}(x_d),\ x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d. \tag{3.6}$$

Applying the argument used in the proof of Theorem 3.3 with $M = 1$ and $A_1 = (0, 1)^d$, we have the following result, which is given in [16] for $d = 1$.

Corollary 3.7 *Let X_i contain $2N_i - 1$ distinct points in $(0, 1)$, $1 \le i \le d$. Then any nonseparable signal $f \in V(B_{\mathbf{N}})$ can be reconstructed, up to a sign, from its phaseless samples taken on the set $X_1 \times \ldots \times X_d + \mathbb{Z}^d$.*

The phaseless sampling set $X_1 \times \ldots \times X_d + \mathbb{Z}^d$ in the above corollary has density $\prod_{i=1}^d (2N_i - 1)$. Applying Theorem 3.5, we may select a phaseless sampling set with smaller sampling density $2\mathcal{N} - 1$, where $\mathcal{N} = \prod_{i=1}^d N_i$.

Corollary 3.8 *For almost all $(y_1, y_2, \ldots, y_{2\mathcal{N}-1}) \in (0, 1)^d \times \cdots \times (0, 1)^d$, it holds that any nonseparable signal $f \in V(B_{\mathbf{N}})$ can be reconstructed, up to a sign, from its phaseless samples taken on the set $\{y_1, y_2, \ldots, y_{2\mathcal{N}-1}\} + \mathbb{Z}^d$.*

4 Stability of Phaseless Sampling and Reconstruction

Stability is of paramount importance in the phaseless sampling and reconstruction problem. In this section, we construct an approximation

$$f_\epsilon = \sum_{k \in \mathbb{Z}^d} c_\epsilon(k)\phi(\cdot - k) \in V(\phi), \tag{4.1}$$

up to a sign, to the original nonseparable signal

$$f = \sum_{k \in \mathbb{Z}^d} c(k)\phi(\cdot - k) \in V(\phi) \tag{4.2}$$

from the noisy phaseless samples

$$z_\epsilon(y) = |f(y)| + \epsilon(y) \ge 0, \;\; y \in \Gamma + \mathbb{Z}^d, \tag{4.3}$$

where $\epsilon = (\epsilon(y))_{y \in \Gamma + \mathbb{Z}^d}$ has the bounded noise level $\|\epsilon\|_\infty = \max_{y \in \Gamma + \mathbb{Z}^d} |\epsilon(y)|$, and the finite set $\Gamma = \cup_{m=1}^M \Gamma_m$ is given either by (3.3) in Theorem 3.3 or by (3.5) in Theorem 3.5.

Let

$$\Omega_m = \{k \in \mathbb{Z}^d : \; \phi(\gamma - k) \ne 0 \text{ for some } \gamma \in \Gamma_m\}, \;\; 1 \le m \le M, \tag{4.4}$$

and define the hard thresholding function H_η, $\eta \ge 0$, by $H_\eta(t) = t\chi_{\mathbb{R}\setminus(-\eta,\eta)}(t)$. Based on the constructive proofs of Theorems 3.3 and 3.5, we propose the following approach with $M_0 \ge 0$ being the phase adjustment threshold value chosen appropriately. Its detailed implementation is given in Sect. 5.

1. For $l \in \mathbb{Z}^d$ and $1 \le m \le M$, let

$$c_{\epsilon,l;m} = (c_{\epsilon,l;m}(k))_{k\in\mathbb{Z}^d} \tag{4.5}$$

take zero components except that $c_{\epsilon,l;m}(k)$, $k \in l + \Omega_m$, are solutions of the local minimization problem

$$\min_{e(k),k\in l+\Omega_m} \sum_{\gamma\in\Gamma_m} \left| \Big| \sum_{k\in l+\Omega_m} e(k)\phi(\gamma + l - k)\Big| - z_\epsilon(\gamma + l)\right|^2. \tag{4.6}$$

2. Adjust phases of $c_{\epsilon,l;m}$, $l \in \mathbb{Z}^d$, $1 \le m \le M$, appropriately so that the resulting vectors $\delta_{l,m}c_{\epsilon,l;m}$ with $\delta_{l,m} \in \{-1, 1\}$ satisfy

$$\langle \delta_{l,m}c_{\epsilon,l;m}, \delta_{l',m'}c_{\epsilon,l';m'}\rangle \ge -M_0 \tag{4.7}$$

for all $l, l' \in \mathbb{Z}^d$ and $1 \le m, m' \le M$.

3. Sew vectors $\delta_{l,m}c_{\epsilon,l;m}$, $l \in \mathbb{Z}^d$, $1 \le m \le M$, together to obtain

$$d_\epsilon(k) = \frac{\sum_{m=1}^M \sum_{l\in\mathbb{Z}^d} \delta_{l,m}c_{\epsilon,l;m}(k)\chi_{l+\Omega_m}(k)}{\sum_{m=1}^M \sum_{l\in\mathbb{Z}^d} \chi_{l+\Omega_m}(k)},\ k \in \mathbb{Z}^d. \tag{4.8}$$

4. Threshold the vector $d_\epsilon = (d_\epsilon(k))_{k\in\mathbb{Z}^d}$,

$$c_\epsilon(k) = H_\eta(d_\epsilon(k)),\ \ k \in \mathbb{Z}^d \tag{4.9}$$

to construct the approximation f_ϵ in (4.1), where $\eta = \sqrt{M_0}$.

Given $l \in \mathbb{Z}^d$ and $1 \le m \le M$, the local minimizers $c_{\epsilon,l;m}(k)$, $k \in l+\Omega_m$, in the first step of the above approach are determined, up to a sign, from noisy phaseless samples $z_\epsilon(\gamma+l)$, $\gamma \in \Gamma_m$, and they provide approximations to amplitudes $c(k)$, $k \in l+\Omega_m$, of the original nonseparable signal f, up to a sign *depending* on $l \in \mathbb{Z}^d$ and $1 \le m \le M$, see (6.20). Due to the above approximation property, we adjust phases of vectors $c_{\epsilon,l;m}$, $l \in \mathbb{Z}^d$, $1 \le m \le M$, in the second step of our approach so that components $\delta_{l,m}c_{\epsilon,l;m}(k)$, $k \in l + \Omega_m$, of the resulting vectors $\delta_{l,m}c_{\epsilon,l;m}$ are close to amplitudes $c(k)$, $k \in l + \Omega_m$ of the original signal f, up to a sign $\delta \in \{-1, 1\}$ *independent* on $l \in \mathbb{Z}^d$ and $1 \le m \le M$. Therefore we can sew the vectors $\delta_{l,m}c_{\epsilon,l;m}$, $l \in \mathbb{Z}^d$, $1 \le m \le M$, together to yield an approximation d_ϵ in the third step of our approach to the amplitude vector c of the original nonseparable signal f, up to a sign δ. The final thresholding procedure in our approach leads to an approximation f_ϵ to the original signal f, up to a sign, with their graphs $\mathcal{G}_{f_\epsilon}$ and $\mathcal{G}_f$ being the same, see (4.16) and (4.18). The mathematical justification of the above signal reconstruction in a noisy scenario is presented in the following theorem, while its implementation will be presented in Sect. 5.

Theorem 4.1 *Let $A_1, \ldots, A_M$ be bounded open sets satisfying* (3.2)*, ϕ be a compactly supported continuous function such that ϕ has local linear independence on A_m, $1 \le m \le M$, and let $\Gamma_m \subset A_m$, $1 \le m \le M$, be so chosen that $\Phi_{A_m}(\gamma)$, $\gamma \in \Gamma_m$, is a* *phase retrievable frame for* $\mathbb{R}^{\#K_{A_m}}$. *Set* $\Gamma = \cup_{m=1}^{M}\Gamma_m$ *and*

$$\big\|\Phi^{-1}\big\|_{\mathrm{P}} = \sup_{\Theta_m\subset\Gamma_m, 1\le m\le M} \Big(\min\big(\sup_{\|d\|_2=1}\|\Phi_{\Theta_m}d\|_2^{-1}, \sup_{\|d\|_2=1}\|\Phi_{\Gamma_m\backslash\Theta_m}d\|_2^{-1}\big)\Big)^{-1}, \tag{4.10}$$

where $\Phi_{\Theta_m} = (\phi(\gamma-k))_{\gamma\in\Theta_m, k\in\Omega_m}$ *for* $\Theta_m \subset \Gamma_m$. *Assume that the original signal* $f = \sum_{k\in\mathbb{Z}^d} c(k)\phi(\cdot - k)$ *is nonseparable, and*

$$F_0 := \inf_{k\in V_f} |c(k)|^2 > 0, \tag{4.11}$$

where $\mathcal{G}_f = (V_f, E_f)$ *is the graph associated with the the signal* f. *If the phase adjustment threshold value* $M_0 \ge 0$ *and the noise level* $\|\epsilon\|_\infty := \sup_{y\in\Gamma+\mathbb{Z}^d}|\epsilon(y)|$ *satisfy*

$$M_0 \le \frac{2F_0}{9} \tag{4.12}$$

and

$$8\#\Gamma\big\|\Phi^{-1}\big\|_{\mathrm{P}}^2\|\epsilon\|_\infty^2 \le M_0, \tag{4.13}$$

then there exists $\delta \in \{-1, 1\}$ *such that the signal* $f_\epsilon = \sum_{k\in\mathbb{Z}^d} c_\epsilon(k)\phi(\cdot - k) \in V(\phi)$ *reconstructed from the proposed approach* (4.5)–(4.9) *satisfies*

$$|c_\epsilon(k) - \delta c(k)| \le 2\sqrt{\#\Gamma}\big\|\Phi^{-1}\big\|_{\mathrm{P}}\|\epsilon\|_\infty,\ k \in V_f \tag{4.14}$$

and

$$c_\epsilon(k) = c(k) = 0,\ k \notin V_f. \tag{4.15}$$

By Theorem 4.1, the reconstructed signal f_ϵ in (4.1) provides an approximation, up to a sign, to the original nonseparable signal f in (4.2),

$$\|f_\epsilon - \delta f\|_\infty \le 2\sqrt{\#\Gamma}\big\|\Phi^{-1}\big\|_{\mathrm{P}}\Big(\sup_{x\in\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\big|\phi(x-k)\big|\Big)\|\epsilon\|_\infty \tag{4.16}$$

and

$$\sup_{y\in\Gamma+\mathbb{Z}^d}\big||f_\epsilon(y)| - |f(y)|\big| \le 2\sqrt{\#\Gamma}\big\|\Phi^{-1}\big\|_{\mathrm{P}}\Big(\sup_{x\in\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\big|\phi(x-k)\big|\Big)\|\epsilon\|_\infty. \tag{4.17}$$

By (4.11), (4.12), (4.13) and (4.14), a vertex in the graph $\mathcal{G}_f$ is also a vertex of the graph $\mathcal{G}_{f_\epsilon}$. This together with (2.3) and (4.15) implies that the graphs $\mathcal{G}_f$ and $\mathcal{G}_{f_\epsilon}$ associated with the original signal f and the reconstructed signal f_ϵ are the same, i.e.,

$$\mathcal{G}_f = \mathcal{G}_{f_\epsilon}. \tag{4.18}$$

The square root of the quantity F_0 in (4.11) is the minimal magnitude of amplitude vector of the original signal f. It can be used to measure the "distance" between the nonseparable signal f to the set of all separable signals in $V(\phi)$. For instance, take a sufficiently small $\alpha \in (0, 1)$ and a nonseparable spline signal

$$f_\alpha(x) = B_2(x) + \alpha B_2(x-1) + B_2(x-2) \in V(B_2) \tag{4.19}$$

of order 2 with $F_0 = \alpha$. Write $g(x) = c_0 B_2(x) + c_1 B_2(x-1) + c_2 B_2(x-2) \in V(B_2)$, one may verify that all signals $g(x)$ satisfying $\max(|c_0 - 1|, |c_1 - \alpha|, |c_2 - 1|) < \sqrt{F_0} = \alpha$ are nonseparable, and also that the signal $g(x)$ with $c_0 = 1, c_1 = 0, c_2 = 1$ is separable and satisfies $\max(|c_0 - 1|, |c_1 - \alpha|, |c_2 - 1|) = \sqrt{F_0}$.

The quantity $\|\Phi^{-1}\|_{\mathrm{P}}$ in (4.10) for phase retrievable frames Φ_{Θ_m}, $1 \le m \le M$, is closely related to their strong complement property [9,12]. Following the argument in [9, Theorem 18], we conclude that the reciprocal of $\|\Phi^{-1}\|_{\mathrm{P}}$ is a Lipschitz lower bound of nonlinear maps $x_m \longmapsto |\Phi_{\Theta_m} x_m|$, $1 \le m \le M$, i.e.,

$$\|\Phi^{-1}\|_P^{-1} \min\left(\|x_m - y_m\|_2, \|x_m + y_m\|_2\right) \le \left\| |\Phi_{\theta_m} x_m| - |\Phi_{\theta_m} y_m| \right\|_2$$

hold for all $x_m, y_m \in \mathbb{R}^{\#\Omega_m}$, $1 \le m \le M$. By (4.16) and (4.17), we see that the quantity $\|\Phi^{-1}\|_{\mathrm{P}}$ in our phaseless sampling and reconstruction plays a similar role to the minimal nonzero singular value of a matrix in finding a least squares solution of a linear system.

Selection of the threshold value $M_0 \ge 0$ is imperative to find a good approximation to the original signal from its phaseless samples, and its inappropriate selection could lead our approach to fail. In the noiseless environment, we may take $M_0 = 0$ and the proposed approach leads to a perfect reconstruction, i.e., $f_\epsilon = \pm f$, when f is nonseparable. In practical applications, the noise level is usually positive and the phase adjustment threshold value M_0 needs to be appropriately selected. For instance, we may require that (4.12) and (4.13) are satisfied if we have some prior information about the amplitude vector of the original signal. From the proof of Theorem 4.1 and also the simulations in the next section, it is observed that phases can not be adjusted to satisfy (4.7) if the threshold value M_0 is far below square of noise level $\|\epsilon\|_\infty$ (for instance, (4.13) is not satisfied), while the phase adjustment (4.7) in the algorithm is not essentially determined and hence the reconstructed signal is not a good approximation of the original signal if the threshold value M_0 is much larger than the square of minimal magnitude of amplitude vector of the original signal (for instance, (4.12) is not satisfied).

Remark 4.2 By Theorem 4.1, there is no resonance phenomenon in the sense that

$$\inf_{\delta \in \{-1,1\}} \|f_\epsilon - \delta f\|_\infty \le C\|\epsilon\|_\infty \tag{4.20}$$

if the noise level is far below the minimal magnitude $\sqrt{F_0}$ of amplitude vector of the original signal f, i.e.,

$$\|\epsilon\|_\infty \le C_0\sqrt{F_0} = C_0 \inf_{k\in V_f} |c(k)| \tag{4.21}$$

for some sufficiently small positive constant C_0. The above requirement (4.21) on the noise level is necessary, since the phaseless sampling and reconstruction problem is ill-posed (and hence the estimate (4.20) is not satisfied) if the ratio between the noise level and minimal magnitude of amplitude vector of the original signal is not small. For instance, taking nonseparable spline signals $f_\alpha(x) = B_2(x)+\alpha B_2(x-1)+B_2(x-2)$ and $\tilde f_\alpha(x) = B_2(x) + \alpha B_2(x-1) - B_2(x-2) \in V(B_2)$ of order 2, we have

$$\min_{\delta\in\{-1,1\}} \|f_\alpha - \delta \tilde f_\alpha\|_\infty = 2 \ \text{ and } \ \big\||f_\alpha| - |\tilde f_\alpha|\big\|_\infty = \frac{2\alpha}{1+\alpha},$$

where $\alpha \in (0, 1)$ is sufficiently small. Hence there does not exist an absolute constant C independent on $\alpha \in (0, 1)$ such that the estimate (4.20) is satisfied.

5 Reconstruction Algorithm and Numerical Simulations

Consider the scenario that available data

$$z_\epsilon(y) = |f(y)| + \epsilon(y),\ \ y \in \Gamma + K, \tag{5.1}$$

are noisy phaseless samples of a nonseparable signal $f = \sum_{k\in\mathbb{Z}^d} c(k)\phi(\cdot-k) \in V(\phi)$ taken on $\Gamma + K$, where $\Gamma = \cup_{m=1}^M \Gamma_m$ is either as in (3.3) or in (3.5), $K \subset \mathbb{Z}^d$ is a finite set, and the additive noise $\epsilon = (\epsilon(y))_{y\in\Gamma+K}$ is bounded,

$$\|\epsilon\|_\infty := \max_{y\in\Gamma+K} |\epsilon(y)| \le \varepsilon \tag{5.2}$$

for some $\varepsilon \ge 0$. Based on the approach in Sect. 4, we propose an algorithm to construct a signal f_ϵ of the form

$$f_\epsilon = \sum_{k\in\widetilde K} c_\epsilon(k)\phi(\cdot - k) \in V(\phi), \tag{5.3}$$

where $\widetilde K = \cup_{l\in K} \cup_{m=1}^M (l + \Omega_m)$ and Ω_m, $1 \le m \le M$, are as in (4.4). Observe that the original signal f and its truncation

$$f_K = \sum_{k\in\widetilde K} c(k)\phi(\cdot - k) \tag{5.4}$$

have the same (phaseless) sampling values on $\Gamma + K$ by the supporting property ϕ. Therefore the signal f_ϵ in (5.3) is an approximation, up to a sign, to the truncation f_K of the original nonseparable signal f by Theorem 4.1.

Algorithm 1 MAPSET Algorithm

Inputs: finite set $K \subset \mathbb{Z}^d$; sampling set $\Gamma = \cup_{m=1}^M \Gamma_m$ either in (3.3) or in (3.5); noisy phaseless sampling data $\big(z_\epsilon(y)\big)_{y\in\Gamma+K}$; index set $\widetilde{K} = \cup_{l\in K} \cup_{m=1}^M (l+\Omega_m) \subset \mathbb{Z}^d$; and the phase adjustment threshold value M_0.
Initials: Start from zero vectors $c_{\epsilon,l;m} = \big(c_{\epsilon,l;m}(k)\big)_{k\in\widetilde{K}}, l \in K, 1 \le m \le M$.
Instructions:
1) Local minimization: For $l \in K$ and $1 \le m \le M$, replace $c_{\epsilon,l;m}(k), k \in l+\Omega_m$, by a solution of the local minimization problem

$$\min_{e(k), k\in l+\Omega_m} \sum_{\gamma\in\Gamma_m} \Big|\Big|\sum_{k\in l+\Omega_m} e(k)\phi(\gamma+l-k)\Big| - z_\epsilon(\gamma+l)\Big|^2.$$

2) Phase adjustment: For $l \in K$ and $1 \le m \le M$, multiply $c_{\epsilon,l;m}$ by $\delta_{l,m} \in \{-1,1\}$ so that $\langle \delta_{l,m} c_{\epsilon,l;m}, \delta_{l',m'} c_{\epsilon,l';m'}\rangle \ge -M_0$ for all $l, l' \in K$ and $1 \le m, m' \le M$.
3) Sewing local approximations:

$$d_\epsilon(k) = \frac{\sum_{m=1}^M \sum_{l\in K} \delta_{l,m} c_{\epsilon,l;m}(k)\chi_{l+\Omega_m}(k)}{\sum_{m=1}^M \sum_{l\in K} \chi_{l+\Omega_m}(k)}, \ k \in \widetilde{K}. \tag{5.5}$$

4) Hard thresholding:

$$c_\epsilon(k) = H_\eta(d_\epsilon(k)), k \in \widetilde{K}, \tag{5.6}$$

where $\eta = \sqrt{M_0}$.
Outputs: Amplitude vector $(c_\epsilon(k))_{k\in\widetilde{K}}$, and the reconstructed signal $f_\epsilon = \sum_{k\in\widetilde{K}} c_\epsilon(k)\phi(\cdot - k)$.

The proposed algorithm contains four parts: **m**inimization, **a**djusting **p**hases, **se**wing and **t**hresholding, and we call it the MAPSET algorithm. For every $l \in K$ and $1 \le m \le M$, the local minimizers $c_{\epsilon,l;m}(k), k \in l+\Omega_m$, in the first step of the MAPSET algorithm are determined, up to a sign, from noisy phaseless samples $z_\epsilon(\gamma+l), \gamma \in \Gamma_m$, by the selection of the sampling set Γ_m, and it can be found by solving a finite family of least squares problems

$$\min_{\delta_\gamma\in\{-1,1\}, \gamma\in\Gamma_m} \min_{e(k), k\in l+\Omega_m} \sum_{\gamma\in\Gamma_m} \Big|\sum_{k\in l+\Omega_m} e(k)\phi(\gamma+l-k) - \delta_\gamma z_\epsilon(\gamma+l)\Big|^2. \tag{5.7}$$

The phase adjustment in the second step of the MAPSET algorithm is not unique in a noisy environment. Our implementation to the second step has three components: (1) we first construct a symmetric sign matrix $B = (b_{l,m;l',m'})_{l,l'\in K, 1\le m,m'\le M}$ by

$$b_{l,m;l',m'} = \begin{cases} 1 & \text{if } \langle c_{\epsilon,l;m}, c_{\epsilon,l';m'}\rangle \ge M_0 \\ -1 & \text{if } \langle c_{\epsilon,l;m}, c_{\epsilon,l';m'}\rangle \le -M_0 \\ 0 & \text{otherwise} \end{cases}$$

for $l, l' \in K, 1 \le m, m' \le M$; (2) we then find a diagonal matrix $D = (\delta_{l,m})_{l\in K, 1\le m\le M}$ with diagonal entries $\delta_{l,m}, l \in K, 1 \le m \le M$, being either 1 or -1 such that DBD has nonnegative entries; and (3) we finally use diagonal entries $\delta_{l,m}$ as signs of our phase adjustment to vectors $c_{\epsilon,l;m}, l \in K, 1 \le m \le M$. The third and fourth steps of the MAPSET algorithm can be implemented as stated.

Set $N = \#K$. By (4.4), we have

$$\#\Omega_m = O(1),\ 1 \le m \le M \ \text{ and } \ \#\tilde{K} = O(N). \tag{5.8}$$

The computational complexity of the first step in the MAPSET algorithm is $O(N)$, since the number of additions and multiplications required to solve the family of least squares problems (5.7) is $O(1)$. Recall that for every $l \in K$ and $1 \le m \le M$, the vector $c_{\epsilon,l;m}$ has $O(1)$ nonzero entries, and observe that every row of the matrix B has $O(1)$ nonzero entries by the supporting property of vectors $c_{\epsilon,l;m}, l \in K, 1 \le m \le M$. Therefore the computational complexity of the second step in the MAPSET algorithm is $O(N)$ too. By (5.5), (5.6) and (5.8), we need $O(N)$ additions and multiplications to implement the third and fourth steps of the MAPSET algorithm. Combining the above arguments, we conclude that the proposed MAPSET algorithm has linear computational complexity to construct an approximation in $V(\phi)$ from noisy phaseless samples of a nonseparable signal f in $V(\phi)$.

In the rest of this section, we demonstrate the performance of the proposed MAPSET algorithm on reconstructing box spline signals from their noisy phaseless samples on discrete sets.

5.1 Nonseparable Spline Signals of Tensor-Product Type

Let $B_{(3,3)}$ be the tensor product of one-dimensional quadratic spline B_3, and set $b_0(s) = s^2/2$, $b_{-1}(s) = (-2s^2 + 2s + 1)/2$ and $b_{-2}(s) = (1-s)^2/2, 0 \le s \le 1$. For $A = (0, 1)^2$ and $\phi = B_{(3,3)}$, one may verify that the set K_A in (2.7) is

$$K_{(0,1)^2} = \{(i, j) : -2 \le i, j \le 0\},$$

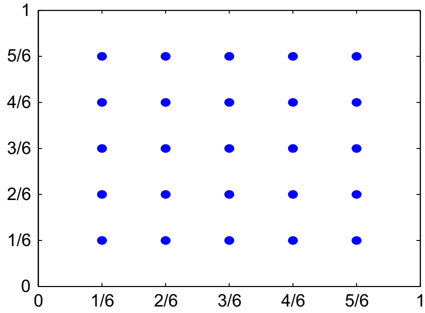
and the vector-valued function Φ_A in (2.6) becomes

$$\Phi_{(0,1)^2}(s, t) = \big(b_i(s)b_j(t)\big)_{(i,j)\in K_{(0,1)^2}},\ (s, t) \in (0, 1)^2 \tag{5.9}$$

which is a 9-dimensional vector-valued polynomial about $s^p t^q, 0 \le p, q \le 2$. This implies that the shift-invariant space generated by $B_{(3,3)}$ are locally linearly independent on $(0, 1)^2$ and it has local complement property on $(0, 1)^2$. By Corollary 3.7, the set

$$\Gamma_0 = \{(i, j)/6, 1 \le i, j \le 5\} \subset (0, 1)^2 \tag{5.10}$$

with cardinality 25 can be selected to be a phaseless sampling set in (3.3), see Fig. 1. For the above uniformly distributed set Γ_0, the corresponding $\|\Phi^{-1}\|_{\mathrm{P}}$ in (4.10) is



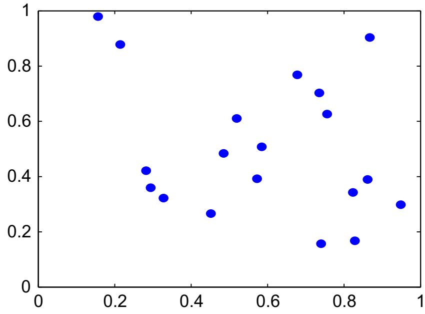


Fig. 1 Plotted on the left is a uniformly distributed set Γ_0 satisfying (3.3), while on the right is a randomly distributed set Γ_1 satisfying (3.5). The corresponding $\|\Phi^{-1}\|_{\mathrm{P}}$ in (4.10) to the above sets are 2.7962×10^3 (left) and 3.2995×10^4 (right), respectively

2.7962×10^3. From the argument used in Corollary 3.8, $\big(\Phi_{(0,1)^2}(s_i, t_i)\big)_{1\le i\le 17}$ is a phase retrievable frame for almost all $(s_i, t_i) \in (0, 1)^2$, $1 \le i \le 17$, however the corresponding $\|\Phi^{-1}\|_{\mathrm{P}}$ in (4.10) is relatively large from our calculation. So we use another randomly distributed set

$$\Gamma_1 = \{(s_i, t_i), 1 \le i \le 19\} \subset (0, 1)^2 \tag{5.11}$$

with cardinality 19 in our simulations, see Fig. 1. The above set satisfies (3.5) and the corresponding $\|\Phi^{-1}\|_{\mathrm{P}}$ in (4.10) is 3.2995×10^4.

In our simulations, the available data $z_\epsilon(y) = |f(y)| + \epsilon(y) \ge 0$, $y \in \Gamma + K$, are noisy phaseless samples of a spline signal

$$f(s, t) = \sum_{0\le m\le K_1, 0\le n\le K_2} c(m, n) B_{(3,3)}(s - m, t - n) \tag{5.12}$$

taken on $\Gamma + K$, where $K = [0, K_1] \times [0, K_2]$ for some positive integers $K_1, K_2 \ge 1$, Γ is either the uniform set Γ_0 in (5.10) or the random set Γ_1 in (5.11), amplitudes of the signal f,

$$c(m, n) \in [-1, 1]\backslash[-0.1, 0.1],\ 0 \le m \le K_1, 0 \le n \le K_2, \tag{5.13}$$

are randomly chosen, and the additive noises $\epsilon(y) \in [-\varepsilon, \varepsilon]$, $y \in \Gamma + K$, with noise level $\varepsilon \ge 0$ are randomly selected. Denote the signal reconstructed by the proposed MAPSET algorithm with phase adjustment threshold value $M_0 = 0.01$, cf. (4.12) with $F_0 = 0.01$, by

$$f_\epsilon(s, t) = \sum_{-2\le m\le K_1, -2\le n\le K_2} c_\epsilon(m, n) B_{(3,3)}(s - m, t - n). \tag{5.14}$$

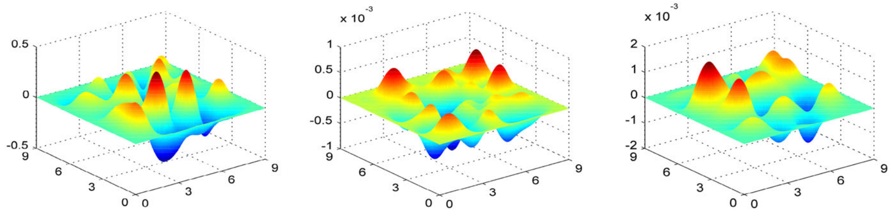


Fig. 2 Plotted on the left is a nonseparable spline signal in (5.12) with $K_1 = K_2 = 7$. In the middle and on the right are the difference between the above spline signal f and the signal f_ϵ reconstructed by the MAPSET algorithm with noise level $\varepsilon = 10^{-4}$ and sampling set Γ being Γ_0 and Γ_1 in Fig. 1, respectively. The maximal amplitude errors $e(\epsilon)$ in (5.15) are 0.0014 (middle) and 0.0030 (right), and the reconstruction errors $\min_{\delta\in\{-1,1\}} \|f_\epsilon - \delta f\|_\infty$ are 7.2567×10^{-4} (middle) and 0.0015 (right), respectively

Define the maximal amplitude error of the MAPSET algorithm by

$$e(\epsilon) := \min_{\delta\in\{-1,1\}} \max_{-2\le m\le K_1, -2\le n\le K_2} |c_\epsilon(m,n) - \delta c(m,n)|. \tag{5.15}$$

As the original spline signal f in (5.12) is nonseparable, the conclusions (4.14) and (4.15) guarantee that the reconstruction signal f_ϵ provides an approximation, up to a sign, to the original signal f if $\|\Phi^{-1}\|_{\mathrm{P}}\epsilon$ is much smaller than a multiple of $\sqrt{M_0}$, where M_0 is the phase adjustment threshold value. Our numerical simulations indicate that the MAPSET algorithm reconstructs phases successfully in 100 trials and the maximal amplitude error $e(\epsilon)$ in (5.15) is about $O(\epsilon)$, provided that $\epsilon \le 2 \times 10^{-3}$ for $\Gamma = \Gamma_0$ and $\varepsilon \le 7 \times 10^{-4}$ for $\Gamma = \Gamma_1$. Presented in Fig. 2 are a nonseparable spline signal f in (5.12), and the difference between the original signal f and the reconstructed signal f_ϵ via the MAPSET algorithm with noise level $\varepsilon = 10^{-4}$.

The signal f_ϵ reconstructed from the MAPSET algorithm may not provide a good approximation, up to a sign, to the original signal f if the noise level ε is much larger than a multiple of $\sqrt{F_0}/\|\Phi^{-1}\|_{\mathrm{P}}$, cf. (4.13) in Theorem 4.1. Our numerical simulations indicate that the MAPSET algorithm sometimes fails to reconstruct the phase of the original signal f when $\varepsilon \ge 3 \times 10^{-3}$ for $\Gamma = \Gamma_0$ and $\varepsilon \ge 8 \times 10^{-4}$ for $\Gamma = \Gamma_1$, where $\sqrt{F_0}/\|\Phi^{-1}\|_{\mathrm{P}}$ are 3.5763×10^{-5} and 3.0307×10^{-6} respectively. Detailed analysis of our simulations shows that the main reason of failures of our MAPSET algorithm at high noise level is that the local minimizer in the first step of the algorithm does not provide a good approximation to amplitudes of the original signal, up to a sign.

5.2 Nonseparable Spline Signals of Non-tensor Product Type

Let M_{Ξ_Z} be the box spline function in (2.9) with $\Xi_Z = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, see [24]. Unlike the spline function $B_{(3,3)}$ of tensor-product type, our numerical result indicates that the shift-invariant space generated by M_{Ξ_Z} does *not* have the local complement property on $(0,1)^2$, cf. Sect. 5.1. Set $A_U := \{(s,t) : 0 < s < t < 1\}$ and $A_L := \{(s,t) : 0 < t < s < 1\}$. One may verify that the triangle regions A_U and A_L satisfy (3.2), and the shift-invariant space generated by M_{Ξ_Z} has local complement property on A_U and on A_L.

For $A = A_U$ and $\phi = M_{\Xi_Z}$, one may verify that the set K_{A_U} in (2.7) is $\{(0,0), (-1,0), (-2,0), (-1,-1), (-2,-1)\}$ and the function $\Phi_{A_U}(s,t)$ in (2.6) is

$$\Phi_{A_U}(s,t) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 4 & 4 \\ -1 & 1 & 2 & -2 & 0 \\ 1 & -1 & -6 & -2 & 8 \end{pmatrix}\begin{pmatrix} s^2 \\ (t-s)^2 \\ s \\ t-s \\ 1 \end{pmatrix}, \ (s,t) \in A_U. \quad (5.16)$$

Observe that the space spanned by the outer products of vectors $(s^2, (t-s)^2, s, t-s, 1)^T$, $(s,t) \in A_U$ has dimension 13. This together with (5.16) implies that the space spanned by the outer products of $\Phi_{A_U}(s,t)$ has dimension 13. Therefore we can select a set $\Gamma_{2,U} \subset A_U$ with cardinality 13 to satisfy (3.3), see Fig. 3. Similarly, for the lower triangle region A_L, a sampling set $\Gamma_{2,L}$ with cardinality 13 can be chosen to satisfy (3.3). For our simulations, we use

$$\Gamma_2 = \Gamma_{2,U} \cup \Gamma_{2,L}$$

as the sampling set contained in $A_U \cup A_L \subset (0,1)^2$, see Fig. 3. For the above set Γ_2, the corresponding $\|\Phi^{-1}\|_{\mathrm{P}}$ in (4.10) is 87.9420.

Recall that $\Phi_{A_U}(s,t)$ is a vector-valued polynomial generated by s^2, $(t-s)^2$, s, $t-s$ and 1, and observe that the matrix $(\Phi_{A_U}(s_i,t_i))_{1\le i\le 5}$ has full rank 5 for almost all $(s_i,t_i) \in A_U$, $1 \le i \le 5$ as its determinant is a nonzero polynomial of (s_i,t_i), $1 \le i \le 5$. Thus $(\Phi_{A_U}(s_i,t_i))_{1\le i\le 9}$ is a phase retrievable frame for almost all $(s_i,t_i) \in A_U$, $1 \le i \le 9$. So we can use randomly distributed sets $\Gamma_{3,U} \subset A_U$ and $\Gamma_{3,L} \subset A_L$ with cardinality 9 that satisfy (3.5), see Fig. 3. Set

$$\Gamma_3 = \Gamma_{3,U} \cup \Gamma_{3,L}.$$

For the above set Γ_3, the corresponding $\|\Phi^{-1}\|_{\mathrm{P}}$ in (4.10) is 761.2227.

In our simulations, the available data $z_\epsilon(y) = |f(y)| + \epsilon(y) \ge 0$, $y \in \Gamma + K$, are noisy phaseless samples of a spline signal

$$f(s,t) = \sum_{0\le m\le K_1, 0\le n\le K_2} c(m,n) M_{\Xi_Z}(s-m, t-n), \quad (5.17)$$

taken on $\Gamma + K$, where $K = [0, K_1] \times [0, K_2]$ for some $1 \le K_1, K_2 \in \mathbb{Z}$, Γ is either Γ_2 or Γ_3 in Fig. 3, amplitudes of the signal f are as in (5.13), and the additive noises $\epsilon(y) \in [-\varepsilon, \varepsilon]$, $y \in \Gamma + K$, with noise level $\varepsilon \ge 0$ are randomly selected. Denote the signal reconstructed by the proposed MAPSET algorithm with phase adjustment threshold value $M_0 = 0.01$ by

$$f_\epsilon(s,t) = \sum_{0\le m\le K_1, 0\le n\le K_2} c_\epsilon(m,n) M_{\Xi_Z}(s-m, t-n). \quad (5.18)$$

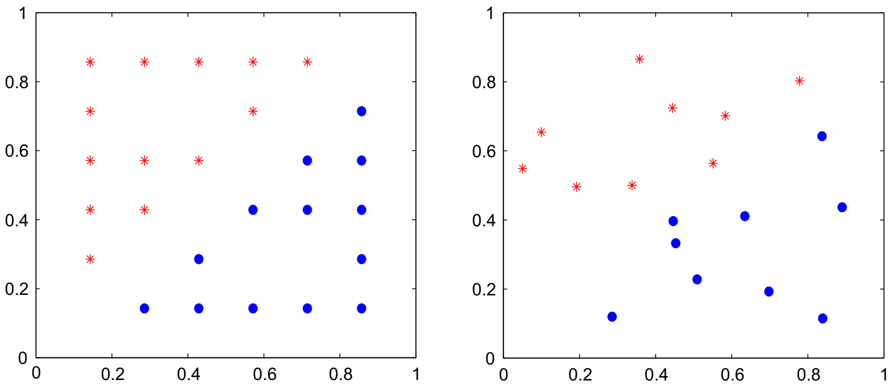


Fig. 3 Plotted on the left are the sampling sets $\Gamma_{2,U} \subset A_U$ (in the red star) and $\Gamma_{2,L} \subset A_L$ (in the blue dot). Plotted on the right are the random sets $\Gamma_{3,U} \subset A_U$ (in the red star) and $\Gamma_{3,L} \subset A_L$ (in the blue dot) that have cardinality 9. The corresponding $\|\Phi^{-1}\|_{\mathrm{P}}$ in (4.10) to the above sets is 87.9420 (left) and 761.2227 (right) respectively

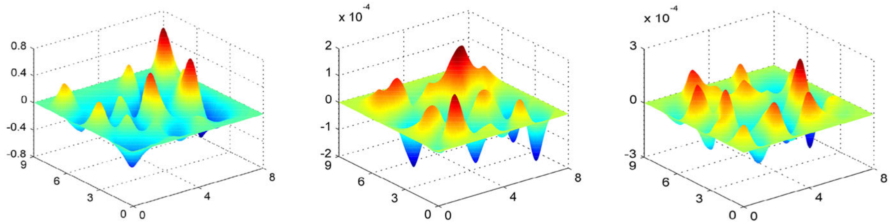


Fig. 4 Plotted on the left is a nonseparable spline signal of the form (5.17), where $K = [0, 7] \times [0, 6]$, and in the middle and on the right are the difference between the above signal f and the signal f_ϵ reconstructed by the MAPSET algorithm with noise level $\varepsilon = 10^{-4}$ and the sampling set Γ being Γ_2 and Γ_3 in Fig. 3, respectively. The maximal amplitude errors $e(\epsilon)$ in (5.15) are 2.4922×10^{-4} (middle) and 3.8975×10^{-4} (right). The reconstruction errors $\min_{\delta \in \{-1,1\}} \|f_\epsilon - \delta f\|_\infty$ are 1.9660×10^{-4} (middle) and 2.9216×10^{-4} (right)

As in Sect. 5.1, the reconstruction signal f_ϵ provides an approximation, up to a sign, to the original signal f. Our numerical simulations indicate that the MAPSET algorithm saves phases in 100 trials and the reconstruction error $e(\epsilon)$ is about $O(\varepsilon)$, provided that $\varepsilon \le 5 \times 10^{-3}$ for $\Gamma = \Gamma_2$ and $\varepsilon \le 3 \times 10^{-3}$ for $\Gamma = \Gamma_3$, where $\sqrt{F_0}/\|\Phi^{-1}\|_{\mathrm{P}}$ are 1.1371×10^{-3} and 1.3137×10^{-4} respectively. Presented in Fig. 4 are a nonseparable spline signal f in (5.17) with $(K_1, K_2) = (9, 8)$, and the difference between the original signal f and the reconstructed signal f_ϵ via the MAPSET algorithm with noise level $\varepsilon = 10^{-4}$.

As in Sect. 5.1, the MAPSET algorithm may not yield a good approximation to the original signal if the noise level ε is not sufficiently small. Our numerical results indicate that the MAPSET algorithm sometimes fails to save the phase of the original signal f when $\varepsilon \ge 6 \times 10^{-3}$ for $\Gamma = \Gamma_2$ and $\varepsilon \ge 4 \times 10^{-3}$ for $\Gamma = \Gamma_3$. As in the phaseless reconstruction of box spline of tensor-product type, our detailed analysis indicates that the main reason of failures of our MAPSET algorithm at high noise level is that the local minimizer in the first step of the algorithm does not provide a good approximation to amplitudes of the original signal, up to a sign.

6 Proofs

In this section, we include the proofs of Theorems 2.4, 3.1, 3.3, 2.6, 3.5, 4.1, and Corollaries 3.7 and 3.8 .

6.1 Proof of Theorem 2.4

Suppose, on the contrary, that $\mathcal{G}_f$ is disconnected. Let W be the set of vertices in a connected component of the graph $\mathcal{G}_f$. Then $W \neq \emptyset$, $V_f \backslash W \neq \emptyset$, and there are no edges between vertices in W and $V_f \backslash W$. Write

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}^d} c(k)\phi(\cdot - k) = \sum_{k\in V_f} c(k)\phi(\cdot - k) \\ &= \sum_{k\in W} c(k)\phi(\cdot - k) + \sum_{k\in V_f\backslash W} c(k)\phi(\cdot - k) =: f_1 + f_2 \end{aligned} \tag{6.1}$$

where $c(k) \in \mathbb{R}$, $k \in \mathbb{Z}^d$. From the global linear independence on ϕ and nontriviality of the sets W and $V_f \backslash W$, we obtain

$$f_1 \neq 0 \ \text{ and } \ f_2 \neq 0. \tag{6.2}$$

Combining (6.1) and (6.2) with nonseparability of the signal f, we obtain that $f_1(x_0)f_2(x_0) \neq 0$ for some $x_0 \in \mathbb{R}^d$. Then by the global linear independence of ϕ, there exist $k \in W$ and $k' \in V_f \backslash W$ such that $\phi(x_0 - k) \neq 0$ and $\phi(x_0 - k') \neq 0$. Hence (k, k') is an edge between $k \in W$ and $k' \in V_f \backslash W$, which contradicts to the construction of the set W.

6.2 Proof of Theorem 3.1

A linear space V on $\mathbb{R}^d$ is said to be *locally finite-dimensional* if it has finite-dimensional restriction on any bounded open set. The shift-invariant space in (1.1) generated by a compactly supported function ϕ is locally finite-dimensional. The reader may refer to [5] and references therein on locally finite-dimensional spaces, see Appendix A for local complement property of a locally finite-dimensional space. In this section, we will prove the following generalization of Theorem 3.1.

Theorem 6.1 *Let V be a locally finite-dimensional shift-invariant space of functions on $\mathbb{R}^d$. Then there exists a finite set $\Gamma \subset (0, 1)^d$ such that any nonseparable signal $f \in V$ is determined, up to a sign, by its phaseless samples on $\Gamma + \mathbb{Z}^d$.*

Proof Let $A = (0, 1)^d$ and $V|_A$ be the space containing restrictions of all signals in V on A. By the shift-invariance, it suffices to find a set $\Gamma \subset A$ and functions $d_\gamma(x)$, $\gamma \in \Gamma$, such that

$$|f(x)|^2 = \sum_{\gamma\in\Gamma} d_\gamma(x)|f(\gamma)|^2, \ x \in A \tag{6.3}$$

hold for all $f \in V$. By the assumption on V, $V|_A$ is finite-dimensional. Let $g_n \in V$, $1 \le n \le N$, be a basis for $V|_A$, and W be the linear space spanned by symmetric matrices

$$G(x) := \big(g_n(x)g_{n'}(x)\big)_{1\le n,n'\le N}, \ x \in A.$$

Then there exists a finite set $\Gamma \subset A$ with cardinality at most $N(N+1)/2$ such that $G(\gamma)$, $\gamma \in \Gamma$, are a basis for the space W. This implies that for any $x \in A$ there exist $d_\gamma(x)$, $\gamma \in \Gamma$, such that

$$G(x) = \sum_{\gamma\in\Gamma} d_\gamma(x)G(\gamma), \ x \in A.$$

For any $f \in V$, we write $f(x) = \sum_{n=1}^N c_n g_n(x)$, $x \in A$. Then

$$\begin{aligned}|f(x)|^2 &= \Big|\sum_{n=1}^N c_n g_n(x)\Big|^2 = \sum_{n,n'=1}^N c_n c_{n'} g_n(x)g_{n'}(x)\\ &= \sum_{n,n'=1}^N c_n c_{n'}\Big(\sum_{\gamma\in\Gamma} d_\gamma(x)g_n(\gamma)g_{n'}(\gamma)\Big) = \sum_{\gamma\in\Gamma} d_\gamma(x)|f(\gamma)|^2, \ x \in A.\end{aligned}$$

This proves (6.3) and hence completes the proof. □

6.3 Proof of Theorem 3.3

The implication (iii) $\Longrightarrow$ (i) is trivial. By (3.2), local linear independence of ϕ on A_m, $1 \le m \le M$, and shift-invariance of the linear space $V(\phi)$, we obtain that the generator ϕ has the global linear independence. Then the implication (i) $\Longrightarrow$ (ii) follows from Theorem 2.4.

Now it remains to prove (ii) $\Longrightarrow$ (iii). Let Γ_m, $1 \le m \le M$, be finite sets constructed in Proposition A.3 with the set A and the space V replaced by A_m and $V(\phi)$ respectively, and set $\Gamma = \cup_{m=1}^M \Gamma_m$. Let $f, g \in V(\phi)$ satisfy

$$|g(y)| = |f(y)| \text{ for all } y \in \Gamma + \mathbb{Z}^d. \tag{6.4}$$

Then it suffices to prove that

$$g = \delta f \tag{6.5}$$

for some $\delta \in \{-1, 1\}$. Take $l \in \mathbb{Z}^d$ and $1 \le m \le M$. By Proposition A.3 and the shift-invariance of the linear space $V(\phi)$, we have

$$|g(x+l)| = |f(x+l)|, \ x \in A_m.$$

This, together with shift-invariance of the linear space $V(\phi)$ and local complement property on A_m, implies the existence of $\delta_{l,m} \in \{-1, 1\}$ such that

$$g(x) = \delta_{l,m} f(x), \ x \in A_m + l. \tag{6.6}$$

Write $f = \sum_{k\in\mathbb{Z}^d} c(k)\phi(\cdot - k)$ and $g = \sum_{k\in\mathbb{Z}^d} d(k)\phi(\cdot - k) \in V(\phi)$. Then it follows from (6.6) and local linear independence of the generator ϕ on A_m that

$$d(k' + l) = \delta_{l,m} c(k' + l), \ k' \in K_{A_m}, \tag{6.7}$$

where K_{A_m} is given in (2.7).

By (6.7), the proof of (6.5) reduces to showing

$$\delta_{l,m} = \delta \tag{6.8}$$

for all $l \in \mathbb{Z}^d$ and $1 \le m \le M$ so that $k' + l \in V_f$ for some $k' \in K_{A_m}$. Recall that $c(k) \neq 0$ for all $k \in V_f$. Then by (6.7) there exists $\delta_k \in \{-1, 1\}$ for all $k \in V_f$ such that

$$\delta_{l,m} = \delta_k$$

for all $l \in \mathbb{Z}^d$ and $1 \le m \le M$ so that $k = k' + l \in V_f$ for some $k' \in K_{A_m}$. Thus it suffices to prove that

$$\delta_k = \delta_{\tilde{k}} \ \text{ for all } \ k, \tilde{k} \in V_f. \tag{6.9}$$

By the connectivity of the graph $\mathcal{G}_f$, the proof of (6.9) reduces further to proving

$$\delta_k = \delta_{\tilde{k}} \tag{6.10}$$

for all edges $(k, \tilde{k})$ of the graph $\mathcal{G}_f$. For an edge $(k, \tilde{k})$ of the graph $\mathcal{G}_f$, we have that

$$S := \{x \in \mathbb{R}^d : \ \phi(x - k)\phi(x - \tilde{k}) \neq 0\} \neq \emptyset.$$

Then there exists $1 \le m \le M$ by (2.5) and (3.2) such that $S \cap (A_m + k) \neq \emptyset$. Thus $k, \tilde{k} \in K_{A_m} + k$, which together with (6.7) and (6.9) implies that $\delta_k = \delta_{k,m} = \delta_{\tilde{k}}$. Hence (6.10) holds. This completes the proof.

6.4 Proof of Theorem 2.6

By Proposition A.6, there are open sets $A_1, \ldots, A_M$ satisfying the requirements in Theorem 3.3. Then the conclusion in Theorem 2.6 follows from Theorem 3.3.

6.5 Proof of Theorem 3.5

Let $f = \sum_{k\in\mathbb{Z}^d} c(k)\phi(\cdot - k)$ and $g = \sum_{k\in\mathbb{Z}^d} d(k)\phi(\cdot - k)$ satisfy

$$|g(y)| = |f(y)| \ \text{ for all } \ y \in \Gamma' + \mathbb{Z}^d,$$

where $\Gamma' = \cup_{m=1}^M \Gamma'_m$ is given in (3.5). Take $l \in \mathbb{Z}^d$ and $1 \le m \le M$. Then

$$\Big|\sum_{k\in K_{A_m}+l} d(k)\phi(\gamma' + l - k)\Big| = \Big|\sum_{k\in K_{A_m}+l} c(k)\phi(\gamma' + l - k)\Big| \quad \text{for all } \gamma' \in \Gamma'_m.$$

By the assumption on $\Phi_{A_m}(\gamma')$, $\gamma' \in \Gamma'_m$, $1 \le m \le M$, there exists $\delta_{l,m} \in \{1, -1\}$ such that

$$d(k) = \delta_{l,m} c(k), \ k \in K_{A_m} + l.$$

Following the same argument as the one used for the implication (ii)$\Longrightarrow$(iii) in Theorem 3.3, we can find $\delta \in \{-1, 1\}$ such that $\delta_{l,m} = \delta$ for all $l \in \mathbb{Z}^d$ and $1 \le m \le M$. This completes the proof.

6.6 Proof of Corollary 3.7

The box spline $B_{\mathbf{N}}$ has local linear independence on $(0, 1)^d$ by the characterization in [19,20,23,31], and the shift-invariant space $V(B_{\mathbf{N}})$ generated by $B_{\mathbf{N}}$ has local complement property on $(0, 1)^d$ since the restriction of a signal in $V(B_{\mathbf{N}})$ on $(0, 1)^d$ is a polynomial of finite degree. Therefore the requirements for the generator $B_{\mathbf{N}}$ in Theorem 3.3 are satisfied with $M = 1$ and $A_1 = (0, 1)^d$.

It is observed that the function $\Phi_{(0,1)^d}$ in (2.6) is a vector-valued polynomial of degree $\mathbf{N} - \mathbf{1}$, and its outer product $\Phi_{(0,1)^d}(x)\Phi_{(0,1)^d}(x)^T$, $x \in (0, 1)^d$ is a matrix-valued polynomial of degree $2\mathbf{N} - \mathbf{2}$. Recall that X_i is a discrete set containing $2N_i - 1$ distinct points in $(0, 1)$, $1 \le i \le d$. Therefore one may verify by induction on dimension d that $\Phi_{(0,1)^d}(y)\Phi_{(0,1)^d}(y)^T$, $y \in X_1 \times \cdots \times X_d$, is a spanning set of the linear space spanned by $\Phi_{(0,1)^d}(x)\Phi_{(0,1)^d}(x)^T$, $x \in (0, 1)^d$. Therefore the conclusion follows by applying the argument used in the proof of Theorem 3.3 with $M = 1$ and $A_1 = (0, 1)^d$.

6.7 Proof of Corollary 3.8

From the argument used in the proof of Corollary 3.7, the requirements for the generator $B_{\mathbf{N}}$ in Theorem 3.5 are satisfied with $M = 1$ and $A_1 = (0, 1)^d$.

By the local linear independence on $(0, 1)^d$ for the box spline $B_{\mathbf{N}}$, there exists a nonsingular matrix A of size $\mathcal{N} \times \mathcal{N}$ such that

$$\Phi_{(0,1)^d}(x) = Ax_{\mathbf{N}} \text{ for all } x \in (0, 1)^d, \tag{6.11}$$

where $x_{\mathbf{N}} = (x^k)_{\mathbb{Z}_+^d \ni k \le \mathbf{N}}$ is a $\mathcal{N}$-dimensional column vector. For $\mathcal{M} \ge \mathcal{N}$, let

$$\Phi_{(0,1)^d}(y_1, \ldots, y_{\mathcal{M}}) = \left[\Phi_{(0,1)^d}(y_1), \ldots, \Phi_{(0,1)^d}(y_{\mathcal{M}})\right]$$

be the matrix of size $\mathcal{N} \times \mathcal{M}$ with columns $\Phi_{(0,1)^d}(y_i)$, $y_i \in (0,1)^d$, $1 \le i \le \mathcal{M}$. Then

$$\det \Phi_{(0,1)^d}(y_1, \ldots, y_{\mathcal{N}}) = \det A \sum_{\mathbf{0} \le \alpha_1, \ldots, \alpha_{\mathcal{N}} \le \mathbf{N}} \epsilon(\alpha_1, \ldots, \alpha_{\mathcal{N}}) y_1^{\alpha_1} \ldots y_{\mathcal{N}}^{\alpha_{\mathcal{N}}} \quad (6.12)$$

is a nonzero polynomial about $y_1, \ldots, y_{\mathcal{N}}$ by (6.11), where the sum is taken over the set of all mutually distinct $\mathbf{0} \le \alpha_i \le \mathbf{N}$, $1 \le i \le \mathcal{N}$, where $\epsilon(\alpha_1, \ldots, \alpha_{\mathcal{N}}) \in \{-1, 1\}$. Hence for almost all $(y_1, \ldots, y_{2\mathcal{N}-1}) \in (0,1)^d \times \ldots \times (0,1)^d$, all $\mathcal{N} \times \mathcal{N}$ submatrices of $\Phi_{(0,1)^d}(y_1, y_2, \ldots, y_{2\mathcal{N}-1})$ are nonsingular. This together with the complement property [7] for frames implies that $\Phi_{(0,1)^d}(y_i)$, $1 \le i \le 2\mathcal{N}-1$, are phase retrievable frames for almost all $(y_1, \ldots, y_{2\mathcal{N}-1}) \in (0,1)^d \times \ldots \times (0,1)^d$. Therefore the conclusion follows from Theorem 3.5.

6.8 Proof of Theorem 4.1

Given $\Gamma \subset \mathbb{R}^d$ and $f = \sum_{k \in \mathbb{Z}^d} c(k)\phi(\cdot - k)$, we define

$$\widetilde{\mathcal{G}}_{f,\Gamma} = (V_f, E_{f,\Gamma}), \quad (6.13)$$

where $(k, k') \in E_{f,\Gamma}$ only if $\phi(y-k)\phi(y-k') \ne 0$ for some $y \in \Gamma + \mathbb{Z}^d$. To prove Theorem 4.1, we need a lemma about the graph $\mathcal{G}_f$.

Lemma 6.2 *Let ϕ, A_m and Γ_m, $1 \le m \le M$, be as in Theorem 4.1. Set $\Gamma = \cup_{m=1}^M \Gamma_m$. Then for any $f \in V(\phi)$, the graph $\mathcal{G}_f$ in (2.3) and $\widetilde{\mathcal{G}}_{f,\Gamma}$ in (6.13) are the same,*

$$\mathcal{G}_f = \widetilde{\mathcal{G}}_{f,\Gamma}. \quad (6.14)$$

Proof Clearly it suffices to prove that an edge in $\mathcal{G}_f$ is also an edge in $\widetilde{\mathcal{G}}_{f,\Gamma}$. Suppose, on the contrary, that there exists an edge (k, k') in $\mathcal{G}_f$ such that

$$\phi(y-k)\phi(y-k') = 0 \text{ for all } y \in \cup_{m=1}^M \Gamma_m + \mathbb{Z}^d. \quad (6.15)$$

Define

$$S = \{x \in \mathbb{R}^d : \ \phi(x-k)\phi(x-k') \ne 0\} \ne \emptyset. \quad (6.16)$$

By (3.2), there exist $l_0 \in \mathbb{Z}^d$ and $1 \le m_0 \le M$ such that

$$S \cap (A_{m_0} + l_0) \ne \emptyset. \quad (6.17)$$

Set $g_\pm(x) = \phi(x + l_0 - k) \pm \phi(x + l_0 - k')$, $x \in A_{m_0}$. Then it follows from (6.15) that

$$|g_\pm(\gamma)| = |\phi(\gamma + l_0 - k)| + |\phi(\gamma + l_0 - k')|,\ \gamma \in \Gamma_{m_0}.$$

By the construction of the set Γ_{m_0}, we get either $g_+ = g_-$ or $g_+ = -g_-$ on A_{m_0}. Therefore either $\phi(x + l_0 - k) \equiv 0$ on A_{m_0} or $\phi(x + l_0 - k') \equiv 0$ on A_{m_0}. This contradicts to the construction of set S in (6.16) and (6.17). □

Now, we continue the proof of Theorem 4.1.

Proof of Theorem 4.1 Take $l \in \mathbb{Z}^d$ and $1 \le m \le M$. For any $\gamma \in \Gamma_m$, there exists $\tilde\delta_{\gamma,l;m} \in \{-1, 1\}$ such that

$$\begin{aligned}
&\Bigg(\sum_{\gamma\in\Gamma_m}\Bigg|\sum_{k\in l+\Omega_m}(c_{\epsilon,l;m}(k) - \tilde\delta_{\gamma,l;m}c(k))\phi(\gamma + l - k)\Bigg|^2\Bigg)^{1/2}\\
&\quad = \Bigg(\sum_{\gamma\in\Gamma_m}\Bigg|\Bigg|\sum_{k\in l+\Omega_m}c_{\epsilon,l;m}(k)\phi(\gamma + l - k)\Bigg|\\
&\qquad -\Bigg|\sum_{k\in l+\Omega_m}c(k)\phi(\gamma + l - k)\Bigg|\Bigg|^2\Bigg)^{1/2}\\
&\quad \le \Bigg(\sum_{\gamma\in\Gamma_m}\Bigg|\Bigg|\sum_{k\in l+\Omega_m}c_{\epsilon,l;m}(k)\phi(\gamma + l - k)\Bigg| - z_\epsilon(\gamma + l)\Bigg|^2\Bigg)^{1/2}\\
&\qquad +\Bigg(\sum_{\gamma\in\Gamma_m}\Bigg|\Bigg|\sum_{k\in l+\Omega_m}c(k)\phi(\gamma + l - k)\Bigg| - z_\epsilon(\gamma + l)\Bigg|^2\Bigg)^{1/2}\\
&\quad \le 2\Bigg(\sum_{\gamma\in\Gamma_m}\Bigg|\Bigg|\sum_{k\in l+\Omega_m}c(k)\phi(\gamma + l - k)\Bigg| - z_\epsilon(\gamma + l)\Bigg|^2\Bigg)^{1/2}\\
&\quad \le 2\sqrt{\#\Gamma_m}\|\epsilon\|_\infty \le 2\sqrt{\#\Gamma}\|\epsilon\|_\infty,
\end{aligned} \tag{6.18}$$

where the second inequality holds by (4.6) and the last inequality follows from

$$z_\epsilon(\gamma + l) = \Bigg|\sum_{k\in l+\Omega_m}c(k)\phi(\gamma + l - k)\Bigg| + \epsilon(\gamma + l),\ \gamma \in \Gamma_m.$$

From the phase retrievable frame property for $\big(\phi(\gamma - k)\big)_{k\in K_{A_m}}$, $\gamma \in \Gamma_m$, we obtain that

$$\Omega_m = K_{A_m},\ 1 \le m \le M. \tag{6.19}$$

Let $P_{l,m} = \{\gamma \in \Gamma_m : \tilde\delta_{\gamma,l;m} = 1\}$. This together with (6.19) and the phase retrievable frame assumption that either $\big(\phi(\gamma - k)\big)_{k\in\Omega_m}$, $\gamma \in P_{l,m}$ or $\big(\phi(\gamma - k)\big)_{k\in\Omega_m}$, $\gamma \in \Gamma_m \backslash P_{l,m}$ is a spanning set for $\mathbb{R}^{\#\Omega_m}$. This together with (6.18) implies that

$$\Big(\sum_{k\in l+\Omega_m}\big|c_{\epsilon,l;m}(k)-\tilde\delta_{l,m}c(k)\big|^2\Big)^{1/2}\le 2\|\Phi^{-1}\|_{\mathrm P}\sqrt{\#\Gamma}\|\epsilon\|_\infty \tag{6.20}$$

for some sign $\tilde\delta_{l,m}\in\{-1,1\}$.

Now we show that phases of $c_{\epsilon,l;m}, l\in\mathbb Z^d, 1\le m\le M$, can be adjusted so that (4.7) holds. Let $\tilde\delta_{l,m}, l\in\mathbb Z^d, 1\le m\le M$, be as in (6.20). Then for any $l,l'\in\mathbb Z^d$ and $1\le m,m'\le M$, set $\Omega_{l,m;l',m'}=(\Omega_m+l)\cap(\Omega_{m'}+l')$. Then

$$\begin{aligned}
\langle\tilde\delta_{l,m}c_{\epsilon,l;m},\tilde\delta_{l',m'}c_{\epsilon,l';m'}\rangle &= \sum_{k\in\Omega_{l,m;l',m'}}\tilde\delta_{l,m}\tilde\delta_{l',m'}c_{\epsilon,l;m}(k)c_{\epsilon,l';m'}(k)\\
\ge \sum_{k\in\Omega_{l,m;l',m'}}|c(k)|^2 &- \sum_{k\in\Omega_{l,m;l',m'}}|c(k)||\tilde\delta_{l',m'}c_{\epsilon,l';m'}(k)-c(k)|\\
&-\sum_{k\in\Omega_{l,m;l',m'}}|\tilde\delta_{l,m}c_{\epsilon,l;m}(k)-c(k)||c(k))|\\
&-\sum_{k\in\Omega_{l,m;l',m'}}|\tilde\delta_{l,m}c_{\epsilon,l;m}(k)-c(k)||\tilde\delta_{l',m'}c_{\epsilon,l';m'}(k)-c(k)|\\
\ge -\frac12\sum_{k\in\Omega_{l,m;l',m'}}&\Big(|\tilde\delta_{l',m'}c_{\epsilon,l';m'}(k)-c(k)|^2+|\tilde\delta_{l,m}c_{\epsilon,l;m}(k)-c(k)|^2\Big)\\
&-\sum_{k\in\Omega_{l,m;l',m'}}|\tilde\delta_{l,m}c_{\epsilon,l;m}(k)-c(k)||\tilde\delta_{l',m'}c_{\epsilon,l';m'}(k)-c(k)|\\
\ge -8\|\Phi^{-1}\|_{\mathrm P}^2\#\Gamma\|\epsilon\|_\infty^2 &\ge -M_0,
\end{aligned}\tag{6.21}$$

where the third inequality follows from (6.20) and the last inequality holds by the assumption (4.13) on the noise level $\|\epsilon\|_\infty$ and the threshold value M_0.

The phase adjustments in (4.7) for $c_{\epsilon,l;m}, l\in\mathbb Z^d, 1\le m\le M$, are not unique. Next we show that they are essentially the phase adjustments in (6.21), i.e., for the phase adjustment $\delta_{l,m}\in\{-1,1\}$ in (4.7) there exists $\delta\in\{-1,1\}$ such that

$$\delta_{l,m}c(k)=\delta\tilde\delta_{l,m}c(k)\quad\text{for all } k\in l+\Omega_m, l\in\mathbb Z^d, 1\le m\le M. \tag{6.22}$$

To prove (6.22), we claim that

$$\tilde\delta_{l,m}/\delta_{l,m}=\delta_{l',m'}/\tilde\delta_{l',m'} \tag{6.23}$$

for all (l,m) and (l',m') with $\Omega_{l,m;l',m'}\cap V_f\ne\emptyset$. Suppose on the contrary that (6.23) does not hold. Then

$$\langle\delta_{l,m}c_{\epsilon,l;m},\delta_{l',m'}c_{\epsilon,l';m'}\rangle=-\langle\tilde\delta_{l,m}c_{\epsilon,l;m},\tilde\delta_{l',m'}c_{\epsilon,l';m'}\rangle.$$

Therefore

$$
\begin{aligned}
&\langle \delta_{l,m} c_{\epsilon,l;m}, \delta_{l',m'} c_{\epsilon,l';m} \rangle \\
&\le - \sum_{k\in\Omega_{l,m;l',m'}} |c(k)|^2 + \sum_{k\in\Omega_{l,m;l',m'}} |c(k)||\tilde\delta_{l',m'} c_{\epsilon,l';m'}(k) - c(k)| \\
&\quad + \sum_{k\in\Omega_{l,m;l',m'}} |\tilde\delta_{l,m} c_{\epsilon,l;m}(k) - c(k)||c(k))| \\
&\quad + \sum_{k\in\Omega_{l,m;l',m'}} |\tilde\delta_{l,m} c_{\epsilon,l;m}(k) - c(k)||\tilde\delta_{l',m'} c_{\epsilon,l';m'}(k) - c(k)| \\
&\le - \sum_{k\in\Omega_{l,m;l',m'}} |c(k)|^2 + 4\sqrt{\#\Gamma}\|\Phi^{-1}\|_{\mathrm P}\Big(\sum_{k\in\Omega_{l,m;l',m'}} |c(k)|^2\Big)^{1/2}\|\epsilon\|_\infty \\
&\quad + 4\#\Gamma\|\Phi^{-1}\|_{\mathrm P}^2\|\epsilon\|_\infty^2 \\
&\le - \sum_{k\in\Omega_{l,m;l',m'}} |c(k)|^2 + \Big(2M_0 \sum_{k\in\Omega_{l,m;l',m'}} |c(k)|^2\Big)^{1/2} + \frac{M_0}{2} < -M_0,
\end{aligned}
$$

where the second inequality follows from (6.20), and the third and fourth inequalities hold by (4.11), (4.12) and (4.13). This contradicts to the requirement (4.7) for the phase adjustment and hence completes the proof of the Claim (6.23).

By (6.23), for any $k \in V_f$ there exists $\delta_k \in \{-1, 1\}$ such that

$$
\delta_{l,m} c(k) = \delta_k \tilde\delta_{l,m} c(k) \quad \text{for all } k \in l + \Omega_m. \tag{6.24}
$$

Let (k_1, k_2) be an edge in $\mathcal{G}_f$. By Lemma 6.2 there exist $l \in \mathbb{Z}^d$ and $1 \le m \le M$ such that $k_1, k_2 \in \Omega_m + l$. Therefore

$$
\delta_{l,m} c(k_1) = \delta_{k_1} \tilde\delta_{l,m} c(k_1) \text{ and } \delta_{l,m} c(k_2) = \delta_{k_2} \tilde\delta_{l,m} c(k_2)
$$

by (6.24). This implies that $\delta_{k_1} = \delta_{k_2}$ for any edge (k_1, k_2) in $\mathcal{G}_f$. Combining it with the connectivity of the graph $\mathcal{G}_f$, we can find $\delta \in \{-1, 1\}$ such that

$$
\delta_k = \delta \ \text{ for all } k \in V_f. \tag{6.25}
$$

Combining (6.24) and (6.25) proves (6.22).

By (6.20) and (6.22), we obtain

$$
\begin{aligned}
|d_\epsilon(k) - \delta c(k)| &\le \frac{\sum_{m=1}^M \sum_{l\in\mathbb{Z}^d} |\delta_{l,m} c_{\epsilon,l;m}(k) - \delta c(k)|\chi_{l+\Omega_m}(k)}{\sum_{m=1}^M \sum_{l\in\mathbb{Z}^d} \chi_{l+\Omega_m}(k)} \\
&= \frac{\sum_{m=1}^M \sum_{l\in\mathbb{Z}^d} |c_{\epsilon,l;m}(k) - \tilde\delta_{l,m} c(k)|\chi_{l+\Omega_m}(k)}{\sum_{m=1}^M \sum_{l\in\mathbb{Z}^d} \chi_{l+\Omega_m}(k)} \\
&\le 2\sqrt{\#\Gamma}\|\Phi^{-1}\|_{\mathrm P}\|\epsilon\|_\infty, \ k \in \mathbb{Z}^d.
\end{aligned} \tag{6.26}
$$

This together with (4.12) and (4.13) implies that

$$|d_\epsilon(k)| \geq \sqrt{2M_0} \text{ for all } k \in V_f \tag{6.27}$$

and

$$|d_\epsilon(k)| \leq \sqrt{M_0/2} \text{ for all } k \notin V_f. \tag{6.28}$$

Combining (4.9), (6.26), (6.27) and (6.28) completes the proof of the desired error estimates (4.14) and (4.15). □

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Appendix A: Local Complement Property

A linear space V on $\mathbb{R}^d$ is said to be *locally finite-dimensional* if it has finite-dimensional restrictions on any bounded open set. Examples of locally finite-dimensional spaces include the space of polynomials of a fixed degrees, the shift-invariant space generated by finitely many compactly supported functions, and their linear subspaces. The reader may refer to [5] and references therein on locally finite-dimensional spaces. In Sect. 6.2, we have discussed phaseless sampling on a locally finite-dimensional space. In this section, we consider the local complement property for a locally finite-dimensional space, cf. Definition 3.2.

Definition A.1 Let V be a linear space of real-valued continuous functions on $\mathbb{R}^d$, and $A \subset \mathbb{R}^d$. We say that V has *local complement property on* A if for any $A' \subset A$ there does not exist $f, g \in V$ such that $f, g \not\equiv 0$ on A, $f \equiv 0$ on A' and $g \equiv 0$ on $A \backslash A'$.

In the following theorem, we establish the equivalence between the local complement property on a bounded open set and complement property for ideal sampling functionals on a finite subset, cf. [16].

Theorem A.2 *Let A be a bounded open set and V be a locally finite-dimensional space of real-valued continuous signals on $\mathbb{R}^d$. Then V has the local complement property on A if and only if there exists a finite set $\Gamma \subset A$ such that for any $\Gamma' \subset \Gamma$ either there does not exist $f \in V$ satisfying*

$$f \not\equiv 0 \ \text{ on } A \ \text{ and } \ f(\gamma') = 0, \ \gamma' \in \Gamma', \tag{A.1}$$

or there does not exist $g \in V$ satisfying

$$g \not\equiv 0 \ \text{ on } A \ \text{ and } \ g(\gamma) = 0, \ \gamma \in \Gamma \backslash \Gamma'. \tag{A.2}$$

To prove the sufficiency of Theorem A.2, we need a proposition.

Proposition A.3 *Let A be a bounded open set and V be a locally finite-dimensional space of real-valued continuous signals on $\mathbb{R}^d$. Then there exist a finite set $\Gamma \subset A$ and functions $d_\gamma(x)$, $\gamma \in \Gamma$, such that*

$$|f(x)|^2 = \sum_{\gamma\in\Gamma} d_\gamma(x)|f(\gamma)|^2, \ x \in A \tag{A.3}$$

hold for all $f \in V$.

Proof Let g_n, $1 \le n \le N$, be a basis of the space $V|_A$, and W be the linear space spanned by symmetric matrices $G(x) := \big(g_n(x)g_{n'}(x)\big)_{1\le n,n'\le N}$, $x \in A$. Then there exists a finite set $\Gamma \subset A$ such that $G(\gamma)$, $\gamma \in \Gamma$, is a basis (or a spanning set) for the space W. With the above set Γ, we can follow the proof of Theorem 6.1 in Sect. 6.2 to prove (A.3). □

Now we prove Theorem A.2.

Proof of Theorem A.2 We prove the necessity by an indirect proof. Suppose, on the contrary, that V does not have the local complementary property on A. Then there exist a subset $A' \subset A$ and nonzero functions $f, g \in V$ on A such that $f(x) = 0$ for all $x \in A'$ and $g(y) = 0$ for all $y \in A\backslash A'$. This leads to a contradiction by taking a finite subset $\Gamma \subset A$ and letting $\Gamma' = \Gamma \cap A'$.

To prove the sufficiency, let Γ be as in Proposition A.3. Suppose, on the contrary, that there exist nonzero functions $f, g \in V$ on A such that (A.1) and (A.2) are satisfied for some set $\Gamma' \subset \Gamma$. Let $f_1 = f + g$ and $f_2 = f - g$. We obtain from (A.1) and (A.2) that $|f_1(\gamma)| = |f_2(\gamma)|$ for all $\gamma \in \Gamma$. This together with Proposition A.3 implies that $|f_1(x)| = |f_2(x)|$ for all $x \in A$. Since $V(\phi)$ has local complement property on A, we have that either $f_1 = f_2$ or $f_1 = -f_2$. Therefore either $f = 0$ or $g = 0$ on A, which contradicts to our assumption on functions f and g.

□

Let g_n, $1 \le n \le N$, be a basis of the space $V|_A$ and let Γ be chosen in the proof of Proposition A.3. Then Theorem A.2 can be reformulated as follows: V has the local complement property on A if and only if for any $\Gamma' \subset \Gamma$ either there does not exist a nonzero vector $(c_0(n))_{1\le n\le N}$ such that $\sum_{n=1}^N c_0(n)g_n(\gamma') = 0$ for all $\gamma' \in \Gamma'$ or there does not exist a nonzero vector $(c_1(n))_{1\le n\le N}$ such that $\sum_{n=1}^N c_1(n)g_n(\gamma) = 0$ for all $\gamma \in \Gamma\backslash\Gamma'$. Thus the linear space V has the local complement property on A if and only if for any $\Gamma' \subset \Gamma$, either $(g_n(\gamma'))_{1\le n\le N}$, $\gamma' \in \Gamma'$ form a frame for $\mathbb{R}^N$ or $(g_n(\gamma))_{1\le n\le N}$, $\gamma \in \Gamma\backslash\Gamma'$ form a frame for $\mathbb{R}^N$. The above characterization together with [7, Theorem 2.8] implies that the following criterion that can be used to verify the local complement property on a bounded open set A in finite steps.

Theorem A.4 *Let g_n, $1 \le n \le N$, be a basis of the space $V|_A$ and let Γ be chosen in the proof of Proposition A.3. Then the linear space V has the local complement property on A if and only if $(g_n(\gamma))_{1\le n\le N}$, $\gamma \in \Gamma$, is a phase retrievable frame for $\mathbb{R}^N$.*

The local complement property for different open sets could be equivalent. Following the argument used in the proof of Theorem A.2, we have

Proposition A.5 *Let A be a bounded open set and V be a locally finite-dimensional space with the local complement property on A. If B is a bounded open subset of A such that signals g and f satisfying $|g(x)| = |f(x)|$ on B have the same magnitude measurements on A, then V has local complement property on B.*

The conclusion in the above proposition is not true in general. A linear space may have the local complement property on a bounded open A, but not on some of its open supsets and subsets. For instance, the shift-invariant space $V(\phi_0)$ in Example 2.7 has the local complement property on $(0, 1)$, but not on its supset $(0, 3/2)$ and its subset $(0, 1/2)$.

We finish the appendix with a proposition about local linear independence and local complement property, which is used in the proof of Theorem 2.6.

Proposition A.6 *Let ϕ be a compactly supported continuous function with local linear independence on any open set. Then there exist A_m, $1 \le m \le M$, such that (3.2) holds and $V(\phi)$ has the local complement property on A_m, $1 \le m \le M$.*

Proof Let S_k, $k \in \mathbb{Z}^d$, be as in (2.5). For a set $T \subset \mathbb{Z}^d$, define

$$S_T = \cap_{k \in T} S_k = \left\{x \in \mathbb{R}^d : \ \phi(x) \neq 0 \text{ and } \phi(x-k) \neq 0 \text{ for all } k \in T\right\}.$$

We say that T is maximal if $S_T \neq \emptyset$ and $S_{T'} = \emptyset$ for all $T' \supsetneq T$. From the definition, there are finitely many maximal sets $T_1, \ldots, T_M$, and denote the corresponding sets by $A_m := S_{T_m}$, $1 \le m \le M$.

Clearly, (3.2) holds for the above selected open sets as

$$\cup_{m=1}^{M} T_m = \left\{k \in \mathbb{Z}^d : \ S_k \neq \emptyset\right\}.$$

Then it remains to prove that $V(\phi)$ has local complement property on A_m, $1 \le m \le M$. Assume that $f, g \in V(\phi)$ satisfy $|f(x)| = |g(x)|$ for all $x \in A_m$, which implies that $(f+g)(x)(f-g)(x) = 0$ for all $x \in A_m$. Write $f + g = \sum_{k \in \mathbb{Z}^d} c_0(k)\phi(\cdot - k)$ and $f - g = \sum_{k \in \mathbb{Z}^d} c_1(k)\phi(\cdot - k)$ for some sequences $(c_0(k))$ and $(c_1(k))$. Set $B_1 = \{x \in A_m : (f+g)(x) \neq 0\}$ and $B_2 = \{x \in A_m : (f-g)(x) \neq 0\}$. Then either $f - g = 0$ on B_1, or $f + g = 0$ on B_2, or $f - g = f + g = 0$ on A_m. Hence either $c_0(k) = 0$ for all $k \in T_m$ or $c_1(k) = 0$ on $k \in T_m$ by the local independence on B_1, or on B_2 or on A_m. Therefore either $f = g$ on A_m, or $f = -g$ on A_m, or $f = g = 0$ on A_m. This completes the proof. □

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