J Fourier Anal Appl (2019) 25:145–166
https://doi.org/10.1007/s00041-018-9599-1

New Weighted Estimates for the Disc Multiplier on Radial Functions

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Received: 21 April 2017 / Revised: 10 October 2017 / Published online: 17 January 2018

Abstract We prove a weighted estimate for the disc multiplier, acting on radial functions, at the extreme points $p_- = \frac{2n}{n+1}$, extending the result of Chanillo (J Funct Anal 55:18–24, 1984). To this end, we prove a restricted weak type weighted estimate for $p = 2$ and then develop a new extrapolation result of independent interest.

Keywords Rubio de Francia Extrapolation · A_p weights · Hardy–Littlewood maximal function · Radial functions · Disc multiplier

Mathematics Subject Classification 42B35 · 46E30

Communicated by Hans G. Feichtinger.

María J. Carro was partially supported by Grants MTM2016-75196-P (MINECO/FEDER, UE) and 2014SGR289. Carmen Ortiz-Caraballo was supported by Grant MTM2016-75196-P (MINECO/FEDER, UE).

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1 Introduction

Let S_n be the disc multiplier on $\mathbb{R}^n$ $(n > 1)$ defined by

$$\widehat{(S_n f)}(\xi) = \chi_{B(0,1)}(\xi)\hat{f}(\xi), \qquad \xi \in \mathbb{R}^n.$$

It is known [13] that S_n is bounded on $L^p(\mathbb{R}^n)$ if and only if $p = 2$. However, in [17] it was proved that, when restricted to radial functions, S_n is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\frac{2n}{n+1} < p < \frac{2n}{n-1}.$$

Moreover, in [19] the authors proved that S_n is not of weak type on the extreme points

$$p_- := \frac{2n}{n+1}, \qquad p_+ := \frac{2n}{n-1},$$

but it holds (see [8]) that S_n is of restricted weak type at these points; that is

$$S_n : L^{p_-,1}_{\text{rad}} \longrightarrow L^{p_-,\infty}, \quad \text{and} \quad S_n : L^{p_+,1}_{\text{rad}} \longrightarrow L^{p_+,\infty}$$

are bounded, where X_{rad} is the set of radial functions in X. We observe that the boundedness on $L^{p_+,1}_{\text{rad}}$ follows of that in $L^{p_-,1}_{\text{rad}}$ by duality.

Later on, several results concerning the boundedness of S_n on radial functions on weighted L^p spaces were developed. In particular, we have to mention the following results:

(1) In [1,22], two (different) sufficient conditions on a radial weight w such that S_n is bounded on $L^2_{\text{rad}}(w)$ were given. In fact, these conditions were necessary when applied to power weights.

(2) In [12], using the characterization in [22], a new sufficient condition on a radial weight is given. In this case, the weights are in a subclass of the Muckenhoupt class A_2 [23], which was important to obtain other weighted results via Rubio de Francia extrapolation theory [26].

Recall that a positive locally integrable function w (called weight) is said to be in the Muckenhoupt class A_r $(r > 1)$ if

$$\|w\|_{A_r} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(r-1)}(x)\,dx\right)^{r-1} < \infty,$$

where Q stands for any cube in $\mathbb{R}^n$ and, we say that $w \in A_1$, if $Mw(x) \leq Cw(x)$, at almost every point $x \in \mathbb{R}^n$ with M is the Hardy–Littlewood maximal operator defined by

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|}\int_B |f(y)|\,dy,$$

where B is a ball in $\mathbb{R}^n$. In this case, $\|w\|_{A_1}$ will be the least constant C satisfying such inequality, and we notice that we use balls instead of cubes since we shall need that if f is radial, so is Mf in Theorem 3.9.

Concerning the boundedness on weighted L^p spaces of S_n, the following result holds:

Theorem 1.1 [12] *If w is a radial function such that $w^n \in A_2(\mathbb{R}^n)$, then*

$$S_n : L^2_{rad}(w) \longrightarrow L^2(w)$$

is bounded.

Then, using a technique based on Rubio de Francia extrapolation theory (Theorem 7.1, [12]) the following result (although not stated in [12]) can be easily obtained. At this point, we should mention that limited range extrapolation results, of the same nature of the following theorem, have been proved by several different authors and we explicitly referred to [3,10].

Theorem 1.2 *Let $p \in (p_-, p_+)$ and let w be a radial function such that*

$$w = u_0^{\alpha_0} u_1^{\alpha_1(1-p)}, \qquad u_j \in (A_1)_{rad}, \; j = 0, 1,$$

with

$$\alpha_0 = 1 - p\Big(\frac{n-1}{2n}\Big), \qquad \alpha_1 = 1 - p'\Big(\frac{n-1}{2n}\Big).$$

Then,

$$S_n : L^p_{rad}(w) \longrightarrow L^p(w)$$

is bounded.

Now, by the result proved in [19] and mentioned above, in the previous theorem p cannot be either p_- or p_+ showing that Rubio de Francia extrapolation theory does not allow to extrapolate to the end-points. However, taking into account the result of Chanillo [8] on restricted weak type boundedness at the end-points, one can conjecture that something similar could be true in the above weighted setting. To show that this conjecture at the extreme point p_- is true is one of the main goals of this paper. Contrary to what happens in the unweighted case, the analogue boundedness result at the extreme point p_+ does not follow from this one using duality.

Theorem 1.3 (Main Theorem) *Let w be a radial function such that $w^{\frac{n+1}{2}} \in A_1(\mathbb{R}^n)$. Then*

$$S_n : L^{p_-,1}_{rad}(w) \longrightarrow L^{p_-,\infty}(w)$$

is bounded. Moreover, the result is optimal in the sense that the exponent in w can not be improved.

At this point we have to emphasize that the proof of our main result is not an easy extension of Theorem 1.2 although it follows the same pattern. Namely, we shall prove an estimate for $p = 2$, analogous to the one in Theorem 1.1, and then prove some new extrapolation results. To this end, we have to work with the class of weights $A_p^{\mathcal{R}}$ for which the Hardy-Littlewood maximal operator in $\mathbb{R}^n$ satisfies that

$$M : L^{p,1}(w) \longrightarrow L^{p,\infty}(w)$$

is bounded, and one of the main difficulties will be the fact that this class does not satisfy the so-called $p - \varepsilon$ property, neither the reverse Hölder's inequalities or the corresponding duality property that $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$.

This new extrapolation result (Theorem 3.7) is interesting by itself since it can be applied to many other situations and it is the second main result of this paper. In short, it states that it is possible to extrapolate down to the end-point p_-, contrary to what happens with the clasical limited extrapolation result as shown in Theorem 1.2.

Concerning the weighted boundedness of S_n at the upper extreme point $p_+ = \frac{2n}{n-1}$, we believe that the following result is also true: if w is a radial weight such that $w^{-\frac{n-1}{2}} \in A_1(\mathbb{R}^n)$, then

$$S_n : L^{p_+,1}_{\text{rad}}(w) \longrightarrow L^{p_+,\infty}(w)$$

is bounded. However, the lack of the duality property (mentioned above) in our class of weights makes things more complicated and, up to now, this result remains as an open question.

As usual, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. Also, if f is a radial function on $\mathbb{R}^n$, f_0 will denote its radial part $f_0(|x|) = f(x)$ defined on $\mathbb{R}^+$, and all over the paper, we shall denote

$$I_k = (2^k, 2^{k+1}), \qquad J_k = (2^{k-1}, 2^{k+2}), \qquad \forall k \in \mathbb{Z}.$$

For later purposes, we need also to recall (see [18]) that $u \in A_p$ if and only if there exists $u_0, u_1 \in A_1$ such that

$$u = u_0 u_1^{1-p}, \qquad \|u_0\|_{A_1} \leq \|u\|_{A_p}, \|u_1\|_{A_1} \leq \|u\|_{A_p}^{\frac{1}{p-1}}, \tag{1.1}$$

and $u \in A_1$ if and only if there exists $h \in L^1_{loc}(\mathbb{R}^n)$ and k such that $k, k^{-1} \in L^\infty$ satisfying that, for some $0 < \delta < 1$, $u = k(Mh)^\delta$.

Finally, the Lorentz spaces $L^{p,q}(u)$ are defined as the set of measurable functions such that

$$\|f\|_{L^{p,q}(u)} = \left(\int_0^\infty y^{q-1} \lambda_f^u(y)^{q/p} dy \right)^{1/q} < \infty,$$

and $L^{p,\infty}(u)$ is defined by the condition

$$\|f\|_{L^{p,\infty}(u)} = \sup_{y>0} y\lambda_f^u(y)^{1/p} < \infty,$$

where $\lambda_f^u(y) = u\big(\{x : |f(x)| > y\}\big)$ is the distribution function of f with respect to u (see [4]). We use the standard notation $u(E) = \int_E u(x)\,dx$ and, if $u = 1$, we shall write $\lambda_f(y)$ and $|E|$.

The paper is organized as follows: Sect. 2 contains the above mentioned weighted L^2 estimate for S_n. The complete proof of this estimate needs several technical results which are formulated in Proposition 2.4 without proof in order to make easier the reading of the paper. The proof will be given in Sect. 4. Finally, the restricted weak type extrapolation theory and the proof of our main theorem will be done in Sect. 3.

Finally, we want to thank the referees for the very useful comments and remarks that have improved the final presentation of this paper.

2 A Weighted L^2 Estimate

As mentioned in the introduction, we need to deal with restricted weak type estimates and hence, we have to work with the class of weights $A_p^{\mathcal{R}}$ for which

$$M : L^{p,1}(w) \longrightarrow L^{p,\infty}(w)$$

is bounded and we recall that $w \in A_p^{\mathcal{R}}$ if and only if (see [20])

$$\|w\|_{A_p^{\mathcal{R}}} = \sup_{E \subset Q} \frac{|E|}{|Q|}\left(\frac{w(Q)}{w(E)}\right)^{1/p} < \infty, \tag{2.1}$$

where the supremum is taken over all cubes Q and all measurable sets $E \subset Q$, or equivalently [9],

$$\|w\|_{A_p^{\mathcal{R}}}^{*} = \sup_{Q} \frac{||\chi_Q||_{L^{p,1}(w)}||w^{-1}\chi_Q||_{L^{p',\infty}(w)}}{|Q|} < +\infty. \tag{2.2}$$

Remark 2.1 At this point, we have to emphasize that since we shall be dealing with radial functions, we shall work in two settings: $\mathbb{R}^n$ and $\mathbb{R}^+$. Hence, $A_p^{\mathcal{R}}$ could be either $A_p^{\mathcal{R}}(\mathbb{R}^n)$ or $A_p^{\mathcal{R}}(\mathbb{R}^+)$. Clearly in the case $\mathbb{R}^+$, $Q = (a, b)$ with $0 \le a < b < \infty$. We shall try to be precise if needed but we shall use the shorter notation $A_p^{\mathcal{R}}$ whenever things are clear. The same will happen in the context of A_p. Also we shall use the letter M to indicate the Hardy–Littlewood maximal operator either on $\mathbb{R}^n$ or on $\mathbb{R}^+$.

In a recent paper (see [7]) the following class of weights was introduced:

Definition 2.2 Set

$$\widehat{A}_p = \left\{u \in L^1_{\text{loc}} : \exists\, f \in L^1_{\text{loc}} \text{ and } \exists\, u_1 \in A_1 : u = (Mf)^{1-p} u_1\right\},$$

with

$$\|u\|_{\widehat{A}_p} = \inf \|u_1\|_{A_1}^{1/p}.$$

And, it was proved that

$$\widehat{A}_p \subset A_p^{\mathcal{R}}. \tag{2.3}$$

Also, by (1.1), and the fact that $u \approx Mu$ for every $u \in A_1$, it is clear that $A_p \subset \widehat{A}_p$. We need to also introduce the following class of radial weights:

Definition 2.3 We define $\widehat{R}_p(\mathbb{R}^n)$ the class of radial weights u on $\mathbb{R}^n$ such that there exists $f \in (L^1_{\text{loc}}(\mathbb{R}^n))_{\text{rad}}$ and $u_1 \in (A_1(\mathbb{R}^n))_{\text{rad}}$ satisfying that

$$u = (Mf)^{1-p} u_1.$$

Taking into account (2.3), it is clear that $\widehat{R}_p \subset (A_p^{\mathcal{R}})_{\text{rad}}$.

The following proposition collects all the properties of the weights which shall be fundamental for our purposes. In order to make things clearer and not introduce too many technicalities, the proof will be postponed to the last section.

Proposition 2.4 (i) *If $u \in A_1(\mathbb{R}^+)$ then, for every $\gamma > 1$, $v(s) = u(s^\gamma)^{1/\gamma} \in A_1(\mathbb{R}^+)$.*

(ii) *Let $u \in A_1(\mathbb{R}^+)$ and $f \neq 0$, $f \in L^1_{loc}(\mathbb{R}^+)$ such that $Mf(x) < \infty$ a.e. Then, for every $\gamma > 1$,*

$$w(s) = (Mf(s^\gamma))^{\frac{-1}{\gamma}} (u(s^\gamma))^{1/\gamma} \in A_2^{\mathcal{R}}(\mathbb{R}^+),$$

with constant independent of f.

(iii) *If $w^n \in \widehat{R}_2(\mathbb{R}^n)$, then $w_0(s^{\frac{2}{n+1}}) s^{\frac{n-1}{n+1}} \in A_2^{\mathcal{R}}(\mathbb{R}^+)$.*

(iv) *If $w^n \in \widehat{R}_2(\mathbb{R}^n)$, then $w_0 \in A_2^{\mathcal{R}}(\mathbb{R}^+)$.*

Remark 2.5 If we consider the maximal operator

$$T_n f(x) = \sup_{R>0} |S_n^R f(x)| \quad \text{with} \quad \widehat{(S_n^R f)}(\xi) = \chi_{B(0,R)}(\xi) \hat{f}(\xi),$$

the boundedness of T_n on $L^p_{\text{rad}}(\mathbb{R}^n)$ was also studied in [24,25]. The proof is based on the following inequality valid for a radial function f:

$$T_n f(x) \lesssim \frac{1}{|x|^{\frac{n-1}{2}}} (M + \widetilde{H} + Q + \widetilde{C})\big(f_0(s) s^{\frac{n-1}{2}}\big)(|x|), \tag{2.4}$$

where

$$\widetilde{H}g(t) = \sup_{\varepsilon>0}\left|\int_{|t-s|>\varepsilon} \frac{g(s)}{t-s}ds\right|$$

is the maximal Hilbert transform,

$$Qg(s) = \int_s^\infty |g(t)|\frac{dt}{t}$$

is the conjugate Hardy operator, and $\widetilde{C}$ is the maximal Carleson operator (we omit the definition of the Carleson operator since it will not be used in this paper).

Now, revisiting the proof of (2.4) in [25] and adapting it to the case of the operator S_n, one can easily see that if f is a radial function, then

$$|S_n f(x)| \lesssim \frac{1}{|x|^{\frac{n-1}{2}}}(M + \widetilde{H} + Q + H_{\mathrm{loc}})(f_0(s)s^{\frac{n-1}{2}})(|x|), \tag{2.5}$$

where

$$H_{\mathrm{loc}}g(t) = p.v.\left|\int_{t/2}^{2t} \frac{g(s)e^{is}}{t-s}ds\right|$$

is a local Hilbert transform. Moreover, it was proved in [25] that, for every $x \in I_k$ and every f such that $\operatorname{supp} f_0 \subset J_k^c$,

$$|S_n f(x)| \lesssim P_n(f)(|x|) + Q_n(f)(|x|), \tag{2.6}$$

where

$$P_n(f)(|x|) = \frac{1}{|x|^{\frac{n+1}{2}}} \int_0^{|x|} |f_0(s)|s^{\frac{n-1}{2}}ds,$$

and

$$Q_n(f)(|x|) = \frac{1}{|x|^{\frac{n-1}{2}}} \int_{|x|}^\infty |f_0(s)|s^{\frac{n-1}{2}}\frac{ds}{s}.$$

Lemma 2.6 *If* $v \in A_2^{\mathcal{R}}(\mathbb{R}^+)$,

$$Q : L^{2,1}(v) \longrightarrow L^{2,\infty}(v)$$

is bounded.

Proof Let $h \in L^{2,1}(v)$ such that $\|h\|_{L^{2,1}(v)} \le 1$. Then

$$\left|\int_0^\infty Q(f)(s)h(s)v(s)ds\right| \le \int_0^\infty |f(s)| \frac{P(hv)(s)}{v(s)} v(s)ds,$$

where $Pf(t) = \frac{1}{t}\int_0^t f(s)ds$ is the Hardy operator and hence $Pf(t) \le Mf(t)$. But it is known (see [5]) that, if $v \in A_2^{\mathcal{R}}(\mathbb{R}^+)$,

$$\left\|\frac{M(hv)}{v}\right\|_{L^{2,\infty}(v)} \lesssim \|h\|_{L^{2,1}(v)},$$

and hence the result follows by duality. □

Lemma 2.7 *For every* $w \in A_2^{\mathcal{R}}(\mathbb{R}^+)$

$$H_{loc} : L^{2,1}(w) \longrightarrow L^{2,\infty}(w)$$

is bounded.

Proof The proof will be an easy modification of the one given in [2] for the strong boundedness using that $M, H : L^{2,1}(w) \longrightarrow L^{2,\infty}(w)$ (see [7]).

Let us take $f = \chi_E$ and $x \in I_k$. Then,

$$H_{\rm loc} f(x) = \left|\left(\int_{2^{k-1}}^{2^{k+2}} - \int_{2^{k-1}}^{x/2} - \int_{2x}^{2^{k+2}}\right)\frac{f(t)e^{it}}{t-x}dt\right| \le I + II + III.$$

where $I = |H(f_k)(x)|$ with $f_k(t) = f(t)e^{it}\chi_{J_k}$ and $II + III \lesssim Mf_k(x)$. Hence,

$$\begin{aligned} w\big(\{x : H_{\rm loc} f(x) > y\}\big) &= \sum_k w\big(\{x \in I_k : H_{\rm loc} f(x) > y\}\big) \\ &\le \sum_k w\big(\{x \in I_k : Mf_k(x) > Cy\}\big) + \sum_k w\big(\{x \in I_k : Hf_k(x) > Cy\}\big) \\ &\lesssim \sum_k \frac{1}{y^2}\|f_k\|^2_{L^{2,1}(w)} = \frac{1}{y^2}\sum_k w(E \cap J_k) \approx \frac{1}{y^2} w(E). \end{aligned}$$

□

Lemma 2.8 *For every radial weight* w *in* $\mathbb{R}^n$ *such that* $w^n \in \widehat{R}_2$,

$$P_n : L^{2,1}_{rad}(w) \longrightarrow L^{2,\infty}(w)$$

is bounded.

Proof Let $v(s) = w_0(s^{\frac{2}{n+1}})s^{\frac{n-1}{n+1}}$ and note that, by Proposition 2.4, $v \in A_2^{\mathcal{R}}(\mathbb{R}^+)$. Then, if f is a radial function,

$$\begin{aligned}\int_{\{x\in\mathbb{R}|^n : P_n f(x)>\lambda\}} w(x)dx &\approx \int_{\left\{s>0: s^{-\frac{n+1}{2}} \int_0^s f_0(t)t^{\frac{n-1}{2}}dt>\lambda\right\}} w_0(s)s^{n-1}ds \\ &\approx \int_{\left\{s>0: \frac{1}{s}\int_0^{s^{\frac{2}{n+1}}} f_0(t)t^{\frac{n-1}{2}}dt>\lambda\right\}} w_0(s^{\frac{2}{n+1}})s^{\frac{n-1}{n+1}}ds \\ &\approx \int_{\left\{s>0: \frac{1}{s}\int_0^s f_0(u^{\frac{2}{n+1}})du>\lambda\right\}} w_0(s^{\frac{2}{n+1}})s^{\frac{n-1}{n+1}}ds \\ &= \int_{\left\{s>0: \frac{1}{s}\int_0^s f_0(u^{\frac{2}{n+1}})du>\lambda\right\}} v(s)ds.\end{aligned}$$

Now, if $f(x) = \chi_E(x)$, E_0 is the radial part of E and $E_0^n = \{s > 0 : s^{\frac{2}{n+1}} \in E_0\}$, we have that

$$\int_{\{x\in\mathbb{R}^n : P_n\chi_E(x)>\lambda\}} w(x)dx \approx \int_{\left\{s>0: \frac{1}{s}\int_0^s \chi_{E_0^n}(u)du>\lambda\right\}} v(s)ds \lesssim \frac{v(E_0^n)}{\lambda^2},$$

where we have used that since $v \in A_2^{\mathcal{R}}$, the Hardy operator is restricted weak type $(2, 2)$. Finally,

$$v(E_0^n) = \int_{\{s>0: s^{\frac{2}{n+1}}\in E_0\}} w_0(s^{\frac{2}{n+1}})s^{\frac{n-1}{n+1}}ds \approx \int_{E_0} w_0(s)s^{n-1}ds \approx w(E),$$

and the result follows. □

In [12] was proved that if f is a radial function on $\mathbb{R}^n$, then

$$Mf(x) \approx \sup_{r_1<|x|<r_2} \frac{1}{r_2^n - r_1^n}\int_{r_1}^{r_2} |f_0(t)|t^{n-1}dt,$$

and hence there exists $\bar{f}$, defined in $\mathbb{R}^+$, such that

$$Mf(x) \approx M\bar{f}(|x|^n). \tag{2.7}$$

Lemma 2.9 *For every w such that $w^n \in \widehat{R}_2$,*

$$Q_n : L^{2,1}_{rad}(w) \longrightarrow L^{2,\infty}(w)$$

is bounded.

Proof The result will follow by duality. As was done in Lemma 2.6, it can be easily see that the result holds if and only if, for every h radial function,

$$\left\| \frac{P_n(hw)}{w} \right\|_{L^{2,\infty}(w)} \lesssim \|h\|_{L^{2,1}(w)}.$$

Now, if $v(s) = w_0(s^{\frac{2}{n+1}})$,

$$\begin{aligned} &\int_{\left\{x\in\mathbb{R}^n : \frac{P_n(hw)(x)}{w(x)} > \lambda\right\}} w(x)dx \\ &\approx \int_{\left\{s>0: \frac{1}{w_0(s)s^{\frac{n+1}{2}}} \int_0^s h_0(t)t^{\frac{n-1}{2}} w_0(t)dt > \lambda\right\}} w_0(s)s^{n-1}ds \\ &\approx \int_{\left\{s>0: \frac{1}{v(s)s} \int_0^s h_0(t^{\frac{2}{n+1}})v(t)dt > \lambda\right\}} v(s)s^{\frac{n-1}{n+1}}ds. \end{aligned}$$

Therefore, if $u(s) = s^{\frac{n-1}{n+1}}$, we have to prove that, if $\bar{h}(t) = h_0(t^{\frac{2}{n+1}})$,

$$\left\| \frac{P(\bar{h}v)}{v} \right\|_{L^{2,\infty}(vu)} \lesssim \|h\|_{L^{2,1}(w)}.$$

In fact, if we prove that $v^{-1}u \in A_2$, the result will easily follows since

$$\begin{aligned} \left\| \frac{P(\bar{h}v)}{v} \right\|_{L^{2,\infty}(vu)} &\lesssim \left\| P(\bar{h}v) \right\|_{L^2(v^{-1}u)} \lesssim \|\bar{h}\|_{L^2(vu)} \\ &= \left(\int_0^\infty h_0(s^{\frac{2}{n+1}})^2 w_0(s^{\frac{2}{n+1}}) s^{\frac{n-1}{n+1}} ds \right)^{1/2} \\ &\approx \left(\int_0^\infty h_0(t)^2 w_0(t) t^{n-1} dt \right)^{1/2} \approx \|h\|_{L^2(w)} \lesssim \|h\|_{L^{2,1}(w)}. \end{aligned}$$

Now, if $w^n \in \widehat{R}_2$, then $w \approx (Mf)^{-1/n}(Mg)^{\delta/n}$, for some radial functions f and g and $0 < \delta < 1$. Hence, using (2.7),

$$\begin{aligned} v^{-1}(s)u(s) &= \left(M\bar{f}(s^{\frac{2n}{n+1}})\right)^{\frac{1}{n}} \left(M\bar{g}(s^{\frac{2n}{n+1}})\right)^{-\frac{\delta}{n}} s^{\frac{n-1}{n+1}} \\ &= \left[\left(M\bar{f}(s^{\frac{2n}{n+1}})\right)^{\frac{n+1}{2n}} \right]^{\frac{2}{n+1}} \left[\left(M\bar{g}(s^{\frac{2n}{n+1}})\right)^{\frac{\delta}{n}} s^{-\frac{n-1}{n+1}} \right]^{-1}. \end{aligned}$$

By Proposition 2.4, $\left[(M\bar{f}(s^{\frac{2n}{n+1}}))^{\frac{n+1}{2n}}\right]^{\frac{2}{n+1}} \in A_1$. On the other hand, we have that $(M\bar{g}(t))^{\frac{2\delta}{n+1}} t^{-\frac{n-1}{n+1}} \in A_1$. To see this, let us take $\delta < \alpha < 1$ and $0 < \nu < 1$ so that

$$\alpha \frac{2}{n+1} + \frac{1}{\nu} \frac{n-1}{n+1} = 1,$$

and write

$$(M\bar{g}(t))^{\frac{2\delta}{n+1}} t^{-\frac{n-1}{n+1}} = [(M\bar{g}(t))^{\frac{\delta}{\alpha}}]^{\frac{2\alpha}{n+1}} [t^{-\nu}]^{\frac{n-1}{(n+1)\nu}}.$$

Since both $(M\bar{g}(t))^{\frac{\delta}{\alpha}}$ and $t^{-\nu}$ are A_1 weights, we obtain the result by Hölder's inequality.

Therefore, again by Proposition 2.4, we obtain that

$$(M\bar{g}(s^{\frac{2n}{n+1}}))^{\frac{\delta}{n}} s^{-\frac{n-1}{n+1}} \in A_1,$$

and, hence, $v^{-1}u \in A_2$. □

Now, we are ready to formulate our new weighted L^2 estimate for S_n:

Theorem 2.10 *If w is a radial function such that $w^n \in \widehat{R}_2$, then*

$$S_n : L^{2,1}_{rad}(w) \longrightarrow L^{2,\infty}(w)$$

is bounded.

Remark 2.11 In the case of radial power weights, that is, $w(x) = |x|^\alpha$, it was proved in [1] that

$$S_n : L^2_{\mathrm{rad}}(w) \longrightarrow L^2(w)$$

if and only if $-1 < \alpha < 1$. We observe, that the above result, in particular, says that in the extreme case $\alpha = 1$ we have indeed a restricted weak type inequality.

Proof Let $f = \chi_E$ be a radial function. Then, its associated radial function $f_0(s) = \chi_{E_0}(s)$, where E_0 is the radial set associated to E; that is $x \in E$ if and only if $|x| \in E_0$. Let us write $f_0 = f_k^1 + f_k^2$ with $f_k^1 = f_0\chi_{J_k}$. Set $C_k = \{x \in \mathbb{R}^n : 2^k \leq |x| \leq 2^{k+1}\}$ and let $f_k^j(x) = f_k^j(|x|)$, for every $x \in \mathbb{R}^n$ $(j = 1, 2)$. Then,

$$\begin{aligned} & w\big(\{x \in \mathbb{R}^n; |S_n f(x)| > 2y\}\big) \\ & \quad \leq \sum_{k\in\mathbb{Z}} w\big(\{x \in C_k; |S_n f_k^1(x)| > y\}\big) + \sum_{k\in\mathbb{Z}} w\big(\{x \in C_k; |S_n f_k^2(x)| > y\}\big) \\ & \quad = I + II. \end{aligned}$$

For the global part II we use (2.6) together with Lemmas 2.8 and 2.9 to obtain that

$$II \lesssim w(\{x \in \mathbb{R}^n; |(P_n + Q_n)(f)(x)| > y\}) \lesssim \frac{w(E)}{y^2}.$$

Let us now estimate the local part I. If w is such that $w^n \in \widehat{R}_2$, we have to use (2.5) and proceed as follows: By Proposition 2.4, we have that $w_0 \in A_2^{\mathcal{R}}(\mathbb{R}^+)$ and hence we have that M and $\widetilde{H}$ are of restricted weak type (2,2) on $L^{2,1}(w_0)$ (see Section 4 in [7]), and by Lemmas 2.6 and 2.7, the same estimate holds for Q and H_{loc} and therefore, if

$$\mathcal{A} := M + \widetilde{H} + Q + H_{\text{loc}},$$

we have, using the boundedness of $\mathcal{A}$ proved in the previous lemmas, that

$$\begin{aligned}
& w\big(\big\{x \in C_k;\, |S_n f_k^1(x)| > y\big\}\big) \\
& \leq w\big(\big\{x \in C_k;\, |\mathcal{A}(f_k^1(r) r^{\frac{n-1}{2}})(x)| > 2^{k\frac{n-1}{2}} y\big\}\big) \\
& \approx 2^{k(n-1)} w_0\big(\big\{s \in I_k;\, |\mathcal{A}(f_k^1(r) r^{\frac{n-1}{2}})(s)| > 2^{k\frac{n-1}{2}} y\big\}\big) \\
& \lesssim \frac{1}{y^2} \Big\| f_k^1(r) r^{\frac{n-1}{2}} \Big\|^2_{L^{2,1}(w_0)} = \frac{1}{y^2} \left(\int_0^\infty w_0(\{r : f_k^1(r) r^{\frac{n-1}{2}} > s\})^{1/2} ds \right)^2 \\
& = \frac{1}{y^2} \left(\int_0^\infty w_0(\{r \in E_0 \cap J_k : r^{\frac{n-1}{2}} > s\})^{1/2} ds \right)^2 \\
& \approx \frac{2^{k(n-1)}}{y^2} w_0(E_0 \cap J_k) \approx \frac{1}{y^2} \int_{(C_{k-1} \cup C_k \cup C_{k+1}) \cap E} w(x) dx.
\end{aligned}$$

Summing in $k \in \mathbb{Z}$, the estimate for I is proved and the result follows. □

3 Limited Restricted Weak Type Extrapolation

In order to prove the main result of this paper (Theorem 1.3), we need to develop a new Rubio de Francia extrapolation result, and this is precisely the first goal of this section. In fact, we just need it for radial functions and radial weights but we shall do it for the general case, and then adapt it easily to our particular case.

The classical result [26] (see also [14–16]), says that if, for some $p \geq 1$ and every $w \in A_p$,

$$T : L^p(w) \longrightarrow L^p(w)$$

is a bounded operator then, for every $q > 1$ and every $w \in A_q$,

$$T : L^q(w) \longrightarrow L^q(w)$$

is also bounded. Moreover, there are examples of operators, for which the hypothesis of Rubio de Francia's theorem holds and they are not of weak type $(1, 1)$ as the operator $M \circ M$ trivially shows.

Since the above result was first proved, many other proofs and improvements have appeared in the literature, but we want to mention the fact that an important property of the A_p weights that makes possible the extrapolation is the factorization property (1.1). We refer to the books [10,14,16] for classical and new results on this theory.

It is not known if the class $A_p^{\mathcal{R}}$ satisfies some factorization property. However, in the recent papers [5,7] a Rubio de Francia extrapolation theory was developed for operators satisfying a restricted weak type boundedness for the class $\widehat{A}_p$. The main advantage of this new class is that allows to obtain boundedness estimates at the end-point $p = 1$.

The restricted weak type Rubio de Francia extrapolation results proved in [7] can be stated as follows:

Theorem 3.1 *Let $1 < p < \infty$ and let T be a sublinear operator. Assume that, for every $v \in \widehat{A}_p$, we have that*

$$T : L^{p,1}(v) \longrightarrow L^{p,\infty}(v)$$

is bounded. Then, for every $v \in A_1$, T is of restricted weak type $(1, 1)$; that is, for every measurable set $E \subset \mathbb{R}^n$,

$$\|T\chi_E\|_{L^{1,\infty}(v)} \lesssim v(E).$$

Moreover, Rubio de Francia's extrapolation theorem was extended in [6,12] to cover the cases of operators which are not bounded for every $p > 1$ but only in a certain interval $(\mathfrak{p}_-, \mathfrak{p}_+)$. In this setting, the authors defined the following class of weights and the following indices:

Definition 3.2 Given $0 \le \alpha \le 1$ and $0 \le \beta \le 1$, let us define

$$A_{p;(\alpha,\beta)} = \left\{v = v_0^{\alpha} v_1^{\beta(1-p)};\, v_j \in A_1\right\}.$$

Definition 3.3 Given $p \in [1, +\infty)$ and $\alpha, \beta \in [0, 1]$, let us define $\mathfrak{p}_-$ and $\mathfrak{p}_+$ as

$$\mathfrak{p}_+ = \frac{p}{1-\alpha}, \quad {\mathfrak{p}_-}' = \frac{p'}{1-\beta}. \tag{3.1}$$

We associate to every $q \in (\mathfrak{p}_-, \mathfrak{p}_+)$ the indices $\alpha(q), \beta(q) \in [0, 1]$ given by

$$\mathfrak{p}_+ = \frac{q}{1-\alpha(q)}, \quad {\mathfrak{p}_-}' = \frac{q'}{1-\beta(q)}. \tag{3.2}$$

So, we have that for any $q \in [\mathfrak{p}_-, \mathfrak{p}_+]$

$$\alpha(q) = 1 - \frac{q}{\mathfrak{p}_+}, \qquad \beta(q) = 1 - \frac{q'}{{\mathfrak{p}_-}'}.$$

Then, the following theorem was proved in [6]:

Theorem 3.4 *Let us assume that*

$$T : L^p(w) \longrightarrow L^{p,\infty}(w)$$

is bounded, for every $w \in A_{p;(\alpha,\beta)}$ *and let* $\mathfrak{p}_-$ *and* $\mathfrak{p}_+$ *be given by* (3.1). *For* $q \in (\mathfrak{p}_-, \mathfrak{p}_+)$*, let* $\alpha(q)$ *and* $\beta(q)$ *be given by* (3.2). *Then for* $v \in A_{q;(\alpha(q),\beta(q))}$*, it holds that*

$$T : L^q(v) \longrightarrow L^{q,\infty}(v)$$

is bounded.

Taking into account these results, our next goal will be to show that a similar result holds true when dealing with restricted weak type estimate and the corresponding class $\widehat{A}_{p;(\alpha_0,\alpha_1)}$ defined as follows:

Definition 3.5 Given $0 \leq \alpha, \beta \leq 1$, let

$$\widehat{A}_{p;(\alpha,\beta)} = \left\{ v : \exists\, g \in L^1_{\text{loc}} \text{ and } \exists\, u \in A_1 : v = u^{\alpha}(Mg)^{\beta(1-p)} \right\}.$$

To prove our next theorem, we will need the following result from [11] (see Theorem 1.3 and Remark 2.2). See also, [21] where very interesting estimates as the one in this proposition have been provided.

Proposition 3.6 *If* $u \in A_1$ *and* $v \in A_\infty$*, then*

$$\left\| \frac{M(fv/u)}{v/u} \right\|_{L^{1,\infty}(v)} \lesssim \|f\|_{L^1(v)}.$$

With this inequality, we can now prove our fundamental extrapolation result that will allow us (together with Theorem 3.9) to prove our main result. We emphasize here that, contrary to what happens with the classical limited range extrapolation, we can obtain here an estimate at the endpoint $\mathfrak{p}_-$.

Theorem 3.7 *Let* T *be an operator satisfying that, for some* $p > 1$ *and every* $w \in \widehat{A}_{p;(\alpha,\beta)}$

$$T : L^{p,1}(w) \longrightarrow L^{p,\infty}(w)$$

is bounded. Then, if $\mathfrak{p}_- > 1$ *and* v *is a weight such that* $v^{\frac{1}{\alpha(\mathfrak{p}_-)}} \in A_1$*,*

$$T : L^{\mathfrak{p}_-,1}(v) \longrightarrow L^{\mathfrak{p}_-,\infty}(v)$$

is bounded.

Proof Let $v = u^{\alpha(\mathfrak{p}_-)}$ with $u \in A_1$, and let

$$\mathcal{M}f = u^{\frac{1}{\mathfrak{p}_+}} M\left(\frac{f^{\mathfrak{p}_-}}{u^{\frac{\mathfrak{p}_-}{\mathfrak{p}_+}}}\right)^{1/\mathfrak{p}_-}.$$

Then,

$$\lambda_{Tf}^{v}(y) \le \lambda_{\mathcal{M}f}^{v}(y) + v(\{|Tf| > y,\ \mathcal{M}f \le y\}$$
$$\le \lambda_{\mathcal{M}f}^{v}(y) + \frac{y^p}{y^{\mathfrak{p}_-}} \int_{\{|Tf|>y\}} (\mathcal{M}f(x))^{-(p-\mathfrak{p}_-)}\, v(x)\, dx.$$

Therefore,

$$\|Tf\|_{L^{\mathfrak{p}_-,\infty}(v)}^{\mathfrak{p}_-} \lesssim \|\mathcal{M}f\|_{L^{\mathfrak{p}_-,\infty}(v)}^{\mathfrak{p}_-} + \sup_{y>0} y^p \int_{\{|Tf|>y\}} (\mathcal{M}f(x))^{-(p-\mathfrak{p}_-)}\, v(x)\, dx$$
$$= \|\mathcal{M}f\|_{L^{\mathfrak{p}_-,\infty}(v)}^{\mathfrak{p}_-} + \sup_{y>0} y^p \int_{\{|Tf|>y\}} v_0(x)\, dx$$
$$= \|\mathcal{M}f\|_{L^{\mathfrak{p}_-,\infty}(v)}^{\mathfrak{p}_-} + \|Tf\|_{L^{p,\infty}(v_0)}^{p},$$

with

$$v_0 = \left(u^{1/\mathfrak{p}_+} M^{1/\mathfrak{p}_-}\left(\frac{f^{\mathfrak{p}_-}}{u^{\mathfrak{p}_-/\mathfrak{p}_+}} \right) \right)^{-(p-\mathfrak{p}_-)} u^{\alpha(\mathfrak{p}_-)}.$$

Now, since

$$\begin{cases} -\dfrac{p-\mathfrak{p}_-}{\mathfrak{p}_-} = 1 - \dfrac{p}{\mathfrak{p}_-} = 1 - p\left(1 - \dfrac{(1-\beta)(p-1)}{p}\right) = \beta(1-p) \\ -\dfrac{p-\mathfrak{p}_-}{\mathfrak{p}_+} + \alpha(\mathfrak{p}_-) = 1 - \dfrac{p}{\mathfrak{p}_+} = 1 - \dfrac{p(1-\alpha)}{p} = \alpha. \end{cases}$$

we have that $v_0 \in \widehat{A}_{p;(\alpha,\beta)}$, and hence, by hypothesis, we get that

$$\|Tf\|_{L^{p,\infty}(v_0)}^{p} \lesssim \|f\|_{L^{p,1}(v_0)}^{p} \approx \left(\int_0^\infty \left(\int_{\{|f|>z\}} v_0(x)\, dx \right)^{1/p} dz \right)^p.$$

Now, if x is such that $|f(x)| > z$, $v_0(x) \le z^{-(p-\mathfrak{p}_-)} v(x)$ and hence,

$$\|Tf\|_{L^{p,\infty}(v_0)}^{p} \lesssim \left(\int_0^\infty z^{\frac{\mathfrak{p}_-}{p}-1} (\lambda_f^v(z))^{1/p} dz \right)^p \approx \|f\|_{L^{\mathfrak{p}_-,\frac{\mathfrak{p}_-}{p}}(v)}^{\mathfrak{p}_-}.$$

On the other hand, using that $\frac{\mathfrak{p}_-}{\mathfrak{p}_+} = 1 - \alpha(\mathfrak{p}_-)$, and Proposition 3.6, we obtain that

$$\|\mathcal{M}f\|_{L^{\mathfrak{p}_-,\infty}(v)}^{\mathfrak{p}_-} = \left\| u^{1/\mathfrak{p}_+} \left(M\left(\frac{f^{\mathfrak{p}_-}}{u^{\mathfrak{p}_-/\mathfrak{p}_+}} \right) \right)^{1/\mathfrak{p}_-} \right\|_{L^{\mathfrak{p}_-,\infty}(u^{\alpha(\mathfrak{p}_-)})}^{\mathfrak{p}_-}$$
$$= \left\| u^{1-\alpha(\mathfrak{p}_-)} M\left(\frac{f^{\mathfrak{p}_-}}{u^{1-\alpha(\mathfrak{p}_-)}} \right) \right\|_{L^{1,\infty}(u^{\alpha(\mathfrak{p}_-)})} \lesssim \|f\|_{L^{\mathfrak{p}_-}(u^{\alpha(\mathfrak{p}_-)})}^{\mathfrak{p}_-}$$
$$= \|f\|_{L^{\mathfrak{p}_-}(v)}^{\mathfrak{p}_-}.$$

Therefore,

$$\|Tf\|^{\mathfrak{p}_-}_{L^{\mathfrak{p}_-,\infty}(v)} \lesssim \|f\|^{\mathfrak{p}_-}_{L^{\mathfrak{p}_-,\frac{\mathfrak{p}_-}{p}}(v)},$$

and the result follows since $\|\chi_E\|^{\mathfrak{p}_-}_{L^{\mathfrak{p}_-,\frac{\mathfrak{p}_-}{p}}(v)} \approx \|\chi_E\|^{\mathfrak{p}_-}_{L^{\mathfrak{p}_-,1}(v)}$, and $L^{\mathfrak{p}_-,\infty}(v)$ is a Banach space. □

Now, if T is an operator that takes radial functions into radial functions and we work with radial weights, the proof of Theorem 3.7 can be adapted to obtain the following results. We will need two new classes of radial weights defined as follows:

Definition 3.8 Given $0 \le \alpha, \beta \le 1$, let

$$R_{p;(\alpha,\beta)}(\mathbb{R}^n) = \left\{v : \exists\, v_0, v_1 \in (A_1)_{\mathrm{rad}} : v = v_0^{\alpha} v_1^{\beta(1-p)}\right\},$$

and

$$\widehat{R}_{p;(\alpha,\beta)}(\mathbb{R}^n) = \left\{v : \exists\, g \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))_{\mathrm{rad}} \text{ and } \exists\, u \in (A_1)_{\mathrm{rad}} : v = u^{\alpha}(Mg)^{\beta(1-p)}\right\}.$$

Theorem 3.9 *Let T be an operator satisfying that, for some $p > 1$ and every $w \in \widehat{R}_{p;(\alpha,\beta)}$*

$$T : L^{p,1}_{rad}(w) \longrightarrow L^{p,\infty}_{rad}(w),$$

is bounded. Then, if $\mathfrak{p}_- > 1$, for every radial weight v such that $v^{\frac{1}{\alpha(\mathfrak{p}_-)}} \in A_1$

$$T : L^{\mathfrak{p}_-,1}_{rad}(v) \longrightarrow L^{\mathfrak{p}_-,\infty}_{rad}(v)$$

is bounded.

With all these results in our hands, we are now ready to prove our MAIN THEOREM.

Proof of Theorem 1.3 If $w^n \in \widehat{R}_2$, we have by Theorem 2.10 that

$$S_n : L^{2,1}_{\mathrm{rad}}(w) \longrightarrow L^{2,\infty}(w)$$

is bounded. Hence, since $w \in \widehat{R}_{2;(\frac{1}{n},\frac{1}{n})}$, we can apply Theorem 3.9 to obtain that, under the conditions on v,

$$S_n : L^{\mathfrak{p}_-,1}_{\mathrm{rad}}(v) \longrightarrow L^{\mathfrak{p}_-,\infty}_{\mathrm{rad}}(v)$$

is bounded and the result follows since $\mathfrak{p}_- = \frac{2n}{n+1}$ and $\alpha(\mathfrak{p}_-) = \frac{2}{n+1}$.

Finally, to prove that the exponent in Theorem 1.3 is optimal, assume that

$$S_n : L_{\text{rad}}^{\frac{2n}{n+1},1}(w) \longrightarrow L^{\frac{2n}{n+1},\infty}(w)$$

is bounded, for every w such that $w^\mu \in A_1(\mathbb{R}^n)$, for some $\mu < \frac{n+1}{2}$. Then, by Theorem 3.4, it holds that

$$S_n : L_{\text{rad}}^{q,1} \longrightarrow L^{q,\infty},$$

is bounded for every $\frac{2n}{n+1} \leq q < \frac{2n\mu}{(\mu-1)(n+1)}$ but since $\frac{2n\mu}{(\mu-1)(n+1)} > \frac{2n}{n-1}$, this is not possible. Therefore, the best exponent μ is precisely $\frac{n+1}{2}$. □

4 Proof of Proposition 2.4

(i) Let $u \in A_1(\mathbb{R}^+)$ and let $r > 1$ such that $u^r \in A_1$. Take $0 < \alpha < 1$ such that $\frac{1}{r\gamma} + \left(1 - \frac{1}{\gamma}\right)\frac{1}{\alpha} = 1$ and let $0 < a < y < b < \infty$. Then, by Hölder's inequality,

$$\begin{aligned}
\frac{1}{b-a}\int_a^b (u(s^\gamma))^{1/\gamma} ds &\approx \frac{1}{b-a}\int_{a^\gamma}^{b^\gamma} (u(t))^{1/\gamma} t^{\frac{1}{\gamma}-1} dt \\
&= \frac{b^\gamma - a^\gamma}{b-a}\left[\frac{1}{b^\gamma - a^\gamma}\int_{a^\gamma}^{b^\gamma} (u^r(t))^{1/r\gamma}\left(\frac{1}{t^\alpha}\right)^{1-\frac{1}{r\gamma}} dt\right] \\
&\leq \frac{b^\gamma - a^\gamma}{b-a}\left[\frac{1}{b^\gamma - a^\gamma}\int_{a^\gamma}^{b^\gamma} u^r(t)\, dt\right]^{1/r\gamma}\left[\frac{1}{b^\gamma - a^\gamma}\int_{a^\gamma}^{b^\gamma} t^{-\alpha} dt\right]^{1-\frac{1}{r\gamma}} \\
&\lesssim (u(y^\gamma))^{1/\gamma},
\end{aligned}$$

where in the last inequality we have used that $u^r \in A_1$ and the easy fact that

$$\frac{b^\gamma - a^\gamma}{b-a}\left[\frac{1}{b^\gamma - a^\gamma}\int_{a^\gamma}^{b^\gamma} t^{-\alpha} dt\right]^{1-\frac{1}{r\gamma}} \lesssim 1.$$

Therefore, if we write $v(s) = (u(s^\gamma))^{1/\gamma}$, we have shown that $Mv(y) \lesssim v(y)$ and hence $v \in A_1$ as we wanted to prove.
(ii) By (2.2), it suffices to prove that, for every $0 \leq a < b$ and every $t > 0$,

$$\left(\int_a^b w\right)^{1/2} t \left(\int_{\{x\in(a,b):w^{-1}(x)>t\}} w\right)^{1/2} \lesssim (b-a),$$

with $w(s) = (Mf(s^\gamma))^{\frac{-1}{\gamma}}(u(s^\gamma))^{1/\gamma}$.

Let us first consider the case with $u = 1$:

Case 1 If $b/a \in (1, 2)$, then

$$\int_a^b (Mf(s^\gamma))^{\frac{-1}{\gamma}} ds \le \left(\int_a^b (Mf(s^\gamma))^{-1} s^{\gamma-1} ds \right)^{1/\gamma} \left(\log \frac{b}{a} \right)^{\frac{\gamma-1}{\gamma}}$$
$$\approx \left(\int_{a^\gamma}^{b^\gamma} (Mf(s))^{-1} ds \right)^{1/\gamma} \left(\frac{b}{a} - 1 \right)^{\frac{\gamma-1}{\gamma}},$$

and similarly,

$$\int_{\{s\in(a,b):Mf(s^\gamma)>t^\gamma\}} (Mf(s^\gamma))^{\frac{-1}{\gamma}} ds$$
$$\lesssim \left(\int_{\{s\in(a^\gamma,b^\gamma):Mf(s)>t^\gamma\}} (Mf(s))^{-1} ds \right)^{1/\gamma} \left(\frac{b}{a} - 1 \right)^{\frac{\gamma-1}{\gamma}}.$$

Since $(Mf(s))^{-1} \in \widehat{A}_2$, we have that

$$\left(\int_{a^\gamma}^{b^\gamma} (Mf(s))^{-1} ds \right)^{1/2} t^\gamma \left(\int_{\{s\in(a^\gamma,b^\gamma):Mf(s)>t^\gamma\}} (Mf(s))^{-1} ds \right)^{1/2}$$
$$\lesssim (b^\gamma - a^\gamma),$$

and hence,

$$\left(\int_a^b (Mf(s^\gamma))^{\frac{-1}{\gamma}} ds \right)^{1/2} t \left(\int_{\{s\in(a,b):Mf(s^\gamma)>t^\gamma\}} (Mf(s^\gamma))^{\frac{-1}{\gamma}} ds \right)^{1/2}$$
$$\lesssim (b^\gamma - a^\gamma)^{1/\gamma} \left(\frac{b}{a} - 1 \right)^{\frac{\gamma-1}{\gamma}} \lesssim (b - a).$$

Case 2 If $b/a \ge 2$, let $I = (a, b)$, $I_\gamma = (a^\gamma, b^\gamma)$ and let $g = f\chi_{(3I_\gamma)^c}$ then there exists $0 < K < +\infty$ such that

$$\frac{K}{2} \le (Mg(s^\gamma))^{1/\gamma} \le K,$$

for every $s \in I$ and it holds that $w(s) \le (Mg(s^\gamma))^{\frac{-1}{\gamma}}$.

Now, if $0 < t \leq 2K$ then

$$\begin{aligned}&\left(\int_a^b w\right)^{1/2} t\left(\int_{\{x\in(a,b):w^{-1}(x)>t\}} w\right)^{1/2}\\ &\quad\leq \left(\int_a^b (Mg(s^\gamma))^{\frac{-1}{\gamma}}\right)^{1/2} t\left(\int_{\{x\in(a,b):w^{-1}(x)>t\}} (Mg(s^\gamma))^{\frac{-1}{\gamma}}\right)^{1/2}\\ &\quad\leq \left(\int_a^b \frac{2}{K}\right)^{1/2} t\left(\int_{x\in(a,b)} \frac{2}{K}\right)^{1/2} \lesssim (b-a).\end{aligned}$$

And, if $t > 2K$, then for every $s \in (a, b)$,

$$\begin{aligned}(Mf(s^\gamma))^{1/\gamma} &\leq (M(f\chi_{3I_\gamma})(s^\gamma))^{1/\gamma} + (Mg(s^\gamma))^{1/\gamma}\\ &\leq (M(f\chi_{3I_\gamma})(s^\gamma))^{1/\gamma} + K < (M(f\chi_{3I_\gamma})(s^\gamma))^{1/\gamma} + \frac{t}{2}.\end{aligned}$$

Therefore, there exists $c > 0$ such that

$$\{s \in (a,b) : w^{-1}(s) > t\} \subset \left\{s \in (a,b) : M(f\chi_{3I})(s^\gamma) > ct^\gamma\right\} := F.$$

Then,

$$\begin{aligned}&t\left(\int_{\{s\in(a,b):w^{-1}(s)>t\}} w\right)^{1/2} \leq t\,(w(F))^{1/2}\\ &\quad\leq t^{1/2}\left(t\int_F (M(f\chi_{3I_\gamma})(s^\gamma))^{\frac{-1}{\gamma}}\right)^{1/2} \lesssim t^{1/2}\left(\int_F ds\right)^{1/2}\\ &\quad\approx t^{1/2}\left(\int_{\left\{s\in(a^\gamma,b^\gamma):M(f\chi_{3I_\gamma})(s)>ct^\gamma\right\}} s^{\frac{1}{\gamma}-1}ds\right)^{1/2}\end{aligned}$$

Now, since $\gamma > 1$, the function $s^{\frac{1}{\gamma}-1}$ is decreasing and hence, for every measurable set E, $\int_E s^{\frac{1}{\gamma}-1}ds \lesssim |E|^{\frac{1}{\gamma}}$. Therefore,

$$\begin{aligned}&t\left(\int_{\{s\in(a,b):w^{-1}(s)>t\}} w\right)^{1/2} \lesssim t^{1/2}\left(\left|\left\{s \in I_\gamma : M(f\chi_{3I_\gamma})(s) > ct^\gamma\right\}\right|^{\frac{1}{\gamma}}\right)^{1/2}\\ &\approx t^{1/2}\left(\frac{1}{t^\gamma}\int_{3I_\gamma} f\right)^{\frac{1}{2\gamma}} \approx \left(\int_{3I_\gamma} f\right)^{\frac{1}{2\gamma}} \approx (b^\gamma - a^\gamma)^{\frac{1}{2\gamma}}\left(\frac{1}{|3I_\gamma|}\int_{3I_\gamma} f\right)^{\frac{1}{2\gamma}}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\left(\int_a^b w\right)^{1/2} t\left(\int_{\{x\in(b,c):w^{-1}(x)>t\}} w\right)^{1/2}\\ &\quad\le (b^\gamma-a^\gamma)^{\frac{1}{2\gamma}}\left(\int_a^b w\right)^{1/2}\left(\frac{1}{|3I_\gamma|}\int_{3I_\gamma} f\right)^{\frac{1}{2\gamma}}\\ &\quad\lesssim (b^\gamma-a^\gamma)^{\frac{1}{2\gamma}}\left(\int_a^b w(s)(Mf(s^\gamma))^{\frac{1}{\gamma}}\right)^{1/2}\lesssim (b^\gamma-a^\gamma)^{\frac{1}{2\gamma}}(b-a)^{1/2}\approx b-a.\end{aligned}$$

Therefore, we have proved that $w(s)=(Mf(s^\gamma))^{\frac{-1}{\gamma}}\in A_2^{\mathcal{R}}(\mathbb{R}^+)$.

Let us now take $u\in A_1(\mathbb{R}^+)$ and set $w(s)=(Mf(s^\gamma))^{\frac{-1}{\gamma}}(u(s^\gamma))^{1/\gamma}$. Let E be a measurable subset of an interval I and let us use that, by (i), $v(s)=(u(s^\gamma))^{1/\gamma}\in A_1$. Then,

$$\int_I (Mf(s^\gamma))^{\frac{-1}{\gamma}}(u(s^\gamma))^{1/\gamma}\,ds\le\frac{1}{\inf_{s\in I}(Mf(s^\gamma))^{\frac{1}{\gamma}}}\int_I (u(s^\gamma))^{1/\gamma}ds=\frac{v(I)}{\inf_{s\in I}(Mf(s^\gamma))^{\frac{1}{\gamma}}}.$$

Let us take $0<\alpha<1$. Then, since

$$\frac{|I|}{\int_I (Mf(s^\gamma))^{\frac{\alpha}{\gamma}}ds}\le\frac{\int_I (Mf(s^\gamma))^{\frac{-\alpha}{\gamma}}ds}{|I|},$$

we have, using that $(Mf(s^\gamma))^{\frac{\alpha}{\gamma}}\in A_1$,

$$\begin{aligned}\frac{1}{\inf_{s\in I}(Mf(s^\gamma))^{\frac{1}{\gamma}}}&=\left(\frac{1}{\inf_{s\in I}(Mf(s^\gamma))^{\frac{\alpha}{\gamma}}}\right)^{\frac{1}{\alpha}}\lesssim\left(\frac{|I|}{\int_I (Mf(s^\gamma))^{\frac{\alpha}{\gamma}}ds}\right)^{\frac{1}{\alpha}}\\ &\le\left(\frac{\int_I (Mf(s^\gamma))^{\frac{-\alpha}{\gamma}}ds}{|I|}\right)^{\frac{1}{\alpha}}\le\frac{\int_I (Mf(s^\gamma))^{\frac{-1}{\gamma}}ds}{|I|}.\end{aligned}$$

Therefore, we obtain that

$$A:=\left(\frac{|E|}{|I|}\right)^2\int_I (Mf(s^\gamma))^{\frac{-1}{\gamma}}u(s^\gamma)^{1/\gamma}\,ds\lesssim\left(\frac{|E|}{|I|}\right)^2\frac{v(I)}{|I|}\int_I (Mf(s^\gamma))^{\frac{-1}{\gamma}}\,ds$$

Now, we have already proved that $(Mf(s^\gamma))^{\frac{-1}{\gamma}}\in A_2^{\mathcal{R}}(\mathbb{R}^+)$, and hence

$$\left(\frac{|E|}{|I|}\right)^2\int_I (Mf(s^\gamma))^{\frac{-1}{\gamma}}\lesssim\int_E (Mf(s^\gamma))^{\frac{-1}{\gamma}}\,ds.$$

Consequently,

$$A \lesssim \left(\int_E (Mf(s^\gamma))^{\frac{-1}{\gamma}} \, ds \right) \frac{v(I)}{|I|} \lesssim \int_E (Mf(s^\gamma))^{\frac{-1}{\gamma}} (u(s^\gamma))^{1/\gamma} \, ds,$$

and the result follows by (2.1).
(iii) If $w^n \in \widehat{R}_2(\mathbb{R}^n)$, then $w^n = (Mf)^{-1}(Mg)^\delta$ for some radial functions f and g and some $0 < \delta < 1$. Hence $w_0(s) = (M\bar{f}(s^n))^{\frac{-1}{n}} (M\bar{g}(s^n))^{\frac{\delta}{n}}$. So

$$w_0(s^{\frac{2}{n+1}}) = (M\bar{f}(s^{\frac{2n}{n+1}}))^{\frac{-1}{n}} (M\bar{g}(s^{\frac{2n}{n+1}}))^{\frac{\delta}{n}},$$

and by (ii), $v_0(s) := (w_0(s^{\frac{2}{n+1}}))^{\frac{n+1}{2}} \in A_2^{\mathcal{R}}(\mathbb{R}^+)$. On the other hand, if $u(s) = s$, then one can immediately see that $u \in A_2^{\mathcal{R}}(\mathbb{R}^+)$. Consequently,

$$M : L^{2,1}(v_0) \longrightarrow L^{2,\infty}(v_0)$$

and

$$M : L^{2,1}(u) \longrightarrow L^{2,\infty}(u)$$

are bounded operators and, by interpolation,

$$M : L^{2,1}(v_0^\theta u^{1-\theta}) \longrightarrow L^{2,\infty}(v_0^\theta u^{1-\theta})$$

is also bounded, for every $0 < \theta < 1$. Therefore, $w_0(s^{\frac{2}{n+1}}) s^{\frac{n-1}{n+1}} = (v_0(s))^{\frac{2}{n+1}} (u(s))^{\frac{n-1}{n+1}} \in A_2^{\mathcal{R}}(\mathbb{R}^+)$.
(iv) If $w^n \in \widehat{R}_2(\mathbb{R}^n)$, then $w = (Mf)^{\frac{-1}{n}} (Mg)^{\frac{\delta}{n}}$ for some radial functions f and g and some $0 < \delta < 1$. Hence, using (ii),

$$w_0(s) = (M\bar{f}(s^n))^{\frac{-1}{n}} (M\bar{g}(s^n))^{\frac{\delta}{n}} \in A_2^{\mathcal{R}}(\mathbb{R}^+).$$

□

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