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On Schrödinger Oscillatory Integrals Associated with the Dunkl Transform

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Abstract In the paper we study the Schrödinger oscillatory integrals $T^t_{\lambda,a} f(x)$ ($\lambda \geq 0$, $a > 1$) associated with the one-dimensional Dunkl transform $\mathscr{F}_\lambda$. If $a = 2$, the function $u(x,t) := T^t_{\lambda,2} f(x)$ solves the free Schrödinger equation associated to the Dunkl operator, with f as the initial data. It is proved that, if f is in the Sobolev spaces $H^s_\lambda(\mathbb{R})$ associated with the Dunkl transform, with the exponents s not less than $1/4$, then $T^t_{\lambda,a} f$ converges almost everywhere to f as $t \to 0$. A counterexample is constructed to show that $1/4$ can not be improved for $a = 2$, and when $1/4 \leq s \leq 1/2$, the Hausdorff dimension of the divergence set of $T^t_{\lambda,a} f$ for $f \in H^s_\lambda(\mathbb{R})$ is proved to be $1 - 2s$ at most.

Keywords Schrödinger oscillatory integral · Sobolev space · Dunkl transform · Dunkl operator · Hausdorff dimension

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1 Introduction and Main Results

For $\lambda \geq 0$ and $1 \leq p < \infty$, we denote by $L^p_\lambda(\mathbb{R})$ the space of measurable functions ϕ on $\mathbb{R}$ satisfying $\|\phi\|_{L^p_\lambda} := \left\{c_\lambda \int_{\mathbb{R}} |\phi(x)|^p |x|^{2\lambda} dx\right\}^{1/p} < \infty$ with $c_\lambda^{-1} = 2^{\lambda+1/2}\Gamma(\lambda + 1/2)$. The Dunkl operator on the line $\mathbb{R}$ is

$$(D\phi)(x) = \phi'(x) + \frac{\lambda}{x}\left[\phi(x) - \phi(-x)\right],$$

involving a reflection part, and the Dunkl transform of a function $\phi \in L^1_\lambda(\mathbb{R})$ is defined by

$$(\mathscr{F}_\lambda \phi)(\rho) := c_\lambda \int_{\mathbb{R}} \phi(x) E_\lambda(-ix\rho)|x|^{2\lambda} dx, \quad \rho \in \mathbb{R}, \tag{1}$$

where E_λ is the Dunkl kernel

$$E_\lambda(z) = j_{\lambda-1/2}(iz) + \frac{z}{2\lambda+1} j_{\lambda+1/2}(iz), \quad z \in \mathbb{C}, \tag{2}$$

and $j_\alpha(z)$ is the normalized Bessel function

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+\alpha+1)}. \tag{3}$$

Since $j_{-1/2}(z) = \cos z$, $j_{1/2}(z) = z^{-1}\sin z$, it follows that $E_0(iz) = e^{iz}$ and $\mathscr{F}_0$ agrees with the usual Fourier transform on $\mathbb{R}$.

For $s \geq 0$, the Sobolev space $H^s_\lambda(\mathbb{R})$ associated with the Dunkl transform is the collection of all $f \in \mathscr{S}'(\mathbb{R})$ (the space of tempered distributions on $\mathbb{R}$) satisfying

$$\|f\|_{H^s_\lambda} := \left\{c_\lambda \int_{\mathbb{R}} |(\mathscr{F}_\lambda f)(\rho)|^2 \left(1+|\rho|^2\right)^s |\rho|^{2\lambda} d\rho\right\}^{1/2} < \infty.$$

The homogeneous analog of $H^s_\lambda(\mathbb{R})$ is denoted by $\dot{H}^s_\lambda(\mathbb{R})$, which consists of the elements $f \in \mathscr{S}'(\mathbb{R})$ satisfying

$$\|f\|_{\dot{H}^s_\lambda} := \left\{c_\lambda \int_{\mathbb{R}} |(\mathscr{F}_\lambda f)(\rho)|^2 \, |\rho|^{2(s+\lambda)} d\rho\right\}^{1/2} < \infty.$$

It is obvious that, for $s \geq 0$, $H^s_\lambda(\mathbb{R}) = L^2_\lambda(\mathbb{R}) \bigcap \dot{H}^s_\lambda(\mathbb{R})$ and $\|f\|_{H^s_\lambda} \asymp \|f\|_{L^2_\lambda} + \|f\|_{\dot{H}^s_\lambda}$.

Given $a > 0$, we concern with the family of operators $T^t_{\lambda,a}$ with $t \in \mathbb{R}$, initially defined for $f \in \mathscr{S}(\mathbb{R})$ (the space of Schwartz functions on $\mathbb{R}$) by

$$T^t_{\lambda,a} f(x) = c_\lambda \int_{\mathbb{R}} (\mathscr{F}_\lambda f)(\rho) e^{it|\rho|^a} E_\lambda(ix\rho)|\rho|^{2\lambda} d\rho, \quad x \in \mathbb{R}, \tag{4}$$

and call $T^t_{\lambda,a} f$ the Schrödinger oscillatory integral associated with the Dunkl transform.

If $a = 2$ and $f \in \mathscr{S}(\mathbb{R})$, then the function $u(x,t) := T^t_{\lambda,a} f(x)$ is the solution to the free Schrödinger equation associated to the Dunkl operator, with f as the initial data, that is,

$$\begin{cases} i\frac{\partial u}{\partial t}(x,t) = \Delta_\lambda u(x,t), & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(x,0) = f(x), \end{cases} \tag{5}$$

where $\Delta_\lambda = D_x^2$, or explicitly, for $\phi \in C^2(\mathbb{R})$,

$$(\Delta_\lambda \phi)(x) = \frac{\partial^2}{\partial x^2}\phi(x) + \frac{2\lambda}{x}\frac{\partial}{\partial x}\phi(x) - \frac{\lambda}{x^2}[\phi(x) - \phi(-x)]. \tag{6}$$

For given f, the solution to the problem (5) may be in symbol written as $u(x,t) = e^{-it\Delta_\lambda} f(x)$.

We remark that, as a family of multiplier operators with multipliers $e^{it|\rho|^a}$, uniformly bounded for $t \in \mathbb{R}$, $T^t_{\lambda,a}$ can be extended to bounded operators on $L^2_\lambda(\mathbb{R})$, by the Plancherel theorem for the Dunkl transform (see Proposition 2.3(v) in Sect. 2).

One of the fundamental problems is, for $f \in H^s_\lambda(\mathbb{R})$ with suitable s, to identify the exponents s for which,

$$\lim_{t\to 0} T^t_{\lambda,a} f(x) = f(x) \qquad \text{for a.e. } x \in \mathbb{R}. \tag{7}$$

The answer to the above question is contained in the following theorem.

Theorem 1.1 *Let $a > 1$ and $\lambda \geq 0$. If $s \geq 1/4$, then for all $f \in H^s_\lambda(\mathbb{R})$, the assertion (7) is true in the sense that, for a.e. $x \in \mathbb{R}$, the function $t \mapsto T^t_{\lambda,a} f(x)$, $t \in \mathbb{R}\setminus\{0\}$, will, after modification on a null set, be continuous with limit $f(x)$ as $t \to 0$.*

The proof of Theorem 1.1 is essentially based on an L^q estimate for the relevant maximal function, with which, an almost sharp result is given in the next theorem.

Theorem 1.2 *Let $a > 1$, $\lambda \geq 0$ and $q = \frac{8\lambda+4}{4\lambda+1}$. Then for all $f \in \mathscr{S}(\mathbb{R})$, we have the following norm estimate*

$$\left(\int_{\mathbb{R}} \left(\sup_{0<|t|<\infty} \left|T^t_{\lambda,a} f(x)\right|\right)^q |x|^{2\lambda} dx\right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{H}^{1/4}_\lambda}. \tag{8}$$

When $\lambda = 0$, (5) is identical with the one spatial dimension case of the initial value problem for the classical Schrödinger equation

$$\begin{cases} i\frac{\partial u}{\partial t}(x,t) = (\Delta u)(x,t), & (x,t) \in \mathbb{R}^d\times\mathbb{R}, \\ u(x,0) = f(x), & x \in \mathbb{R}^d. \end{cases} \tag{9}$$

For a Schwartz function f on $\mathbb{R}^d$, if $\hat{f}$ denotes its Fourier transform, i.e., $\hat{f}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-i\langle x,\xi\rangle}dx$, the solution to (9), formally written as $u(x,t) = e^{-it\Delta}f(x)$, is given by

$$u(x,t) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\hat{f}(\xi)e^{i\left(\langle x,\xi\rangle+t|\xi|^2\right)}d\xi. \tag{10}$$

As usual, the Sobolev space $H^s(\mathbb{R}^d)$ for $s \geq 0$ consists of all $\phi \in L^2(\mathbb{R}^d)$ satisfying $\|\phi\|_{H^s}^2 := \int_{\mathbb{R}^d}|\hat{\phi}(\xi)|^2(1+|\xi|^2)^s d\xi < \infty$. For $f \in H^s(\mathbb{R}^d)$, the problem to find the exponents s for which,

$$\lim_{t\to 0} e^{-it\Delta}f(x) = f(x) \qquad \text{for a.e. } x \in \mathbb{R}^d, \tag{11}$$

has been studied by various authors. Carleson [7] yielded the first conclusion about the problem, proving convergence for $s \geq 1/4$ when $d = 1$. Dahlberg and Kenig [8] showed that this result is sharp. In dimensions $d \geq 2$, Sjölin [23] and Vega [28] established independently convergence for $s > 1/2$, while similar examples as considered in [8] show failure of convergence for $s < 1/4$.

The problem for $d = 2$ has been made more progress. The proof of convergence for some $s < 1/2$ appears first in Bourgain's papers [3,4]. Subsequently, the required threshold was lowered by Moyua et al. [18] and Tao and Vargas [26,27]; and in Lee [13], $s > 3/8$ was proved to be sufficient. The strongest result to date appears in Du et al. [10] which asserts a.e. convergence for $f \in H^s(\mathbb{R}^2)$, $s > 1/3$.

If the problem (11) is restricted to radial initial data f on $\mathbb{R}^d$, $d \geq 2$, Prestini [19] proved that $s \geq 1/4$ is sufficient to guarantee a.e. convergence of $e^{it\Delta}f(x)$, and Sjölin [24] obtained a sharp maximal estimate like (8) with $q = 4d/(2d-1)$.

One seemed to have reason to speculate that, the correct condition in arbitrary dimension d should be $s \geq 1/4$. After a longer time when no advances were achieved, the problem for $d > 2$ was driven by progress in Bourgain's paper [5], where, that $f \in H^s(\mathbb{R}^d)$, $s > 1/2 - 1/(4d)$, was proved to be sufficient for a.e. convergence. However, another result in the same paper [5] of Bourgain breaks the illusion of the exponent $s = 1/4$ in higher dimension, that is, the requirement $s > 1/2 - 1/d$ is necessary when $d > 4$. In this direction, the range was improved by Lucà, Rogers [15], to $s > d/2(d+2)$ for all $d \geq 2$, and furthermore, Bourgain [6] proved that $s \geq d/2(d+1)$ is necessary for $d \geq 2$.

Combining the assertions in Bourgain [6] and Du et al. [10] shows that $s = 1/3$ is optimal for $d = 2$ up to the endpoint. In the higher dimensions, there is some evidence to suggest that the exponent $s = d/2(d+1)$ could be sharp too.

The paper is organized as follows. Section 2 contains some preliminaries about the Dunkl transform and the Dunkl operator which is necessary subsequently; the key lemmas about the estimates for an associated oscillatory integral involving the Dunkl kernel are given in Sect. 3; the main results, i.e., Theorems 1.1 and 1.2, are proved in Sect. 4; and in Sect. 5, we construct a function f_0 to show that Theorem 1.1 with $a = 2$ is not true for any $s < 1/4$, and when $1/4 \leq s \leq 1/2$, we prove that the Hausdorff dimension of the divergence set of $T^t_{\lambda,a}f$ for $f \in H^s_\lambda(\mathbb{R})$ is $1-2s$ at most.

There are closing comments in Sect. 6, about higher dimensional counterparts of the free Schrödinger equation associated to the Dunkl operators.

Throughout the paper, $X \lesssim Y$ or $Y \gtrsim X$ means that $X \leq cY$ for some positive constant c independent of variables, functions, etc., but possibly dependent of some fixed parameters λ, s, a and q.

2 The Dunkl Transform and the Dunkl Operator on $\mathbb{R}$

Lemma 2.1 *Assume that* $\lambda \geq 0$.

(i) *The Dunkl operator* D *leaves* $\mathscr{D}(\mathbb{R})$ *(the space of* C^∞*-functions on* $\mathbb{R}$ *with compact supports) and* $\mathscr{S}(\mathbb{R})$ *invariant.*

(ii) *If* $\phi, \psi \in C^1[a, b]$, *then*

$$\int_a^b (D\phi)(x)\psi(x)|x|^{2\lambda}dr = \phi(x)\psi(x)|x|^{2\lambda}\Big|_{x=a}^b - \int_a^b \phi(x)(D\psi)(x)|x|^{2\lambda}dx \\ - \int_a^b \frac{\lambda}{x}\left(\phi(x)\psi(-x) + \phi(-x)\psi(x)\right)|x|^{2\lambda}dx; \quad (12)$$

and in particular, if $\phi, \psi \in C^1[\mathbb{R}]$ *and* $\phi(x)\psi(x)|x|^{2\lambda}$ *vanishes at infinity, then*

$$c_\lambda \int_{\mathbb{R}} (D\phi)(x)\psi(x)|x|^{2\lambda}dx = -c_\lambda \int_{\mathbb{R}} \phi(x)(D\psi)(x)|x|^{2\lambda}dx \quad (13)$$

provided one of the two integral exists.

(iii) *For* $\phi, \psi \in C^1(\mathbb{R})$,

$$[D(\phi\psi)](x) = (D\phi)(x)\psi(x) + \phi(x)(D\psi)(x) \\ - \frac{\lambda}{x}[\phi(x) - \phi(-x)][\psi(x) - \psi(-x)]; \quad (14)$$

and in particular, if ψ *is even, then*

$$[D(\phi\psi)](x) = (D\phi)(x)\psi(x) + \phi(x)(D\psi)(x). \quad (15)$$

Part (i) follows from the equalities

$$(Df)(x) = f'(x) + \lambda \int_{|s|\leq 1} f'(xs)ds = f'(x) - \lambda \int_{|s|\geq 1} f'(xs)ds.$$

For part (ii), we begin at the left hand side of (13) and split the integral into a sum of $\int_{-\infty}^0$ and $\int_0^\infty$. After integrating each term by parts, the equality (13) follows from a simple change of variables. Part (iii) is verified by direct calculations.

Lemma 2.2 *Assume that* $\lambda \geq 0$.

(i) $E_\lambda(ix\rho)$ *is the eigenfunction of* D_x *with* $i\rho$ *as its eigenvalue, that is*

$$D_x[E_\lambda(ix\rho)] = i\rho E_\lambda(ix\rho), \quad \text{and} \quad E_\lambda(ix\rho)|_{x=0} = 1. \tag{16}$$

(ii) $|E_\lambda(ix)| \lesssim (1+|x|)^{-\lambda}$, $x \in (-\infty, \infty)$.

To show part (i), we apply the formula $\left(z^{\alpha+1}J_{\alpha+1}(z)\right)' = z^{\alpha+1}J_\alpha(z)$ (see [12, 7-2(50)]) to obtain

$$\begin{aligned}\left(z^{-\alpha}J_{\alpha+1}(z)\right)' &= \left(z^{-2\alpha-2}\cdot z^{\alpha+1}J_{\alpha+1}(z)\right)' \\ &= z^{-\alpha}J_\alpha(z) - (2\alpha+1)z^{-\alpha-1}J_{\alpha+1}(z),\end{aligned}$$

which may be written as

$$(zj_{\alpha+1}(z))' + (2\alpha+1)j_{\alpha+1}(z) = 2(\alpha+1)j_\alpha(z).$$

Furthermore, from [12, 7-2(51)] we have $(j_\alpha(z))' = -zj_{\alpha+1}(z)/(2\alpha+2)$. Taking action of D_x to the real and the imagine parts of $E_\lambda(ix\rho)$ respectively, and applying the two equalities above, we prove the first formula in part (i). The second one is obvious, by (2) and (3).

From [12, 7-3(4)], $|j_\alpha(x)| \lesssim 1$ for $|x| \leq 1$, and from [12, 7-13(3)], $|j_\alpha(x)| \lesssim |x|^{-\alpha-1/2}$ for $|x| \geq 1$, so that

$$|j_\alpha(x)| \lesssim (1+|x|)^{-\alpha-1/2}, \qquad \text{for} \quad x \in (-\infty, \infty). \tag{17}$$

Thus part (ii) follows immediately.

The Dunkl transform shares many of the important properties with the usual Fourier transform, part of which are listed as follows.

Proposition 2.3 (cf. [14]) *Assume that* $\lambda \geq 0$.

(i) *If* $\phi \in L^1_\lambda(\mathbb{R})$, *then* $\mathscr{F}_\lambda\phi \in C_0(\mathbb{R})$ *and* $\|\mathscr{F}_\lambda\phi\|_\infty \leq \|\phi\|_{L^1_\lambda}$.

(ii) (Inversion) *If* $\phi \in L^1_\lambda(\mathbb{R})$ *such that* $\mathscr{F}_\lambda\phi \in L^1_\lambda(\mathbb{R})$, *then* $\phi(x) = [\mathscr{F}_\lambda(\mathscr{F}_\lambda\phi)](-x)$ *for a.e.* $x \in \mathbb{R}$.

(iii) *For* $\phi \in \mathscr{S}(\mathbb{R})$ *or* $\phi \in C^1(\mathbb{R})$ *having compact support, we have*

$$[\mathscr{F}_\lambda(D\phi)](\rho) = i\rho(\mathscr{F}_\lambda\phi)(\rho), \quad [\mathscr{F}_\lambda(x\phi)](\rho) = i[D_\rho(\mathscr{F}_\lambda\phi)](\rho), \quad \rho \in \mathbb{R};$$

and $\mathscr{F}_\lambda$ *is a topological automorphism on* $\mathscr{S}(\mathbb{R})$.

(iv) (product formula) *For all* $\phi, \psi \in L^1_\lambda(\mathbb{R})$, *we have*

$$c_\lambda \int_{\mathbb{R}} (\mathscr{F}_\lambda\phi)(x)\psi(x)|x|^{2\lambda}dx = c_\lambda \int_{\mathbb{R}} \phi(x)(\mathscr{F}_\lambda\psi)(x)|x|^{2\lambda}dx.$$

(v) (Plancherel) *There exists a unique extension of* $\mathscr{F}_\lambda$ *to* $L^2_\lambda(\mathbb{R})$ *with* $\|\mathscr{F}_\lambda\phi\|_{L^2_\lambda} = \|\phi\|_{L^2_\lambda}$ *and* $(\mathscr{F}_\lambda^{-1}\phi)(x) = (\mathscr{F}_\lambda\phi)(-x)$.

(vi) (Hausdorff-Young) *If* $1 \le p \le 2$*, then* $\mathscr{F}_\lambda \phi$ *for* $\phi \in L^p_\lambda(\mathbb{R})$ *is well defined and* $\|\mathscr{F}_\lambda \phi\|_{L^{p'}_\lambda} \le \|\phi\|_{L^p_\lambda}$*, where* $1/p + 1/p' = 1$.

It follows from Lemmas 2.1(i), 2.2(i), and Proposition 2.3(ii) that, for $f \in \mathscr{S}(\mathbb{R})$, the function $u(x,t) := T^t_{\lambda,2} f(x)$ is the solution to the initial value problem (5).

If $(x, y) \neq (0,0)$, the generalized translation $(\tau_y f)(x)$ of $f \in L^p_\lambda(\mathbb{R})$ $(1 \le p < \infty)$ associated to the Dunkl transform $\mathscr{F}_\lambda$ is defined by (cf. [20])

$$(\tau_y f)(x) = c'_\lambda \int_0^\pi \left(f_e(\langle x,y\rangle_\theta) + f_o(\langle x,y\rangle_\theta) \frac{x+y}{\langle x,y\rangle_\theta} \right) (1+\cos\theta) \sin^{2\lambda-1}\theta d\theta, \tag{18}$$

where $c'_\lambda = \Gamma(\lambda+1/2)/(\Gamma(\lambda)\Gamma(1/2))$, $f_e(x) = (f(x)+f(-x))/2$, $f_o(x) = (f(x)-f(-x))/2$, $\langle x,y\rangle_\theta = \sqrt{x^2+y^2+2xy\cos\theta}$; and if $(x,y) = (0,0)$, $(\tau_y f)(x) = f(0)$. For two appropriate functions f and g on $\mathbb{R}$, their generalized convolution $f *_\lambda g$ is defined by

$$(f *_\lambda g)(x) = c_\lambda \int_{\mathbb{R}} f(y)(\tau_x g)(-y)|y|^{2\lambda} dy, \qquad x \in \mathbb{R}. \tag{19}$$

The associated Poisson integral of a function f, for $\epsilon > 0$, is given by $(P_\epsilon f)(x) = (f *_\lambda P_\epsilon)(x)$, where $P_\epsilon(x) = m_\lambda \epsilon(\epsilon^2 + x^2)^{-\lambda-1}$ with $m_\lambda = 2^{\lambda+1/2}\Gamma(\lambda+1)/\sqrt{\pi}$.

We shall also need the Dunkl-Stieltjies transform for $d\mu \in \mathfrak{B}_\lambda(\mathbb{R})$ (Borel measure on $\mathbb{R}$ with $\|d\nu\|_{\mathfrak{B}_\lambda} := c_\lambda \int_{\mathbb{R}} |x|^{2\lambda} |d\nu(x)| < \infty$) defined by

$$[\mathscr{F}_\lambda(d\mu)](\xi) := c_\lambda \int_{\mathbb{R}} E_\lambda(-ix\xi)|x|^{2\lambda} d\mu(x), \quad \xi \in \mathbb{R},$$

and the generalized convolution $f *_\lambda (d\mu)$ of $f \in C_0(\mathbb{R})$ and $d\mu \in \mathfrak{B}_\lambda(\mathbb{R})$, by

$$[f *_\lambda (d\mu)](x) = c_\lambda \int_{\mathbb{R}} (\tau_x f)(-y)|y|^{2\lambda} d\mu(y), \qquad x \in \mathbb{R}. \tag{20}$$

Proposition 2.4 (cf. [14]) *Assume that* $\lambda \ge 0$*. The following statements are valid for* τ *and* $*_\lambda$*:*

(i) *If* $f \in L_{\lambda,\mathrm{loc}}(\mathbb{R})$*, then for all* $x, y \in \mathbb{R}$*,* $(\tau_y f)(x) = (\tau_x f)(y)$*,* $(\tau_y \tilde{f})(x) = \widetilde{(\tau_{-y} f)}(x)$*, where* $\tilde{f}(x) = f(-x)$*.*

(ii) *For all* $1 \le p \le \infty$ *and* $f \in L^p_\lambda(\mathbb{R})$*,* $\|\tau_y f\|_{L^p_\lambda} \le 4\|f\|_{L^p_\lambda}$ *with* $y \in \mathbb{R}$*.*

(iii) *If* $f \in L^p_\lambda(\mathbb{R})$*,* $1 \le p \le 2$*, and* $y \in \mathbb{R}$*, then* $[\mathscr{F}_\lambda(\tau_y f)](\xi) = E_\lambda(iy\xi)(\mathscr{F}_\lambda f)(\xi)$ *for almost every* $\xi \in \mathbb{R}$*.*

(iv) (Young inequality) *If* $p, q, r \in [1, \infty]$ *and* $1/p + 1/q = 1 + 1/r$*, then* $\|f *_\lambda g\|_{L^r_\lambda} \le 4\|f\|_{L^p_\lambda}\|g\|_{L^q_\lambda}$ *for* $f \in L^p_\lambda(\mathbb{R})$*,* $g \in L^q_\lambda(\mathbb{R})$*.*

(v) *Assume that* $p, q, r \in [1, 2]$ *and* $1/p + 1/q = 1 + 1/r$*. Then for* $f \in L^p_\lambda(\mathbb{R})$*,* $g \in L^q_\lambda(\mathbb{R})$*,* $[\mathscr{F}_\lambda(f *_\lambda g)](\xi) = (\mathscr{F}_\lambda f)(\xi)(\mathscr{F}_\lambda g)(\xi)$*. In particular* $*_\lambda$ *is associative in* $L^1_\lambda(\mathbb{R})$*.*

(vi) *For* $1 \le p < \infty$ *and* $d\mu \in \mathfrak{B}_\lambda(\mathbb{R})$, *the mapping* $f \mapsto f *_\lambda (d\mu)$ *has an extension from* $L^p_\lambda(\mathbb{R})$ *to itself and satisfies* $\|f *_\lambda (d\mu)\|_{L^p_\lambda} \le 4\|f\|_{L^p_\lambda}\|d\mu\|_{\mathfrak{B}_\lambda}$. *If* f *is even, then the constant* 4 *in (ii), (iv) and (vi) may be replaced by* 1.

(vii) *For* $f \in L^p_\lambda(\mathbb{R})$, $1 \le p \le 2$, *and* $d\mu \in \mathfrak{B}_\lambda(\mathbb{R})$, $[\mathscr{F}_\lambda(f *_\lambda (d\mu))](\xi) = (\mathscr{F}_\lambda f)(\xi)[\mathscr{F}_\lambda(d\mu)](\xi)$.

Proposition 2.5 (cf. [14]) *Assume that* $\lambda \ge 0$. *Then for* $f \in \mathscr{S}(\mathbb{R})$ *and given* $m > 0$,

$$|(\tau_y|f|)(x)| \le \frac{c_m(1+x^2+y^2)^{-\lambda}}{(1+||x|-|y||^2)^m}, \qquad x, y \in \mathbb{R}.$$

Proposition 2.6 (cf. [14]) *Assume that* $\lambda \ge 0$ *and* $\epsilon > 0$. *If* $f \in L^p_\lambda(\mathbb{R})$, $1 \le p \le 2$, *then* $[\mathscr{F}_\lambda(P_\epsilon f)](\xi) = e^{-\epsilon|\xi|}(\mathscr{F}_\lambda f)(\xi)$, *and for* $x \in \mathbb{R}$,

$$(P_\epsilon f)(x) = c_\lambda \int_{\mathbb{R}} e^{-\epsilon|\xi|}(\mathscr{F}_\lambda f)(\xi)E_\lambda(ix\xi)|\xi|^{2\lambda}d\xi.$$

3 The Key Lemmas

The key to the proofs of Theorems 1.1 and 1.2 is the following lemma. The lemma contains more information than what is need in the paper, but may be used elsewhere.

Lemma 3.1 *Suppose that* $a > 1$, $1/2 \le s < 1$, *and* $\psi \in C^\infty(\mathbb{R})$ *is even and with compact support. Then for* $b \in \mathbb{R}$, $x, y \in \mathbb{R}\setminus\{0\}$, $x \ne y$,

$$\left|\int_{-\infty}^{\infty} e^{ib|\rho|^a}E_\lambda(ix\rho)E_\lambda(-iy\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{2\lambda-s}d\rho\right| \lesssim |xy|^{-\lambda}|x-y|^{s-1}. \qquad (21)$$

The proof of Lemma 3.1 is based on the following two lemmas.

Lemma 3.2 *Suppose that* $a > 1$, $0 < k_1 < 1 < k_2$, *and* $\theta \in C^\infty(\mathbb{R})$ *is even, nonnegative, bounded, with support contained in* $\mathbb{R}\setminus\{0\}$ *and satisfies* $\int_{-\infty}^{\infty}|\theta'(\rho)|d\rho < \infty$. *Then for* $b, x, y \in \mathbb{R}\setminus\{0\}$, $x \ne y$, *satisfying*

$$[k_1c, k_2c]\bigcap \mathrm{supp}\,\theta = \emptyset \qquad \text{with} \quad c = \left(\frac{|x-y|}{ab}\right)^{\frac{1}{a-1}},$$

we have, for all $h > 0$,

$$\left|\int_{-h}^{h}\theta(\rho)e^{ib|\rho|^a}E_\lambda(ix\rho)E_\lambda(-iy\rho)|\rho|^{2\lambda}d\rho\right| \lesssim \frac{|xy|^{-\lambda}}{|x-y|}. \qquad (22)$$

Lemma 3.3 *Suppose that $a > 1$, and $\theta \in C^\infty(\mathbb{R})$ is even, nonnegative, bounded, and satisfies $\int_{-\infty}^{\infty} |\theta'(\rho)|d\rho < \infty$. Then for $b, x, y \in \mathbb{R} \setminus \{0\}$, $x \neq y$, and for all $h > 0$,*

$$\left|\int_{-h}^{h} \theta(\rho) e^{ib|\rho|^a} E_\lambda(ix\rho) E_\lambda(-iy\rho) |\rho|^{2\lambda} d\rho\right| \lesssim \frac{|xy|^{-\lambda}}{|x-y|} \left(\frac{|x-y|^a}{b} + 1\right)^{\frac{1}{2(a-1)}}. \tag{23}$$

Proof of Lemma 3.2 Let Ω denote the integral on the left hand side in (22) and assume that $b > 0$ without loss of generality. We shall need the following equality

$$\begin{aligned} i(ab|\rho|^{a-2}\rho + x - y) e^{ib|\rho|^a} E_\lambda(ix\rho) E_\lambda(-iy\rho) &= D_\rho \left[e^{ib|\rho|^a} E_\lambda(ix\rho) E_\lambda(-iy\rho)\right] \\ &+ \frac{4\lambda e^{ib|\rho|^a}}{(2\lambda+1)^2} \rho x y j_{\lambda+\frac{1}{2}}(x\rho) j_{\lambda+\frac{1}{2}}(y\rho), \quad \text{for } \rho \neq 0. \end{aligned} \tag{24}$$

Indeed, by (15),

$$\begin{aligned} &D_\rho \left[e^{ib|\rho|^a} E_\lambda(ix\rho) E_\lambda(-iy\rho)\right] \\ &\quad = iab|\rho|^{a-2}\rho e^{ib|\rho|^a} E_\lambda(ix\rho) E_\lambda(-iy\rho) + e^{ib|\rho|^a} D_\rho \left[E_\lambda(ix\rho) E_\lambda(-iy\rho)\right], \end{aligned}$$

and by (14) and Lemma 2.2(i),

$$\begin{aligned} D_\rho \left[E_\lambda(ix\rho) E_\lambda(-iy\rho)\right] &= i(x-y) E_\lambda(ix\rho) E_\lambda(-iy\rho) \\ &\quad - \frac{\lambda}{\rho} \left[E_\lambda(ix\rho) - E_\lambda(-ix\rho)\right] \left[E_\lambda(-iy\rho) - E_\lambda(iy\rho)\right]. \end{aligned}$$

Combining the two equalities proves (24).

Applying (24) to Ω and appealing to (12), then Ω may be written as the sum of Ω_1, Ω_2, and Ω_3, where

$$\begin{aligned} \Omega_1 &:= i \int_{-h}^{h} D_\rho \left[\frac{\theta(\rho)}{ab|\rho|^{a-2}\rho + x - y}\right] \cdot e^{ib|\rho|^a} E_\lambda(ix\rho) E_\lambda(-iy\rho) |\rho|^{2\lambda} d\rho, \\ \Omega_2 &:= \frac{-4i\lambda}{(2\lambda+1)^2} \int_{-h}^{h} \frac{\theta(\rho) e^{ib|\rho|^a}}{ab|\rho|^{a-2}\rho + x - y} \rho x y j_{\lambda+\frac{1}{2}}(x\rho) j_{\lambda+\frac{1}{2}}(y\rho) |\rho|^{2\lambda} d\rho, \\ \Omega_3 &:= -i \left(\frac{\theta(\rho) e^{ib|\rho|^a}}{ab|\rho|^{a-2}\rho + x - y} \cdot E_\lambda(ix\rho) E_\lambda(-iy\rho) |\rho|^{2\lambda}\right)\Bigg|_{-h}^{h}. \end{aligned}$$

It is noted that for $\rho \in \operatorname{supp}\theta$, if $|\rho| \le k_1 c$, then

$$\left|ab|\rho|^{a-2}\rho + x - y\right| \ge |x-y| - ab|\rho|^{a-1} \ge \left(1 - k_1^{a-1}\right)|x-y|; \tag{25}$$

and if $|\rho| \geq k_2 c$, then

$$\left|ab|\rho|^{a-2}\rho + x - y\right| \geq ab|\rho|^{a-1} - |x-y| \geq \left(k_2^{a-1} - 1\right)|x-y|. \tag{26}$$

It follows from Lemma 2.2(ii), (25) and (26) that $|\Omega_3| \lesssim |xy|^{-\lambda}|x-y|^{-1}$. As for Ω_1, by Lemma 2.2(ii) and (15) we have

$$\begin{aligned}|\Omega_1| &\lesssim |xy|^{-\lambda}\int_{-h}^{h}\left|D_\rho\left[\frac{\theta(\rho)}{ab|\rho|^{a-2}\rho + x - y}\right]\right|d\rho \\ &\lesssim |xy|^{-\lambda}\left(\int_{-h}^{h}\left|D_\rho\left[\frac{1}{ab|\rho|^{a-2}\rho + x - y}\right]\right|\theta(\rho)d\rho \right.\\ &\quad\left.+\int_{-h}^{h}\frac{|\theta'(\rho)|d\rho}{|ab|\rho|^{a-1} - |x-y||}\right).\end{aligned}$$

It follows from (25) and (26) that the later integral is bounded by a multiple of

$$|x-y|^{-1}\int_{-\infty}^{\infty}|\theta'(\rho)|d\rho \lesssim |x-y|^{-1};$$

furthermore, since

$$\left|D_\rho\left[\frac{1}{ab|\rho|^{a-2}\rho + x - y}\right]\right| \lesssim \frac{ab|\rho|^{a-2}}{(ab|\rho|^{a-1} - |x-y|)^2},$$

we have

$$|\Omega_1| \lesssim |xy|^{-\lambda}\int_0^h \frac{ab\rho^{a-2}\theta(\rho)}{(ab\rho^{a-1} - |x-y|)^2}d\rho + \frac{|xy|^{-\lambda}}{|x-y|}. \tag{27}$$

Taking partial integration to the integral above and in virtue of (25) and (26), we obtain

$$\begin{aligned}|\Omega_1| &\lesssim |xy|^{-\lambda}\int_0^h \frac{|\theta'(\rho)|}{|ab\rho^{a-1} - |x-y||}d\rho + \frac{|xy|^{-\lambda}}{|x-y|} \\ &\lesssim |xy|^{-\lambda}|x-y|^{-1}.\end{aligned}$$

For Ω_2, taking the substitution $\rho \to -\rho$ and summing with the original form, by (17) we get

$$\begin{aligned}|\Omega_2| &\lesssim |xy|^{-\lambda}\int_{-h}^{h}\frac{ab|\rho|^{a-2}\theta(\rho)}{|(ab|\rho|^{a-1})^2 - (x-y)^2|}d\rho \\ &\lesssim |xy|^{-\lambda}\int_0^h \frac{ab\rho^{a-2}\theta(\rho)}{\left(ab\rho^{a-1} - |x-y|\right)^2}d\rho,\end{aligned}$$

which is the same as the first term on the right hand side of (27). Therefore

$$|\Omega_2| \lesssim |xy|^{-\lambda}|x-y|^{-1}.$$

Finally combining the estimates for Ω_1, Ω_2 and Ω_3 proves (22). The proof of Lemma 3.2 is completed. □

Proof of Lemma 3.3 Assume that $b > 0$. Since, by Lemma 2.2(ii), the integrand above is dominated in absolute value by a multiple of $|xy|^{-\lambda}$, the desired estimate for $0 < h \lesssim |x-y|^{-1}$ follows immediately.

To evaluate the integral in (23) for larger h, take an even function $\psi_0 \in C^\infty(\mathbb{R})$ satisfying

$$\operatorname{supp}\psi_0 \subset]-1,1[, \quad 0 \le \psi_0(\rho) \le 1, \quad \text{and} \quad \psi_0(\rho) \equiv 1 \text{ for } \rho \in]-2,2[.$$

It suffices to prove that

$$\left|\int_{-h}^{h} \theta(\rho)\psi_0(|x-y|\rho)e^{ib|\rho|^a} E_\lambda(ix\rho)E_\lambda(-iy\rho)|\rho|^{2\lambda}d\rho\right| \lesssim \frac{|xy|^{-\lambda}}{|x-y|}\left(\frac{|x-y|^a}{b}+1\right)^{\frac{1}{2(a-1)}}. \tag{28}$$

Let Ω denote the integral on the left hand side above. If $|x-y|^a \le b/2$, then for $|x-y||\rho| \ge 1$,

$$|\rho|^{a-1} \ge \frac{1}{|x-y|^{a-1}}\frac{2|x-y|^a}{b} = 2ac^{a-1}, \tag{29}$$

where c is given in Lemma 3.2, so that the support of the function

$$\theta_0(\rho) := \theta(\rho)\psi_0(|x-y|\rho)$$

is contained in $[k_2c, \infty) \cup (-\infty, -k_2c]$ with $k_2 = (2a)^{1/(a-1)}$. Thus by Lemma 3.2, $|\Omega| \lesssim |xy|^{-\lambda}|x-y|^{-1}$.

In what follows, we assume that $|x-y|^a > b/2$. □

Put

$$\gamma = \frac{1}{|x-y|}\left(\frac{|x-y|^a}{b}\right)^{\frac{1}{2(a-1)}}.$$

We split the integral Ω on the left hand side of (28) into two parts: $\Omega = \Omega_4 + \Omega_5$, where

$$\Omega_4 = \int_{-h}^{h} \left(1 - \psi_0\left(\frac{\rho+c}{\gamma}\right)\psi_0\left(\frac{\rho-c}{\gamma}\right)\right)\theta_0(\rho)e^{ib|\rho|^a}E_\lambda(ix\rho)E_\lambda(-iy\rho)|\rho|^{2\lambda}d\rho;$$
$$\Omega_5 = \int_{-h}^{h} \psi_0\left(\frac{\rho+c}{\gamma}\right)\psi_0\left(\frac{\rho-c}{\gamma}\right)\theta_0(\rho)e^{ib|\rho|^a}E_\lambda(ix\rho)E_\lambda(-iy\rho)|\rho|^{2\lambda}d\rho.$$

It is easy to see that the support of the integrand in Ω_4 is contained in $[-c-2\gamma, -c+2\gamma]\cup[c-2\gamma, c+2\gamma]$; and then by Lemma 2.2(ii), we have

$$|\Omega_4| \lesssim \gamma|xy|^{-\lambda}.$$

As in the proof of Lemma 3.2, applying (24) to Ω_5 and appealing to (12), then Ω_5 may be written as the sum of Ω_6, Ω_7, and Ω_8, where

$$\begin{aligned}\Omega_6 &:= i\int_{-h}^{h} D_\rho\left[\psi_0\left(\frac{\rho+c}{\gamma}\right)\psi_0\left(\frac{\rho-c}{\gamma}\right)\right.\\ &\qquad\left.\times\frac{\theta_0(\rho)}{ab|\rho|^{a-2}\rho+x-y}\right]\cdot e^{ib|\rho|^a}E_\lambda(ix\rho)E_\lambda(-iy\rho)|\rho|^{2\lambda}d\rho,\\ \Omega_7 &:= \frac{-4i\lambda}{(2\lambda+1)^2}\int_{-h}^{h}\psi_0\left(\frac{\rho+c}{\gamma}\right)\psi_0\left(\frac{\rho-c}{\gamma}\right)\\ &\qquad\times\frac{\theta_0(\rho)e^{ib|\rho|^a}}{ab|\rho|^{a-2}\rho+x-y}\rho xy\, j_{\lambda+\frac{1}{2}}(x\rho)j_{\lambda+\frac{1}{2}}(y\rho)|\rho|^{2\lambda}d\rho,\\ \Omega_8 &:= -i\left(\psi_0\left(\frac{\rho+c}{\gamma}\right)\psi_0\left(\frac{\rho-c}{\gamma}\right)\right.\\ &\qquad\left.\times\frac{\theta_0(\rho)e^{ib|\rho|^a}}{ab|\rho|^{a-2}\rho+x-y}\cdot E_\lambda(ix\rho)E_\lambda(-iy\rho)|\rho|^{2\lambda}\right)\Bigg|_{-h}^{h}.\end{aligned}$$

By Lemma 2.2(ii),

$$|\Omega_6| \lesssim |xy|^{-\lambda}\int_{-h}^{h}\left|D_\rho\left[\psi_0\left(\frac{\rho+c}{\gamma}\right)\psi_0\left(\frac{\rho-c}{\gamma}\right)\frac{\theta_0(\rho)}{ab|\rho|^{a-2}\rho+x-y}\right]\right|d\rho,$$

and from (15), Ω_6 is dominated by a multiple of the sum of Ω_6' and Ω_6'', where

$$\Omega_6' = |xy|^{-\lambda}\int_{-h}^{h}\left|D_\rho\left[\frac{1}{ab|\rho|^{a-2}\rho+x-y}\right]\right|\psi_0\left(\frac{\rho+c}{\gamma}\right)\psi_0\left(\frac{\rho-c}{\gamma}\right)\theta_0(\rho)d\rho,$$
$$\Omega_6'' = |xy|^{-\lambda}\int_{-h}^{h}\left|D_\rho\left[\psi_0\left(\frac{\rho+c}{\gamma}\right)\psi_0\left(\frac{\rho-c}{\gamma}\right)\theta_0(\rho)\right]\right|\frac{d\rho}{|ab|\rho|^{a-1}-|x-y||}.$$

Since $\left|D_\rho\left[(ab|\rho|^{a-2}\rho+x-y)^{-1}\right]\right| \lesssim ab|\rho|^{a-2}/\left(ab|\rho|^{a-1}-|x-y|\right)^2$, it follows that

$$\Omega_6' \lesssim |xy|^{-\lambda}\left(\int_{|x-y|^{-1}}^{c-\gamma}+\int_{c+\gamma}^{h}\right)\frac{ab\rho^{a-2}}{\left(ab\rho^{a-1}-|x-y|\right)^2}d\rho \tag{30}$$

$$\lesssim \frac{|xy|^{-\lambda}}{|x-y|-ab(c-\gamma)^{a-1}}+\frac{|xy|^{-\lambda}}{ab(c+\gamma)^{a-1}-|x-y|}. \tag{31}$$

We note that, if $\gamma \geq c$, the first term above disappears and

$$ab(c+\gamma)^{a-1}-|x-y| \geq \left(2^{a-1}-1\right)|x-y|,$$

so that $\Omega_6' \lesssim |xy|^{-\lambda}|x-y|^{-1} \lesssim |xy|^{-\lambda}\gamma$; if $0<\gamma<c$, then

$$|x-y|-ab(c-\gamma)^{a-1} = |x-y|\left(1-(1-\gamma/c)^{a-1}\right) \gtrsim |x-y|\gamma/c,$$
$$ab(c+\gamma)^{a-1}-|x-y| = |x-y|\left((1+\gamma/c)^{a-1}-1\right) \gtrsim |x-y|\gamma/c,$$

so that $\Omega_6' \lesssim |xy|^{-\lambda}|x-y|^{-1}c/\gamma \lesssim |xy|^{-\lambda}\gamma$.

For Ω_6'', we first note that the value of ρ in the support of the integrand satisfies $|\rho+c| \geq \gamma$, $|\rho-c| \geq \gamma$, so that $|ab|\rho|^{a-1}-|x-y||$ is bounded below by $|x-y|-ab(c-\gamma)^{a-1} \gtrsim |x-y|\gamma/c$ or $ab(c+\gamma)^{a-1}-|x-y| \gtrsim |x-y|\gamma/c$. Thus we have

$$\Omega_6'' \lesssim \frac{|xy|^{-\lambda}}{|x-y|}\frac{c}{\gamma}\left(\int_{-2|x-y|^{-1}}^{2|x-y|^{-1}}|x-y|d\rho+\int_{-c-\gamma}^{-c+\gamma}\gamma^{-1}d\rho+\int_{c-\gamma}^{c+\gamma}\gamma^{-1}d\rho\right)$$
$$\lesssim \frac{|xy|^{-\lambda}}{|x-y|}\frac{c}{\gamma} \lesssim |xy|^{-\lambda}\gamma,$$

and so

$$\Omega_6 \lesssim |xy|^{-\lambda}\gamma.$$

For Ω_7, taking the substitution $\rho \to -\rho$ and summing with the original form, we get

$$|\Omega_7| \lesssim |xy|^{-\lambda}\int_{-h}^{h}\psi_0\left(\frac{\rho+c}{\gamma}\right)\psi_0\left(\frac{\rho-c}{\gamma}\right)\frac{\psi_0(|x-y|\rho)ab|\rho|^{a-2}}{|(ab|\rho|^{a-1})^2-(x-y)^2|}d\rho$$
$$\lesssim |xy|^{-\lambda}\left(\int_{|x-y|^{-1}}^{c-\gamma}+\int_{c+\gamma}^{h}\right)\frac{ab\rho^{a-2}}{\left(ab\rho^{a-1}-|x-y|\right)^2}d\rho,$$

which is the same as in (30) for Ω_6'. Therefore

$$|\Omega_7| \lesssim |xy|^{-\lambda}\gamma.$$

Finally for Ω_8, by Lemma 2.2(ii) it is easy to see that it has a bound similar to (31) for Ω_6', and so

$$|\Omega_8| \lesssim |xy|^{-\lambda}\gamma.$$

Combining the estimates for Ω_6, Ω_7 and Ω_8 shows that $|\Omega_5| \lesssim |xy|^{-\lambda}\gamma$, and in conjunction with that for Ω_4, one has $|\Omega| \lesssim |xy|^{-\lambda}\gamma$ which proves (28) when $|x-y|^a > b/2$.

Proof of Lemma 3.1 Assume that $b > 0$. Denote by Λ the integral on the left hand side of (21) and define the function $\psi_0 \in C^\infty(\mathbb{R})$ as in the proof of Lemma 3.3

We then write Λ into the sum of Λ_0 and Λ_1, where

$$\Lambda_0 = \int_{-\infty}^{\infty} \left(1 - \psi_0(|x-y|\rho)^2\right) e^{ib|\rho|^a} E_\lambda(ix\rho)E_\lambda(-iy\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{2\lambda-s}d\rho, \tag{32}$$

$$\Lambda_1 = \int_{-\infty}^{\infty} \psi_0(|x-y|\rho)^2 e^{ib|\rho|^a} E_\lambda(ix\rho)E_\lambda(-iy\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{2\lambda-s}d\rho. \tag{33}$$

The estimate for Λ_0 is trivial, since by Lemma 2.2(ii),

$$|\Lambda_0| \lesssim |xy|^{-\lambda}\int_0^{2|x-y|^{-1}} \rho^{-s}d\rho \lesssim |xy|^{-\lambda}|x-y|^{s-1}.$$

To evaluate Λ_1, Two cases are to be considered separately. □

Case I: $|x-y|^a \le b/2$.

As in the former part of the proof of Lemma 3.3, in this case, the deduction in (29) for $|x-y||\rho| \ge 1$ implies that the support of the function

$$\theta_1(\rho) := \psi_0(|x-y|\rho)$$

is contained in $[k_2 c, \infty) \cup (-\infty, -k_2 c]$ with $k_2 = (2a)^{1/(a-1)}$. Thus by Lemma 3.2, the function

$$\Omega(\rho) := \int_{-\rho}^{\rho} \theta_1(\eta)e^{ib|\eta|^a} E_\lambda(ix\eta)E_\lambda(-iy\eta)|\eta|^{2\lambda}d\eta$$

satisfies, for all $\rho > 0$, $|\Omega(\rho)| \lesssim |xy|^{-\lambda}|x-y|^{-1}$. Now taking partial integration

$$\begin{aligned}\Lambda_1 &= \frac{1}{2}\int_{-\infty}^{\infty} \theta_1(\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{-s}d\Omega(\rho)\\ &= -\frac{1}{2}\int_{-\infty}^{\infty} \Omega(\rho)\left(\theta_1(\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{-s}\right)_\rho' d\rho,\end{aligned}$$

and hence

$$|\Lambda_1| \lesssim \frac{|xy|^{-\lambda}}{|x-y|}\left(|x-y|^s \int_{|x-y|^{-1}}^{\infty} \left|\left(\theta_1(\rho)\psi\left(\frac{\rho}{N}\right)\right)'_\rho\right| d\rho + \int_{|x-y|^{-1}}^{\infty} \rho^{-s-1} d\rho\right) \lesssim |xy|^{-\lambda}|x-y|^{s-1}.$$

Case II: $|x-y|^a > b/2$.
Since

$$1 \equiv \psi_0\left(\frac{\rho}{2c}\right)\left(2-\psi_0\left(\frac{\rho}{2c}\right)\right) + \left(1-\psi_0\left(\frac{4\rho}{c}\right)^2\right) + \left(1-\psi_0\left(\frac{\rho}{2c}\right)\right)^2 \psi_0\left(\frac{4\rho}{c}\right)^2,$$

we split the integral for Λ_1 into two parts: $\Lambda_1 = \Lambda_{11} + \Lambda_{12}$, where

$$\Lambda_{11} = \int_{-\infty}^{\infty} \theta_1(\rho)\theta_2(\rho)e^{ib|\rho|^a} E_\lambda(ix\rho)E_\lambda(-iy\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{2\lambda-s}d\rho,$$
$$\Lambda_{12} = \int_{-\infty}^{\infty} \theta_3(\rho)^2 e^{ib|\rho|^a} E_\lambda(ix\rho)E_\lambda(-iy\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{2\lambda-s}d\rho,$$

with

$$\theta_2(\rho) = \left[\psi_0\left(\frac{\rho}{2c}\right)\left(2-\psi_0\left(\frac{\rho}{2c}\right)\right) + \left(1-\psi_0\left(\frac{4\rho}{c}\right)^2\right)\right]\psi_0(|x-y|\rho),$$
$$\theta_3(\rho) = \left(1-\psi_0\left(\frac{\rho}{2c}\right)\right)\psi_0\left(\frac{4\rho}{c}\right)\psi_0(|x-y|\rho).$$

For Λ_{11}, we take partial integration to get

$$\Lambda_{11} = \frac{1}{2}\int_{-\infty}^{\infty} \theta_1(\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{-s} d\Omega(\rho) = -\frac{1}{2}\int_{-\infty}^{\infty} \Omega(\rho)\left(\theta_1(\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{-s}\right)'_\rho d\rho,$$

here

$$\Omega(\rho) := \int_{-\rho}^{\rho} \theta_2(\eta)e^{ib|\eta|^a} E_\lambda(ix\eta)E_\lambda(-iy\eta)|\eta|^{2\lambda}d\eta.$$

It is easy to see that

$$\operatorname{supp}\theta_2 \subset \{\rho : |\rho| \le \frac{c}{2}\} \cup \{\rho : |\rho| \ge 2c\},$$

and θ_2 satisfies the conditions in Lemma 3.2. Hence $|\Omega(\rho)| \lesssim |xy|^{-\lambda}|x-y|^{-1}$ for all $\rho > 0$ and

$$|\Lambda_{11}| \lesssim |xy|^{-\lambda}|x-y|^{s-1}.$$

As for Λ_{12}, we again take partial integration to get

$$\begin{aligned}\Lambda_{12} &= \frac{1}{2}\int_{-\infty}^{\infty} \theta_3(\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{-s} d\Omega(\rho) \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} \Omega(\rho)\left(\theta_3(\rho)\psi\left(\frac{\rho}{N}\right)|\rho|^{-s}\right)'_{\rho} d\rho,\end{aligned} \tag{34}$$

where

$$\Omega(\rho) := \int_{-\rho}^{\rho} \theta_3(\eta)e^{ib|\eta|^a} E_\lambda(ix\eta)E_\lambda(-iy\eta)|\eta|^{2\lambda}d\eta.$$

It is noted that

$$\operatorname{supp}\theta_3 \subset \left\{\rho : \frac{c}{4} \le |\rho| \le 4c\right\}.$$

Now we apply Lemma 3.3 to obtain

$$\begin{aligned}|\Lambda_{12}| &\lesssim \frac{|xy|^{-\lambda}}{|x-y|}\left(\frac{|x-y|^a}{b}\right)^{\frac{1}{2(a-1)}}\left(c^{-s}\int_{\frac{c}{4}}^{4c}\left|\left(\theta_3(\rho)\psi\left(\frac{\rho}{N}\right)\right)'_{\rho}\right|d\rho + \int_{\frac{c}{4}}^{4c}\rho^{-s-1}d\rho\right) \\ &\lesssim \frac{|xy|^{-\lambda}}{|x-y|}\left(\frac{|x-y|^a}{b}\right)^{\frac{1}{2(a-1)}} c^{-s},\end{aligned}$$

so that, for $1/2 \le s < 1$, $|x-y|^a > b/2$,

$$|\Lambda_{12}| \lesssim \frac{|xy|^{-\lambda}}{|x-y|^{1-s}}\left(\frac{b}{|x-y|^a}\right)^{\left(s-\frac{1}{2}\right)\frac{1}{a-1}} \lesssim \frac{|xy|^{-\lambda}}{|x-y|^{1-s}}. \tag{35}$$

The proof of Lemma 3.1 is finished.

4 Proofs of the Main Results

We shall prove a more general theorem than Theorem 1.2.

Theorem 4.1 *Let $a > 1$, $\lambda \ge 0$, $1/4 \le s < 1/2$, and $q = \frac{4\lambda+2}{2\lambda+1-2s}$. Then for all $f \in \mathscr{S}(\mathbb{R})$, we have the following norm estimate*

$$\left(\int_{\mathbb{R}}\left(\sup_{0<|t|<\infty}\left|T^t_{\lambda,a}f(x)\right|\right)^q |x|^{2\lambda}dx\right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{H}^s_\lambda}. \tag{36}$$

Obviously Theorem 1.2 is the particular case of Theorem 4.1 with $s = 1/4$.

Proof of Theorem 4.1 We first note that, to show (36), it is sufficient to prove the inequality taking integration over $[-M, M]$ for each $M > 0$, instead of over $\mathbb{R}$. Further, it suffices to prove that, for all $A > 0$,

$$\left(\int_{-M}^{M}\left(\sup_{0<t\le A}\left|T_{\lambda,a}^{t}f(x)\right|\right)^{q}|x|^{2\lambda}dx\right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{H}_{\lambda}^{s}},$$

since $\sup_{0<t\le A}|T_{\lambda,a}^{t}f(x)|$ increases and tends to $\sup_{0<t<\infty}|T_{\lambda,a}^{t}f(x)|$ as $A\to\infty$. Moreover, for $f\in\mathscr{S}(\mathbb{R})$, one may manipulate the linearization of the maximal function $\sup_{0<t\le A}|T_{\lambda,a}^{t}f(x)|$ instead of itself, in terms of a measurable function $t = t(x)$ with $0 < t(x) \le A$, that is, we shall show that

$$\left(\int_{-M}^{M}\left|T_{\lambda,a}^{t(x)}f(x)\right|^{q}|x|^{2\lambda}dx\right)^{\frac{1}{q}} \lesssim \|f\|_{\dot{H}_{\lambda}^{s}}, \qquad q=\frac{4\lambda+2}{2\lambda+1-2s}. \tag{37}$$

If we set $g(\rho) = |\rho|^{s}(\mathscr{F}_{\lambda}f)(\rho)$, then (37) is equivalent to

$$\left(\int_{-M}^{M}\left|U_{\lambda,a}g(x)\right|^{q}|x|^{2\lambda}dx\right)^{\frac{1}{q}} \lesssim \|g\|_{L_{\lambda}^{2}}, \tag{38}$$

where

$$U_{\lambda,a}g(x) = c_{\lambda}\int_{\mathbb{R}} g(\rho)e^{it(x)|\rho|^{a}}E_{\lambda}(ix\rho)|\rho|^{-s}|\rho|^{2\lambda}d\rho, \quad x\in\mathbb{R}.$$

By duality, (38) is further equivalent to

$$\left(c_{\lambda}\int_{\mathbb{R}}\left|U_{\lambda,a}^{*}h(\rho)\right|^{2}|\rho|^{2\lambda}d\rho\right)^{\frac{1}{2}} \lesssim \left(\int_{-M}^{M}|h(x)|^{q'}|x|^{2\lambda}dx\right)^{\frac{1}{q'}}, \quad q'=\frac{4\lambda+2}{2\lambda+1+2s}, \tag{39}$$

where

$$U_{\lambda,a}^{*}h(\rho) = |\rho|^{-s}\int_{-1}^{1} h(x)e^{-it(x)|\rho|^{a}}E_{\lambda}(-ix\rho)|x|^{2\lambda}dx, \quad \rho\in\mathbb{R}. \tag{40}$$

We take an even function $\psi\in C^{\infty}(\mathbb{R})$ satisfying

$$\operatorname{supp}\psi\subset[-2,2], \quad 0\le\psi(\rho)\le 1, \quad \text{and} \quad \psi(\rho)\equiv 1 \text{ for } \rho\in[-1,1]. \tag{41}$$

It follows that, for large $N > 0$,

$$\int_{\mathbb{R}}\psi\left(\frac{\rho}{N}\right)\left|U_{\lambda,a}^{*}h(\rho)\right|^{2}|\rho|^{2\lambda}d\rho = \int_{\mathbb{R}}\psi\left(\frac{\rho}{N}\right)U_{\lambda,a}^{*}h(\rho)\overline{U_{\lambda,a}^{*}h(\rho)}|\rho|^{2\lambda}d\rho$$

$$= \int_{-M}^{M} \int_{-M}^{M} \left[\int_{\mathbb{R}} e^{i(t(x)-t(y))|\rho|^a} E_\lambda(ix\rho) E_\lambda(-iy\rho) \right.$$
$$\left. \times \psi\left(\frac{\rho}{N}\right) |\rho|^{2\lambda-2s} d\rho \right] \overline{h(x)} h(y) |x|^{2\lambda} |y|^{2\lambda} \, dx \, dy,$$

and then by Lemma 3.1, we have

$$\int_{\mathbb{R}} \psi\left(\frac{\rho}{N}\right) \left|U^*_{\lambda,a} h(\rho)\right|^2 |\rho|^{2\lambda} d\rho \lesssim \int_{-M}^{M} \int_{-M}^{M} \frac{|h(x)h(y)|}{|x-y|^{1-2s}} |x|^\lambda |y|^\lambda \, dx \, dy.$$

Now letting $N \to \infty$, Fatou's lemma asserts that

$$\int_{\mathbb{R}} \left|U^*_{\lambda,a} h(\rho)\right|^2 |\rho|^{2\lambda} d\rho \lesssim \int_{-M}^{M} \int_{-M}^{M} \frac{|h(x)h(y)|}{|x-y|^{1-2s}} |x|^\lambda |y|^\lambda \, dx \, dy. \qquad (42)$$

Finally we need to apply a lemma from [25] to the right hand side above.

Lemma 4.2 ([25, Lemma 1.7]) *Assume that* $0 < \beta < 1$, $2/(2-\beta) \le p \le 2$, $\gamma = 1 - \beta/2 - 1/p$. *Then for measurable functions* h,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|h(x)||h(y)|}{|x-y|^\beta |x|^\gamma |y|^\gamma} dx \, dy \lesssim \left(\int_{\mathbb{R}} |h(x)|^p dx \right)^{2/p}.$$

We rewrite the right hand side of (42) as

$$\int_{-M}^{M} \int_{-M}^{M} \frac{|h(x)||x|^{2\lambda/q'} \cdot |h(y)||y|^{2\lambda/q'}}{|x-y|^{1-2s} |x|^{2\lambda/q'-\lambda} |y|^{2\lambda/q'-\lambda}} \, dx \, dy,$$

and then, putting $\beta = 1 - 2s$, $p = q'$ (given in (39)), and $\gamma = 1 - \beta/2 - 1/p = 2\lambda/q' - \lambda$, this double integral is bounded by a multiple of

$$\left[\int_{-M}^{M} \left(|h(x)||x|^{2\lambda/q'} \right)^{q'} \, dx \right]^{2/q'} = \left(\int_{-M}^{M} |h(x)|^{q'} |x|^{2\lambda} \, dx \right)^{2/q'}.$$

Thus the inequality (39) is proved, and then, the proof of Theorem 4.1 is completed. □

Now we turn to the proof of Theorem 1.1.

We restrict the discussion to $t \in [0, \infty)$.

An immediate consequence of Theorem 1.2 is that, for each $M > 0$ and $A > 0$,

$$\int_{-M}^{M} \sup_{0<t\le A} \left|T^t_{\lambda,a} f(x)\right| |x|^{2\lambda} dx \le c_M \|f\|_{H^{1/4}_\lambda}, \qquad f \in \mathscr{S}(\mathbb{R}), \qquad (43)$$

where the constant $c_M > 0$ is independent of f. Since $H^s_\lambda(\mathbb{R}) \subseteq H^{1/4}_\lambda(\mathbb{R})$ when $s \ge 1/4$, it suffices to assume $f \in H^{1/4}_\lambda(\mathbb{R})$ in proving Theorem 1.1. In what follows, we work with a procedure as in [22]. Given such an f, we take a sequence $\{f_k : k \geq 1\} \subset \mathscr{S}(\mathbb{R})$ satisfying

$$\|f_k - f\|_{H_\lambda^{1/4}} \leq 2^{-k}.$$

It then follows from (43) that

$$\int_{-M}^{M} \sum_{k=1}^{\infty} \sup_{0<t\leq A} \left|T_{\lambda,a}^t f_k(x) - T_{\lambda,a}^t f_{k+1}(x)\right| |x|^{2\lambda} dx \leq c_M \sum_{k=1}^{\infty} \|f_k - f_{k-1}\|_{H_\lambda^{1/4}} \leq 2c_M.$$

It is noted that each function $(x, t) \mapsto T_{\lambda,a}^t f_k(x)$ is continuous in $\mathbb{R} \times \mathbb{R}$ with values $f_k(x)$ when $t = 0$, so that the supremum can be taken over $t \in [0, A]$. We have that, for almost all $x \in [-M, M]$,

$$\sum_{k=1}^{\infty} \sup_{0\leq t\leq A} \left|T_{\lambda,a}^t f_k(x) - T_{\lambda,a}^t f_{k+1}(x)\right| < \infty,$$

which implies, for x with the property, that the sequence of functions $t \mapsto T_{\lambda,a}^t f_k(x)$ converges uniformly in $t \in [0, A]$, to a continuous function, $v_x(t)$, says. Obviously $v_x(t)$ is well defined for a.e. $x \in \mathbb{R}$, and continuous in $t \in [0, A]$. However, since f_k tends to f as $k \to \infty$ in $L_\lambda^2(\mathbb{R})$ norm, it follows that

$$\lim_{k\to\infty} \int_{\mathbb{R}} |x|^{2\lambda} \int_0^A \left|T_{\lambda,a}^t f_k(x) - T_{\lambda,a}^t f(x)\right|^2 dt\, dx = 0,$$

and hence, there is a subsequence $k_j \to \infty$ such that, for a.e. $x \in \mathbb{R}$, the sequence of functions $t \mapsto T_{\lambda,a}^t f_{k_j}(x)$ converges in the $L^2([0, A])$ norm to the function $t \mapsto T_{\lambda,a}^t f(x)$. We conclude that, except x in a null set in $\mathbb{R}$, $T_{\lambda,a}^t f(x) = v_x(t)$ for a.e. $t \in [0, A]$. By choosing $A_\ell \to \infty$, the functions $t \mapsto v_x(t)$ are continuously extended to $[0, \infty)$, for a.e. $x \in \mathbb{R}$.

It remains to show that $v_x(0) = f(x)$ for a.e. $x \in \mathbb{R}$. Indeed, since $\|T_{\lambda,a}^t f - f\|_{L_\lambda^2} \to 0$ as $t \to 0$, then, for a sequence $t_j \to 0$, $T_{\lambda,a}^{t_j} f(x) \to f(x)$ almost everywhere, and so does $v_x(t_j) \to f(x) = v_x(0)$.

5 The Divergence Set of $T_{\lambda,a}^t f$

5.1 Another Representation of $T_{\lambda,2}^t f(x)$

We shall need the following equality (cf. [29, §§13.31])

$$2c_\lambda \int_0^\infty e^{-z\xi^2} j_{\lambda-1/2}(x\xi) j_{\lambda-1/2}(y\xi)\xi^{2\lambda} d\xi$$
$$= \frac{1}{(2z)^{\lambda+1/2}} \exp\left(-\frac{x^2+y^2}{4z}\right) j_{\lambda-1/2}\left(i\frac{xy}{2z}\right) \tag{44}$$

for $\lambda > -1/2$ and $\arg z \in (-\pi/2, \pi/2)$.

For $f \in \mathscr{D}(\mathbb{R})$ and $x \in \mathbb{R}$, from (4) we have

$$T^t_{\lambda,2} f(x) = \lim_{\epsilon\to 0+} c_\lambda \int_{\mathbb{R}} (\mathscr{F}_\lambda f)(\rho) e^{(it-\epsilon)\rho^2} E_\lambda(ix\rho)|\rho|^{2\lambda} d\rho$$
$$= \lim_{\epsilon\to 0+} c_\lambda \int_{\mathbb{R}} f(y) K^t_\epsilon(x,y)|y|^{2\lambda} dy, \tag{45}$$

where, for $\epsilon > 0$,

$$K^t_\epsilon(x,y) = c_\lambda \int_{\mathbb{R}} e^{(it-\epsilon)\rho^2} E_\lambda(ix\rho) E_\lambda(-iy\rho)|\rho|^{2\lambda} d\rho. \tag{46}$$

It follows from (2) that

$$K^t_\epsilon(x,y) = 2c_\lambda \int_0^\infty e^{(it-\epsilon)\rho^2} \Big[j_{\lambda-1/2}(x\rho) j_{\lambda-1/2}(y\rho)$$
$$+ \frac{xy\rho^2}{(2\lambda+1)^2} j_{\lambda+1/2}(x\rho) j_{\lambda+1/2}(y\rho) \Big] \rho^{2\lambda} d\rho,$$

and then, by (44),

$$K^t_\epsilon(x,y) = \frac{1}{(2z)^{\lambda+1/2}} \exp\left(-\frac{x^2+y^2}{4z}\right) E_\lambda\left(\frac{xy}{2z}\right), \qquad \text{with} \quad z = \epsilon - it. \tag{47}$$

Substituting this expression into (45) and applying the dominated convergence theorem, we obtain, for $f \in \mathscr{D}(\mathbb{R})$ and $t \neq 0$,

$$T^t_{\lambda,2} f(x) = c_\lambda \int_{\mathbb{R}} f(y) K^t(x,y)|y|^{2\lambda} dy, \tag{48}$$

where

$$K^t(x,y) = \left(\frac{i}{2t}\right)^{\lambda+1/2} \exp\left(-i\frac{x^2+y^2}{4t}\right) E_\lambda\left(i\frac{xy}{2t}\right). \tag{49}$$

5.2 A Counterexample in $H^s_\lambda(\mathbb{R})$ for $s < 1/4$

Assume that $s < 1/4$. We shall construct a function $f_0 \in H^s_\lambda(\mathbb{R})$ for which, in a set of positive measure, $T^t_{\lambda,2} f$ diverges as $t \to 0+$, even after a modification on a null set of t.

Let ϕ be a nonzero element in $\mathscr{D}(\mathbb{R})$, supported in $(-\infty, -1)$. We consider the functions, for $0 < t < 1$,

$$\phi_t(x) = \frac{1}{t^{2\lambda}}\phi\left(\frac{x}{t}\right)E_\lambda\left(-i\frac{x}{t^2}\right), \qquad x \in \mathbb{R}.$$

Since, from (1),

$$(\mathscr{F}_\lambda\phi_t)(\rho) := c_\lambda t\int_{\mathbb{R}}\phi(-x)E_\lambda\left(ixt^{-1}\right)E_\lambda(ixt\rho)|x|^{2\lambda}dx, \quad \rho \in \mathbb{R},$$

then by Propositions 2.3(ii) and 2.4(iii),

$$(\mathscr{F}_\lambda\phi_t)(\rho) = t\left[\tau_{t^{-1}}(\mathscr{F}_\lambda\phi)\right](t\rho).$$

Applying Proposition 2.5 to $\mathscr{F}_\lambda\phi$ we have

$$\begin{aligned}\|\phi_t\|^2_{H^s_\lambda} &= c_\lambda t^{1-2\lambda-2s}\int_{\mathbb{R}}\left|\left[\tau_{t^{-1}}(\mathscr{F}_\lambda\phi)\right](\rho)\right|^2(t^2+\rho^2)^s|\rho|^{2\lambda}d\rho\\ &\lesssim t^{1-2\lambda-2s}\int_{\mathbb{R}}\frac{(1+\rho^2+t^{-2})^{-2\lambda}}{(1+||\rho|-t^{-1}|^2)^2}(t^2+\rho^2)^s|\rho|^{2\lambda}d\rho\\ &\lesssim t^{1-2s}\int_{\mathbb{R}}\frac{(t^2+\rho^2)^s}{(1+||\rho|-t^{-1}|^2)^2}d\rho,\end{aligned}$$

so that

$$\|\phi_t\|_{H^s_\lambda} \lesssim t^{(1-4s)/2} \qquad \text{for} \quad 0 < t < 1. \tag{50}$$

For $x \in (0, 1)$, we choose $t(x) = t^2x/2$. From (48) and (49), we have

$$\begin{aligned}T^{t(x)}_{\lambda,2}\phi_t(x) = \frac{c_\lambda e^{i\pi(\lambda+1/2)/2}}{t^{2\lambda}x^{\lambda+1/2}}e^{-ix/(2t^2)}\\ \times\int_{\mathbb{R}}\phi(y)\exp\left(-i\frac{y^2}{2x}\right)E_\lambda\left(-i\frac{y}{t}\right)E_\lambda\left(i\frac{y}{t}\right)|y|^{2\lambda}dy.\end{aligned} \tag{51}$$

We shall need the following equality

$$E_\lambda(iy)E_\lambda(-iy) = \frac{|y|^{-2\lambda}}{2\pi c_\lambda^2}\left(1+O\left(\frac{1}{y}\right)\right), \qquad \text{for} \quad |x| \geq c > 0. \tag{52}$$

Indeed, from (2) we have

$$E_\lambda(iy)E_\lambda(-iy) = j_{\lambda-1/2}(y)^2 + \frac{y^2}{(2\lambda+1)^2}j_{\lambda+1/2}(y)^2,$$

and then, (52) follows by making use of (cf. [12, (7-13(3))])

$$j_{\lambda-1/2}(y)=\frac{|y|^{-\lambda}}{\sqrt{2\pi}c_\lambda}\left[\cos\left(|y|-\frac{\pi}{2}\lambda\right)+O\left(\frac{1}{y}\right)\right],\qquad \text{for}\quad |x|\geq c>0.$$

Applying (52) to (51) we obtain, for $x\in(0,1)$ and $0<t<1$,

$$T_{\lambda,2}^{t(x)}\phi_t(x)=\frac{e^{i\pi(\lambda+1/2)/2}}{2\pi c_\lambda x^{\lambda+1/2}}e^{-ix/(2t^2)}\left[\psi(x)+O(t)\right],\tag{53}$$

where

$$\psi(z)=\int_{\mathbb{R}}\phi(y)\exp\left(-i\frac{y^2}{2z}\right)dy,\qquad z\neq 0.$$

Since $\psi(z)$ is a nonzero holomorphic function in $\mathbb{C}\setminus\{0\}$, there exist an interval $I\subset(1/2,1)$, $t_0\in(0,1)$, and a constant $c_0>0$, such that for $x\in I$ and $t\in(0,t_0)$,

$$\left|T_{\lambda,2}^{t(x)}\phi_t(x)\right|>c_0.\tag{54}$$

We now adopt the procedure in [22] to finish the construction of f_0. We shall define recursively a sequence $\{t_j\}\subset(0,t_0)$, converging to zero so fast that it satisfies $\sum_{j=1}^{\infty}jt_j^{(1-4s)/2}<\infty$, and for all $x\in I$,

$$\left|T_{\lambda,2}^{t_k(x)}\phi_{t_j}(x)\right|<2^{-j},\qquad \text{for}\quad j\neq k,\tag{55}$$

where $t_k(x)=t_k^2x/2$.

We start with a number $t_1\in(0,t_0)$ and assume that $t_1,\ldots,t_{m-1}$ have been chosen so that $0<t_j<\min\{t_0,2^{-j+1}\}$ $(1\leq j<m)$ and (55) is satisfied for $j,k<m$. Since for each j $(1\leq j<m)$, $\phi_{t_j}\in\mathscr{D}(\mathbb{R})$ and $\operatorname{supp}\phi_{t_j}\subset(-\infty,0)$, $T_{\lambda,2}^{t(x)}\phi_{t_j}(x)$ converges to zero uniformly for $x\in I$ as $t\to 0+$, and from (48), (49), and Lemma 2.2(ii), we have, for each j $(1\leq j<m)$,

$$\left|T_{\lambda,2}^{t_j(x)}\phi_t(x)\right|\lesssim\frac{t}{(t_j\sqrt{|x|})^{2\lambda+1}}\int_{\mathbb{R}}|\phi(y)||y|^{2\lambda}dy,$$

which, again, converges to zero uniformly for $x\in I$ as $t\to 0+$. Hence one may choose t_m, $0<t_m<\min\{t_0,2^{-m+1}\}$, so that, for $j=1,\ldots,m-1$, and for $x\in I$,

$$\left|T_{\lambda,2}^{t_m(x)}\phi_{t_j}(x)\right|<2^{-j}\qquad\text{and}\qquad\left|T_{\lambda,2}^{t_j(x)}\phi_{t_m}(x)\right|<2^{-m}.$$

We define $f_0 = \sum_{j=1}^{\infty} j\phi_{t_j}$. It follows from (50) that $f_0 \in H_\lambda^s(\mathbb{R})$ and

$$T_{\lambda,2}^t f_0(x) = \sum_{j=1}^{\infty} j T_{\lambda,2}^t \phi_{t_j}(x), \qquad x \in \mathbb{R}.$$

For $x \in I$ and each k, by (54) and (55) we have

$$\left| T_{\lambda,2}^{t_k(x)} f_0(x) \right| \geq c_0 k - \sum_{j \neq k} j 2^{-j}, \tag{56}$$

which shows divergence of $T_{\lambda,2}^{t_k(x)} f_0(x)$ as $k \to \infty$.

Finally we remark that, by continuity, (54) is also true if $t(x)$ is replaced by any number in a small neighborhood of $t(x)$. Similarly, (55) still holds when $t_k(x)$ is replaced by numbers close to $t_k(x)$, and so does (56).

5.3 The Divergence Set of $T_{\lambda,a}^t f$

Assume that $a > 1$ and $\lambda \geq 0$. Theorem 1.1 shows that, if $f \in H_\lambda^s(\mathbb{R})$ with $s \geq 1/4$, then the set $\{x \in \mathbb{R} : T_{\lambda,a}^t f(x) \nrightarrow f(x) \text{ as } t \to 0\}$ is of zero measure in Lebesgue sense. A refinement of this question in the classical case (i.e. for $\lambda = 0$) was studied in [1,2,21], by considering the Hausdorff dimension of the divergence sets. This subsection will extend the methods of [1,2] to determine the Hausdorff dimension of the divergence set of $T_{\lambda,a}^t f$.

As in [1], discrete times will be taken to avoid measurability issues. Let

$$\alpha_\lambda(s) = \sup_{\{t_k\}, f} \dim_H \left\{ x \in \mathbb{R} : T_{\lambda,a}^{t_k} f(x) \nrightarrow f(x) \text{ as } k \to \infty \right\},$$

where $\dim_H$ denotes the Hausdorff dimension, and the supremum is taken over all sequence $t_k \to 0+$ and all $f \in H_\lambda^s(\mathbb{R})$. $\alpha_\lambda(s)$ may have no sense when considering the space $H_\lambda^s(\mathbb{R})$ itself, since the functions are only defined up to a set of zero measure of Lebesgue. However each equivalence class of the Sobolev space $H_\lambda^s(\mathbb{R})$ has a representative as $f = g *_\lambda G_s$ with some $g \in L_\lambda^2(\mathbb{R})$, where G_s, with the Dunkl transform $(1+|\rho|^2)^{-\frac{s}{2}}$, is the Bessel potential kernel associated to the Dunkl transform. The definition of $\alpha_\lambda(s)$, and also Lemma 5.1, Lemma 5.2, and Theorem 5.3 below, will be restricted to these representatives.

To make sense of $T_{\lambda,a}^t f$ for all $f \in H_\lambda^s(\mathbb{R})$, it may be defined as the pointwise limit

$$T_{\lambda,a}^t f = \lim_{N \to \infty} T_{\lambda,a}^{t,N} f$$

whenever the limit exists, where

$$T_{\lambda,a}^{t,N} f(x) = c_\lambda \int_{\mathbb{R}} \psi\left(\frac{|\rho|}{N}\right) (\mathscr{F}_\lambda f)(\rho) e^{it|\rho|^a} E_\lambda(ix\rho)|\rho|^{2\lambda} d\rho, \quad x \in \mathbb{R}, \tag{57}$$

$\psi \in C^\infty(\mathbb{R})$ satisfying the conditions in (41).

A positive Borel measure μ on $\mathbb{R}$ is said to be α-dimensional, $0 \leq \alpha \leq 1$, if

$$c_\alpha(\mu) := \sup_{x\in\mathbb{R}, r>0} \frac{\mu((x-r, x+r))}{r^\alpha} < \infty.$$

$\mathcal{M}^\alpha(\mathbb{A}_m)$ denotes the set of α-dimensional probability measures supported in the "annulus" $\mathbb{A}_m = \{x \in \mathbb{R} : 1/m \leq |x| \leq m\}$.

Lemma 5.1 *Assume that* $0 < s \leq 1/2$ *and* $1 - 2s < \alpha \leq 1$*. If* $f \in H_\lambda^s(\mathbb{R})$ *and* $\mu \in \mathcal{M}^\alpha(\mathbb{A}_m)$*, then*

$$\|f\|_{L^1(d\mu)} \lesssim m^{3\lambda+(\alpha-1+2s)/2} \sqrt{c_\alpha(\mu)} \|f\|_{H_\lambda^s}. \tag{58}$$

Proof For $s > 0$, put

$$G_s(x) = \frac{2^{-\lambda-1/2}}{\Gamma(s/2)} \int_0^\infty e^{-\delta - x^2/(4\delta)} \delta^{\frac{s}{2}-\lambda-\frac{3}{2}} d\delta, \qquad x \in \mathbb{R}. \tag{59}$$

Applying Fubini's theorem shows that $c_\lambda \int_{\mathbb{R}} G_s(x)|x|^{2\lambda} dx = 1$; and furthermore we have

$$(\mathscr{F}_\lambda G_s)(\rho) = (1 + |\rho|^2)^{-\frac{s}{2}}, \qquad \rho \in \mathbb{R}. \tag{60}$$

Indeed, since $(\mathscr{F}_\lambda \phi)(\rho) = e^{-\delta\rho^2}$ for $\phi(x) = (2\delta)^{-\lambda-1/2} e^{-x^2/(4\delta)}$ (taking $\epsilon = 1/(4\delta)$, $t = x = 0$ in (46) and (47), Proposition 2.3(iv) implies that

$$c_\lambda \int_{\mathbb{R}} e^{-\delta x^2} \psi(x)|x|^{2\lambda} dx = c_\lambda \int_{\mathbb{R}} (2\delta)^{-\lambda-1/2} e^{-x^2/(4\delta)} (\mathscr{F}_\lambda \psi)(x)|x|^{2\lambda} dx,$$

whenever $\psi \in \mathscr{S}(\mathbb{R})$. We integrate both sides with respect to $e^{-\delta}\delta^{\frac{s}{2}-1} d\delta$ and change the order of integration, to get

$$c_\lambda \int_{\mathbb{R}} (1 + |x|^2)^{-\frac{s}{2}} \psi(x)|x|^{2\lambda} dx = c_\lambda \int_{\mathbb{R}} G_s(x)(\mathscr{F}_\lambda \psi)(x)|x|^{2\lambda} dx$$

for all $\psi \in \mathscr{S}(\mathbb{R})$. This proves (60).

Appealing to G_s and by Proposition 2.4(v), proving (58) is equivalent to showing, for $g \in L_\lambda^2(\mathbb{R})$,

$$\|g *_\lambda G_s\|_{L^1(d\mu)} \lesssim m^{3\lambda+(\alpha-1+2s)/2} \sqrt{c_\alpha(\mu)} \|g\|_{L_\lambda^2}. \tag{61}$$

We observe that, from (19) and (20),

$$\|g *_\lambda G_s\|_{L^1(d\mu)} \le c_\lambda \int_{\mathbb{R}} \int_{\mathbb{R}} |g(y)|\,|(\tau_x G)(-y)|\,|y|^{2\lambda} dy\, d\mu(x) \\ \le m^{2\lambda} c_\lambda^{-1} \|G_s *_\lambda (d\mu)\|_{L^2_\lambda} \|g\|_{L^2_\lambda},$$

where we have used the fact $(\tau_x G)(-y) = (\tau_y G)(-x)$ by Proposition 2.4(i), and so, (61) is a consequence of the inequality

$$\|G_s *_\lambda (d\mu)\|^2_{L^2_\lambda} \lesssim m^{2\lambda+\alpha-1+2s} c_\alpha(\mu). \tag{62}$$

To prove (62), we first note that, by Propositions 2.3(v) and 2.4(vii),

$$\|G_s *_\lambda (d\mu)\|^2_{L^2_\lambda} = \|\mathscr{F}_\lambda G_s \cdot \mathscr{F}_\lambda (d\mu)\|^2_{L^2_\lambda} = \|\mathscr{F}_\lambda \left(G_{2s} *_\lambda (d\mu)\right) \cdot \overline{\mathscr{F}_\lambda (d\mu)}\|_{L^1_\lambda}.$$

Applying Lebesgue's monotone convergence theorem, Fubini's theorem, and Proposition 2.6 successively, we obtain

$$\|G_s *_\lambda (d\mu)\|^2_{L^2_\lambda} = \lim_{\epsilon\to 0+} c_\lambda \int_{\mathbb{R}} e^{-\epsilon|\rho|} \left[\mathscr{F}_\lambda \left(G_{2s} *_\lambda (d\mu)\right)\right](\rho) \cdot \overline{\mathscr{F}_\lambda (d\mu)}(\rho) |\rho|^{2\lambda} d\rho \\ = \lim_{\epsilon\to 0+} c_\lambda^2 \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\epsilon|\rho|} \left[\mathscr{F}_\lambda \left(G_{2s} *_\lambda (d\mu)\right)\right](\rho) \\ E_\lambda(ix\rho)|\rho|^{2\lambda} d\rho\, |x|^{2\lambda} d\mu(x) \\ = \lim_{\epsilon\to 0+} c_\lambda \int_{\mathbb{R}} \left[P_\epsilon \left(G_{2s} *_\lambda (d\mu)\right)\right](x) |x|^{2\lambda} d\mu(x).$$

In what follows, we shall show that, for $\mu \in \mathcal{M}^\alpha(\mathbb{A}_m)$ with $1 \ge \alpha > 1 - 2s \ge 0$,

$$\left(G_{2s} *_\lambda (d\mu)\right)(x) \lesssim m^{\alpha-1+2s} c_\alpha(\mu), \qquad x \in \mathbb{R}, \tag{63}$$

which implies, by Proposition 2.4(iv), $\left[P_\epsilon \left(G_{2s} *_\lambda (d\mu)\right)\right](x) \lesssim m^{\alpha-1+2s} c_\alpha(\mu)$ for all $x \in \mathbb{R}$, so that (62) is proved.

Since, by (59),

$$G_s(x) \lesssim \int_0^\infty e^{-x^2/(4\delta)} \delta^{\frac{s}{2}-\lambda-\frac{3}{2}} d\delta \lesssim |x|^{s-2\lambda-1}, \qquad x \ne 0,$$

it follows from (18) that

$$(\tau_x G_{2s})(-y) \lesssim \int_0^\pi (x^2 + y^2 - 2xy\cos\theta)^{s-\lambda-1/2} (1+\cos\theta) \sin^{2\lambda-1}\theta d\theta. \tag{64}$$

If $xy > 0$, then

$$\begin{aligned}(\tau_x G_{2s})(-y) &\lesssim \int_0^{\pi/2} \left[(x-y)^2 + 4xy\sin^2\frac{\theta}{2}\right]^{s-\lambda-\frac{1}{2}} \sin^{2\lambda-1}\theta d\theta + (x^2+y^2)^{s-\lambda-\frac{1}{2}} \\ &\lesssim \int_0^{\pi/2} \left(|x-y| + \sqrt{xy}\theta\right)^{2s-2\lambda-1} \theta^{2\lambda-1} d\theta + (x^2+y^2)^{s-\lambda-\frac{1}{2}} \\ &\lesssim \frac{|xy|^{-\lambda}}{|x-y|^{1-2s}} \int_0^A (1+r)^{2s-2\lambda-1} r^{2\lambda-1} dr + (x^2+y^2)^{s-\lambda-\frac{1}{2}},\end{aligned}$$

where $A = (\pi/2)\sqrt{|xy|}/|x-y|$. Since, for $0 < s \le 1/2$,

$$\int_0^A (1+r)^{2s-2\lambda-1} r^{2\lambda-1} dr \lesssim \left(\frac{A}{A+1}\right)^{2\lambda} \log(A+2),$$

it follows that, for $xy > 0$,

$$(\tau_x G_{2s})(-y) \lesssim \frac{(|x|+|y|)^{-2\lambda}}{|x-y|^{1-2s}} \log\left(\frac{|x|+|y|}{|x-y|} + 1\right). \tag{65}$$

It is remarked that only for $s = 1/2$, the factor log appears in (65). If $xy < 0$, then

$$(x^2+y^2)\cos^2\frac{\theta}{2} \le x^2 + y^2 - 2xy\cos\theta \le 2(x^2+y^2),$$

so that from (64), $(\tau_x G_{2s})(-y) \lesssim (x^2+y^2)^{s-\lambda-\frac{1}{2}}$, which shows that (65) hold also for $xy < 0$.

Now for $0 < s \le 1/2$, $1-2s < \alpha \le 1$, and for $\mu \in \mathcal{M}^\alpha(\mathbb{A}_m)$, applying (65) gives

$$(G_{2s} *_\lambda (d\mu))(x) \lesssim \int_{\mathbb{R}} \frac{(|x|+|y|)^{-2\lambda}}{|x-y|^{1-2s}} \log\left(\frac{|x|+|y|}{|x-y|} + 1\right) |y|^{2\lambda} d\mu(y).$$

Choosing $\alpha' \in (1-2s, \alpha)$ and noting that $r^{\alpha'-1+2s}\log(r^{-1}+1) \lesssim 1$ for $0 < r \le 1$, it follows that

$$(G_{2s} *_\lambda (d\mu))(x) \lesssim \int_{\mathbb{R}} \frac{(|x|+|y|)^{\alpha'-1+2s}}{|x-y|^{\alpha'}} d\mu(y). \tag{66}$$

If $|x| \le 2m$, decomposing dyadically around x shows that

$$\begin{aligned}(G_{2s} *_\lambda (d\mu))(x) &\lesssim m^{\alpha'-1+2s} \sum_{j=-2}^{\infty} (m2^{-j})^{\alpha-\alpha'} \frac{\mu(x-m2^{-j}, x+m2^{-j})}{(m2^{-j})^\alpha} \\ &\lesssim m^{\alpha-1+2s} c_\alpha(\mu);\end{aligned} \tag{67}$$

and if $|x| > 2m$, then $(G_{2s} *_\lambda (d\mu))(x) \lesssim |x|^{2s-1}\mu(-m,m) \lesssim m^{\alpha-1+2s}c_\alpha(\mu)$. Thus (63) is proved. □

Lemma 5.2 *Let $a > 1$, $\lambda \geq 0$, and $1/4 \leq s < 1/2$. Then for $1-2s < \alpha \leq 1$ and a real sequence $\{t_k\}$, we have*

$$\left\| \sup_{k\geq 1} \sup_{N\geq 1} \left| T_{\lambda,a}^{t_k,N} f \right| \right\|_{L^1(d\mu)} \lesssim m^{\lambda+(\alpha-1+2s)/2}\sqrt{c_\alpha(\mu)}\|f\|_{H_\lambda^s} \tag{68}$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{A}_m)$ and $f \in H_\lambda^s(\mathbb{R})$.

Proof By linearization of the maximal function, it suffices to prove

$$\left| \int_{\mathbb{A}_m} \left(T_{\lambda,a}^{t(x),N(x)} f \right)(x)\, g(x) d\mu(x) \right| \lesssim m^{\lambda+(\alpha-1+2s)/2}\sqrt{c_\alpha(\mu)}\|f\|_{H_\lambda^s}\|g\|_{L^\infty(d\mu)} \tag{69}$$

uniformly in the measurable functions $t = t(x)$ and $N = N(x)$, whenever $\mu \in \mathcal{M}^\alpha(\mathbb{A}_m)$, $f \in H_\lambda^s(\mathbb{R})$, and $g \in L^\infty(d\mu)$.

From (57), Fubini's theorem and the Cauchy-Schwarz inequality give us

$$\left| \int_{\mathbb{A}_m} \left(T_{\lambda,a}^{t(x),N(x)} f \right)(x)\, g(x) d\mu(x) \right| \leq \Theta \|f\|_{H_\lambda^s}, \tag{70}$$

where

$$\Theta^2 = c_\lambda \int_{\mathbb{R}} \left| \int_{\mathbb{A}_M} \psi\left(\frac{|\rho|}{N(x)}\right) e^{it(x)|\rho|^a} E_\lambda(ix\rho) g(x) d\mu(x) \right|^2 |\rho|^{2\lambda-2s} d\rho.$$

Again by Fubini's theorem,

$$\Theta^2 \leq c_\lambda \|g\|_{L^\infty(d\mu)}^2 \int_{\mathbb{A}_m}\int_{\mathbb{A}_m} \Omega(x,y) d\mu(x) d\mu(y),$$

where

$$\Omega(x,y) = \left| \int_{\mathbb{R}} \psi\left(\frac{|\rho|}{N(x)}\right)\psi\left(\frac{|\rho|}{N(y)}\right) e^{i(t(x)-t(y))|\rho|^a} E_\lambda(ix\rho) E_\lambda(-iy\rho)|\rho|^{2\lambda-2s} d\rho \right|.$$

But by Lemma 3.1, $\Omega(x,y) \lesssim |xy|^{-\lambda}|x-y|^{2s-1}$, and hence, in a way similar to (66) and (67),

$$\begin{aligned} \Theta^2 &\lesssim \|g\|_{L^\infty(d\mu)}^2 \int_{\mathbb{A}_m}\int_{\mathbb{A}_m} \frac{|xy|^{-\lambda}}{|x-y|^{1-2s}} d\mu(x) d\mu(y) \\ &\lesssim \|g\|_{L^\infty(d\mu)}^2 m^{2\lambda+\alpha-1+2s} c_\alpha(\mu). \end{aligned}$$

Substituting this into (70) proves (69) □

Theorem 5.3 *Let $a > 1$ and $\lambda \geq 0$. Then for $1/4 \leq s \leq 1/2$, we have*

$$\alpha_\lambda(s) = 1 - 2s.$$

Proof We first assume that $1/4 \leq s < 1/2$. To show $\alpha_\lambda(s) \leq 1 - 2s$, it suffices to prove that, for $1 - 2s < \alpha \leq 1$ and each $m \geq 1$,

$$\mathcal{H}^\alpha\left(\left\{x \in \mathbb{A}_m : T^{t_k}_{\lambda,a} f(x) \nrightarrow f(x) \text{ as } k \to \infty\right\}\right) = 0, \tag{71}$$

whenever $f \in H^s_\lambda(\mathbb{R})$ and $t_k \to 0+$, where $\mathcal{H}^\alpha$ denotes the α-Hausdorff measure. Furthermore, by Frostman's lemma (cf. [16]), (71) is asserted once we prove

$$\mu\left(\left\{x : T^{t_k}_{\lambda,a} f(x) \nrightarrow f(x) \text{ as } k \to \infty\right\}\right) = 0, \tag{72}$$

for $f \in H^s_\lambda(\mathbb{R})$, $\mu \in \mathcal{M}^\alpha(\mathbb{A}_m)$ with $1 - 2s < \alpha \leq 1$, and $t_k \to 0+$.

It is first noted that, if $f \in \mathscr{S}(\mathbb{R})$, $T^{t_k}_{\lambda,a} f(x)$ tends to $f(x)$ for every $x \in \mathbb{R}$. For general $f \in H^s_\lambda(\mathbb{R})$ and a given $\epsilon > 0$, let $f_0 \in \mathscr{S}(\mathbb{R})$ satisfy $\|f - f_0\|_{H^s_\lambda} < \epsilon$. Since

$$|T^{t,N}_{\lambda,a} f - f| \leq |T^{t,N}_{\lambda,a}(f - f_0)| + |T^{t,N}_{\lambda,a} f_0 - f_0| + |f_0 - f|,$$

it follows that, for $\delta > 0$,

$$\begin{aligned}
&\mu\left(\left\{x : \limsup_{k\to\infty} \limsup_{N\to\infty} |T^{t_k,N}_{\lambda,a} f - f| > \delta\right\}\right) \\
&\quad\leq \mu\left(\left\{x : \sup_{k\geq 1} \sup_{N\geq 1} |T^{t_k,N}_{\lambda,a}(f - f_0)| > \frac{\delta}{3}\right\}\right) + \mu\left(\left\{x : |f_0 - f| > \frac{\delta}{3}\right\}\right).
\end{aligned}$$

Thus by Lemmas 5.1 and 5.2,

$$\mu\left(\left\{x : \limsup_{k\to\infty} \limsup_{N\to\infty} |T^{t_k,N}_{\lambda,a} f - f| > \delta\right\}\right) \lesssim \frac{1}{\delta}\|f - f_0\|_{H^s_\lambda} < \frac{\epsilon}{\delta},$$

and then, (72) is concluded by letting $\epsilon \to 0+$ and $\delta \to 0+$ successively.

The reverse inequality $\alpha_\lambda(s) \geq 1 - 2s$ is a consequence of the existence of functions like $f = g *_\lambda G_s$ with some $g \in L^2_\lambda(\mathbb{R})$, which are singular on sets of dimension α when $0 < \alpha < 1 - 2s$. Indeed. as in [2], we may take $E = E_0 \bigcap [1/2, 3/2]$, where E_0 is the generalized Cantor set with $\dim_H E_0 = \alpha$, and define, for $s < \gamma < (1 - \alpha)/2$,

$$g_0(x) = \chi_{(0,2)}(x) d(x, E)^{-\gamma},$$

where $d(x, E)$ denotes the distance from x to E. We then have $\int_{\mathbb{R}} |g_0(x)|^2 |x|^{2\lambda} dx \leq 2^{2\lambda} \int_0^2 d(x, E)^{-2\gamma} dx < \infty$ by [17, Lemma 3.6] (cf. [30, Lemma 1] also). It remains to show that $g_0 *_\lambda G_s$ is singular on E.

If $|x| \leq 4$, then, from (59), $G_s(x) \gtrsim \int_0^1 e^{-x^2/(4\delta)} \delta^{\frac{s}{2}-\lambda-\frac{3}{2}} d\delta \gtrsim |x|^{s-1-2\lambda}$, and so for $x, y \in (0,2)$, we have, by (18),

$$\begin{aligned}(\tau_x G_s)(-y) &\gtrsim \int_0^\pi (x^2+y^2-2xy\cos\theta)^{\frac{s}{2}-\lambda-\frac{1}{2}}(1+\cos\theta)\sin^{2\lambda-1}\theta d\theta \\ &\gtrsim \int_0^{\pi/2} \left(|x-y|+\sqrt{xy}\theta\right)^{s-2\lambda-1}\theta^{2\lambda-1}d\theta \\ &\gtrsim (x+y)^{-2\lambda}|x-y|^{s-1}.\end{aligned}$$

Now for $x \in (0,2)$, let $x_0 \in E$ be such that $|x-x_0| = d(x,E)$. We have

$$\begin{aligned}(g_0 *_\lambda G_s)(x) &\gtrsim \int_0^2 (x+y)^{-2\lambda}|x-y|^{s-1}d(y,E)^{-\gamma}|y|^{2\lambda}dy \\ &\gtrsim \int_{|y-x_0|\leq|x-x_0|/8} |x-y|^{s-1}|y-x_0|^{-\gamma}dy \\ &\gtrsim |x-x_0|^{s-\gamma},\end{aligned}$$

that means, $(g_0 *_\lambda G_s)(x) \gtrsim 1/d(x,E)^{\gamma-s}$ and $g_0 *_\lambda G_s$ is singular on E.

If $s = 1/2$ and $f \in H_\lambda^{1/2}(\mathbb{R})$, we choose, for $0 < \alpha \leq 1$, $s' \in [1/4, 1/2)$ such that $1-2s' < \alpha \leq 1$. Then, since $f \in H_\lambda^{1/2}(\mathbb{R}) \subseteq H_\lambda^{s'}(\mathbb{R})$, (72) holds for all $\mu \in \mathcal{M}^\alpha(\mathbb{A}_m)$ and $t_k \to 0+$. Therefore, for $0 < \alpha \leq 1$, (71) is true for all $f \in H_\lambda^{1/2}(\mathbb{R})$ and $t_k \to 0+$, which shows that $\alpha_\lambda(1/2) = 0$. □

6 Closing Comments

6.1. Here we give a short description on the free Schrödinger equation associated to the Dunkl operators on $\mathbb{R}^d \times \mathbb{R}$ for $d \geq 2$, which is left for further work.

For given $\lambda_k \geq 0$, $k = 1, \ldots, d$, we put

$$\lambda = (\lambda_1, \ldots, \lambda_d)$$

as a multiplicity vector. For a differentiable function f on $\mathbb{R}^d$, the Dunkl operators are defined by

$$\mathcal{D}_k f(x) := \frac{\partial}{\partial x_k} f(x) + \frac{\lambda_k}{x_k}(f(x) - f(x\sigma_k)), \quad k = 1, \ldots, d,$$

where $x\sigma_k = (x_1, \ldots, -x_k \ldots, x_d)$. The associated Laplacian is $\Delta_\lambda = \sum_{k=1}^d \mathcal{D}_k^2$, or explicitly, for a twice differentiable function f,

$$(\Delta_\lambda f)(x) = \Delta f(x) + \sum_{k=1}^d \frac{2\lambda_k}{x_k}\frac{\partial}{\partial x_k} f(x) - \sum_{k=1}^d \frac{\lambda_k}{x_k^2}(f(x) - f(x\sigma_k)).$$

For each k, the adjoint of the Dunkl operator $\mathcal{D}_k$ is $-\mathcal{D}_k$, as a densely defined operator from the L^2 space on $\mathbb{R}^d$ to itself, which is associated with the measure $w_\lambda(x)dx$, where

$$w_\lambda(x) = \prod_{k=1}^{d} |x_k|^{2\lambda_k}.$$

In general, for $1 \le p < \infty$, we denote by $L^p_\lambda(\mathbb{R}^d)$ the space of measurable functions f on $\mathbb{R}^d$ satisfying $\|f\|_{L^p_\lambda(\mathbb{R}^d)} := \left\{c_\lambda \int_{\mathbb{R}^d} |f(x)|^p w_\lambda(x)dx\right\}^{1/p} < \infty$, where $c_\lambda = \prod_{k=1}^{d} c_{\lambda_k}$ with $c_\gamma^{-1} = 2^{\gamma+1/2}\Gamma(\gamma+1/2)$.

The Schrödinger equation associated to the Dunkl operators is $i\partial_t u(x,t) = \Delta_\lambda u(x,t)$, $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, and we consider its initial value problem with an initial data f,

$$\begin{cases} i\partial_t u(x,t) = \Delta_\lambda u(x,t), & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ \quad u(x,0) = f(x), & \quad x \in \mathbb{R}^d. \end{cases} \tag{73}$$

If $f \in \mathscr{S}(\mathbb{R}^d)$, the solution u to the problem (73) would have an explicit representation in terms of the multiple Dunkl transform. From the details, we define the Dunkl transform of a function $f \in L^1_\lambda(\mathbb{R}^d)$ by

$$(\mathscr{F}_\lambda f)(\xi) := c_\lambda \int_{\mathbb{R}^d} f(x) E_\lambda(-ix \circ \xi) w_\lambda(x)dx, \qquad \xi \in \mathbb{R}^d,$$

where $x \circ \xi = (x_1\xi_1, \ldots, x_d\xi_d)$, E_λ is the Dunkl kernel given by

$$E_\lambda(ix \circ \xi) = \prod_{k=1}^{d} E_{\lambda_k}(ix_k\xi_k),$$

with $E_{\lambda_k}(z)$ is the one-dimensional Dunkl kernel given in (2). It is noted that $E_0(-ix \circ \xi) = e^{-i\langle x,\xi\rangle}$.

For $f \in \mathscr{S}(\mathbb{R}^d)$, the solution u to (73) is given by

$$u(x,t) = c_\lambda \int_{\mathbb{R}^d} (\mathscr{F}_\lambda f)(\xi) e^{it|\xi|^2} E_\lambda(ix \circ \xi) w_\lambda(\xi)d\xi, \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}.$$

6.2. The general setting of the Dunkl theory is on the study of multivariable analytic structures associated with finite reflection groups, of which the basic tools are the Dunkl transform and the Dunkl operators invariant under a given group. During the last decades, it has gained considerable interest in various fields of mathematics and also in physical applications (cf. [11]); for example, the Dunkl operators for the symmetric group S_d on $\mathbb{R}^d$ are naturally connected with the analysis of quantum many body systems of Calogero–Moser–Sutherland type, which describe algebraically integrable systems in one dimension.

The free Schrödinger equation studied in the present paper and its higher dimensional counterparts described above are associated to the abelian groups $\mathbb{Z}_2$ on $\mathbb{R}$ and $\mathbb{Z}_2^d$ on $\mathbb{R}^d$ for $d \geq 2$ respectively.

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References

1. Barceló, J.A., Bennett, J., Carbery, A., Rogers, K.M.: On the dimension of divergence sets of dispersive equations. Math. Ann. **349**, 599–622 (2011)
2. Bennett, J., Rogers, K.M.: On the size of divergence sets for the Schrödinger equation with radial data. Indiana Univ. Math. J. **61**, 1–13 (2012)
3. Bourgain, J.: A remark on Schrödinger operators. Isr. J. Math. **77**, 1–16 (1992)
4. Bourgain, J.: Some new estimates on oscillatory integrals. In: Essays on Fourier Analysis in Honor of Elias M. Stein: Proc. Conf. (Princeton, NJ, 1991). Princeton Math. Ser., vol. 42, pp. 83–112. Princeton Univ. Press, Princeton, NJ (1995)
5. Bourgain, J.: On the Schrödinger maximal function in higher dimension. Proc. Steklov Inst. Math. **280**, 46–60 (2013)
6. Bourgain, J.: A note on the Schrödinger maximal function. J. Anal. Math. **130**, 393–396 (2016)
7. Carleson, L.: Some analytic problems related to statistical mechanics. In: Euclidean Harmonic Analysis: Proc. Semin. (Univ. Maryland, 1979). Lect. Notes Math., vol. 779, pp. 5–45. Springer, Berlin (1980)
8. Dahlberg, B.E.J., Kenig, C.E.: A note on the almost everywhere behavior of solutions to the Schrödinger equation. In: Harmonic Analysis: Proc. Conf. (Minneapolis, 1981). Lect. Notes Math. vol. 908, pp. 205–209. Springer, Berlin (1982)
9. Demeter, C., Guo, S.: Schrödinger maximal function estimates via the pseudoconformal transformation. arXiv:1608.07640v1
10. Du, X., Guth, L., Li, X.: A sharp Schrödinger maximal estimate in $\mathbb{R}^2$. arXiv:1612.08946v2
11. Dunkl, C.F.: Reflection groups in analysis and applications. Jpn. J. Math. **3**(2), 215–246 (2008)
12. Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G.: Higher Transcendental Functions, Vols. I and II. McGraw-Hill, New York (1953)
13. Lee, S.: On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$. Int. Math. Res. Not. (2006). arXiv:0711.0717v3
14. Li, Zh.-K., Liao, J.-Q.: Harmonic analysis associated with the one-dimensional Dunkl transform. Constr. Approx. **37**, 233–281 (2013)
15. Lucà, R., Rogers, K.M.: An improved necessary condition for the Schrödinger maximal estimate. arXiv:1506.05325v1
16. Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability. Cambridge Univ. Press, Cambridge (1995)
17. Mou, L.: Removability of singular sets of harmonic maps. Arch. Ration. Mech. Anal. **127**, 199–217 (1994)
18. Moyua, A., Vargas, A., Vega, L.: Schrödinger maximal function and restriction properties of the Fourier transform. Int. Math. Res. Not. **16**, 793–815 (1996)
19. Prestini, E.: Radial functions and regularity of solutions to the Schrödinger equation. Monatsh. Math. **109**, 135–143 (1990)
20. Rösler, M.: Bessel-type signed hypergroups on $\mathbb{R}$. In: Heyer, H., Mukherjea, A. (eds.) Probability Measures on Groups and Related Structures XI, pp. 292–304. World Scientific, Singapore (1995)
21. Sjögren, P., Sjölin, P.: Convergence properties for the time dependent Schrödinger equation. Ann. Aca. Sci. Fen. **14**, 13–25 (1989)
22. Sjögren, P., Torrea, J.L.: On the boundary convergence of solutions to the Hermite-Schrödinger equation. Colloq. Math. **118**, 161–174 (2010)
23. Sjölin, P.: Regularity of solutions to the Schrödinger equation. Duke Math. J. **55**(3), 699–715 (1987)
24. Sjölin, P.: Radial functions and maximal estimates for solutions to the Schrödinger equation. J. Aust. Math. Soc. Ser. A **59**, 134–142 (1995)

25. Sjölin, P.: Radial functions and maximal operators of Schrödinger type. Indiana Univ. Math. J. **60**, 143–159 (2011)
26. Tao, T., Vargas, A.: A bilinear approach to cone multipliers. I: restriction estimates. Geom. Funct. Anal. **10**(1), 185–215 (2000)
27. Tao, T., Vargas, A.: A bilinear approach to cone multipliers. II: applications. Geom. Funct. Anal. **10**(1), 216–258 (2000)
28. Vega, L.: Schrödinger equations: pointwise convergence to the initial data. Proc. Am. Math. Soc. **102**(4), 874–878 (1988)
29. Watson, G.N.: A Treatise on the Theory of Bessel Functions, 2nd edn. Cambridge Univ. Press, Cambridge (1952)
30. Žubrinić, D.: Singular sets of Sobolev functions. C. R. Math. Acad. Sci. Paris **334**, 539–544 (2002)