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Non-harmonic Cones are Heisenberg Uniqueness Pairs for the Fourier Transform on $\mathbb{R}^n$

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Abstract In this article, we prove that a cone is a Heisenberg uniqueness pair corresponding to sphere as long as the cone does not completely recline on the level surface of any homogeneous harmonic polynomial on $\mathbb{R}^n$. We derive that $\left(S^2,\ \text{paraboloid}\right)$ and $\left(S^2,\ \text{geodesic of } S_r(o)\right)$ are Heisenberg uniqueness pairs for a class of certain symmetric finite Borel measures in $\mathbb{R}^3$. Further, we correlate the problem of Heisenberg uniqueness pairs to the sets of injectivity for the spherical mean operator.

Keywords Bessel function · Fourier transform · Spherical harmonics

Mathematics Subject Classification Primary 42A38 · Secondary 44A35

1 Introduction

A Heisenberg uniqueness pair is a pair (Γ, Λ), where Γ is a surface in $\mathbb{R}^n$ and Λ is a subset of $\mathbb{R}^n$ such that any finite Borel measure μ which is supported on Γ and absolutely continuous with respect to the surface measure on Γ, whose Fourier transform $\hat{\mu}$ vanishes on Λ, implies $\mu = 0$.

In general, the existence of Heisenberg uniqueness pair (HUP) is a question of asking about the determining properties of the finite Borel measures which are supported on some lower dimensional entities whose Fourier transform also vanishes on lower dimensional entities. In fact, the main contrast in the HUP problem to the known

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results on determining sets for measures [10] is that the set Λ has also been considered as a very thin set. In particular, if Γ is compact, then $\hat{\mu}$ is real analytic, having exponential growth, and hence $\hat{\mu}$ can vanishes on a very delicate set. Thus, the HUP problem becomes little easier in this case. However, this problem becomes immensely difficult when the measure is supported on a non-compact entity. It appears that the HUP problem is a natural invariant of the theme of the uncertainty principle for the Fourier transform.

In addition, the concept of determining the Heisenberg uniqueness pair for a class of finite measures has also a significant similarity with the celebrated result due to M. Benedicks (see [11]). That is, support of a function $f \in L^1(\mathbb{R}^n)$ and its Fourier transform $\hat{f}$ both cannot be of finite measure simultaneously. Later, various analogues of the Benedicks theorem has been investigated in different aspects including the Heisenberg group and Euclidean motion groups (see [17,21,22]).

However, our main objective in this article is to discuss the concept of HUP, which was first introduced by Hedenmalm and Montes-Rodríguez in 2011. In the article [14], Hedenmalm and Montes-Rodríguez have shown that the pair (hyperbola, some discrete set) is a Heisenberg uniqueness pair. As a dual problem, a weak* dense subspace of $L^\infty(\mathbb{R})$ has been constructed to solve the Klein–Gordon equation. Further, a complete characterization of the Heisenberg uniqueness pairs corresponding to any two parallel lines has been given by Hedenmalm and Montes-Rodríguez (see [14]). Thereafter, a considerable amount of work has been done pertaining to the Heisenberg uniqueness pair in the plane as well as in the Euclidean spaces.

Recently, Lev [16] and Sjölin [23] have independently shown that circle and certain system of lines are HUP corresponding to the unit circle S^1. Further, Vieli [32] has generalized HUP corresponding to circle in the higher dimension and shown that a sphere whose radius does not lie in the zero set of the Bessel functions $J_{(n+2k-2)/2};\ k \in \mathbb{Z}_+$, the set of non-negative integers, is a HUP corresponding to the unit sphere S^{n-1}. Vieli [33] has worked out some HUPs corresponding to the paraboloid in the higher dimensions.

Further, Sjölin [24] has investigated some of the Heisenberg uniqueness pairs corresponding to the parabola. Subsequently, Babot [9] has given a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of three parallel lines. Thereafter, the authors in [12] have given some necessary and sufficient conditions for the Heisenberg uniqueness pairs corresponding to a system of four parallel lines. In the latter case, a phenomenon of three totally disconnected interlacing sets that are given by zero sets of three trigonometric polynomials has been observed. However, an exact analogue for the finitely many parallel lines as compared to three lines result [9] is still unsolved. In [12], the authors have also investigated some of the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle and the exponential curves.

In a major development, Jaming and Kellay [15] have given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to the hyperbola, polygon, ellipse and graph of the functions $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$. Further, Gröchenig and Jaming [13] have worked out some of the Heisenberg uniqueness pairs corresponding to the quadratic surface.

Let Γ be a finite disjoint union of smooth curves in $\mathbb{R}^2$. Let $X(\Gamma)$ be the space of all finite complex-valued Borel measure μ in $\mathbb{R}^2$ which is supported on Γ and absolutely continuous with respect to the arc length measure on Γ. For $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of μ is defined by

$$\hat{\mu}(\xi, \eta) = \int_{\Gamma} e^{-i\pi(x\cdot\xi + y\cdot\eta)} d\mu(x, y).$$

In the above context, the function $\hat{\mu}$ becomes a uniformly continuous bounded function on $\mathbb{R}^2$. Thus, we can analyze the pointwise vanishing nature of the function $\hat{\mu}$.

Definition 1.1 Let Λ be a set in $\mathbb{R}^2$. The pair (Γ, Λ) is called a Heisenberg uniqueness pair for $X(\Gamma)$ if any $\mu \in X(\Gamma)$ satisfies $\hat{\mu}|_\Lambda = 0$, implies $\mu = 0$.

Since the Fourier transform is invariant under translation and rotation, one can easily deduce the following invariance properties about the Heisenberg uniqueness pair.

(i) Let $u_o, v_o \in \mathbb{R}^2$. Then the pair (Γ, Λ) is a HUP if and only if the pair $(\Gamma + u_o, \Lambda + v_o)$ is a HUP.
(ii) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear transform whose adjoint is denoted by T^*. Then (Γ, Λ) is a HUP if and only if $\left(T^{-1}\Gamma, T^*\Lambda\right)$ is a HUP.

Now, we would like to state the first known result about the Heisenberg uniqueness pair due to Hedenmalm and Montes-Rodríguez [14].

Theorem 1.2 [14] *Let Γ be the hyperbola $x_1x_2 = 1$ and $\Lambda_{\alpha,\beta}$ a lattice-cross defined by*

$$\Lambda_{\alpha,\beta} = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z}),$$

where α, β are positive reals. Then $\left(\Gamma, \Lambda_{\alpha,\beta}\right)$ is a Heisenberg uniqueness pair if and only if $\alpha\beta \leq 1$.

For $\xi \in \Lambda$, defining a function e_ξ on Γ by $e_\xi(x) = e^{i\pi x\cdot\xi}$. As a dual problem to Theorem 1.3, Hedenmalm and Montes-Rodríguez [14] have proved the following density result which in turn solve the one-dimensional Klein–Gordon equation.

Theorem 1.3 [14] *The pair (Γ, Λ) is a Heisenberg uniqueness pair if and only if the set $\{e_\xi : \xi \in \Lambda\}$ is a weak* dense subspace of $L^\infty(\Gamma)$.*

Remark 1.4 In particular, the HUP problem has another formulation. That is, if Γ is the zero set of a polynomial P on $\mathbb{R}^2$, then $\hat{\mu}$ satisfies the PDE $P(-i\partial)\hat{\mu} = 0$ with initial condition $\hat{\mu}|_\Lambda = 0$. This may help potentially in determining the geometrical structure of the set $Z(\hat{\mu})$, the zero set of the function $\hat{\mu}$. If we consider Λ to be contained in $Z(\hat{\mu})$, then (Γ, Λ) is not a HUP. Hence the question of the HUP arises when Λ has located away from $Z(\hat{\mu})$.

Definition 1.5 A set C in $\mathbb{R}^n$ $(n \geq 2)$ which satisfies the scaling condition $\lambda C \subseteq C$, for all $\lambda \in \mathbb{R}$, is called a cone.

Let S^{n-1} denote unit sphere in $\mathbb{R}^n$. In this article, we prove that the pair $\left(S^{n-1}, C\right)$ is a Heisenberg uniqueness pair as long as the cone C does not recline on the level surface of any homogeneous harmonic polynomial on $\mathbb{R}^n$. We will call such cones as ***non-harmonic*** cones.

An example of such a cone has been produced by Armitage (see [8]). Let $0 < \alpha < 1$ and let G_l^{λ} denote the Gegenbauer polynomial of degree l and order λ. Then

$$K_\alpha = \left\{x \in \mathbb{R}^n : |x_1|^2 = \alpha^2 |x|^2\right\}$$

is a non-harmonic cone if and only if $D^m G_l^{\frac{n-2}{2}}(\alpha) \neq 0$ for all $0 \leq m \leq l-2$, where D^m denotes the m-th derivative.

2 Notation and Preliminaries

In this section, we recall certain standard facts about spherical harmonics. For more details see [31, p. 12].

Let $K = SO(n)$ be the special orthogonal group and $M = SO(n-1)$. Let $\hat{K}_M$ denote the set of all the equivalence classes of irreducible unitary representations of K which have a nonzero M-fixed vector. It is well known that each representation in $\hat{K}_M$ has, in fact, a unique nonzero M-fixed vector, up to a scalar multiple.

For a $\sigma \in \hat{K}_M$, which is realized on V_σ, let $\{e_1, \ldots, e_{d(\sigma)}\}$ be an orthonormal basis of V_σ, with e_1 as the M-fixed vector. Let $t_\sigma^{ij}(k) = \langle e_i, \sigma(k) e_j \rangle$, whenever $k \in K$. By the Peter-Weyl Theorem for the representations of a compact group, it follows that $\{\sqrt{d(\sigma)}\, t_k^{1j} : 1 \leq j \leq d(\sigma), \sigma \in \hat{K}_M\}$ is an orthonormal basis of $L^2(K/M)$.

We also need a concrete realization of the representations in $\hat{K}_M$, which can be done in the following way.

Let $\mathbb{Z}_+$ denote the set of all non-negative integers. For $l \in \mathbb{Z}_+$, let P_l denote the space of all homogeneous polynomials P in n variables of degree l. Let $H_l = \{P \in P_l : \Delta P = 0\}$, where Δ is the standard Laplacian on $\mathbb{R}^n$. The elements of H_l are called solid spherical harmonics of degree l. It is easy to see that the natural action of K leaves the space H_l invariant. In fact, the corresponding unitary representation π_l is in $\hat{K}_M$. Moreover, $\hat{K}_M$ can be identified, up to unitary equivalence, with the collection $\{\pi_l : l \in \mathbb{Z}_+ .\}$

Define the spherical harmonics on the sphere S^{n-1} by $Y_{lj}(\omega) = \sqrt{d_l}\, t_{\pi_l}^{1j}(k)$, where $\omega = k.e_n \in S^{n-1}$, $k \in K$ and d_l is the dimension of H_l. Then the set $\widetilde{H}_l = \left\{Y_{lj} : 1 \leq j \leq d_l, l \in \mathbb{Z}_+\right\}$ forms an orthonormal basis for $L^2(S^{n-1})$. Thus, we can expand a suitable function g on S^{n-1} as

$$g(\omega) = \sum_{l=0}^{\infty} \sum_{j=1}^{d_l} a_{lj} Y_{lj}(\omega). \tag{2.1}$$

For each fixed $\xi \in S^{n-1}$, define a linear functional on $\widetilde{H}_l$ by $\xi \mapsto Y_l(\xi)$. Then there exists a unique spherical harmonic, say $Z_\xi^{(l)} \in H_l$ such that

$$Y_l(\xi) = \int_{S^{n-1}} Z_\xi^{(l)}(\eta) Y_l(\eta) d\sigma(\eta). \tag{2.2}$$

The spherical harmonic $Z_\xi^{(l)}$ is a K bi-invariant real-valued function which is constant on the geodesics orthogonal to the line joining the origin and ξ. The spherical harmonic $Z_\xi^{(l)}$ is called the zonal harmonic of the space $\widetilde{H}_l$ around the point ξ for the above and the various other peculiar reasons. For more details, see [29, p. 143].

Let f be a function in $L^1(S^{n-1})$. For each $l \in \mathbb{Z}_+$, we define the l^{th} spherical harmonic projection of the function f by

$$\Pi_l f(\xi) = \int_{S^{n-1}} Z_\xi^{(l)}(\eta) f(\eta) d\sigma(\eta). \tag{2.3}$$

Then the function $\Pi_l f$ is a spherical harmonic of degree l. If for a $\delta > (n-2)/2$, we denote $A_l^m(\delta) = \binom{m-l+\delta}{\delta}\binom{m+\delta}{\delta}^{-1}$, then the spherical harmonic expansion $\sum\limits_{l=0}^{\infty} \Pi_l f$ of the function $f \in L^1(\mathbb{R}^n)$ is δ - Cesaro summable to f. That is,

$$f = \lim_{m\to\infty} \sum_{l=0}^{m} A_l^m(\delta) \Pi_l f, \tag{2.4}$$

where limit on the right-hand side of (2.4) exists in $L^1\left(S^{n-1}\right)$. For more details see [25].

We would like to mention that the proof of our main result is carried out by restricting the problem to the unit sphere S^{n-1} in terms of averages of its geodesic spheres. This is possible because the cone C is closed under scaling.

For $\omega \in S^{n-1}$ and $t \in (-1, 1)$, the set $S_\omega^t = \left\{\nu \in S^{n-1} : \omega \cdot \nu = t\right\}$ is a geodesic sphere on S^{n-1} with pole at ω. Let f be an integrable function on S^{n-1}. Then by Fubini's Theorem, we can define the geodesic spherical means of the function f by

$$\tilde{f}(\omega, t) = \int_{S_\omega^t} f d\nu_{n-2},$$

where ν_{n-2} is the normalized surface measure on the geodesic sphere S_ω^t.

Since the zonal harmonic $Z_\xi^{(l)}(\eta)$ is K bi-invariant, there exists a nice function F in $(-1, 1)$ such that $Z_\xi^{(l)}(\eta) = F(\xi \cdot \eta)$. Hence the extension of the formula (2.2) becomes inevitable. An extension of formula (2.2) for the functions F in $L^1(-1, 1)$ was obtained. This is known as the Funk–Hecke Theorem. That is,

$$\int_{S^{n-1}} F(\xi \cdot \eta) Y_l(\eta) d\sigma(\eta) = C_l Y_l(\xi), \tag{2.5}$$

where the constant C_l is given by

$$C_l = \alpha_l \int_{-1}^{1} F(t) G_l^{\frac{n-2}{2}}(t)(1-t^2)^{\frac{n-3}{2}}\, dt$$

and G_l^{β} stands for the Gegenbauer polynomial of degree l and order β. As a consequence of the Funk–Hecke Theorem, it can be deduced that the geodesic mean of a spherical harmonic Y_l can be expressed as

$$\widetilde{Y}_l(\omega, t) = D_l (1-t^2)^{\frac{n-2}{2}} G_l^{\frac{n-2}{2}}(t) Y_l(\omega), \tag{2.6}$$

where the constant $D_l = |S^{n-2}|/G_l^{\frac{n-2}{2}}(1)$ and $|S^{n-2}|$ denotes the surface area of the unit sphere in $\mathbb{R}^{n-1}$. For more details see [7, p. 459]. In order to prove the main result of this article, we need the following lemma, which percolates the geodesic mean vanishing condition of $f \in L^1(S^{n-1})$ to each spherical harmonic component of f. For the class of continuous functions $C(S^{n-1})$, this lemma has been worked in [5]. We prove in this article for $L^1(S^{n-1})$ using δ - Cesaro summation technique described above.

Lemma 2.1 *Let $f \in L^1(S^{n-1})$. Then $\tilde{f}(\omega, t) = 0$ for all $t \in (-1, 1)$ if and only if $\Pi_l f(\omega) = 0$ for all $l \in \mathbb{Z}_+$.*

Notice that as a corollary to Lemma 2.1, it can be deduced that if $\tilde{f}(\omega, t) = 0$ for all $t \in (-1, 1)$, then $f = 0$ a.e. on S^{n-1} if and only if ω is not contained in the zero set of any homogeneous harmonic polynomial.

Proof By the hypothesis, we have $\tilde{f}(\omega, t) = 0$ for all $t \in (-1, 1)$. Now, by taking geodesic mean in (2.4) and then using (2.6), we arrive at

$$\lim_{m\to\infty} \sum_{k=0}^{m} A_l^m(\delta) C_l G_l^{\frac{n-2}{2}}(t) \Pi_l f(\omega) = 0. \tag{2.7}$$

Since the set $\left\{ G_l^{\frac{n-2}{2}} : l \in \mathbb{Z}_+ \right\}$ form an orthonormal set on $(-1, 1)$ with weight $\left(1-t^2\right)^{-1/2}$, from (2.7) it follows that

$$\lim_{m\to\infty} A_l^m(\delta) C_l \left\| G_l^{\frac{n-2}{2}} \right\|_2^2 \Pi_l f(\omega) = 0.$$

By using the fact that for each fixed l, we have $\lim_{m\to\infty} A_l^m(\delta) = 1$, we conclude that $\Pi_l f(\omega) = 0$ for all $l \in \mathbb{Z}_+$. In particular, if ω is not contained in $Y_l^{-1}(0)$ for all $l \in \mathbb{Z}_+$, then $f(\omega) = 0$ a.e. on S^{n-1}. This completes the proof of Lemma 2.1. □

3 Proofs of the Main Result

In this section, we first prove that a non-harmonic cone is a Heisenberg uniqueness pair corresponding to the unit sphere.

Theorem 3.1 *Let $\Lambda = C$ be a cone in $\mathbb{R}^n$. Then $\left(S^{n-1}, \Lambda\right)$ is a Heisenberg uniqueness pair if and only if Λ is not contained in $P^{-1}(0)$, whenever $P \in H_l$ and $l \in \mathbb{Z}_+$.*

Proof Since μ is absolutely continuous with respect to the surface measure on S^{n-1}, by Radon-Nikodym theorem, there exists a function f in $L^1\left(S^{n-1}\right)$ such that $d\mu = f(\eta)d\sigma(\eta)$, where $d\sigma$ is the normalized surface measure on S^{n-1}. Suppose $\hat{\mu}|_\Lambda = 0$. Then

$$\hat{\mu}(\xi) = \int_{S^{n-1}} e^{-i\xi\cdot\eta} f(\eta)d\sigma(\eta) = 0 \tag{3.1}$$

for all $\xi \in S^{n-1}$. Let $\xi = r\omega$, where $r > 0$ and $\omega \in S^{n-1}$. By decomposing the integral in (3.1) into the geodesic spheres at pole ω, we get

$$\int_{-1}^{1}\left(\int_{S_\omega^t} e^{-ir\omega\cdot\nu} f(\nu)d\sigma_{n-2}(\nu)\right)dt = 0,$$

where $S_\omega^t = \left\{\nu \in S^{d-1} : \omega\cdot\nu = t\right\}$. That is,

$$\int_{-1}^{1} e^{irt}\tilde{f}(\omega, t)dt = 0, \tag{3.2}$$

for all $r > 0$. Since $f \in L^1\left(S^{n-1}\right)$, the geodesic mean $\tilde{f}(\omega, t)$ will be a continuous function on $(-1, 1)$. Thus for each fixed ω, the left-hand side of (3.2) can be viewed as the Fourier transform of the compactly supported function $\tilde{f}(\omega, .)$ on $\mathbb{R}$. Hence, it can be extended holomorphically to $\mathbb{C}$. Then, in this case, the Fourier transform of $\tilde{f}(\omega, .)$ can vanish at most on a countable set. Thus, by the continuity of $\tilde{f}(\omega, .)$ it follows that $\tilde{f}(\omega, t) = 0$ for all $t \in (-1, 1)$. Hence, in view of Lemma 2.1, we conclude that $f = 0$ a.e. on S^{n-1} if and only if ω is not contained in $Y_l^{-1}(0)$ for all $l \in \mathbb{Z}_+$. Since the cone Λ is closed under scaling, we infer that $f = 0$ a.e. if and only if Λ is not contained in $P^{-1}(0)$ for any $P \in H_l$ and for all $l \in \mathbb{Z}_+$. Thus $\mu = 0$.

Conversely, suppose the cone C is contained in the zero set of a homogeneous harmonic polynomial, say $P_j \in H_l$. Then, we can construct a finite complex Borel measure μ in $\mathbb{R}^n$ such that $d\mu = Y_j(\eta)d\sigma(\eta)$, where $Y_j \in \widetilde{H}_l$.

Using the Funk–Hecke Theorem, it has been shown that for spherical harmonic $Y_j \in \widetilde{H}_l$, the following identity holds.

$$\int_{S^{n-1}} e^{-ix\cdot\eta} Y_j(\eta)d\sigma(\eta) = i^j\,\frac{J_{j+(n-2)/2}(r)}{r^{(n-2)/2}} Y_j(\xi), \tag{3.3}$$

where $x = r\xi$, for some $r > 0$. For a proof of identity (3.3), see [7, p. 464]. This in turn implies that $\hat{\mu}|_C = 0$. □

Remark 3.2 **(a)** A set which is determining set for any real analytic function is called NA - set. For instance, the spiral is an NA - set in the plane (see [20]). The set

$$\Lambda_\varphi = \left\{ (x_1, x_2, x_3) : x_3 \left(x_1^2 + x_2^2 \right) = x_1 \varphi(x_3) \right\},$$

where function φ is given by $\varphi(x_3) = \exp \dfrac{1}{x_3^2 - 1}$, for $|x_3| < 1$ and 0 otherwise. The set Λ_φ is an NA - set. For more details see [20]. Since the Fourier transform of a finite Borel measure μ which is supported on the boundary $\partial\Omega$ of a bounded domain Ω in $\mathbb{R}^n$ can be extended holomorphically to $\mathbb{C}^n$, the pair $(\partial\Omega$, NA - set) is a Heisenberg uniqueness pair. However, the converse is not true. Hence, all together with the result of Vieli [32], it is an interesting question to examine, whether the exceptional sets for the HUPs corresponding to $\Gamma = S^{n-1}$, are eventually contained in the zero sets of all homogeneous harmonic polynomials and the countably many spheres whose radii are contained in the zero set of the certain class of Bessel functions. We leave it open for the time being.
(b) For $\Gamma = S^{n-1}$, it is easy to verify that $\hat{\mu}$ satisfies Helmholtz's equation

$$\Delta \hat{\mu} + \hat{\mu} = 0 \tag{3.4}$$

with initial condition $\hat{\mu}|_\Lambda = 0$. For a continuous function f on $\mathbb{R}^n$ $(n \geq 2)$, the spherical mean Rf of f over the sphere $S_r(x) = \{y \in \mathbb{R}^n : |x - y| = r\}$ is defined by

$$Rf(x, r) = \int_{S_r(x)} f(y) d\sigma_r(y),$$

where $d\sigma_r$ is the normalized surface measure on the sphere $S_r(x)$. Then $\hat{\mu}$ will satisfy the functional equation

$$R\hat{\mu}(x, r) = c_n \frac{J_{(n-2)/2}(r)}{r^{(n-2)/2}} \hat{\mu}(x). \tag{3.5}$$

Thus, we infer that $\hat{\mu}(x) = 0$ if and only if $R\hat{\mu}(x, r) = 0$ for all $r > 0$.

In an interesting article by Zalcman et al. [5], it is shown that for f to be continuous function on $\mathbb{R}^n$ if $Rf(x, r) = 0$ for all $r > 0$ and for all $x \in C$, then $f \equiv 0$ if and only if C is a non-harmonic cone in $\mathbb{R}^n$. In integral geometry, such sets are called sets of injectivity for the spherical means. We do not digress here to give more history of sets of injectivity for the spherical means in various set ups, still, we would like to refer to [1–6, 18, 19, 26–28]. However, this is an incomplete list of the articles on the sets of injectivity.

Thus, in view of the above result, it follows that $\hat{\mu} \equiv 0$ if and only if C is a non-harmonic cone in $\mathbb{R}^n$. As μ is a signed measure, we again need to go through the proof of Theorem 3.1, in order to show that $\mu = 0$.

Now, consider Λ to be an arbitrary set in $\mathbb{R}^n$. Then, it is clear that (S^{n-1}, Λ) is HUP if and only if Λ is a set of injectivity for spherical mean over a class of certain real analytic functions. However, the latter problem is yet not settled.

4 Some Observations for a Special Class of Measures in $\mathbb{R}^3$

In this section, we shall prove that the paraboloid is a HUP corresponding to the unit sphere S^2 in $\mathbb{R}^3$ for a class of finite Borel measure which are given by certain symmetric functions in $L^1(S^2)$. Further, we prove that a geodesic on the sphere $S_R(o)$ is a HUP corresponding to S^2 for the above class of measures.

We need the following lemma for proofs of our results of this section.

Lemma 4.1 *Let* $f \in L^1\left(S^{n-1}\right)$ *be such that* $\int\limits_{S^{n-1}} e^{-ix\cdot\eta} f(\eta)d\sigma(\eta) = 0$. *Then*

$$\lim_{m\to\infty} \sum_{k=0}^{m} i^k A_k^m \frac{J_{k+(n-2)/2}(r)}{r^{(n-2)/2}} \Pi_k f(\xi) = 0, \tag{4.1}$$

where $x = r\xi$, *for some* $r > 0$ *and* $\xi \in S^{n-1}$.

Proof We have

$$\begin{aligned}
&\left|\sum_{k=0}^{m} A_k^m \int_{S^{n-1}} e^{-ix\cdot\eta} \Pi_k f(\eta) d\sigma(\eta)\right| \\
&\quad = \left|\sum_{k=0}^{m} \int_{S^{n-1}} e^{-ix\cdot\eta} (A_k^m \Pi_k f(\eta) - f(\eta))\, d\sigma(\eta)\right| \\
&\quad \le \sum_{k=0}^{m} \int_{S^{n-1}} \left|(A_k^m \Pi_k f(\eta) - f(\eta))\right| d\sigma(\eta).
\end{aligned}$$

In view of (2.4), it follows that

$$\lim_{m\to\infty} \sum_{k=0}^{m} A_k^m \int_{S^{n-1}} e^{-ix\cdot\eta} \Pi_k f(\eta) d\sigma(\eta) = 0. \tag{4.2}$$

This in turn, from (3.3) implies that (4.1) holds. □

We know that for $n = 3$, a typical spherical harmonic of degree k can be expressed as $Y_k^l(\theta, \varphi) = e^{il\varphi} P_k^l(\cos\theta)$, where P_k^l's are the associated Legendre functions. In fact, the set $\{Y_k^l : -k \le l \le k\}$ forms an orthonormal basis for $\widetilde{H}_k$, (see [30, p. 91]). Hence, the k-th spherical harmonic projection $\Pi_k f$ can be expressed as

$$\Pi_k f(\theta, \varphi) = \sum_{l=-k}^{k} C_k^l(f) e^{il\varphi} P_k^l(\cos\theta),$$

where $0 \leq \theta < \pi$ and $0 \leq \varphi < 2\pi$. Thus, an integrable function f on S^2 has the spherical harmonic expansion as

$$f(\theta, \varphi) = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} C_k^l(f) e^{il\varphi} P_k^l(\cos\theta). \tag{4.3}$$

Let $L^1_{\text{sym}}(S^2)$ denotes the space of all those functions f in $L^1(S^2)$ that satisfy a set of *symmetric-coefficient conditions* $C_k^l(f) = C_{k'}^l(f)$, for $|l| \leq \min\{k, k'\}$.

Theorem 4.2 *Let* $\Lambda = \{(x_1, x_2, x_3) : \ x_3 = x_1^2 + x_2^2\}$. *Then* $\left(S^2, \Lambda\right)$ *is a Heisenberg uniqueness pair with respect to* $L^1_{sym}(S^2)$.

Proof Since μ is absolutely continuous with respect to the surface measure on S^2, there exists a function $f \in L^1_{\text{sym}}\left(S^2\right)$ such that $d\mu = f(\eta)d\sigma(\eta)$, where $d\sigma$ is the normalized surface measure on S^2. Suppose $\widehat{\mu}|_\Lambda = 0$. Then

$$\int_{S^2} e^{-i\xi\cdot\eta} f(\eta) d\sigma(\eta) = 0 \tag{4.4}$$

for all $\xi \in S^2$. Now, consider the spherical polar co-ordinates $x_1 = r\sin\theta\cos\varphi$, $x_2 = r\sin\theta\sin\varphi$ and $x_3 = r\cos\theta$, where $0 \leq \theta < \pi$ and $0 \leq \varphi < 2\pi$. Then, in view of Lemma 4.1, Eq. (4.4) becomes

$$\lim_{m\to\infty} \sum_{k=0}^{m} i^k A_k^m \ J_{\frac{k+1}{2}}(r) \ \Pi_k f(\theta, \varphi) = 0 \tag{4.5}$$

for all $\varphi \in [0, 2\pi)$. Notice that the rotation φ is independent of the choice of r, because, the paraboloid is completely determined by $\cos\theta = r\sin^2\theta$. Since the set $\{e^{il\varphi} : \ l \in \mathbb{Z}_+\}$ form an orthonormal set in $L^2[0, 2\pi)$ and $f \in L^1_{\text{sym}}(S^2)$, a simple calculation gives

$$\int_0 \Pi_k f(\theta, \varphi) \overline{\Pi_d f(\theta, \varphi)} d\varphi = \begin{cases} \|\Pi_k f(\theta, .)\|_2^2, & \text{if } k < d \\ \|\Pi_d f(\theta, .)\|_2^2, & \text{if } k \geq d. \end{cases} \tag{4.6}$$

After multiplying (4.5) by $\overline{\Pi_d f(\theta, \varphi)}$ and using (4.6), we conclude that

$$\lim_{m\to\infty} \left[\sum_{k=0}^{d-1} A_k^m \left| J_{\frac{k+1}{2}}(r) \right|^2 \|\Pi_k f(\theta, .)\|_2^2 + \sum_{k=d}^{m} A_k^m \left| J_{\frac{k+1}{2}}(r) \right|^2 \|\Pi_d f(\theta, .)\|_2^2 \right] = 0.$$

Thus, using the fact that $\lim_{m\to\infty} A_k^m = 1$ and the second sum goes to zero as $d \to \infty$, we obtain that

$$\sum_{l=0}^{\infty} \left| J_{\frac{l+1}{2}}(r) \right|^2 \|\Pi_l f(\theta, .)\|_2^2 = 0.$$

That is, $|J_{\frac{l+1}{2}}(r)| \, \|\Pi_l f(\theta, .)\|_2 = 0$ for all $r > 0$. Since the Bessel functions can have at most countably many zeros, it follows that

$$\Pi_l f(\theta, \varphi) = \sum_{d=-l}^{l} C_d^l(f) e^{id\varphi} P_l^d(\cos\theta) = 0.$$

This in turn, because of orthogonality of the set $\{e^{il\varphi} : l \in \mathbb{Z}_+\}$, implies that $C_d^l(f) P_l^d(\cos\theta) = 0$. However, on the paraboloid, we have $\cos\theta = r\sin^2\theta$, which gives $\cos\theta = \frac{-1+\sqrt{1+4r^2}}{2r}$. Since the Legendre functions can vanish only at countably many points, it follows that $C_d^l(f) = 0$ for all d with $-l \le d \le l$. That is, $\Pi_l f = 0$ for all $l \in \mathbb{Z}_+$. Thus $f = 0$ a.e. This completes the proof. □

Remark 4.3 We observe that Theorem 4.2 could be extended to higher dimensions in a similar way. However, to avoid the complexities of notation and calculation, we prove the result for $n = 3$.

Next, we prove that a geodesic sphere which is parallel to the equator of the sphere $S_R(o)$ is a HUP corresponding to the unit sphere S^2 with respect to $L^1_{\text{sym}}(S^2)$.

Theorem 4.4 *Let $\Lambda_{\alpha,R} = \{(\alpha, \varphi) : R\cos\alpha = r$ and $0 \le \varphi < 2\pi\}$. Then $\left(S^2, \Lambda_{\alpha,R}\right)$ is a HUP if and only if $J_{\frac{l+1}{2}}(R) \ne 0$ for all $l \in \mathbb{Z}_+$ and the ratio r/R is not contained in the zero set of any Legendre function.*

Proof Suppose $\widehat{\mu}|_{\Lambda_{\alpha,R}} = 0$. Then similarly the proof of Theorem 4.2, we reach the conclusion that $|J_{\frac{l+1}{2}}(R)| \, \|\Pi_l f(\alpha, .)\|_2 = 0$. Then $\|\Pi_l f(\alpha, .)\|_2 = 0$ for all $l \in \mathbb{Z}_+$ if $|J_{\frac{l+1}{2}}(R)| \ne 0$ for all $l \in \mathbb{Z}_+$. That is,

$$\Pi_l f(\alpha, \varphi) = \sum_{d=-l}^{l} C_d^l(f) e^{id\varphi} P_l^d(\cos\alpha) = 0.$$

By the uniqueness of the Fourier series, it follows that $C_d^l(f) P_l^d\left(\frac{r}{R}\right) = 0$. Then $C_d^l(f) = 0$ if $P_l^d\left(\frac{r}{R}\right) \ne 0$. Under the assumptions of the hypothesis, we conclude that $\Pi_l f = 0$ for all $l \in \mathbb{Z}_+$. Thus $f = 0$.

Conversely, if either of the conditions of Theorem 4.4 fails, then for the measure $d\mu = e^{il\varphi} P_k^l(\cos\theta) d\sigma(\theta, \varphi)$, it follows from the Funk–Hecke identity (3.3) that $\widehat{\mu}|_{\Lambda_{\alpha,R}} = 0$. This complete the proof. □

Remark 4.5 It is reasonable to mention that if Theorem 4.4 can be extended to a general class of finite Borel measures, then this result would have a sharp contrast, in terms of the topological dimension of the pairing set, with the known results for HUP corresponding to sphere.

5 Concluding Remarks

In this article, we have shown that (S^{n-1}, C) is a HUP as long as the cone C is not contained in the zero set of any homogeneous harmonic polynomial. Now, it is natural to consider a compact subgroup K of $SO(n)$ with K_o the orbit of K around the origin. Let $\Gamma_K = K/K_o$. We know that a unitary irreducible representation of $SO(n)$ can be decomposed into finitely many irreducible representations of K. Thus, the action of the group K on a spherical harmonic Y_l on the unit sphere S^{n-1} will decompose Y_l uniquely into a finite sum of spherical harmonics. Therefore, it would be an interesting question to find out the possibility that (Γ_K, C) is a HUP as long as the cone C does not recline on the level surface of any K—invariant homogeneous polynomial. We leave this question open for the time being.

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