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Letter to the Editor: Proof of the HRT Conjecture for Almost Every (1,3) Configuration

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Abstract
We prove that for almost every (1,3) configuration, there is no linear dependence between the associated time-frequency translates of any $f \in L^2(\mathbb{R})\backslash\{0\}$.

Keywords HRT conjecture · Configuration · Continued fraction expansion

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1 Introduction

For a measurable function $f : \mathbb{R} \to \mathbb{C}$ and a subset $\Lambda \subset \mathbb{R}^2$, the associated Gabor system is given by

$$\mathcal{G}(f, \Lambda) = \{M_b T_a f : (a, b) \in \Lambda\},$$

where

$$M_b T_a f(x) = e^{2\pi i b x} f(x - a).$$

We call $M_b T_a f$ a time-frequency translate of f.

The Heil–Ramanathan–Topiwala (HRT) conjecture [10] asserts that finite Gabor systems in $L^2(\mathbb{R})$ are linearly independent (also see [8]). That is

The HRT Conjecture Let $\Lambda \subset \mathbb{R}^2$ be a finite set. Then there is no non-trivial function $f \in L^2(\mathbb{R})$ such that the associated Gabor system $\mathcal{G}(f, \Lambda)$ is linearly dependent in $L^2(\mathbb{R})$.

Here are some examples to show that the $L^2(\mathbb{R})$ property of function f is essential.

1 For any trigonometric polynomial f, there exists a finite subset $\Lambda \subset \mathbb{R}^2$ such that the associated Gabor system $\mathcal{G}(f, \Lambda)$ is linearly dependent.

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2 Let $f(x) = \frac{1}{2^n}$ for $x \in [n-1, n)$. Then $f \in L^2(\mathbb{R}^+)$ but $f \notin L^2(\mathbb{R})$. It is easy to see that $\{f(x+1), f(x)\}$ is linearly dependent.

Since the formulation of the HRT conjecture, some results were obtained (see [9] and references therein) under further restrictions on the behavior of function $f(x)$ at $x = \infty$ [1,3,4,10] or the structure of the time-frequency translates Λ [2,5,6,10,12]. Recall that we call Λ an (n, m) configuration if there exist 2 distinct parallel lines containing Λ such that one of them contains exactly n points of Λ, and the other one contains exactly m points of Λ. The following results hold without restriction on $f \in L^2(\mathbb{R})\backslash\{0\}$.

- $\mathcal{G}(f, \Lambda)$ is linearly independent if $\#\Lambda \leq 3$ or Λ is colinear [10].
- $\mathcal{G}(f, \Lambda)$ is linearly independent if Λ is a finite subset of a translate of a lattice in $\mathbb{R}^2$ [12]. See [2,6] for alternative proofs.
- $\mathcal{G}(f, \Lambda)$ is linearly independent if Λ is a (2,2) configuration [5,7].
- $\mathcal{G}(f, \Lambda)$ is linearly independent if Λ is a (1,3) configuration with certain arithmetic restriction [5]. See Theorem 1.1.

In this paper, we consider $(1, 3)$ configurations. In [5], Demeter proved

Theorem 1.1 *The HRT conjecture holds for special* $(1, 3)$ *configurations*

$$\Lambda = \{(0,0), (\alpha, 0), (\beta, 0), (0,1)\},$$

(a) *if there exists* $\gamma > 1$ *such that*

$$\liminf_{n\to\infty} n^\gamma \min\left\{\left\|n\frac{\beta}{\alpha}\right\|, \left\|n\frac{\alpha}{\beta}\right\|\right\} < \infty, \tag{1}$$

(b) *if at least one of* α, β *is rational.*

It is known that $\{x \in \mathbb{R} :$ there exists some $\gamma > 1$ such that $\liminf_{n\to\infty} n^\gamma ||nx|| < \infty\}$ is a set of zero Lebesgue measure (e.g., [11, Theorem 32]). Thus Theorem 1.1 holds for a measure zero subset of parameters, and it has been an open problem to extend it to other $(1, 3)$ configurations.

Our main result is

Theorem 1.2 *The HRT conjecture holds for special* $(1, 3)$ *configurations*

$$\Lambda = \{(0,0), (\alpha, 0), (\beta, 0), (0,1)\},$$

(a) *if*

$$\liminf_{n\to\infty} n \ln n \min\left\{\left\|n\frac{\beta}{\alpha}\right\|, \left\|n\frac{\alpha}{\beta}\right\|\right\} < \infty, \tag{2}$$

(b) *if at least one of* α, β *is rational.*

Remark 1.3 (b) of Theorem 1.2 is the same statement as (b) in Theorem 1.1. We list here for completeness.

It is well known that $\{x \in \mathbb{R} : \liminf_{n\to\infty} n \ln n||nx|| < \infty\}$ is a set of full Lebesgue measure (e.g., [11, Theorem 32]). Then using metaplectic transformations, we have the following theorem

Theorem 1.4 *Given any line l in $\mathbb{R}^2$, let (a_2, b_2) and (a_3, b_3) be any two points lying in l. Let (a_1, b_1) be an any point not lying in l. Then for almost every point (a_4, b_4) in l, the HRT conjecture holds for the configuration $\Lambda = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)\}$.*

Proof By metaplectic transformations (see [10] for details), we can assume l is x-axis, and $(a_1, b_1) = (0, 1)$, $(a_2, b_2) = (0, 0)$ and $(a_3, b_3) = (\alpha, 0)$. By Theorem 1.2 and the fact that $\{x \in \mathbb{R} : \liminf_{n\to\infty} n \ln n||nx|| < \infty\}$ is a set of full Lebesgue measure, we have for almost every β, the HRT conjecture holds for $\Lambda = \{(0, 1), (0, 0), (\alpha, 0), (\beta, 0)\}$. We finish the proof. □

2 The Framework of the Proof of Theorem 1.2

If $\frac{\alpha}{\beta}$ is a rational number, it reduces to the lattice case, which has been proved [12]. Thus we also assume $\frac{\alpha}{\beta}$ is irrational.

Assume Theorem 1.2 does not hold. Then there exists some function f satisfying

$$\lim_{|n|\to\infty} |f(x+n)| = 0 \text{ a.e. } x \in [0, 1) \tag{3}$$

and nonzero $C_i \in \mathbb{C}$ such that

$$f(x+1) = f(x)\Big(C_0 + C_1 e^{2\pi i\alpha x} + C_2 e^{2\pi i\beta x}\Big) \text{ a.e. } x \in \mathbb{R}. \tag{4}$$

(Theorem 1.2 is covered by the known results if $C_i = 0$ for some $i = 0, 1, 2$)

Let

$$P(x) = C_0 + C_1 e^{2\pi i\alpha x} + C_2 e^{2\pi i\beta x}.$$

For $n > 0$, define

$$P_n(x) = \prod_{j=0}^{n} P(x+j),$$

$$P_{-n}(x) = \prod_{j=-n}^{-1} P(x+j).$$

Notice that $P(x+n)$ is an almost periodic function. Thus for almost every $x \in [0, 1)$,

$$P(x+n) \neq 0 \text{ for any } n \in \mathbb{Z}.$$

Iterating (4) n times on both sides (positive and negative), we have for $n > 0$,

$$f(x+n) = P_n(x) f(x) \text{ a.e. } x \in [0, 1), \tag{5}$$

and

$$f(x-n) = P_{-n}(x)^{-1} f(x) \text{ a.e. } x \in [0,1). \tag{6}$$

This implies that the value of function f on $\mathbb{R}$ can be determined uniquely by its value on $[0, 1)$ and function $P(x)$.

By Egoroff's theorem and conditions (3), (5) and (6), there exists some positive Lebesgue measure set $S \subset [0, 1)$ and $d > 0$, such that

$$\lim_{|n|\to\infty} f(x+n) = 0 \text{ uniformly for } x \in S, \tag{7}$$

$$d < |f(x)| < d^{-1} \text{ for all } x \in S, \tag{8}$$

$$f(x+n) = P_n(x) f(x) \text{ for all } x \in S, \tag{9}$$

and

$$f(x-n) = P_{-n}(x)^{-1} f(x) \text{ for all } x \in S. \tag{10}$$

Demeter constructed a sequence $\{n_k\} \subset \mathbb{Z}^+$, such that

$$|P_{n_k}(x_k) P_{-n_k}(x_k')^{-1}| \geq C \tag{11}$$

for some $x_k, x_k' \in S$. This contradicts (7)–(10).

In order to complete the construction of (11), growth condition (1) was necessary in [5]. In the present paper, we follow the approach of [5]. The novelty of our work is in the subtler Diophantine analysis. This allows to make the restriction weak enough to obtain the result for a full Lebesgue measure set of parameters, and significantly simplifies the proof.

The rest of the paper is organized as follows. In Sect. 3, we will give some basic facts. In Sect. 4, we give the proof of Theorem 1.2.

3 Preliminaries

We start with some basic notations. Denote by $[x]$, $\{x\}$, $\|x\|$ the integer part, the fractional part and the distance to the nearest integer of x. Let $\langle x\rangle$ be the unique number in $[-1/2, 1/2)$ such that $x - \langle x\rangle$ is an integer. For a measurable set $A \subset \mathbb{R}$, denote by $|A|$ its Lebesgue measure.

For any irrational number $\alpha \in \mathbb{R}$, we define

$$a_0 = [\alpha], \alpha_0 = \alpha,$$

and inductively for $k > 0$,

$$a_k = \left[\alpha_{k-1}^{-1}\right], \alpha_k = \alpha_{k-1}^{-1} - a_k. \tag{12}$$

We define

$$p_0 = a_0, q_0 = 1,$$

$$p_1 = a_0 a_1 + 1, q_1 = a_1,$$

and inductively,

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2}, \\ q_k &= a_k q_{k-1} + q_{k-2}. \end{aligned} \tag{13}$$

Recall that $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of denominators of best approximations of irrational number α, since it satisfies

$$\forall 1 \le k < q_{n+1}, \|k\alpha\| \ge ||q_n\alpha||. \tag{14}$$

Moreover, we also have the following estimate,

$$\frac{1}{2q_{n+1}} \le \|q_n\alpha\| \le \frac{1}{q_{n+1}}. \tag{15}$$

Lemma 3.1 *Let $k_1 < k_2 < k_3 < \cdots < k_m$ be a monotone integer sequence such that $k_m - k_1 < q_n$. Suppose for some $x \in \mathbb{R}$*

$$\min_{j=1,2\cdots,m} ||k_j\alpha - x|| \ge \frac{1}{4q_n}. \tag{16}$$

Then

$$\sum_{j=1,2\cdots,m} \frac{1}{||k_j\alpha - x||} \le Cq_n \ln q_n.$$

Proof Recall that $\langle x\rangle$ is the unique number in $[-1/2, 1/2)$ such that $x - \langle x\rangle$ is an integer. Thus $||x|| = |\langle x\rangle|$. In order to prove the Lemma, it suffices to show that

$$\sum_{j=1,2\cdots,m} \frac{1}{|\langle k_j\alpha - x\rangle|} \le Cq_n \ln q_n. \tag{17}$$

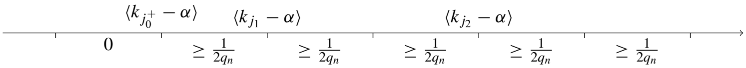
Let $S^+ = \{j : j = 1, 2, \cdots, m, \langle k_j\alpha - x\rangle > 0\}$. Let j_0^+ be such that $j_0^+ \in S^+$, and

$$\langle k_{j_0^+}\alpha - x\rangle = \min_{j\in S^+} \langle k_j\alpha - x\rangle. \tag{18}$$

By (14) and (15), one has for $i \ne j$ and $i, j \in S^+$,

$$|\langle k_i\alpha - x\rangle - \langle k_j\alpha - x\rangle| = ||(k_i\alpha - x) - (k_j\alpha - x)|| \ge \frac{1}{2q_n}.$$

It implies the gap between any two points $\langle k_i\alpha - x\rangle$ and $\langle k_j\alpha - x\rangle$ with $i, j \in S^+$ is larger than $\frac{1}{2q_n}$. See the following figure.



It easy to see that the upper bound of $\sum_{j\in S^+}\frac{1}{||k_j\alpha - x||}$ is achieved if all the gaps are exactly $\frac{1}{2q_n}$. In this case, the gap between the ith closest points of $\langle k_j\alpha - x\rangle$ with $j \in S^+$ to $\langle k_{j_0^+}\alpha - x\rangle$ is exactly $\frac{i}{2q_n}$. Thus by (16), we have

$$\begin{aligned}\sum_{j\in S^+}\frac{1}{||k_j\alpha - x||} &= \frac{1}{||k_{j_0^+}\alpha - x||} + \sum_{j\in S^+, j\neq j_0^+}\frac{1}{||k_j\alpha - x||}\\ &= \frac{1}{\langle k_{j_0}\alpha - x\rangle} + \sum_{j\in S^+, j\neq j_0^+}\frac{1}{\langle k_j\alpha - x\rangle}\\ &\leq 4q_n + \sum_{1\leq j\leq q_n}\frac{2q_n}{j}\\ &\leq Cq_n\ln q_n. \end{aligned}\tag{19}$$

Similarly, letting $S^- = \{j : j = 1, 2, \cdots, m, \langle k_j\alpha - x\rangle < 0\}$, one has

$$\sum_{j\in S^-}\frac{1}{||k_j\alpha - x||} \leq Cq_n\ln q_n. \tag{20}$$

By (19) and (20), we finish the proof. □

Now we give two lemmas which can be found in [5].

Lemma 3.2 ([5, Lemma 2.1]) *Let* $C_0, C_1, C_2 \in \mathbb{C}\backslash\{0\}$. *The polynomial* $p(x, y) = C_0 + C_1e^{2\pi ix} + C_2e^{2\pi iy}$ *has at most two real zeros* $(\gamma_1^{(j)}, \gamma_2^{(j)}) \in [0, 1)^2$, $j \in \{1, 2\}$ *and there exists* $t = t(C_0, C_1, C_2) \in \mathbb{R}\setminus\{0\}$ *such that*

$$|p(x, y)| \geq C(C_0, C_1, C_2)\min_{j=1,2}(\|x-\gamma_1^j+t\langle y-\gamma_2^j\rangle\|+\|x-\gamma_1^j\|^2+\|y-\gamma_2^j\|^2), \tag{21}$$

for each $x, y \in \mathbb{R}$.

Remark 3.3 In (21), we assume $p(x, y)$ has two zeros. If $p(x, y)$ has one or no zeros, we can proceed with our proof by replacing (21) with

$$|p(x, y)| \geq C(C_0, C_1, C_2)(\|x - \gamma_1 + t\langle y - \gamma_2\rangle\| + \|x - \gamma_1\|^2 + \|y - \gamma_2\|^2),$$

or

$$|p(x, y)| \geq C(C_0, C_1, C_2).$$

Lemma 3.4 ([5, Lemma 4.1]) *Let $x_1, x_2, \ldots, x_N$ be N not necessarily distinct real numbers. Then for each $N \in \mathbb{Z}^+$ and each $\delta > 0$, there exists a set $E_{N,\delta} \subset [0, 1)$ with $|E_{N,\delta}| \le \delta$, such that*

$$\sum_{n=1}^{N} \frac{1}{\|x - x_n\|} \le C(\delta) N \log N, \tag{22}$$

and

$$\sum_{n=1}^{N} \frac{1}{\|x - x_n\|^2} \le C(\delta) N^2, \tag{23}$$

for each $x \in [0, 1) \setminus E_{N,\delta}$.

4 Proof of Theorems 1.2

In this section, q_k, p_k, a_k are always the coefficients of the continued fraction expansion of $\frac{\alpha}{\beta}$ as given in (12) and (13). Then condition (2) holds iff

$$\limsup_k \frac{a_k}{\ln q_k} > 0, \tag{24}$$

and also iff

$$\limsup_k \frac{q_{k+1}}{q_k \ln q_k} > 0.$$

Lemma 4.1 *Suppose $\frac{\alpha}{\beta}$ is irrational and satisfies condition (2). Then for any $s \in (0, 1)$, there exists a sequence N_k such that*

(i)

$$N_k = m_{n_k} q_{n_k}, m_{n_k} \le C(s), \tag{25}$$

(ii)

$$\left\| N_k \frac{\alpha}{\beta} \right\| \le \frac{C(s)}{N_k \ln N_k}, \tag{26}$$

and

(iii)

$$\left\{ \frac{N_k}{\beta} \right\} \le s. \tag{27}$$

Proof By (24), there exists a sequence n_k such that $a_{n_k} \ge c \ln q_{n_k}$. For any $s \in (0, 1)$, let $m_{n_k} \in \mathbb{Z}^+$ be such that $1 \le m_{n_k} \le 1/s + 1$ and $N_k = m_{n_k} q_{n_k}$ satisfies (iii) (this can be done by the pigeonhole principle). It is easy to check that N_k satisfies condition (ii) by the fact $a_{n_k} \ge c \ln q_{n_k}$. □

Lemma 4.2 *Let $C_0, C_1, C_2 \in \mathbb{C}\backslash\{0\}$ and α, β be such that $\frac{\alpha}{\beta}$ is irrational. Let Q_k be a sequence such that $\gamma q_{n_k} \le Q_k \le \hat{\gamma} q_{n_k}$, where q_n is the continued fraction expansion of $\frac{\alpha}{\beta}$ and $\gamma, \hat{\gamma}$ are constants. Define*

$$P(x) = C_0 + C_1 e^{2\pi i \alpha x} + C_2 e^{2\pi i \beta x}.$$

Then for each $\delta > 0$*, there exists a set* $E_{Q_k,\delta} \subset [0,1)$ *such that*

$$|E_{Q_k,\delta}| < \delta$$

and

$$\sum_{n=0}^{Q_k-1} \frac{1}{|P(x+n)|} \le C(\gamma, \hat{\gamma}, \delta, C_0, C_1, C_2, \alpha, \beta) Q_k \ln Q_k$$

for each $x \in [0,1) \setminus E_{Q_k,\delta}$.

Proof In order to make the proof simpler, we will use C for constants depending on $\gamma, \hat{\gamma}, \delta, C_0, C_1, C_2, \alpha, \beta$.

Let (γ_1, γ_2) be a zero of the polynomial $p(x, y) = C_0 + C_1 e^{2\pi i x} + C_2 e^{2\pi i y}$, and let t be the real number given by Lemma 3.2. Define

$$\begin{aligned} A_n(x) := &\|\alpha(x+n) - \gamma_1 + t\langle\beta(x+n) - \gamma_2\rangle\| \\ &+\|\alpha(x+n) - \gamma_1\|^2 + \|\beta(x+n) - \gamma_2\|^2. \end{aligned}$$

By Lemma 3.2, it suffices to find a set with $|E_{Q_k,\delta}| \le \delta$, such that

$$\sum_{n=0}^{Q_k-1} \frac{1}{A_n(x)} \le C Q_k \ln Q_k, \tag{28}$$

for each $x \in [0,1) \setminus E_{Q_k,\delta}$.

We distinguish between two cases.

Case 1 $\alpha + t\beta \ne 0$

In this case, one has

$$\begin{aligned} &\|\alpha(x+n) - \gamma_1 + t\langle\beta(x+n) - \gamma_2\rangle\| \\ &\quad = \|(\alpha + t\beta)x + (\alpha + t\beta)n - \gamma_1 - t\gamma_2 - t[\beta(x+n) - \gamma_2] + mt\|, \end{aligned}$$

where $m = -1$ if $\{\beta(x+n) - \gamma_2\} > 1/2$ and $m = 0$ otherwise. We remind that $[\beta(x+n) - \gamma_2]$ is the integer part of $\beta(x+n) - \gamma_2$.

Note that the set

$$\begin{aligned} S :=&\{(\alpha + t\beta)n - \gamma_1 - t\gamma_2 - t[\beta(x+n) - \gamma_2] \\ &+mt : \ x \in [0,1), \ 0 \le n \le Q_k - 1, \ m \in \{0, -1\}\} \end{aligned}$$

has $O(Q_k)$ elements. By (22) there exists some $E^1_{Q_k,\delta}$ with $|E^1_{Q_k,\delta}| < \delta/2$ such that

$$\sum_{y \in S} \frac{1}{\|(\alpha + t\beta)x + y\|} \le C Q_k \ln Q_k$$

for each $x \in [0,1) \setminus E^1_{Q_k,\delta}$. This implies (28).

Case 2 $\alpha + t\beta = 0$.

In this case, one has

$$\|\alpha(x+n) - \gamma_1 + t\langle\beta(x+n) - \gamma_2\rangle\| = \left\| -\gamma_1 - t\gamma_2 + mt + \frac{\alpha}{\beta}[\beta(x+n) - \gamma_2]\right\|,$$

where m is as before. Let ξ be either $\gamma_1 + t\gamma_2$ or $\gamma_1 + t\gamma_2 + t$, depending on whether $m = 0$ or -1. From Lemma 3.1, we have that for each $x \in [0, 1)$

$$\sum_{\substack{n=0 \\ \|\frac{\alpha}{\beta}[\beta(x+n)-\gamma_2]-\xi\| \geq \frac{1}{4q_{n_k}}}}^{Q_k-1} \frac{1}{\|\frac{\alpha}{\beta}[\beta(x+n) - \gamma_2] - \xi\|} \leq C \sum_{\substack{n=0 \\ \|\frac{\alpha}{\beta}n-\xi\| \geq \frac{1}{4q_{n_k}}}}^{CQ_k} \frac{1}{\|\frac{\alpha}{\beta}n - \xi\|}$$
$$\leq CQ_k \ln Q_k. \tag{29}$$

Let $S(\xi)$ (not depending on x) be the set of those $0 \leq n \leq Q_k - 1$ such that $\|\frac{\alpha}{\beta}[\beta(x+n) - \gamma_2] - \xi\| \leq \frac{1}{4q_{n_k}}$ for some $x \in [0, 1)$. It is easy to see that $\#S(\xi) \leq C$ by (14) and (15). For $n \in S(\xi)$, we will use an alternative estimate

$$A_n(x) \geq \|\alpha(x+n) - \gamma_1\|^2.$$

By (23), there exists some set $E^2_{Q_k,\delta} \subset [0, 1)$ such that $|E^2_{Q_k,\delta}| \leq \frac{\delta}{2}$ and

$$\sum_{\substack{n=0 \\ \|\frac{\alpha}{\beta}[\beta(x+n)-\gamma_2]-\xi\| \leq \frac{1}{4q_{n_k}}}}^{Q_k-1} \frac{1}{A_n(x)} \leq C \sum_{\substack{n=0 \\ \|\frac{\alpha}{\beta}[\beta(x+n)-\gamma_2]-\xi\| \leq \frac{1}{4q_{n_k}}}}^{Q_k-1} \frac{1}{\|\alpha(x+n) - \gamma_1\|^2}$$
$$\leq C(\delta), \tag{30}$$

for each $x \in [0, 1) \setminus E^2_{Q_k,\delta}$. Thus in this case, (28) follows from (29) and (30). Putting two cases together, we finish the proof. □

Theorem 4.3 *Under the conditions of Lemma 4.2, let N_k be a sequence such that* (i)*,* (ii) *and* (iii) *in Lemma 4.1 hold. Define $P_k := \frac{N_k}{\beta}$ for $\beta > 0$ and $P_k := -\frac{N_k}{\beta}$ for $\beta < 0$. Given $\delta > 0$, there exists $E_{k,\delta} \subset [0, 1)$ with $|E_{k,\delta}| \leq \delta$ such that for each x, y satisfying $x \in [0, 1) \setminus E_{k,\delta}$ and $x = y + P_k$, we have*

$$\left|\prod_{n=0}^{[P_k]-1} P(y+n)\right| \leq C(\delta, s, C_0, C_1, C_2, \alpha, \beta) \left|\prod_{n=0}^{[P_k]-1} P(x+n)\right|.$$

Proof We write C for $C(\delta, s, C_0, C_1, C_2, \alpha, \beta)$ again. Without loss of generality, we only consider the case $\beta > 0$.

By (26) we have

$$\left|e^{2\pi i\alpha x}-e^{2\pi i\alpha y}\right|=\left|e^{2\pi i N_k\frac{\alpha}{\beta}}-1\right|\leq\frac{C}{P_k\ln P_k}$$

and

$$\left|e^{2\pi i\beta x}-e^{2\pi i\beta y}\right|=0.$$

Thus, for each $n\in\mathbb{Z}^+$, one has

$$|P(y+n)|\leq|P(x+n)|+\frac{C}{P_k\ln P_k}.$$

By the fact $1+x\leq e^x$ for $x>0$, we get

$$|P(y+n)|\leq|P(x+n)|e^{\frac{C}{P_k\ln P_k|P(x+n)|}},$$

and thus

$$\left|\prod_{n=0}^{[P_k]-1}P(y+n)\right|\leq\left|\prod_{n=0}^{[P_k]-1}P(x+n)\right|e^{\frac{C}{P_k\ln P_k}\sum_{n=0}^{[P_k]-1}\frac{1}{|P(x+n)|}}.$$

Now Theorem 4.3 follows from Lemma 4.2. □

Proof of Theorem 1.2 Suppose Theorem 1.2 is not true. As argued in Sect. 2, there there exist some function f, a positive Lebesgue measure set $S\subset[0,1)$ and $d>0$ such that (7)–(10) hold. By the continuity of Lebesgue measure, there exists $\varepsilon=\varepsilon(S)>0$ such that

$$|S\cup(\{P_k\}+S)|\leq\frac{101}{100}|S|,$$

for $\{P_k\}\leq\varepsilon$. Let $\delta=\frac{|S|}{100}$. Then $(S\setminus E_{k,\delta})\cap(\{P_k\}+S)\neq\emptyset$. Let $s=\varepsilon$. Applying Theorem 4.3 with s and δ, we have

$$\left|\prod_{n=0}^{[P_k]-1}P(y+n)\right|\leq C\left|\prod_{n=0}^{[P_k]-1}P(x+n)\right|, \tag{31}$$

for each $x\in[0,1)\setminus E_{k,\delta}$ and $x=y+P_k$.

Now we can choose $x_k\in S\setminus E_{k,\delta}$ such that $x_k-\{P_k\}\in S$. Let $y_k=x_k'-[P_k]=x_k-P_k$. Then

$$\prod_{n=0}^{[P_k]-1}P(y_k+n)=\prod_{n=1}^{[P_k]}P(x_k'-n). \tag{32}$$

By (31) and (32), we get

$$\left|\prod_{n=1}^{[P_k]}P(x_k'-n)\right|\leq C\left|\prod_{n=0}^{[P_k]-1}P(x_k+n)\right|.$$

Applying (9) and (10) with $x_k, x'_k \in S$, one has

$$f(x_k + [P_k]) = f(x_k) \prod_{n=0}^{[P_k]-1} P(x_k + n), \tag{33}$$

and

$$f(x'_k - [P_k]) = f(x'_k) \left(\prod_{n=1}^{[P_k]} P(x'_k - n) \right)^{-1}. \tag{34}$$

By (8), (33) and (34), we obtain that

$$|f(x_k + [P_k]) f(x'_k - [P_k])| \geq \frac{d^2}{C}.$$

This is contradicted by (7), if we let $k \to \infty$. □

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